Logic and Logical Philosophy
Volume 9 (2001), 91–107

S. P. Odintsov

LOGIC OF CLASSICAL REFUTABILITY AND CLASS OF EXTENSIONS OF MINIMAL LOGIC*

Introduction

This article continues the investigation of paraconsistent extensions of minimal logic **Lj** started in [6, 7]. The name "logic of classical refutability" is taken from the H.Curry monograph [1], where it denotes the logic **Le** obtained from **Lj** by adding the Peirce law $((p \supset q) \supset p) \supset p$. In [1, Ch. 6, Sec. A] one can find the discussion of name "logic of classical refutability". Due to Curry, the logic **Le** was introduced by Kripke in his unpublished work [5], who investigated to which extent one can strengthen minimal logic so that the resulting system does not contain intuitionistic logic **Li**. The reference to this work can be found in [1]. In [11], K. Segerberg studied the Kripke-style semantics for numerous extensions of minimal logic, and among them for the system $\mathbf{Lj} + \{p \vee (p \supset q)\}$ equivalent to **Le** [11, p. 46]. It was noted in [11] that **Le** was treated, with different motivation, in 1955 by S. Kanger [3], whose paper also contains a reference to an earlier discussion of Gentzen-style deductive system equivalent to **Le** by P. Bernays. Kanger's reason for defining **Le** is that "it constitutes a weakened classical calculus in the same sense as the minimal calculus is a weakened intuitionistic calculus".

* This work is supported by RFFR grant No. 96-01-01552.

In Section 2 of the present article, we consider the Grzegorczyk-style semantics for **Le** which vividly demonstrates that minimal logic **Lj** relates to logic of classical refutability **Le** essentially in the same way as intuitionistic logic **Li** and classical logic **Lk** are related.

The special role, which **Le** plays in the class of **Lj**-extensions, is determined by the following fact. In [6], we stated that the class of all non trivial extensions of **Lj** partition into three disjoint subclasses. These are the class of intermediate logics, the class of negative logics containing formula $\neg p$, and the class of proper paraconsistent extensions of **Lj** consisting of logics not belonging to the first two classes. The negative logics have degenerate negation in a sense that any negated formula is provable. Thus, the third class includes all non-trivial cases of paraconsistent negations. The greatest logic of this class is **Le**. The above decomposition of the class of **Lj**-extensions motivates an effort to describe the class of proper paraconsistent **Lj**-extensions in terms of classes of intermediate and negative logics, which was extensively studied. Note that the class of negative logics is definitionally equivalent to the class of positive logics.

In [7], isomorphs of classical logic and maximal negative logic $\mathbf{Lmn} = \mathbf{Lj} + \{\neg p, ((p \supset q) \supset p) \supset p\}$ were studied, which was inspired by A. Karpenko article [4] considering isomorphs of classical logic in three-valued Bochvar's logic $\mathbf{B_3}$. It turns out that isomorphs of **Lmn** and **Lk** in **Le** define translations of these logics into **Le**. And these translations can be generalized to translations of arbitrary negative and intermediate logics into proper paraconsistent extensions of the logic $\mathbf{Le'} = \mathbf{Lj} + \{\bot \vee (\bot \supset p)\}$. Note that $\mathbf{Le'}$ is axiomatized by the partial case of the generalized law of excluded middle $p \vee (p \supset q)$, which axiomatizes **Le** modulo **Lj**. In fact, it was proved in [7] that there exists an effective one-to-one correspondence between the interval $[\mathbf{Le'}, \mathbf{Le}]$ and the direct product of classes of intermediate and negative logics. Taking into account that [7] was published only in Russian and for the article to be self-contained we include some results of [7] in sections 3 and 4.

However, the interval $[\mathbf{Le'}, \mathbf{Le}]$ constitute a relatively small part of the class of proper paraconsistent **Lj**-extensions. It does not contain many interesting paraconsistent systems. One of them, so-called Glivenko's logic, the weakest logic in which $\neg\neg\varphi$ is provable whenever $\mathbf{Lk} \vdash \varphi$, is treated in Section 5.

Finally, in Section 6 we define for every proper paraconsistent extension $\boldsymbol{L}$ of **Lj** its intuitionistic and negative counterparts and their translations into $\boldsymbol{L}$. We consider classes of logic with fixed intuitionistic and negative counterparts. Each of such classes will be an interval in the lattice of **Lj**-

extensions. These intervals are pairwise disjoint, and their upper points constitute the interval [$\mathbf{Le}'$, $\mathbf{Le}$]. In this way, we reduce the study of the lattice of **Lj**-extensions to the study of intervals consisting of logics with fixed intuitionistic and negative counterparts.

1. Preliminaries

We only consider logics in the propositional language $\{\vee, \wedge, \supset, \bot\}$ with the constant "contradiction" $\bot$. The negation $\neg$ is assumed to be a definable symbol, $\neg\varphi = \varphi \supset \bot$. By a logic we mean a set of formulas closed under *modus ponens* and the substitution rule. Except for the logics mentioned in the introduction we need also positive logic **Lp**, negative logic $\mathbf{Ln} = \mathbf{Lj} + \{\neg p\}$, and trivial logic $\mathcal{F}$ consisting of all formulas.

Below we collect some facts and definitions concerning algebraic semantics for different logics. If it is necessary, further details can be found in [9, 10].

Let $\boldsymbol{A}$ be an algebra of the language $\{\vee, \wedge, \supset, \bot, 1\}$. A mapping $V\colon \mathsf{Prop} \to A$ from the set of propositional variables into the universe of $\boldsymbol{A}$ is called an $\boldsymbol{A}$*-valuation*. We say that a formula φ is *true in* $\boldsymbol{A}$, or is an *identity of* $\boldsymbol{A}$, and write $\boldsymbol{A} \vDash \varphi$ if $V(\varphi) = 1$ for every $\boldsymbol{A}$-valuation V. The set of formulas $\mathsf{L}\,\boldsymbol{A} = \{\varphi \mid \boldsymbol{A} \vDash \varphi\}$ forms a logic, the *logic of* $\boldsymbol{A}$. The *logic of* a class of algebras is an intersection of logics of algebras from the class.

Given a logic $\boldsymbol{L}$, an algebra $\boldsymbol{A}$ is a (*characteristic*) *model for* $\boldsymbol{L}$ if $\boldsymbol{L} \subseteq \mathsf{L}\,\boldsymbol{A}$ ($\boldsymbol{L} = \mathsf{L}\,\boldsymbol{A}$). If $\boldsymbol{L} \subseteq \mathsf{L}\,\boldsymbol{A}$, we also write $\boldsymbol{A} \vDash \boldsymbol{L}$. Every logic has a characteristic model [10, Ch. VIII].

An *implicative lattice* $\boldsymbol{A} = \langle A, \vee, \wedge, \supset, 1\rangle$ is a lattice with the greatest element 1 and the pseudo-complement (or implication) operation $\supset$. All implicative lattices form a variety whose logic is exactly **Lp** [9, Theorem XI.2.2].

We call a j-*algebra* an implicative lattice treated in the language $\langle \vee, \wedge, \supset, \bot, 1\rangle$, where $\bot$ is interpreted as an arbitrary element of the lattice. A *Heyting algebra* is a j-algebra with the least element $\bot$, and a *negative algebra* is a j-algebra with the greatest element $\bot$, i.e., $1 = \bot$. The logics of the varieties of j-algebras, Heyting algebras, and negative algebras are **Lj**, **Li**, and **Ln**, respectively.

An implicative lattice satisfying the identity $((p \supset q) \supset p) \supset p$ is called a *Peirce algebra*, and a j-algebra satisfying the same identity is called a *Peirce-Johansson algebra* (pj-*algebra*). The variety of Peirce algebras defines the positive fragment of classical logic, and the variety of pj-algebras — the logic **Le** [8].

We consider a two-element Heyting algebra $\mathbf{2} = \langle \{0, 1\}, \vee, \wedge, \supset 0, 1\rangle$ and a two-element negative algebra $\mathbf{2'} = \langle \{0, 1\}, \vee, \wedge, \supset, 1, 1\rangle$. It is well-known that $\mathbf{L}\,\mathbf{2} = \mathbf{Lk}$ and $\mathbf{L}\,\mathbf{2'} = \mathbf{Lmn}$.

Now we take a four-element pj-algebra $\mathbf{4'} = \langle \{0, 1, a, -1\}, \vee, \wedge, \supset, 0, 1\rangle$, where $-1 \leq 0 \leq 1$, $-1 \leq a \leq 1$, $\perp = 0$, and the elements 0 and a are incomparable. That algebra exemplifies a characteristic model for **Le** [8]. We note also that $\mathbf{4'} \cong \mathbf{2} \times \mathbf{2'}$ and $\mathbf{Le} = \mathbf{Lk} \cap \mathbf{Lmn}$.

Let **Jhn** denote the class of all non-trivial extensions of minimal logic **Lj**, **Int** the class of all intermediate logics, **Neg** the class of all negative logics, and $\mathbf{Par} \rightleftharpoons \mathbf{Jhn} \setminus (\mathbf{Int} \cup \mathbf{Neg})$ the class of all proper paraconsistent extensions of **Lj**. For each $\boldsymbol{L} \in \mathbf{Jhn}$ we have the following equivalences:

$$\boldsymbol{L} \in \mathbf{Int} \iff \mathbf{Li} \subseteq \boldsymbol{L} \subseteq \mathbf{Lk}\,,$$
$$\boldsymbol{L} \in \mathbf{Neg} \iff \mathbf{Ln} \subseteq \boldsymbol{L} \subseteq \mathbf{Lmn}\,,$$
$$\boldsymbol{L} \in \mathbf{Par} \iff \mathbf{Lj} \subseteq \boldsymbol{L} \subseteq \mathbf{Le}\,.$$

Thus, **Jhn** splits into a union of three disjoint intervals.

In conclusion of this section, we note that $\mathbf{Le'} = \mathbf{Ln} \cap \mathbf{Li} \neq \mathbf{Lj}$. Indeed, $\mathbf{Le'}$ is axiomatized relative to **Lj** by a disjunction of formulas $\perp$ and $\perp \supset p$ axiomatizing **Ln** and **Li** relative to **Lj**, therefore, $\mathbf{Le'}$ equals to an intersection of intuitionistic and negative logics. At the same time, **Lj** has the disjunctive property and does not contain these formulas, which immediately implies the above inequality.

2. Grzegorczyk-style semantics for minimal logic

Grzegorczyk [2] suggested a semantics for intuitionistic logic which demonstrates that classical and intuitionistic logics are related as the logic of ontological thought and the logic of scientific research (positivistically conceived). A scientific research was formally presented as a triple

$$R = \langle J, \circ, P\rangle,$$

where J denotes an information set, i.e. the set of all possible experimental data, $\circ$ is an initial information, and P a function of possible prolongations of the information. In propositional case, elements of J are finite tuples of propositional variables. Define $\alpha \prec^* \beta \rightleftharpoons \beta \in P(\alpha)$. Passing to the transitive closure of this relation yields a partial ordering $\prec$ on the information set J, which allows in turn to define in a natural way an intuitionistic forcing relation $\alpha \rhd_R \varphi$ between informations and formulas. The following is a propositional version of Theorem 1 [2].

Proposition 2.1 ([2]). *A formula φ is an intuitionistic tautology if and only if φ is forced by each information α of every research R.*

By a j-research we mean a triple $R = \langle J, \circ, P\rangle$ defined as above except for the following. Elements of J are tuples of atomic formulas, i.e. the constant $\bot$ may occur in elements of J. For the given j-research R we define a relation $\prec$ as above and a minimal forcing relation $\alpha \rhd_R \varphi$ as follows.

If $\varphi = \bot$ or φ is a propositional variable, then

$$\alpha \rhd_R \varphi \iff \varphi \in \alpha\,.$$

Further,

$$\begin{aligned} \alpha \rhd_R \varphi \wedge \psi &\iff \alpha \rhd_R \varphi \wedge \alpha \rhd_R \psi\,,\\ \alpha \rhd_R \varphi \vee \psi &\iff \alpha \rhd_R \varphi \vee \alpha \rhd_R \psi\,,\\ \alpha \rhd_R \varphi \supset \psi &\iff \forall\beta(\alpha \prec \beta \Rightarrow (\beta \rhd_R \varphi \Rightarrow \alpha \rhd_R \psi)\,. \end{aligned}$$

In particular,

$$\alpha \rhd_R \neg\varphi \iff \forall\beta(\alpha \prec \beta \Rightarrow (\beta \rhd_R \varphi \Rightarrow \alpha \rhd_R \bot).$$

This forcing relation differs from the intuitionistic one only in cases of $\bot$ and $\neg$. We have the following modification of the theorem cited above.

Proposition 2.2. *A formula φ is a minimal tautology if and only if φ is forced by each information α of every j-research R.*

Proof. The proof can be easily reduced to the characterization of minimal logic in terms of Kripke semantics [9, 11]. We observe that any j-research can be transformed in a natural way to a Kripke model $\langle W, \leq, S, v\rangle$ for **Lj** satisfying the following conditions.

1. For any $\alpha \in W$, α has only finitely many successors relative to $\leq$.
2. For any $\alpha \in W$, the set $\{\, p_i \mid p_i \in v(\alpha) \,\}$ is finite.

The converse statement is also valid. Any Kripke model satisfying 1 and 2 is transformed to a j-research.

The correctness of the presented semantics for **Lj** can be easily verified. On the other hand, as is known any φ, $\mathbf{Lj} \nvdash \varphi$, is refuted on a finite Kripke model. Any Kripke model satisfies condition 1. Restricting v to variables occurring φ we also meet condition 2. In this way, we obtain a j-research which does not force φ. □

Now, we recall that $\mathbf{Le} = \mathbf{Lk} \cap \mathbf{Lmn}$. This equality leads to the valuation semantics for **Le** defined as follows. An *e-valuation* is a map $V\colon \{p_0, p_1, \ldots\} \cup$

$\{\bot\} \to \{0,1\}$. Every e-valuation extends in a natural way to the set of all formulas. For any formula φ we have the equivalence:

$$\mathbf{Le} \vdash \varphi \quad \text{if and only if} \quad V(\varphi) = 1 \text{ for every } e\text{-valuation } V.$$

Comparing the semantics for **Lj** and for **Le** defined above with the Grzegorczyk semantics for **Li** and the ordinary valuation semantics for **Lk** we can see that **Lj** is related to **Le** exactly in the same way as **Li** and **Lk** are related. Indeed, classical logic can be defined as the logic characterized by the class of intuitionistic researches R with trivial prolongation function, i.e., $P(\alpha) = \{\alpha\}$ for all $\alpha \in J$. In the same way, **Le** is characterized by the class of j-researches with trivial prolongation function. Therefore, we may consider minimal logic as a positivistic approximation of logic of classical refutability and suppose that classical logic and **Le** have common ontological presuppositions except for the indefinite status of the contradiction $\bot$ in **Le**. More precisely, we may consider statements of **Le** as well as statements of **Lk** as judgements which agree or do not agree the reality. To compute the truth value for the statement of classical logic we need only know the truth values of propositional variables occurring the statement. It is not enough for **Le**-statements, we must also know the status of contradiction $\bot$. In this way, we explain the sense of truth values of **Le** as follows.

The set of truth-values for **Le** contains four elements $\{1, 0, -1, a\}$ [8]. Due to the above considerations, 1 and -1 can be identified with classical "truth" and "falsehood", whereas a and 0 are, in a sense, the two opposite indefinite values. And this indefiniteness is completely determined by the status of $\bot$. In view of the equalities $a \supset 0 = 0$ and $0 \supset a = a$, if a situation is consistent $(0 = -1)$, statements with the value a become "true" and with the value 0 "false", and vice versa in an inconsistent situation $(0 = 1)$.

3. Paraconsistent extensions of Le′

In this section, we state that proper paraconsistent extensions of **Le′** are exactly intersections of two logics, one of which is intermediate and the other is negative. Prior to do it we study the algebraic semantics for logics extending **Le′**.

For an implicative lattice $\boldsymbol{A} = \langle A, \vee, \wedge, \supset, 1\rangle$ and $a \in A$, we put $A^a \rightleftharpoons \{b \in A \mid b \geq a\}$ and $A_a \rightleftharpoons \{b \in A \mid b \leq a\}$. The set A^a is obviously closed under the operations of $\boldsymbol{A}$, and we can define an implicative sublattice $\boldsymbol{A}^a$ of $\boldsymbol{A}$ with the universe A^a. Except for the case $a = 1$, the set A_a defines a sublattice, but not an implicative sublattice of $\boldsymbol{A}$, because A_a is not closed under the implication. However, the operation $x \supset_a y \rightleftharpoons (x \supset y) \wedge a$ turns A_a into an implicative lattice with unit element a. Denote this implicative lattice by $\boldsymbol{A}_a$.

If $\boldsymbol{A}$ is a j-algebra and $a = \bot$, $\boldsymbol{A}^a$ can be treated as a Heyting algebra and $\boldsymbol{A}_a$ as a negative one.

We recall one well-known fact of the lattice theory. Let $\boldsymbol{A}$ be a distributive lattice, a an arbitrary element of $\boldsymbol{A}$, and let sublattices $\boldsymbol{A}^a$ and $\boldsymbol{A}_a$ be defined as above. The mappings $\varepsilon(x) \rightleftharpoons x \vee a$ and $\tau(x) \rightleftharpoons x \wedge a$ are epimorphisms of $\boldsymbol{A}$ onto $\boldsymbol{A}^a$ and $\boldsymbol{A}_a$, respectively. And the mapping $\lambda(x) \rightleftharpoons (x \vee a, x \wedge a)$ gives an embedding of $\boldsymbol{A}$ into a direct product of lattices $\boldsymbol{A}^a$ and $\boldsymbol{A}_a$.

These facts does not generally hold for implicative lattices. As before, the mapping τ is an epimorphism of implicative lattices. But $\varepsilon\colon \boldsymbol{A} \to \boldsymbol{A}^a$ and $\lambda\colon \boldsymbol{A} \to \boldsymbol{A}^a \times \boldsymbol{A}_a$ are an epimorphism and an embedding of implicative lattices only if some additional condition is imposed on $\boldsymbol{A}$. More precisely, the following assertions hold.

PROPOSITION 3.1. *For an implicative lattice $\boldsymbol{A}$ and $a \in \boldsymbol{A}$, the mapping $\tau\colon \boldsymbol{A} \to \boldsymbol{A}_a$, $\tau(x) = x \wedge a$, is an epimorphism of implicative lattices.*

PROOF. The proof immediately follows from the definition of implication in $\boldsymbol{A}_a$ and the identity $(x \supset y) \wedge z = ((x \wedge z) \supset (y \wedge z)) \wedge z$ satisfied in all implicative lattices. □

PROPOSITION 3.2. *Let $\boldsymbol{A}$ be an implicative lattice and $a \in A$. The following three conditions are equivalent:*

(a) *For all $x, y \in \boldsymbol{A}$ we have*

$$(x \vee a) \supset (y \vee a) \leq (x \supset y) \vee a\,.$$

(b) *The map $\varepsilon\colon \boldsymbol{A} \to \boldsymbol{A}^a$ given by the rule $\varepsilon(x) = x \vee a$ is an epimorphism of implicative lattices.*

(c) *The map $\lambda\colon \boldsymbol{A} \to \boldsymbol{A}^a \times \boldsymbol{A}_a$ given by the rule $\lambda(x) = (x \vee a, x \wedge a)$ is an isomorphism of $\boldsymbol{A}$ onto a direct product of implicative lattices.*

PROOF. "(a) ⇒ (b)" We check that ε preserves the implication, i.e., that the equality $(x \supset y) \vee a = (x \vee a) \supset (y \vee a)$ holds. We have $\mathbf{Lp} \vdash ((p \supset q) \vee r) \supset ((p \vee r) \supset (q \vee r))$, whence the inequality $(x \supset y) \vee a \leq (x \vee a) \supset (y \vee a)$ is valid in any implicative lattice. The inverse inequality holds by assumption.

"(b) ⇒ (c)" It follows easily by assumption that λ is a homomorphism of $\boldsymbol{A}$ into $\boldsymbol{A}^a \times \boldsymbol{A}_a$. We prove the injectivity of λ. Take an element $b \in \boldsymbol{A}$, it is a complement of a in the interval $[b \wedge a, b \vee a]$. Assuming $\lambda(c) = \lambda(b)$ for some $c \in \boldsymbol{A}$ yields that c is a complement of a in the same interval $[b \wedge a, b \vee a]$. And we have $b = c$ since complements are unique in distributive lattices. Thus, it remains to prove that λ maps $\boldsymbol{A}$ onto $\boldsymbol{A}^a \times \boldsymbol{A}_a$.

For $x \in \boldsymbol{A}^a$ and $y \in \boldsymbol{A}_a$, we set $z = (a \supset y) \wedge x$. The direct computation shows that $z \wedge a = y$ and $z \vee a = ((a \supset y) \vee a) \wedge x$. Further, $(a \supset y) \vee a = (a \vee a) \supset (y \vee a) = a \supset a = 1$ in view of the assumption that ε is a homomorphism, whence $z \vee a = x$. We have thereby proved $\lambda(z) = (x, y)$.

"(c) $\Rightarrow$ (a)" Obviously, 3 implies 2. Therefore, the desired equality follows from the fact that ε preserves the implication. $\square$

Let us consider the following formulas:

(**P**) $$((p \supset q) \supset p) \supset p\,,$$

(**E**) $$p \vee (p \supset q)\,,$$

(**D**) $$((p \vee r) \supset (q \vee r)) \supset ((p \supset q) \vee r)\,.$$

We have
$$\mathbf{Lk}^{+} = \mathbf{Lp} + \{\mathbf{P}\} = \mathbf{Lp} + \{\mathbf{E}\} = \mathbf{Lp} + \{\mathbf{D}\},$$
where $\mathbf{Lk}^{+}$ is the positive fragment of classical logic. It is well-known that $\mathbf{Lk}^{+}$ is axiomatized relative to positive logic by the Peirce law (**P**) or by the generalized law of excluded middle (**D**). The last equality can be verified directly. We have thus obtained that (**D**) axiomatizes $\mathbf{Lk}^{+}$ modulo **Lp**. Combining this fact and Proposition 3.2 yields a characterization of Peirce algebras in terms of mappings ε and λ.

Proposition 3.3. *Let $\boldsymbol{A}$ be an implicative lattice. The following conditions are equivalent:*

(a) *$\boldsymbol{A}$ is a Peirce algebra.*

(b) *For any $a \in \boldsymbol{A}$, the mapping $\varepsilon_a(x) = x \vee a$ defines an epimorphism of $\boldsymbol{A}$ onto $\boldsymbol{A}^a$.*

(c) *For any $a \in \boldsymbol{A}$, the mapping $\lambda_a(x) = (x \vee a, x \wedge a)$ defines an isomorphism of $\boldsymbol{A}$ and $\boldsymbol{A}^a \times \boldsymbol{A}_a$.*

Now we turn to the subsystem $\mathbf{Le}'$ of **Le**, which can be axiomatized relative to **Lj** by each of the following substitutional instances of (**E**) and (**D**):

($\mathbf{E}'$) $$\bot \vee (\bot \supset p)\,,$$

($\mathbf{D}'$) $$((p \vee \bot) \supset (q \vee \bot)) \supset ((p \supset q) \vee \bot)\,.$$

The equality $\mathbf{Lj}+\{\mathbf{E'}\} = \mathbf{Lj}+\{\mathbf{D'}\}$ is easily verified. The logic $\mathbf{Le'}$ presented as $\mathbf{Lj}+\{\mathbf{E'}\}$ was treated in [11], where it was proved that $\mathbf{Le'}$ is characterized by the class of so-called closed frames. In this work, we consider an algebraic semantics for this logic. We note also the curious fact that the instance of the Peirce law

$$(\mathbf{P'}) \qquad ((p \supset \bot) \supset p) \supset p = (\neg p \supset p) \supset p\,,$$

which is known as the Clavius law, is not equivalent to the above formulas relative to $\mathbf{Lj}$. Indeed, $\mathbf{Lj} \vdash (\mathbf{P'}) \equiv (p \vee \neg p)$, and the logics $\mathbf{Lj} + (\mathbf{P'})$ and $\mathbf{Le'}$ are incomparable in the lattice of $\mathbf{Lj}$-extensions.

Proposition 3.4. *Let $\boldsymbol{A}$ be a j-algebra. $\boldsymbol{A}$ is a model for $\mathbf{Le'}$ if and only if one of the following equivalent conditions holds.*

1. *The mapping $\varepsilon(x) = x \vee \bot$ defines an epimorphism of j-algebra $\boldsymbol{A}$ onto Heyting algebra $\boldsymbol{A}^{\bot}$.*
2. *The mapping $\lambda(a) = (a \vee \bot, a \wedge \bot)$ determines an isomorphism of j-algebras $\boldsymbol{A}$ and $\boldsymbol{A}^{\bot} \times \boldsymbol{A}_{\bot}$.*
3. *For any $a, b \in \boldsymbol{A}$ with $a \leq \bot \leq b$, $\bot$ has a complement in the interval $[a, b]$.*

Proof. The inclusion $\mathbf{Le'} \subseteq \mathbf{L}\,\boldsymbol{A}$ is equivalent to the fact that $(\mathbf{D'})$ is an identity of $\boldsymbol{A}$, which is equivalent in its own right to item 1 of Proposition 3.2 for $a = \bot$. In this way, Proposition 3.2 implies that each of the conditions 1,2 characterizes models for $\mathbf{Le'}$. Proving Proposition 3.2 we stated, in fact, that 2 implies 3. Now we check the inverse implication, which completes the proof. □

Assume that 3 holds. It is not hard to prove that λ is an isomorphism of distributive lattices $\boldsymbol{A}$ and $\boldsymbol{A}^{\bot} \times \boldsymbol{A}_{\bot}$. The implication is defined in terms of the ordering preserved by λ, consequently, λ also preserves the implication.

Corollary 3.5. *Let $\boldsymbol{L}$ be an extension of $\mathbf{Lj}$. Then $\mathbf{Le'} \subseteq \boldsymbol{L} \subseteq \mathbf{Le}$ if and only if $\boldsymbol{L} = \boldsymbol{L}_1 \cap \boldsymbol{L}_2$, where $\boldsymbol{L}_1 \in \mathbf{Int}$ and $\boldsymbol{L}_2 \in \mathbf{Neg}$.*

Proof. Let $\boldsymbol{L}$ be an intersection of intermediate and negative logics $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$. Then $\mathbf{Li} \subseteq \boldsymbol{L}_1$ and $\mathbf{Ln} \subseteq \boldsymbol{L}_2$, whence $\mathbf{Le'} = \mathbf{Li} \cap \mathbf{Ln} \subseteq \boldsymbol{L}$. It is clear that the $\boldsymbol{L}$ is neither intermediate, nor negative, therefore, $\boldsymbol{L} \in \mathbf{Par}$ and $\boldsymbol{L} \subseteq \mathbf{Le}$.

Conversely, let $\mathbf{Le'} \subseteq \boldsymbol{L} \subseteq \mathbf{Le}$ and let $\boldsymbol{A}$ be a characteristic model for $\boldsymbol{L}$. By the above proposition, $\boldsymbol{A}$ is presented as a direct product of Heyting algebra $\boldsymbol{A}^{\bot}$ and negative algebra $\boldsymbol{A}_{\bot}$, hence, $\boldsymbol{L} = \mathbf{L}\,\boldsymbol{A} = \mathbf{L}\,\boldsymbol{A}^{\bot} \cap \mathbf{L}\,\boldsymbol{A}_{\bot}$. It remains to note that $\mathbf{L}\,\boldsymbol{A}^{\bot}$ is an intermediate logic and $\mathbf{L}\,\boldsymbol{A}_{\bot}$ is a negative one. □

4. Translations of intermediate and negative logics in paraconsistent extensions of $\mathbf{Le'}$

First we state one more property of models for $\mathbf{Le'}$.

Letting $\boldsymbol{A}$ be a j-algebra, $\mathbf{Le'} \subseteq \mathsf{L}\,\boldsymbol{A}$, put $C_{\boldsymbol{A}}(\bot) = \{a \in \boldsymbol{A} \mid a \vee \bot = 1\}$ and decompose $\boldsymbol{A}$ into a direct product $\boldsymbol{A}^{\bot} \times \boldsymbol{A}_{\bot}$. Then $C_{\boldsymbol{A}}(\bot) = \{(1, b) \mid b \in \boldsymbol{A}_{\bot}\}$. Indeed, for $a = (x, y) \in \boldsymbol{A}^{\bot} \times \boldsymbol{A}_{\bot}$, we have $1 = a \vee \bot \Leftrightarrow (1, 1) = (x, y) \vee (0, 1) = (x, 1) \Leftrightarrow x = 1$. It follows immediately that the set $C_{\boldsymbol{A}}(\bot)$ is closed under $\vee$, $\wedge$, and $\supset$. We will consider $C_{\boldsymbol{A}}(\bot)$ as a negative algebra with operations induced from $\boldsymbol{A}$ and $1 = \bot$.

Proposition 4.1. *Let a j-algebra $\boldsymbol{A}$ be a model for $\mathbf{Le'}$. Then the mapping $\delta(x) = \bot \supset x$ defines an epimorphism of j-algebra $\boldsymbol{A}$ onto negative algebra $C_{\boldsymbol{A}}(\bot)$.*

Proof. Again, we need a presentation of $\boldsymbol{A}$ as direct product of Heyting and negative algebras. For $a = (b, c) \in \boldsymbol{A}^{\bot} \times \boldsymbol{A}_{\bot}$, we have $\delta(a) = \delta(b, c) = (0, 1) \supset (b, c) = (0 \supset b, 1 \supset c) = (1, c)$. Consequently, for $* \in \{\vee, \wedge, \supset\}$ and for any $(a, b), (c, d) \in \boldsymbol{A}^{\bot} \times \boldsymbol{A}_{\bot}$,

$$\delta((a, b) * (c, d)) = \delta((a * c, b * d)) = (1, b * d) = (1, b) * (1, d) = \delta(a, b) * \delta(c, d).$$

It remains to note that $\delta(\bot) = 1$ and $\delta(a) = a$ for any $a \in C_{\mathbf{A}}(\bot)$. □

Now we are ready to define translations of intermediate and negative logics into proper paraconsistent extensions of $\mathbf{Le'}$.

Theorem 4.2. *Let $\boldsymbol{L}$ extend $\mathbf{Le'}$, $\boldsymbol{L} \in \mathbf{Par}$, and let $\boldsymbol{A}$ be a characteristic model for $\boldsymbol{L}$. Set $\boldsymbol{L}_1 = \mathsf{L}\,\boldsymbol{A}^{\bot}$ and $\boldsymbol{L}_2 = \mathsf{L}\,\boldsymbol{A}_{\bot}$. Then for arbitrary formula φ, the following equivalences hold.*

1. $\boldsymbol{L}_1 \vdash \varphi \iff \boldsymbol{L} \vdash \varphi \vee \bot$.
2. $\boldsymbol{L}_2 \vdash \varphi \iff \boldsymbol{L} \vdash \bot \supset \varphi$.

Proof. 1. Assume $\boldsymbol{L}_1 \vdash \varphi$ and for an $\boldsymbol{A}$-valuation V, compute the value $V(\varphi \vee \bot)$. By Proposition 3.2, $\varepsilon\colon \boldsymbol{A} \to \boldsymbol{A}^{\bot}$ is an epimorphism, from which we have $V(\varphi \vee \bot) = \varepsilon V(\varphi)$. Here εV denotes an $\boldsymbol{A}^{\bot}$-valuation obtained as a composition of V and ε. By definition of $\boldsymbol{L}_1$, $\varepsilon V(\varphi) = 1$. We have thus proved that $V(\varphi \vee \bot) = 1$ for any $\boldsymbol{A}$-valuation V, i.e., $\boldsymbol{L} \vdash \varphi \vee \bot$.

Conversely, let $\boldsymbol{L} \vdash \varphi \vee \bot$. Every $\boldsymbol{A}^{\bot}$-valuation V can be treated as an $\boldsymbol{A}$-valuation with the property $\varepsilon V = V$. As above, we have $V(\varphi) = \varepsilon V(\varphi) = V(\varphi \vee \bot) = 1$, which immediately implies $\boldsymbol{L}_1 \vdash \varphi$.

2. This proof is similar to the previous one with ε replaced by δ. □

As it follows from the theorem the logics $\boldsymbol{L}_1 \rightleftharpoons \mathsf{L}\,\boldsymbol{A}^{\perp}$ and $\boldsymbol{L}_2 \rightleftharpoons \mathsf{L}\,\boldsymbol{A}_{\perp}$ does not depend on the choice of characteristic model $\boldsymbol{A}$ for the logic $\boldsymbol{L}$ extending $\mathbf{Le'}$. Indeed,

$$\boldsymbol{L}_1 = \{\varphi \mid \boldsymbol{L} \vdash \varphi \vee \perp\}, \quad \boldsymbol{L}_2 = \{\varphi \mid \boldsymbol{L} \vdash \perp \supset \varphi\}.$$

It is clear that $\boldsymbol{L}_1 \in \mathbf{Int}$ and $\boldsymbol{L}_2 \in \mathbf{Neg}$. We call the logics $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ defined as above *intuitionistic* and *negative counterparts* of $\boldsymbol{L} \supseteq \mathbf{Le'}$ and denote them $\boldsymbol{L}_{\mathbf{int}}$ and $\boldsymbol{L}_{\mathbf{neg}}$, respectively. We have $\boldsymbol{L} = \boldsymbol{L}_{\mathbf{int}} \cap \boldsymbol{L}_{\mathbf{neg}}$.

Let, on the contrary, $\boldsymbol{L} = \boldsymbol{L}_1 \cap \boldsymbol{L}_2$, where $\boldsymbol{L}_1 \in \mathbf{Int}$ and $\boldsymbol{L}_2 \in \mathbf{Neg}$. For suitable Heyting algebra $\boldsymbol{B}$ and negative algebra $\boldsymbol{C}$, we have $\boldsymbol{L}_1 = \mathsf{L}\,\boldsymbol{B}$ and $\boldsymbol{L}_2 = \mathsf{L}\,\boldsymbol{C}$. The direct product $\boldsymbol{A} = \boldsymbol{B} \times \boldsymbol{C}$ is a characteristic model for $\boldsymbol{L}$ since $\mathsf{L}(\boldsymbol{B} \times \boldsymbol{C}) = \mathsf{L}\,\boldsymbol{B} \cap \mathsf{L}\,\boldsymbol{C} = \boldsymbol{L}_1 \cap \boldsymbol{L}_2 = \boldsymbol{L}$. Moreover, $\boldsymbol{B} \cong \boldsymbol{A}^{\perp}$ and $\boldsymbol{C} \cong \boldsymbol{A}_{\perp}$, consequently, $\boldsymbol{L}_1 = \boldsymbol{L}_{\mathbf{int}}$ and $\boldsymbol{L}_2 = \boldsymbol{L}_{\mathbf{neg}}$. In this way, we arrive at the following statement.

PROPOSITION 4.3. *The mapping* $\boldsymbol{L} \mapsto (\boldsymbol{L}_{\mathbf{int}}, \boldsymbol{L}_{\mathbf{neg}})$ *defines a lattice isomorphism between* $[\mathbf{Le'}, \mathbf{Le}]$ *and the direct product* $\mathbf{Int} \times \mathbf{Neg}$*, and the inverse mapping is given by the rule* $(\boldsymbol{L}_1, \boldsymbol{L}_2) \mapsto \boldsymbol{L}_1 \cap \boldsymbol{L}_2$.

PROOF. In fact, it was stated above that the mapping under consideration is a bijection. It remains to check that it preserves the lattice operations. The only non-trivial case is that of disjunction. We will denote the least upper bound of logics $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ as $\boldsymbol{L}_1 \cap^* \boldsymbol{L}_2$. Clearly, $\boldsymbol{A}$ is a model for a disjunction of logics if and only if it is a model for each of the disjunctive terms. Moreover, for any paraconsistent $\boldsymbol{L} \supseteq \mathbf{Le'}$, $\boldsymbol{A} \models \boldsymbol{L}$ if and only if $\boldsymbol{A}^{\perp} \models \boldsymbol{L}_{\mathbf{int}}$ and $\boldsymbol{A}_{\perp} \models \boldsymbol{L}_{\mathbf{neg}}$.

Let $\boldsymbol{L}^1, \boldsymbol{L}^2 \in [\mathbf{Le'}, \mathbf{Le}]$. By the above, $\boldsymbol{A} \models \boldsymbol{L}^1 \cup^* \boldsymbol{L}^2$ is equivalent to the conjunction

$$\boldsymbol{A}^{\perp} \models \boldsymbol{L}^1_{\mathbf{int}} \wedge \boldsymbol{A}_{\perp} \models \boldsymbol{L}^1_{\mathbf{neg}} \wedge \boldsymbol{A}^{\perp} \models \boldsymbol{L}^2_{\mathbf{int}} \wedge \boldsymbol{A}_{\perp} \models \boldsymbol{L}^2_{\mathbf{neg}},$$

i.e., $\boldsymbol{A}^{\perp} \models \boldsymbol{L}^1_{\mathbf{int}} \cup^* \boldsymbol{L}^1_{\mathbf{int}}$ and $\boldsymbol{A}_{\perp} \models \boldsymbol{L}^1_{\mathbf{neg}} \cup^* \boldsymbol{L}^1_{\mathbf{neg}}$. In view of arbitrary choice of a model for $\boldsymbol{L}^1 \cup^* \boldsymbol{L}^2$, we stated the equalities

$$(\boldsymbol{L}^1 \cup^* \boldsymbol{L}^2)_{\mathbf{int}} = \boldsymbol{L}^1_{\mathbf{int}} \cup^* \boldsymbol{L}^2_{\mathbf{int}}, (\boldsymbol{L}^1 \cup^* \boldsymbol{L}^2)_{\mathbf{neg}} = \boldsymbol{L}^1_{\mathbf{neg}} \cup^* \boldsymbol{L}^2_{\mathbf{neg}}\,,$$

which completes the proof. □

Thus, the class of paraconsistent extensions of $\mathbf{Le'}$ is completely described in terms of intermediate and negative logics. It should be emphasized that the mapping defined in Proposition 4.3 has essentially effective character. Theorem 4.2 allows effectively reconstruct intuitionistic and negative counterparts from the given paraconsistent $\boldsymbol{L}$, whereas the $\boldsymbol{L}$ itself is simply an intersection of its counterparts, i.e. a formula is proved in $\boldsymbol{L}$ if and only if it is proved in both $\boldsymbol{L}_{\mathbf{int}}$ and $\boldsymbol{L}_{\mathbf{neg}}$.

However, the logics from the interval $[\mathbf{Le'}, \mathbf{Le}]$ constitute relatively small part of the class **Par** of all paraconsistent extensions of **Lj**. There are many interesting logics which does not belong to this interval. One of them is the Glivenko logic treated in the next section.

5. Glivenko's logic

Consider the following substitutional instance of the Peirce law:

$$(\mathbf{P'}) \qquad ((\bot \supset p) \supset \bot) \supset \bot = \neg\neg(\bot \supset p).$$

We call the logic $\mathbf{Ljp'} \rightleftharpoons \mathbf{Lj} + \{\mathbf{P'}\}$ *Glivenko's logic*. It was mentioned in [11, p. 46] that Glivenko's logic is the weakest one in which $\neg\neg\varphi$ is derivable whenever φ is derivable in classical logic. Unfortunately, this work contains neither the proof of this assertion, nor any further reference. In this section, we present a natural algebraic proof of this statement, which we call the generalized Glivenko theorem. We also show that $\mathbf{Ljp'}$ is a proper subsystem of $\mathbf{Le'}$.

For a Heyting algebra $\boldsymbol{A}$, we denote by $\nabla_{\boldsymbol{A}}$ its filter of dense elements and by $\mathcal{R}(\boldsymbol{A})$ the Boolean algebra of its regular elements. Recall that $\nabla_{\boldsymbol{A}} = \{a \in \boldsymbol{A} \mid \neg\neg a = 1\}$, $\mathcal{R}(\boldsymbol{A}) = \{a \in \boldsymbol{A} \mid a \vee \neg a = 1\}$, and $\mathcal{R}(\boldsymbol{A}) \cong \boldsymbol{A}/\nabla_{\boldsymbol{A}}$.

PROPOSITION 5.1. 1. *Let $\boldsymbol{A}$ be a j-algebra. Then $\boldsymbol{A}$ is a model for $\mathbf{Ljp'}$ if and only if $\bot \vee (\bot \supset a) \in \nabla_{\boldsymbol{A}^{\bot}}$ for any $a \in \boldsymbol{A}$.*

2. *Let $\boldsymbol{A}$ be a model for $\mathbf{Ljp'}$ and $\nabla \rightleftharpoons \nabla_{\boldsymbol{A}^{\bot}}$. Then the mapping $\pi(a) = (a \vee \bot)/\nabla$ defines an epimorphism of $\boldsymbol{A}$ onto $\mathcal{R}(\boldsymbol{A})$.*

PROOF. 1. Immediately follows from the properties of j-algebras.

2. In fact, we need only check that π preserves the implication, i.e., that $\pi(a \supset b) = \pi(a) \supset \pi(b)$. The last equality is equivalent to

$$(a \supset b) \vee \bot/\nabla = (a \vee \bot) \supset (b \vee \bot)/\nabla\,.$$

We have $((a \supset b) \vee \bot) \supset ((a \vee \bot) \supset (b \vee \bot)) = 1 \in \nabla$ since the corresponding formula is provable in **Lj**. Further, it can be verified directly that

$$\mathbf{Lj} \vdash (\bot \supset q) \supset (((p \vee \bot) \supset (q \vee \bot)) \supset ((p \supset q) \vee \bot)).$$

In view of $\mathbf{Lj} \vdash (p \supset q) \supset (\neg\neg p \supset \neg\neg q)$ we obtain

$$\mathbf{Lj} \vdash \neg\neg(\bot \supset q) \supset \neg\neg(((p \vee \bot) \supset (q \vee \bot)) \supset ((p \supset q) \vee \bot)),$$

i.e.,

$$\mathbf{Ljp'} \vdash \neg\neg(((p \vee \bot) \supset (q \vee \bot)) \supset ((p \supset q) \vee \bot)).$$

By the assumption, $\boldsymbol{A}$ is a model for $\mathbf{Ljp'}$, consequently,

$$((a \vee \bot) \supset (b \vee \bot)) \supset ((a \supset b) \vee \bot) \in \nabla\,,$$

which completes the proof. □

THEOREM 5.2 (the generalized Glivenko theorem). *For every logic $\boldsymbol{L} \supseteq \mathbf{Lj}$ the following conditions are equivalent:*

(a) *For any φ,*

$$\mathbf{Lk} \vdash \varphi \Longleftrightarrow \boldsymbol{L} \vdash \neg\neg\varphi.$$

(b) $\boldsymbol{L} \supseteq \mathbf{Ljp'}$ *and* $\boldsymbol{L} \notin \mathbf{Neg}$.

PROOF. "(a) ⇒ (b)" This implication is trivial.

"(b) ⇒ (a)" Let $\boldsymbol{L} = \mathbf{L}\,\boldsymbol{A}$. By condition $\boldsymbol{L}$ is not negative, hence Heyting algebra $\boldsymbol{A}^{\bot}$ and Boolean algebra $\mathcal{R}(\boldsymbol{A}^{\bot})$ are non-trivial, in particular, $\mathbf{L}\,\mathcal{R}(\boldsymbol{A}^{\bot}) = \mathbf{Lk}$. Using the properties of epimorphism π from Proposition 5.1.2 we complete the proof in the same way as was done in Theorem 4.2. □

For a j-algebra $\boldsymbol{A}$ and a Heyting algebra $\boldsymbol{B}$ we denote by $\boldsymbol{A} + \boldsymbol{B}$ the direct sum of these algebras. It is a j-algebra in which unit element of $\boldsymbol{A}$ is identified with zero of $\boldsymbol{B}$, and for any $a \in \boldsymbol{A}$ and $b \in \boldsymbol{B}$, we have $a \leq b$.

PROPOSITION 5.3. *Let $\boldsymbol{A}$ be a model for $\mathbf{Le'}$, and $\boldsymbol{B}$ a Heyting algebra. Then $\boldsymbol{A} + \boldsymbol{B}$ is a model for $\mathbf{Le'}$.*

PROOF. The proof follows from two facts. For all $a \in \boldsymbol{A} \times \boldsymbol{B}$, we have $\bot \vee (\bot \supset a) \in \boldsymbol{B}$, and all elements of $\boldsymbol{B}$ are dense in $(\boldsymbol{A} + \boldsymbol{B})^{\bot}$. □

Proposition 5.3 implies, in particular, that the inclusion $\mathbf{Ljp'} \supseteq \mathbf{Le'}$ is proper. Indeed, $\boldsymbol{A} + \boldsymbol{B}$ is not a model for $\mathbf{Le'}$ if $\boldsymbol{B}$ is non-trivial.

6. Intuitionistic and negative counterparts of Lj-extensions

In conclusion, we define intuitionistic and negative counterparts for any extension of minimal logic. It will be done so that an intermediate (negative) logic coincides with its intuitionistic (negative) counterpart, while its negative (intuitionistic) counterpart is trivial. For extensions of $\mathbf{Le'}$, our definitions will be equivalent to that of Section 4.

We define the following translation

$$\mathrm{I}(\bot) = \bot, \quad \mathrm{I}(p) = p \vee \bot, \quad \mathrm{I}(\varphi * \psi) = \mathrm{I}(\varphi) * \mathrm{I}(\psi),$$

where p is a propositional variable, φ and ψ arbitrary formulas, and $* \in \{\vee, \wedge, \supset\}$. In other words, if $\varphi(p_0, \dots, p_n)$ is a formula in propositional variables $p_0, \dots, p_n$, then $\mathrm{I}(\varphi) = \varphi(p_0 \vee \bot, \dots, p_n \vee \bot)$. We will see that I may be considered as translation of intuitionistic formulas into formulas of minimal logic.

For $\boldsymbol{L}$ extending $\mathbf{Lj}$, we define

$$\boldsymbol{L}_{\mathbf{int}} \rightleftharpoons \{\varphi \mid \boldsymbol{L} \vdash \mathrm{I}(\varphi)\}, \quad \boldsymbol{L}_{\mathbf{neg}} \rightleftharpoons \{\varphi \mid \boldsymbol{L} \vdash \bot \supset \varphi\}.$$

It can be easily seen that $\boldsymbol{L}_{\mathbf{int}}$ and $\boldsymbol{L}_{\mathbf{neg}}$ are logics. We call $\boldsymbol{L}_{\mathbf{int}}$ and $\boldsymbol{L}_{\mathbf{neg}}$ *intuitionistic* and *negative counterparts* of the logic $\boldsymbol{L}$, respectively.

The definition of negative counterpart is exactly the same as in Section 4. As for $\boldsymbol{L}_{\mathbf{int}}$, using formula $(\mathbf{D'})$ we can easily prove in $\mathbf{Le'}$ the equivalence $(\varphi \vee \bot) \equiv \mathrm{I}(\varphi)$ for any formula φ. Therefore, if $\boldsymbol{L}$ extends $\mathbf{Le'}$, $\boldsymbol{L}_{\mathbf{int}}$ coincides with the intuitionistic counterpart defined in Section 4.

In the following proposition we lists some simple properties of the notions introduced.

Proposition 6.1. 1. *For any* $\boldsymbol{L} \supseteq \mathbf{Lj}$, $\boldsymbol{L}_{\mathbf{int}} \in \mathbf{Int}$, $\boldsymbol{L}_{\mathbf{neg}} \in \mathbf{Neg}$, *and* $\boldsymbol{L} \subseteq \boldsymbol{L}_{\mathbf{int}} \cap \boldsymbol{L}_{\mathbf{neg}}$. *The last inclusion is not proper if and only if* $\boldsymbol{L}$ *extends* $\mathbf{Le'}$.

2. $\boldsymbol{L} \in \mathbf{Int}$ *if and only if* $\boldsymbol{L} \neq \mathcal{F}$, $\boldsymbol{L} = \boldsymbol{L}_{\mathbf{int}}$, *and* $\boldsymbol{L}_{\mathbf{neg}} = \mathcal{F}$.
3. $\boldsymbol{L} \in \mathbf{Neg}$ *if and only if* $\boldsymbol{L} \neq \mathcal{F}$, $\boldsymbol{L} = \boldsymbol{L}_{\mathbf{neg}}$, *and* $\boldsymbol{L}_{\mathbf{int}} = \mathcal{F}$.
4. *If* $\mathbf{Lj} \subseteq \boldsymbol{L}^1 \subseteq \boldsymbol{L}^2$, *then* $\boldsymbol{L}^1_{\mathbf{int}} \subseteq \boldsymbol{L}^2_{\mathbf{int}}$ *and* $\boldsymbol{L}^1_{\mathbf{neg}} \subseteq \boldsymbol{L}^2_{\mathbf{neg}}$.
5. *If* $\boldsymbol{L} \subseteq \boldsymbol{L}_1 \in \mathbf{Int}$, *then* $\boldsymbol{L}_{\mathbf{int}} \subseteq \boldsymbol{L}_1$.
6. *If* $\boldsymbol{L} \subseteq \boldsymbol{L}_1 \in \mathbf{Neg}$, *then* $\boldsymbol{L}_{\mathbf{neg}} \subseteq \boldsymbol{L}_1$.

Proof. We only prove the last two assertions.

5. If $\boldsymbol{L} \vdash \mathrm{I}(\varphi)$, then also $\boldsymbol{L}_1 \vdash \mathrm{I}(\varphi)$. Since $\boldsymbol{L}_1$ is intermediate, we have $\boldsymbol{L}_1 \vdash \mathrm{I}(\varphi) \supset \varphi$, and so $\boldsymbol{L}_1 \vdash \varphi$, which implies the desired inclusion.

6. Again, from $\boldsymbol{L}_1 \vdash \bot \supset \varphi$ we conclude $\boldsymbol{L}_1 \vdash \varphi$ since $\bot$ belongs to any negative logic. □

We have thus proved, in particular, that $\boldsymbol{L}_{\mathbf{int}}$ is the least intermediate logic containing $\boldsymbol{L}$, and $\boldsymbol{L}_{\mathbf{neg}}$ is the least negative logic with the same property. In the following proposition we describe intuitionistic and negative counterparts of the given logic in semantical terms.

PROPOSITION 6.2. *Let $\boldsymbol{L} \supseteq \mathbf{Lj}$, and let $\boldsymbol{A}$ be a characteristic model for $\boldsymbol{L}$. Then the following equations hold.*

1. $\mathbf{L}\,\boldsymbol{A}^{\bot} = \boldsymbol{L}_{\mathbf{int}}$.
2. $\mathbf{L}\,\boldsymbol{A}_{\bot} = \boldsymbol{L}_{\mathbf{neg}}$.

PROOF. 1. Assume $\boldsymbol{A}^{\bot} \models \varphi$ and prove $\boldsymbol{A} \models \mathrm{I}(\varphi)$. For an $\boldsymbol{A}$-valuation V define an $\boldsymbol{A}^{\bot}$-valuation V' by the rule $V'(p) \rightleftharpoons V(p) \vee \bot$. Then it easily follows that $V(\mathrm{I}(\varphi)) = V'(\varphi)$, which immediately implies the desired conclusion.

Conversely, let $\boldsymbol{A} \models \mathrm{I}(\varphi)$. For any $\boldsymbol{A}^{\bot}$-valuation V we have $V = V'$, in particular, $V(\mathrm{I}(\varphi)) = V(\varphi)$, which completes the proof.

2. We will use the mapping $\tau(x) = x \wedge \bot$, which is an epimorphism of $\boldsymbol{A}$ onto $\boldsymbol{A}_{\bot}$ by Proposition 3.1. Note also that $\bot \supset \varphi$ is equivalent to $\bot \supset (\varphi \wedge \bot)$ in $\mathbf{Lj}$.

Assuming $\boldsymbol{A}_{\bot} \models \varphi$ we take an $\boldsymbol{A}$-valuation V and consider the composition τV, which is an $\boldsymbol{A}_{\bot}$-valuation. In view of the fact that τ is an epimorphism, $V(\varphi \wedge \bot) = \tau V(\varphi)$. But $\tau V(\varphi) = \bot$ by assumption, which yields $V(\bot \supset (\varphi \wedge \bot)) = 1$. Thus, $\boldsymbol{A} \models \bot \supset (\varphi \wedge \bot)$ by the arbitrary choice of V.

Now, we let $\boldsymbol{A} \models \bot \supset (\varphi \wedge \bot)$. Clearly, $V = \tau V$ for any $\boldsymbol{A}_{\bot}$-valuation V. By assumption, $V(\bot) \leq V(\varphi \wedge \bot) = \tau V(\varphi) = V(\varphi)$, $\bot$ is the greatest element of $\boldsymbol{A}_{\bot}$, whence, $V(\varphi) = \bot$. In this way, $\boldsymbol{A}_{\bot} \models \varphi$. □

Further, we consider the classes of logics with the given intuitionistic and negative counterparts. For $\boldsymbol{L}_1 \in \mathbf{Int}$ and $\boldsymbol{L}_2 \in \mathbf{Neg}$, we define

$$\mathrm{Spec}(\boldsymbol{L}_1, \boldsymbol{L}_2) \rightleftharpoons \{\boldsymbol{L} \supseteq \mathbf{Lj} \mid \boldsymbol{L}_{\mathbf{int}} = \boldsymbol{L}_1,\ \boldsymbol{L}_{\mathbf{neg}} = \boldsymbol{L}_2\}.$$

It is clear that for any pair of intermediate and negative logics, $(\boldsymbol{L}_1, \boldsymbol{L}_2)$, the set $\mathrm{Spec}(\boldsymbol{L}_1, \boldsymbol{L}_2)$ is non-empty, it contains at least the intersection $\boldsymbol{L}_1 \cap \boldsymbol{L}_2$. Moreover, in view of Proposition 6.1.1 $\boldsymbol{L}_1 \cap \boldsymbol{L}_2$ is the greatest element of $\mathrm{Spec}(\boldsymbol{L}_1, \boldsymbol{L}_2)$. It turns out this set contains also the least element and forms an interval in the lattice of $\mathbf{Lj}$-extensions. Let

$$\boldsymbol{L}_1 * \boldsymbol{L}_2 \rightleftharpoons \mathbf{Lj} + \{\mathrm{I}(\varphi),\ \bot \supset \psi \mid \varphi \in \boldsymbol{L}_1,\ \psi \in \boldsymbol{L}_2\},$$

where $\boldsymbol{L}_1 \in \mathbf{Int}$ and $\boldsymbol{L}_2 \in \mathbf{Neg}$.

Proposition 6.3. *Let* $\boldsymbol{L}_1 \in \mathbf{Int}$ *and* $\boldsymbol{L}_2 \in \mathbf{Neg}$. *Then*

$$\mathrm{Spec}(\boldsymbol{L}_1, \boldsymbol{L}_2) = [\boldsymbol{L}_1 * \boldsymbol{L}_2, \boldsymbol{L}_1 \cap \boldsymbol{L}_2].$$

Proof. Let $\boldsymbol{L}^* \rightleftharpoons \boldsymbol{L}_1 * \boldsymbol{L}_2$. It follows from definition that $\boldsymbol{L}_1 \subseteq \boldsymbol{L}^*_{\mathbf{int}}$ and $\boldsymbol{L}_2 \subseteq \boldsymbol{L}^*_{\mathbf{neg}}$. On the other hand, we can see that for any $\boldsymbol{L} \in \mathrm{Spec}(\boldsymbol{L}_1, \boldsymbol{L}_2)$, $\boldsymbol{L}^* \subseteq \boldsymbol{L}$. Indeed, $\boldsymbol{L}$ contains all axioms of $\boldsymbol{L}^*$. As was noted above $\boldsymbol{L}_1 \cap \boldsymbol{L}_2$ is the greatest element of $\mathrm{Spec}(\boldsymbol{L}_1, \boldsymbol{L}_2)$, whence by Proposition 6.1.4, $\boldsymbol{L}^*$ and all logics intermediate between $\boldsymbol{L}^*$ and $\boldsymbol{L}_1 \cap \boldsymbol{L}_2$ also belongs to $\mathrm{Spec}(\boldsymbol{L}_1, \boldsymbol{L}_2)$. □

The next proposition allows to write axioms for $\boldsymbol{L}_1 * \boldsymbol{L}_2$ relative to **Lj** given the axiomatics of $\boldsymbol{L}_1$ relative to **Li** and of $\boldsymbol{L}_2$ relative to **Ln**.

Proposition 6.4. *Let* $\boldsymbol{L}_1 \in \mathbf{Int}$, $\boldsymbol{L}_1 = \mathbf{Li} + \{\varphi_i \mid i \in I\}$ *and* $\boldsymbol{L}_2 \in \mathbf{Neg}$, $\boldsymbol{L}_2 = \mathbf{Ln} + \{\psi_j \mid j \in J\}$. *Then*

$$\boldsymbol{L}_1 * \boldsymbol{L}_2 = \mathbf{Lj} + \{\ \mathrm{I}(\varphi_i),\ \bot \supset \psi_j \mid i \in I,\ j \in J\ \}.$$

Proof. Denote the right-hand side of the last equality by D. The inclusion $D \subseteq \boldsymbol{L}_1 * \boldsymbol{L}_2$ is trivial. To state the inverse inclusion we show that $\boldsymbol{L}_1 \subseteq D_{\mathbf{int}}$ and $\boldsymbol{L}_2 \subseteq D_{\mathbf{neg}}$.

We argue for $\boldsymbol{L}_2 \subseteq D_{\mathbf{neg}}$. Note that $\mathbf{Ln} = \mathbf{Lj}_{\mathbf{neg}}$, i.e., $\mathbf{Ln} \vdash \varphi$ iff $\mathbf{Lj} \vdash \bot \supset \varphi$. Assume $\psi \in \boldsymbol{L}_2$, then $\mathbf{Ln} \vdash (\psi_{j_1} \wedge \cdots \wedge \psi_{j_n}) \supset \psi$ for suitable $j_1, \ldots, j_n \in J$. By the above

$$\mathbf{Lj} \vdash \bot \supset ((\psi_{j_1} \wedge \cdots \wedge \psi_{j_n}) \supset \psi).$$

The last formula implies in **Lj** the formula

$$((\bot \supset \psi_{j_1}) \wedge \cdots \wedge (\bot \supset \psi_{j_n})) \supset (\bot \supset \psi),$$

from which we infer $\bot \supset \psi \in D$. Consequently, $\boldsymbol{L}_2 \subseteq D_{\mathbf{neg}}$.

The remaining inclusion follows in the same way from the equality $\mathbf{Li} = \mathbf{Lj}_{\mathbf{int}}$. □

As we can see from Proposition 6.3 the class of **Lj**-extensions decomposes into a union of disjoint intervals

$$\mathbf{Jhn} = \bigcup\{\mathrm{Spec}(\boldsymbol{L}_1, \boldsymbol{L}_2) \mid \boldsymbol{L}_1 \in \mathbf{Int},\ \boldsymbol{L}_2 \in \mathbf{Neg}\}.$$

It is interesting that the upper points of these intervals also form an interval in **Jhn**, $[\mathbf{Le'}, \mathbf{Le}]$.

In this way, the investigation of the class of **Lj**-extensions is reduced to the problem what is the structure of the interval $\mathrm{Spec}(\boldsymbol{L}_1, \boldsymbol{L}_2)$ for the given intermediate logic $\boldsymbol{L}_1$ and negative logic $\boldsymbol{L}_2$. This problem will be treated in the subsequent article.

References

[1] Curry, H. B., *Foundations of Mathematical Logic*, McGrow-Hill Book Company, New York, 1963.

[2] Grzegorczyk, A., "A philosophically plausible formal interpretation of intuitionistic logic", Indagationes Mathematicæ 26, No. 5, 596–601 (1964).

[3] Kanger, S., "A note on partial postulate sets for propositional logic", Theoria 21, 99–104 (1955).

[4] Karpenko, A. S., "Two three-valued isomorphs of classical propositional logic and their combinations", First World Congress on Paraconsistency, Abstracts, Ghent, 92–94 (1997).

[5] Kripke, S., "The system **LE**", unpublished.

[6] Odintsov, S. P., "Maximal paraconsistent extension of Johansson logic", First World Congress on Paraconsistency, Abstracts, Ghent, 111–113 (1997).

[7] Odintsov, S. P., "Isomorphs of the logic of classical refutability and their generalizations", Proceedings of the seminar of logical center, Inst. of Philosophy of RAS, Moscow, 1998.

[8] Odintsov, S. P., "Maximal paraconsistent extension of Johansson logic", to appear in Proceedings of First World Congress on Paraconsistency.

[9] Rasiowa, H., *An Algebraic Approach to Non-Classical Logics*, Amsterdam, North-Holland, 1974.

[10] Rautenberg, W., *Klassische und nichtclassische Aussagenlogik*, Braunschweig, Vieweg, 1979.

[11] Segerberg, K., "Propositional logics related to Heyting's and Johansson's", Theoria 34, 26–61 (1968).

S. P. ODINTSOV
Institute of Mathematics
Novosibirsk, 630090, Russia
odintsov@math.nsc.ru