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A COMPLETENESS PROOF IN FULL DDL

Dynamic doxastic logicians — not a large community — have trodden gingerly within the area of full DDL or, as with the present author, have not trodden at all. However, the latter, after having written up the final version of [6], realised that the proof given in that paper for two varieties of basic DDL can be extended to cover full DDL; in fact, the «full» proof is simpler than the «basic» one. The extended proof is outlined in Section 1–3. In Section 4, the relationship to AGM is considered. Section 5 puts the importance of the proof into perspective.

This note — an extended abstract rather than a full-fledged paper — should be read as an appendix to [6]. Although some definitions are repeated here, many are not. Readers who require more detail are referred to [6], a copy of which they should have on hand.

1. Syntax

The formulæ of the revision fragment of DDL are those that can be built from propositional letters, Boolean connectives, the doxastic operators $\mathbf{B}$ and $\mathbf{K}$ and the change operators $[*\varphi]$, where φ is formula. The duals of the non-Boolean operators are $\mathbf{b}$, $\mathbf{k}$ and $<*\varphi>$. The difference between basic and full DDL is that, in basic DDL, for $\mathbf{B}\,\varphi$ and $\mathbf{K}\,\varphi$ and $[*\varphi]\,\chi$ to be well-formed it is necessary that φ is purely Boolean (that is, built from propositional letters and Boolean connectives); in full DDL there is no such limitation. (Note that the revision operator $*$ is our only dynamic doxastic operator — there is no expansion operator and no contraction operator. From a strictly formal point of view, it would not be difficult to extend the analysis to include them.)

Consider the following axiom system. The rules are three — Modus Ponens, Necessitation and a Rule of Congruence:

(MP) If φ and $\varphi \supset \psi$ are theorems, then ψ is a theorem.

($\bigcirc$N) If φ is a theorem, then $\bigcirc \varphi$ is a theorem, where $\bigcirc$ is $\mathbf{B}$ or $\mathbf{K}$ or $[* \theta]$, for some θ.

(RC) If $\varphi \equiv \psi$ is a theorem, then $[* \varphi]\, \chi \equiv [* \psi]\, \chi$ is a theorem.

The axioms are all tautologies plus all instances of the «Kripke schema»

($\bigcirc$K) $\bigcirc(\varphi \supset \psi) \supset (\bigcirc \varphi \supset \bigcirc \psi)$, where $\bigcirc$ is $\mathbf{B}$ or $\mathbf{K}$ or $[* \theta]$, for some θ.

Theories or logics providing all these postulates are said to be *normal*. Notice that the Rule of Replacement of Provable Equivalents holds in all normal theories and logics:

(RPE) Suppose γ and γ' are formulæ that are identical except that γ contains an occurrence of a formula φ in a place where γ' contains an occurrence of a formula ψ. If $\varphi \equiv \psi$ is a theorem, then γ is a theorem only if γ' is a theorem.

By DDL–AGM we mean the smallest normal logic providing the following special postulates:

($*$2) $[* \varphi]\, \mathbf{B}\, \varphi$,

($*$3) $[* \top]\, \mathbf{B}\, \chi \supset \mathbf{B}\, \chi$,

($*$4) $\mathbf{b}\, \top \supset (\mathbf{B}\, \chi \supset [* \top]\, \mathbf{B}\, \chi)$,

($*$5) $[* \varphi]\, \mathbf{B} \perp \supset \mathbf{K} \neg\varphi$,

($*$6) $\mathbf{K}(\varphi \equiv \psi) \supset ([* \varphi]\, \mathbf{B}\, \chi \equiv [* \psi]\, \mathbf{B}\, \chi)$,

($*$7) $[* (\varphi \supset \psi)]\, \mathbf{B}\, \chi \supset [* \varphi]\, \mathbf{B}(\psi \supset \chi)$,

($*$8) $< * \varphi >\, \mathbf{b}\, \psi \supset [* \varphi]\, \mathbf{B}(\psi \supset \chi) \supset [* (\varphi \supset \psi)]\, \mathbf{B}\, \chi)$,

($*$D) $[* \varphi]\, \chi \supset < * \varphi >\, \chi$,

($*$F) $< * \varphi >\, \chi \supset [* \varphi]\, \chi$,

(**KB**) $\mathbf{K}\, \varphi \supset \mathbf{B}\, \varphi$,

$(\mathbf{K}{*_1})$ $\quad \mathbf{k}\,\varphi \supset (\mathbf{K}\,\chi \equiv [*\,\varphi]\,\mathbf{K}\,\chi)$,

$(\mathbf{K}{*_2})$ $\quad \mathbf{K}\,\neg\varphi \supset [*\,\varphi]\,\mathbf{K}\,\bot$.

Notice that schemata (c) and (d) of the following lemma are closely related to those named $(*3)$ and $(*4)$ in [6].

LEMMA 1.1. *The following theorem schemata are derivable in DDL-AGM:*

(a) $\quad \mathbf{K}\,\neg\varphi \supset [*\,\varphi]\,\mathbf{B}\,\bot$,

(b) $\quad <*\,\varphi>\,\mathbf{b}\,\top \equiv \mathbf{k}\,\varphi$,

(c) $\quad [*\,\varphi]\,\mathbf{B}\,\chi \supset \mathbf{B}(\varphi \supset \chi)$,

(d) $\quad \mathbf{b}\,\varphi \supset (\mathbf{B}\,\chi \supset [*\,\varphi]\,\mathbf{B}\,\chi)$,

(e) $\quad [*\,\varphi]\,[*\,\varphi]\,\mathbf{B}\,\chi \supset [*\,\varphi]\,\mathbf{B}\,\chi$,

(f) $\quad [*\,\varphi]\,\mathbf{B}\,\chi \supset [*\,\varphi]\,[*\,\varphi]\,\mathbf{B}\,\chi$.

PROOF. In addition to (RPE), which is needed for (c) and (d), the following special postulates are used: for (a), (**KB**) and $(\mathbf{K}{*_1})$; for (b), $(*5)$, drawing on (a); for (c), $(*3)$ and $(*7)$; for (d), $(*3)$, $(*4)$ and $(*8)$; for (e), $(*2)$, drawing on (c); for (f), $(*\mathrm{F})$, (**KB**) $(\mathbf{K}{*_1})$ and $(\mathbf{K}\,{*_2})$, drawing on (d). ☐

2. Semantics

Let (U, T) be a Stone space. Suppose that R is a function assigning a binary relation $R^{\bigcirc}$ in U to each operator $\bigcirc$; that is, R assigns a relation $R^{\mathbf{B}}$ to $\mathbf{B}$, a relation $R^{\mathbf{K}}$ to $\mathbf{K}$, and a relation R^{*P} to each clopen set P. We say that y is a *strong* or a *weak doxastic alternative* x if $(x, y) \in R^{\mathbf{B}}$ or $(x, y) \in R^{\mathbf{K}}$ respectively, and that y is a *possible result of revision by* P if $(x, y) \in R^{*P}$. (U, T, R) is a *revision space* or just a *frame* if, for all $\bigcirc \in \{\mathbf{B}, \mathbf{K}, *P : P$ is clopen$\}$, $\mathsf{L}^{\bigcirc} Q$ and $\mathsf{M}^{\bigcirc} Q$ are clopen whenever Q is clopen, where

$$\begin{aligned} \mathsf{L}^{\bigcirc} Q &= \{x \in U : \forall y((x, y) \in R^{\bigcirc} \;\Rightarrow\; y \in Q)\}, \\ \mathsf{M}^{\bigcirc} Q &= \{x \in U : \exists y((x, y) \in R^{\bigcirc} \;\&\; y \in Q)\}. \end{aligned}$$

A *valuation* in a frame is an assignment of a clopen set to each propositional letter. A *model* is a frame with a valuation. If $\mathscr{F}$ is a frame with universe U, x a point in U, $\mathscr{M}$ a model on $\mathscr{F}$, and φ a formula, then *truth of* φ *at* x *in* $\mathscr{M}$, in symbols $\mathscr{M} \models_x \varphi$, is defined in a pleasingly diaphanous way:

the crucial conditions are those for the operators $\bigcirc \in \{\mathbf{B}, \mathbf{K}, [*\theta] : \theta$ is a formula$\}$, and they are uniform:

$$\mathscr{M} \vDash_x \bigcirc\chi \quad \text{iff} \quad \forall y((x, y) \in R^{\bigcirc} \Rightarrow \mathscr{M} \vDash_y \chi).$$

A formula is *valid* in a class of frames if true at all points in all models on all the frames. A formula set is *satisfiable* in a frame if there is a model on the frame and a point in the universe of the frame such that the formulæ of the set are true at that point in that model.

A revision space is *serial* if, for every clopen set P,

$$\forall x \exists y R^{*P}(x, y)$$

and *functional* if, for every clopen set P,

$$\forall x \forall y \forall z(((x, y) \in R^{*P} \ \& \ (x, z) \in R^{*P}) \Rightarrow y = z).$$

An *onion* is a nonempty set O of subsets of U satisfying two conditions. One is our version of David Lewis's famous limit condition (LIMIT):

$$\forall P(P \text{ is clopen } \& \ P \cap \bigcup O \neq \emptyset \Rightarrow \exists Z \in O(Z \,\mu\, O \Cap P)).$$

Here, $O \Cap P = \{X \in O : X \cap P \neq \emptyset\}$. Furthermore, μ is short-hand for "is the smallest element of". Thus $Z \,\mu\, O \Cap P$ means that $Z \in O \Cap P$ and, for all $Y \in O \Cap P$, $Z \subseteq Y$. The other condition (NESTED) is that the elements of O are nested, that is, linearly ordered under set inclusion:

$$\forall X \forall Y((X \in O \ \& \ Y \in O) \Rightarrow (X \subseteq Y \vee Y \subseteq X)).$$

A *Lewis onion* is an onion that satisfies two further conditions: closure under arbitrary nonempty intersection (AINT) and closure under arbitrary nonempty union (AUNI):

$$\forall C(\emptyset \neq C \subseteq O \Rightarrow \bigcap C \in O),$$
$$\forall C(\emptyset \neq C \subseteq O \Rightarrow \bigcup C \in O).$$

We refer to $\bigcap O$ as the *belief set* of O and to $\mathsf{C}(\bigcup O)$ as the *commitment set* of O, where C represents topological closure. An *onion determiner* is a function D assigning to each point $x \in U$ an onion D_x. The structure (U, T, R, D) is an *onion frame* if (U, T, R) is a revision space and D is an onion determiner. Furthermore, an onion frame (U, T, R, D) is called an *AGM onion frame* if it is serial, functional and satisfies the following conditions:

(o1) $\forall x \forall y((x, y) \in R^{\mathbf{B}} \iff y \in \bigcap(D_x)),$

(o2) $\forall x \forall y((x, y) \in R^{\mathbf{K}} \iff y \in \mathsf{C}(\bigcup(D_x))),$

and, for all clopen sets P,

(o3) $\forall x \forall y((x, y) \in R^{*P} \Rightarrow (P \cap \bigcup(D_x) \neq \emptyset \Rightarrow$ $\exists Z(Z\,\mu\,(D_x \Cap P)\ \&\ \bigcap(D_y) = P \cap Z))),$

(o4) $\forall x \forall y((x, y) \in R^{*P} \Rightarrow (P \cap \bigcup(D_x) \neq \emptyset \Rightarrow \bigcup(D_x) = \bigcup(D_y)),$

(o5) $\forall x \forall y((x, y) \in R^{*P} \Rightarrow (P \cap \bigcup D_x) = \emptyset \Rightarrow D_y = \{\emptyset\})).$

We can express these conditions in words as follows. Conditions (o1) and (o2) require the set of strong doxastic alternatives and the set of weak doxastic alternatives to coincide with, respectively, the agent's belief set and the agent's commitment set. Condition (o3) — the heart of AGM revision! — places a necessary condition on the onion of a point that is the result of nontrivial revision by P, that is, when P intersects the union of the onion; then the new belief set is to be the intersection of P with the smallest P-intersecting sphere of the old onion. Condition (o4) implies that, even though the agent's belief set may vary, the commitment set remains the same in all nontrivial cases. Finally, condition (o5) specifies what happens in the trivial case, which is when P does not intersect any element of the onion: then the new onion is degenerate, having only the empty set as an element.

The last condition is worth a comment. Classical AGM gives no guidance on what to do in the pathological case to which (o5) applies. One possibility is to hang on, come hell or high water, to the background theory that governs the **K**-perator and accept as a condition

$$\forall x \forall y \forall z((x, z) \in R^{\mathbf{K}} \iff (y, z) \in R^{\mathbf{K}}).$$

However, if you revise yourself out of your onion (and so, as Bertie Wooster might say, you must be off your onion), there is no support for a nonempty commitment set. It is not unreasonable to consider that in such an extreme case you are beyond everything and have to accept the empty set as your commitment set; once you have reached this belief state, no further revision can change it. Here we have followed the latter course. This means that, in effect, we have chosen to cast the belief state corresponding to the degenerate onion $\{\emptyset\}$ in the rôle of what Gärdenfors has termed “epistemic hell”. End of comment.

A *selector* is a function f from the set of clopen sets to subsets of U satisfying the following three conditions:

(i) $fP \subseteq P$,

(ii) $P \subseteq Q \Rightarrow (fP \neq \emptyset \Rightarrow fQ \neq \emptyset)$,

(iii) $P \subseteq Q \Rightarrow (P \cap fQ \neq \emptyset \Rightarrow fP = P \cap fQ)$.

A selector f is *trivial* if, for all clopen sets P, $fP = \emptyset$, otherwise *nontrivial.* We refer to fP as the *segment* of P under f. One segment is particularly important: fU, also called the *belief* set of f. The definition of the *commitment set* of f is more complex: $\mathsf{C}(\bigcup\{fP : P \text{ is a clopen set}\})$. (Again, C represents topological closure.) Some mnemonic aid to talk about conditions (i)–(iii) will be useful. For condition (i), INCLUSION is a natural name. Condition (ii) guarantees a kind of MOnotonicity for NonEmptY Segments, and we accordingly introduce the artificial abbreviation MONEYS for this condition. Condition (iii), finally, is a condition introduced into modal logic by Bengt Hansson in [2] but discussed much earlier by Kenneth Arrow in the theory of social choice; so let us refer to it as ARROW.

A *superselector* is a function F assigning to each $x \in U$ a selector F_x. If (U, T, R) is a revision space, the structure (U, T, R, F) is a *selector frame* if F is a superselector. A serial, functional selector frame (U, T, R, F) is an *AGM selector frame* if

(s1) $\forall x \forall y((x, y) \in R^{\mathbf{B}} \iff y \in F_x U)$,

(s2) $\forall x \forall y((x, y) \in R^{\mathbf{K}} \iff y \in \mathsf{C}(\bigcup\{F_x P : P \text{ is clopen}\}))$,

and, for all clopen sets P,

(s3) $\forall x \forall y((x, y) \in R^{*P} \Rightarrow F_x P = F_y U)$,

(s4) $\forall x \forall y((x, y) \in R^{*P} \Rightarrow (F_x P \neq \emptyset \Rightarrow \bigcup\{F_x Q : Q \text{ is clopen}\} = \bigcup\{F_y Q : Q \text{ is clopen}\}))$,

(s5) $\forall x \forall y((x, y) \in R^{*P} \Rightarrow (F_x P = \emptyset \Rightarrow \forall Q(F_y Q = \emptyset)))$.

Also these conditions can be explained in words. Conditions (s1) and (s2) relate the sets of strong and weak doxastic alternatives to the belief set and to the commitment set, respectively. Condition (s3) is the fundamental requirement that the new belief set shall be the set selected by the current selector. Conditions (s4) and (s5) regulate the change in the commitment set: in «normal» cases it remains the same, in «nonnormal» ones it vanishes.

Let (U, T, R, F) be a given AGM selector frame. If u is an element of U and hence F_u a selector, then call a subset X of U a *sphere under* F_u if

(i) $\forall w \in X \exists P(w \in fP)$,

(ii) $\forall P(P \cap X \neq \emptyset \Rightarrow fP \subseteq X)$,

(iii) $X \neq \emptyset$.

(Here and in the following lemma P is a parameter over the set of clopen sets.)

LEMMA 2.1. *If $P \cap X \neq \emptyset$ then $fP \neq \emptyset$.*

PROOF. Assume that $P \cap X \neq \emptyset$. Then there is some point $w \in P \cap X$. By condition (i) there is some clopen set Q such that $w \in fQ$. By INCLUSION, $w \in Q$. Hence $P \cap Q \cap fQ \neq \emptyset$. Since $P \cap Q \subseteq Q$, ARROW yields $f(P \cap Q) = P \cap Q \cap fQ$. In other words, $f(P \cap Q) \neq \emptyset$. Hence $fP \neq \emptyset$ by MONEYS. □

By (i) and (ii), X has a certain "all-or-nothing" property: whenever a clopen set P overlaps X, the corresponding segment — nonempty by Lemma 2.1 — is completely included in X. Thus X is simply a nonempty union of segments. Define a function D^F on U by the condition

$$D^F_u = \{X : X \text{ is a sphere under } F_u\} \cup \{\emptyset : F_u P = \emptyset, \text{ for all } P\}.$$

The following result — drawing on David Lewis's classic work in [3] — is an important link in the over all completeness proof:

PROPOSITION 2.2. *If (U, T, R, F) is an AGM selector frame, then (U, T, R, D^F) is an AGM onion frame.*

PROOF. Let (U, T, R, F) be an AGM selector frame. We note that R is serial and functional. The proof is naturally divided into two parts: one to prove that D^F_u is an onion, for each $u \in U$; the other to prove that conditions (o1)–(o5) are satisfied.

First we prove that D^F_u is an onion, for $u \in U$; in fact, a Lewis onion. If F_u is trivial, then $D^F_u = \{\emptyset\}$ — a Lewis onion if not a very interesting one. Suppose therefore that F_u is nontrivial. There are four conditions to check: NESTED, AINT, AUNI and LIMIT.

NESTED. We give a reductio argument. Suppose that $X \not\subseteq Y$ and $Y \not\subseteq X$, for some $X, Y \in D^F_u$. Then there are points $v \in X - Y$ and $w \in Y - X$. By condition (i) there are clopen sets P and Q such that $v \in F_u P$ and $w \in F_u Q$. From INCLUSION we gather that $v \in X \cap P$ and $w \in Y \cap Q$. Hence by (ii) we get $F_u P \subseteq X$ and $F_u Q \subseteq Y$. Notice that $P \cup Q$ is clopen. Since $(P \cup Q) \cap X \neq \emptyset$, it follows by condition (ii) that $F_u(P \cup Q) \subseteq X$.

By the same token, $F_u(P \cup Q) \subseteq Y$. Furthermore, $F_u(P \cup Q) \neq \emptyset$ by MONEYS. By INCLUSION therefore $(P \cup Q) \cap F_u(P \cup Q) \neq \emptyset$. There are two alternatives. One is that $P \cap F_u(P \cup Q) \neq \emptyset$. Then $F_uP = P \cap F_u(P \cup Q)$ by ARROW, implying hat $F_uP \subseteq F_u(P \cup Q) \subseteq Y$, which is impossible. The other alternative is that $Q \cap F_u(P \cup Q) \neq \emptyset$, which is similarly impossible (the same kind of argument leads to conclusion that $F_uQ \subseteq X$, something known to be absurd). Thus in either case we encounter contradiction.

AINT. Let C be any nonempty collection of spheres under F_u. Suppose $w \in \bigcap C$. Take any $X \in C$ (since C is nonempty, the existence of such an element is guaranteed). Since condition (i) holds for X there is some P such that $w \in F_uP$. Thus $\bigcap C$ satisfies condition (i). For (ii), take any clopen P such that $\bigcap C \cap P \neq \emptyset$. For any $X \in C$ we have $X \cap P \neq \emptyset$. Since condition (ii) holds for X, $F_uP \subseteq X$ and hence $F_uP \subseteq \bigcap C$. Thus $\bigcap C$ satisfies condition (ii). Suppose, finally, that $\bigcap C = \emptyset$. Since F_u is nontrivial, $F_uP \neq \emptyset$ for some clopen P. Hence $F_uU \neq \emptyset$ by MONEYS. Take any $X \in C$. By (iii), $X \neq \emptyset$. Therefore $X \cap U \neq \emptyset$. Hence $F_uU \subseteq X$ by (ii). This shows that $F_uU \subseteq \bigcap C$ and so $\bigcap C \neq \emptyset$. Thus $\bigcap C$ satisfies condition (iii).

AUNI. This case is similar to that of AINT.

LIM. Suppose that the family $C = \{X \in D_u^F : X \cap P \neq \emptyset\}$ is not empty. Then $\bigcap C \in D_u^F$ by AINT (which we established above). Furthermore, for each $X \in C$, $F_uP \subseteq X$; hence $F_uP \subseteq \bigcap C$. Thus $\bigcap C$ is the smallest sphere under F_u that intersects P.

Now we turn to the five conditions for onion-frame-hood. Condition (o1) is implied by condition (s1) since F_uU is the smallest sphere in D_u^F. (That F_uU is a sphere follows readily from conditions (i)–(iii). To see that it is the smallest, let X be any sphere in D_u^F. By (iii), $X \neq \emptyset$ and so of course $U \cap X \neq \emptyset$. Hence by (ii), $F_uU \subseteq X$.)

For condition (o2) it is enough, in view of condition (s2), to observe that, for every sphere X, there is a nonempty set E_X of clopen sets such that $X = \bigcup\{F_uP : P \in E_X\}$ and that $\bigcup D_u^F = \bigcup\{X : X \in D_u^F\} = \bigcup\{\bigcup\{F_uP : P \in E_X\} : X \in D_u^F\} = \bigcup\{F_uP : P \text{ is clopen}\}$.

To verify that (o3) holds requires some effort. Assume, for any clopen set P, that $(u, v) \in R^{*P}$ and $P \cap \bigcup D_u^F \neq \emptyset$. We have already shown that D_u^F is an onion, so we may infer that there exists a sphere $Z \, \mu \, (D_u^F \Cap P)$; that is, Z is the smallest sphere in D_u^F to intersect P. By condition (ii) above, $F_uP \subseteq Z$. Hence, with the help of INCLUSION, $F_uP \subseteq P \cap Z$. We must now prove that $F_uP = P \cap Z$. To this end, define

$$X = \bigcup\{F_uQ : Q \text{ is clopen } \ \& \ P \subseteq Q\}.$$

We contend that X is a sphere under F_u. There are three conditions to check. Condition (i): Take any $w \in X$. Then, for some Q such that $P \subseteq Q$, $w \in F_u Q$. Condition (ii): Suppose $R \cap X \neq \emptyset$, for some clopen R. Then $R \cap F_u Q \neq \emptyset$, for some clopen Q such that $P \subseteq Q$. Note that, by INCLUSION, the fact that $F_u Q \neq \emptyset$ implies that $F_u(Q \cup R) \neq \emptyset$. Suppose that $R \cap F_u(Q \cup R) = \emptyset$. Then, again because of INCLUSION, $F_u(Q \cup R) \subseteq Q$. Hence ARROW yields $F_u Q = Q \cap F_u(Q \cup R) = F_u(Q \cup R)$. This is in contradiction with the fact that $R \cap F_u Q \neq \emptyset$. Consequently, $R \cap F_u(Q \cup R) \neq \emptyset$. By ARROW we can infer that $F_u R = R \cap F_u(Q \cup R)$. But $F_u(Q \cup R) \subseteq X$, since the fact that $P \subseteq Q$ implies that $P \subseteq Q \cup R$. Hence $F_u R \subseteq X$, as we wanted. Condition (iii): We saw above that Z intersects P; say $v \in P \cap Z$. Since Z is a sphere under F_u there is some clopen set Q such that $v \in F_u Q$. By MONEYS, $F_u(P \cup Q) \neq \emptyset$. By definition of X, $F_u(P \cup Q) \subseteq X$. Hence $X \neq \emptyset$. The proof that X is a sphere under F_u is now complete. Finally, we observe that $P \cap X = P \cap \bigcup\{F_u Q : P \subseteq Q\} = \bigcup\{P \cap F_u Q : P \subseteq Q\}$. Using ARROW, we conclude that $P \cap X = F_u P$. Thus $X = Z$. This proves that $F_u P = P \cap Z$, and so condition (o3) is satisfied.

Condition (o4) follows from condition (s4), as does (o5) from (s5). □

3. Completeness

THEOREM 3.1. *Let Σ be a set of formulæ of full DDL. The following statements are equivalent:*

(i) *Σ is consistent in DDL-AGM,*

(ii) *Σ is satisfiable in a closed-onion AGM frame,*

(iii) *Σ is satisfiable in a selector AGM frame.*

PROOF. The equivalence of clauses (ii) and (iii) can be proved by extending the analysis given in [3]. That each clause implies (i) is clear: all theorems of DDL-AGM are valid in all AGM closed-onion frames as well as all AGM selector frames. Hence it will be enough to show that (i) implies (iii). This is the task to which the rest of this section is devoted.

As usual in canonical model proofs we give it in a general form. Let L be any finitary normal logic. We designate by U_L the set of maximal L-consistent formula sets. For all formulæ φ, define

$$|\varphi|_L = \{\Gamma \in U_L : \varphi \in \Gamma\}.$$

Let T_L be the topology for which the family of sets $|\varphi|_L$, where φ is a formula — call those sets the *propositions* in T_L — is a base. It is clear that T_L is Stone and that the set of propositions coincides with the set of clopen subsets of U_L. Define

$$\begin{aligned} R_L^{\mathbf{B}} &= \{(\Gamma, \Delta) : \forall\chi(\mathbf{B}\,\chi \in \Gamma \Rightarrow \chi \in \Delta)\}, \\ R_L^{\mathbf{K}} &= \{(\Gamma, \Delta) : \forall\chi(\mathbf{K}\,\chi \in \Gamma \Rightarrow \chi \in \Delta)\}, \end{aligned}$$

and, for every formula φ,

$$\begin{aligned} R_L^{*\varphi} &= \{(\Gamma, \Delta) : \forall\chi([*\,\varphi]\,\chi \in \Gamma \Rightarrow \chi \in \Delta)\}, \\ F_L^{\Gamma}|\varphi|_L &= \{\Delta : \forall\chi([*\,\varphi]\,\mathbf{B}\,\chi \in \Gamma \Rightarrow \chi \in \Delta)\}. \end{aligned}$$

(It is easy to prove that, thanks to the fact that L is closed under replacement of provable equivalents, the definitions are formally correct. In other words, if $\varphi \equiv \psi$ is a theorem of L then $|\varphi|_L = |\psi|_L$ and $R_L^{*\varphi} = R_L^{*\psi}$ and $F_L^{\Gamma}|\varphi|_L = F_L^{\Gamma}|\psi|_L$.) Define $\mathscr{F}_L = (U_L, T_L, R_L, F_L)$. It is clear that $\mathscr{F}_L$ is a selector frame. The condition $V_L(\mathrm{P}) = |\mathrm{P}|_L$, for all propositional letters, defines a valuation in (U_L, T_L). Let $\mathscr{M}_L$ be the model defined on $\mathscr{F}_L$ by V_L. By a familiar argument, first used in modal logic by E. J. Lemmon and Dana Scott, it follows that, for all maximal L-consistent sets Σ and all formulæ φ,

$$\mathscr{M}_L \vDash_{\Sigma} \varphi \quad \text{if and only if} \quad \varphi \in \Sigma.$$

Now let us assume that L, still normal and finitary, is an extension of DDL-AGM. It is easy to see that this assumption makes (U_L, T_L, R_L) serial and functional. Thus all we need to do is to verify that $\mathscr{F}_L$ satisfies the special AGM conditions (s1)–(s5) above.

In order to improve readability, from now on we will drop the subscript from expressions such as R_L, F_L, and $|\varphi|_L$. Upper case Greek letters Γ, Δ, Θ will be used to denote maximal L-consistent sets.

Condition (s1). Assume that $(\Gamma, \Delta) \in R^{\mathbf{B}}$. First suppose that $(\Gamma, \Theta) \in R^{\mathbf{B}}$. Suppose that $[*\,\top]\,\mathbf{B}\,\chi \in \Gamma$. Then $\mathbf{B}\,\chi \in \Gamma$ by postulate $(*3)$. Hence $\chi \in \Theta$, which shows that $\Theta \in F^{\Gamma}|\top|$. Conversely, suppose that $\Theta \in F^{\Gamma}|\top|$. Take any χ such that $\mathbf{B}\,\chi \in \Gamma$. Thanks to the assumption, $\mathbf{b}\,\top \in \Gamma$. By postulate $(*4)$, therefore, $[*\,\top]\,\mathbf{B}\,\chi \in \Gamma$. Hence $\chi \in \Theta$, which shows that $(\Gamma, \Theta \in R^{\mathbf{B}}$.

Condition (s2). Assume that $\Delta \notin \mathsf{C}(\bigcup\{F^{\Gamma}|\varphi| : \varphi \text{ is a formula}\})$. Then there is a formula ψ such that $\mathsf{C}(\bigcup\{F^{\Gamma}|\varphi| : \varphi \text{ is a formula}\}) \subseteq |\psi|$ but $\Delta \notin |\psi|$. In other words, $F^{\Gamma}|\varphi| \subseteq |\psi|$, for every formula φ. In particular, $F^{\Gamma}|\neg\psi| \subseteq |\psi|$, so in fact, thanks to postulate $(*2)$, $F^{\Gamma}|\neg\psi| = \emptyset$. Suppose, for a reductio argument, that $\mathbf{K}\,\psi \notin \mathit{\Gamma}$. Then $[*\,(\neg\psi)]\,\mathbf{B}\,\bot \notin \mathit{\Gamma}$, by postulate $(*5)$. Hence $<*\,(\neg\psi)>\,\mathbf{b}\,\top \in \mathit{\Gamma}$, and so by a familiar argument there is some maximal L-consistent set $\mathit{\Theta}$ such that, for all χ, $[*\,\varphi]\,\mathbf{B}\,\chi \in \mathit{\Gamma}$ only if $\chi \in \mathit{\Theta}$. In other words, $\mathit{\Theta} \in F^{\mathit{\Gamma}}|\neg\psi|$, which is impossible. Consequently, $\mathbf{K}\,\psi \in \mathit{\Gamma}$. But $\psi \notin \mathit{\Delta}$! Hence $(\mathit{\Gamma}, \mathit{\Delta}) \notin R^{\mathbf{K}}$.

Conversely, assume that $\mathit{\Delta} \in \bigcup\{F^{\mathit{\Gamma}}|\varphi| : \varphi \text{ is a formula}\}$. Then there is a particular formula φ such that $\mathit{\Delta} \in F^{\mathit{\Gamma}}|\varphi|$. Suppose that $\mathbf{K}\,\chi \in \mathit{\Gamma}$. First suppose that $\mathbf{k}\,\varphi \in \mathit{\Gamma}$. Then $[*\,\varphi]\,\mathbf{K}\,\chi \in \mathit{\Gamma}$, by postulate $(\mathbf{K}\,*_1)$. Hence $[*\,\varphi]\,\mathbf{B}\,\chi \in \mathit{\Gamma}$, by postulate $(\mathbf{KB})$ and modal logic. Next suppose that $\mathbf{K}\,\neg\varphi \in \mathit{\Gamma}$. Then $[*\,\varphi]\,\mathbf{K}\,\bot \in \mathit{\Gamma}$ by postulate $(\mathbf{K}\,*_2)$. By $(\mathbf{KB})$ and modal logic we conclude, first that $[*\,\varphi]\,\mathbf{B}\,\bot \in \mathit{\Gamma}$ and then that $[*\,\varphi]\,\mathbf{B}\,\chi \in \mathit{\Gamma}$. Thus in either case $[*\,\varphi]\,\mathbf{B}\,\chi \in \mathit{\Gamma}$, which implies that $\chi \in \mathit{\Delta}$. This argument shows that $(\mathit{\Gamma}, \mathit{\Delta}) \in R^{\mathbf{K}}$. Thus we have established that $\{\mathit{\Theta} : (\mathit{\Gamma}, \mathit{\Theta}) \in R^{\mathbf{K}}\}$ includes $\bigcup\{F^{\mathit{\Gamma}}|\varphi| : \varphi \text{ is a formula}\}$ as a subset. But $\{\mathit{\Theta} : (\mathit{\Gamma}, \mathit{\Theta}) \in R^{\mathbf{K}}\} = \bigcap\{|\chi| : \mathbf{K}\,\chi \in \mathit{\Gamma}\}$ is a closed set; therefore $\{\mathit{\Theta} : (\mathit{\Gamma}, \mathit{\Theta}) \in R^{\mathbf{K}}\}$ includes $\mathsf{C}(\bigcup\{F^{\mathit{\Gamma}}|\varphi| : \varphi \text{ is a formula}\})$ as well.

Condition (s3). Assume that $(\mathit{\Gamma}, \mathit{\Delta}) \in R^{*\varphi}$.

First suppose that $\mathit{\Theta} \in F^{\mathit{\Gamma}}|\varphi|$. Say that $[*\,\top]\,\mathbf{B}\,\chi \in \mathit{\Delta}$. Then $\mathbf{B}\,\chi \in \mathit{\Delta}$, by postulate $(*3)$. Hence $<*\,\varphi>\,\mathbf{B}\,\chi \in \mathit{\Gamma}$, whence $[*\,\varphi]\,\mathbf{B}\,\chi \in \mathit{\Gamma}$, by postulate $(*\mathrm{F})$. Hence $\chi \in \mathit{\Theta}$. This shows that $\mathit{\Theta} \in F^{\mathit{\Delta}}|\top|$.

Conversely, suppose that $\mathit{\Theta} \in F^{\mathit{\Delta}}|\top|$. Since $\bot \notin \mathit{\Theta}$, $<*\,\top>\,\mathbf{B}\,\bot \notin \mathit{\Delta}$. By maximality of $\mathit{\Delta}$, $<*\,\top>\,\mathbf{b}\,\top \in \mathit{\Delta}$. By Lemma 1.1.(b), $\mathbf{k}\,\top \in \mathit{\Delta}$, so $\mathbf{K}\,\bot \notin \mathit{\Delta}$. Hence $[*\,\varphi]\,\mathbf{K}\,\bot \notin \mathit{\Gamma}$, whence $\mathbf{K}\,\bot \notin \mathit{\Gamma}$ by postulate $(\mathbf{K}\,*_2)$. Hence $\mathbf{k}\,\top \in \mathit{\Gamma}$, and therefore $<*\,\varphi>\,\mathbf{b}\,\chi \in \mathit{\Gamma}$ by $(*5)$ and modal logic, and so $[*\,\varphi]\,\mathbf{b}\,\top \in \mathit{\Gamma}$ by $(*\mathrm{F})$. Now, by postulate $(*4)$ and modal logic, $\mathit{\Gamma}$ contains the formula $[*\,\varphi]\,\mathbf{b}\,\top \supset ([*\,\varphi]\,\mathbf{B}\,\chi \supset [*\,\varphi][\,[*\,\top]]\,\mathbf{B}\,\chi)$. Modus Ponens, twice applied, yields $[*\,\varphi]\,[*\,\top]\,\mathbf{B}\,\chi \in \mathit{\Gamma}$, whence $[*\,\top]\,\mathbf{B}\,\chi \in \mathit{\Delta}$. Hence $\chi \in \mathit{\Theta}$. This argument shows that $\mathit{\Theta} \in F^{\mathit{\Gamma}}|\varphi|$.

Condition (s4). Assume that $(\mathit{\Gamma}, \mathit{\Delta}) \in R^{*\varphi}$. First suppose that $\mathit{\Theta} \in F^{\mathit{\Gamma}}|\varphi|$. Say $[*\,\varphi]\,\mathbf{B}\,\chi \in \mathit{\Delta}$. Then, with the help of postulate $(*\mathrm{F})$, $[*\,\varphi]\,[*\,\varphi]\,\mathbf{B}\,\chi \in \mathit{\Gamma}$. Hence $[*\,\varphi]\,\mathbf{B}\,\chi \in \mathit{\Gamma}$, by Lemma 1.1.(e). Consequently, $\chi \in \mathit{\Theta}$. This argument shows that $\mathit{\Theta} \in F^{\mathit{\Delta}}|\varphi|$.

Next assume that $\mathit{\Theta} \in F^{\mathit{\Delta}}|\varphi|$. Say that $[*\,\varphi]\,\mathbf{B}\,\chi \in \mathit{\Gamma}$. By Lemma 1.1.(f), $[*\,\varphi]\,[*\,\varphi]\,\mathbf{B}\,\chi \in \mathit{\Gamma}$. Hence $[*\,\varphi]\,\mathbf{B}\,\chi \in \mathit{\Delta}$, and so $\chi \in \mathit{\Theta}$. This argument shows that $\mathit{\Theta} \in F^{\mathit{\Gamma}}|\varphi|$.

Putting the two arguments together, we have shown that $F^{\mathit{\Gamma}}|\varphi| = F^{\mathit{\Delta}}|\varphi|$. (This result is even stronger than the one we needed to prove.)

Condition (s5). Assume that $(\mathit{\Gamma}, \mathit{\Delta}) \in R^{*\varphi}$ and $F^{\mathit{\Gamma}}|\varphi| = \emptyset$. Hence the set $\{\chi : [*\,\varphi]\,\mathbf{B}\,\chi \in \mathit{\Gamma}\}$ is L-inconsistent. Using compactness, we infer that

$[*\varphi]\,\mathbf{B}\perp\in\Gamma$. Hence $\mathbf{K}\neg\varphi\in\Gamma$ by postulate $(*5)$. Hence $[*\varphi]\,\mathbf{K}\perp\in\Gamma$ by postulate $(\mathbf{K}*_2)$. Therefore $\mathbf{K}\perp\in\Delta$. Then, for all ψ, $\mathbf{K}\,\psi\in\Delta$, and in particular, for all ψ, $\mathbf{K}\,\neg\psi\in\Delta$. Thus, by Lemma 1.1.(a), $[*\psi]\,\mathbf{B}\perp\in\Delta$, for all ψ. Consequently, $F^{\Delta}|\psi|=\emptyset$, for all ψ.

This ends the proof that clause (iii) of Theorem 3.1 implies clause (i). □

4. Comparison with AGM

Notwithstanding the title of this paper, it cannot be claimed that the modelling presented here really reflects the work of Alchourrón, Gärdenfors and Makinson; there are certainly features of our system that have no foundation in AGM. The most that can be claimed is that our modelling is a reasonable extrapolation. What makes comparison difficult is that in full DDL all combinations of change operators and doxastic operators are allowed. In their classic papers the three fathers of the logic of theory change pay little attention to the question of iterated change; to the question of nested belief they pay even less. Indeed, nested belief is difficult to discuss in their favoured idiom. Consider the following «dictionary», where T is a set of formulæ and φ and χ are formulæ:

$\chi\in T$	$\mathbf{B}\,\chi$	the agent believes that χ
$\chi\in T*\varphi$	$[*\varphi]\,\mathbf{B}\,\chi$	after revising his beliefs by φ, the agent believes that χ.

The first and second columns are how AGM and DDL, respectively, render the informal condition in the third column. There is no problem for AGM to talk about iterated change: the expression $\chi\in(T*\varphi)*\psi$ is meaningful, corresponding to the formula $[*\varphi]\,[*\psi]\,\mathbf{B}\,\chi$ of DDL. But as long as $\mathbf{B}$ is not an operator of the object language, a DDL formula such as $\mathbf{B}\,\mathbf{B}\,\varphi$ has no natural counterpart in AGM — to resort to expressions like $(\varphi\in T)\in T$ invites complication, if not worse.

Nevertheless, the system presented here has one great advantage over the (iterated) basic version presented in [6]: it avoids the embarrassment over what was there called the Postulate of Unique Methodology. If onions represent belief states, belief change consists in going from one onion to another. In basic DDL the points that make up the universes of our frames are interpreted as «states of the world», while in full DDL we think of them as «possible worlds». The difference is that in basic DDL the agent's beliefs are seen as outside the world, whereas in full DDL they are part of it. It is not surprising that, in full DDL, each possible world determines the onion (the belief state) — it is what you would expect! But if AGM is to be rendered in (iterative) basic DDL, then it is the belief set that must determine the next onion, something that is philosophically implausible and technically awkward. So, in some ways, the modelling presented here may be more faithful to the spirit of AGM than those of [6].

5. Conclusion

The system presented above provides the schema known as Preservation (schema (d) of Lemma 1.1):

$$(*\mathrm{P}) \qquad \mathbf{b}\,\varphi \supset (\mathbf{B}\,\chi \supset [*\,\varphi]\,\mathbf{B}\,\chi).$$

However, this schema is known to give rise to Moore's paradox. For example, putting $\neg\,\mathbf{B}\,\varphi$ or $\mathbf{B}\,\neg\varphi$ for χ in $(*\mathrm{P})$ yields the schemata

$$\mathbf{b}\,\varphi \supset (\mathbf{B}\,\neg\,\mathbf{B}\,\varphi \supset [*\,\varphi]\,\mathbf{B}\,\neg\,\mathbf{B}\,\varphi),$$
$$\mathbf{b}\,\varphi \supset (\mathbf{B}\,\mathbf{B}\,\neg\varphi \supset [*\,\varphi]\,\mathbf{B}\,\mathbf{B}\,\neg\varphi),$$

which against the background of $(*2)$ and modal logic is unintuitive. The undesirability of $(*\mathrm{P})$ in full DDL was noted already by van Linder, van der Hoek and Meyer ([4], p. 114); ways of avoiding it were discussed at length by Lindström and Rabinowicz ([5], p. 139f.).

The validity of $(*\mathrm{P})$ deprives our result of some of the interest it would otherwise have. Nevertheless, there are two reasons for publishing it. One is to show that full DDL, like other modal logics, can yield to the methods of traditional modal logic. The other — related to the first — is the hope of finding more acceptable modellings for which completeness proofs can be given along the lines of the completeness proof outlined here. The author hopes to pursue this topic in a future paper.*

References

[1] Alchourrón, Carlos, Peter Gärdenfors & David Makinson. "On the logic of theory change." *The Journal of Symbolic Logic*, vol. 50 (1985), pp. 510–530.

[2] Hansson, Bengt. "An analysis of some deontic logics." *Noûs*, vol. 3 (1969), pp. 373–398.

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[3] Lewis, David. *Counterfactuals*. Oxford: Blackwell, 1973.

[4] van Linder, Bernd, Wiebe van der Hoek & John-Jules Meyer. "Actions that make you change your mind." In *Knowledge and belief in philosophy and artificial intelligence*, edited by Armin Laux & Heinrich Wansing, pp. 103–146. Berlin: Akademie Verlag GmbH, 1995.

[5] Lindström, Sten & Wlodek Rabinowicz. "Extending dynamic doxastic logic: accommodating iterated beliefs and Ramsey conditionals within DDL." In *For good measure*, edited by Lars Lindahl, Paul Needham & Rysiek Sliwinski, pp. 126–153. Uppsala Philosophical Studies, vol. 46, 1997.

[6] Segerberg, Krister. "The basic dynamic doxastic logic of AGM." In *Frontiers in belif revision*, edited by Marg-Anne Williams & Hans Rott, pp. 57–84. Dordrecht: Kluwer, 2001.

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