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A COMPARISON OF TWO APPROACHES TO PARAINCONSISTENCY: Flemish and Polish

Abstract. In this paper we present a comparison of certain inconsistency adaptive logics and Jaśkowski's logic.

Introduction

One of the first formal ways of handling inconsistency was presented in [11]. The approach opened the way to further investigations concerning paraconsistent logic. In time the importance of so called *M*-fragments of a given modal logic was also discovered.

Another approach is given by D. Batens. The whole project started with the paper [1]. The idea is very attractive and can be applied to many different logical calculi.

1. Inconsistency adaptive logics

The adaptive logics are built with the help of two consequences relations: a weaker one (in the case of inconsistency adaptive logics it is a paraconsistent logic) and a stronger (classical logic). You can always derive a conclusion from given premises using the first kind of consequences relation, while in some cases we can use the second one. The cases in which the stronger consequence relation is allowed are determined by a given strategy. Logics



ACLuN1 and **ACLuN2** are the most famous. In what follows, we will refer only to the propositional part of these logics, since the most important features of the predicative formulations are retained in the propositional versions.

In both these logics, the logic **CLuN** is the lower limit logic. The propositional case of the logic **CLuN** is defined as the full positive classical logic plus the law of excluded middle.

Let us recall a very important theorem from [2] which was originally expressed for the propositional case.

$\vdash_{\mathbf{CL}} A$ iff there are C_1, \dots, C_n ($n \geq 0$) that $\vdash_{\mathbf{CLuN}} \text{DEK}(C_1, \dots, C_n) \vee A$.

This suggests rules of inference for inconsistency adaptive logics.

In the proofs of inconsistency adaptive logics every formula is added under assumption of consistent behavior of some set of formulas. If the set of assumptions under which a given formula is added to the proof, is the empty set, we say that the formula appears unconditionally in the proof.

We only recall the meta-rules which govern the inferences for **ACLuN1** logic. It is enough for our purposes since in the case of consistent sets of premisses both consequences coincide, while in general if $X \vdash_{\mathbf{ACLuN1}} A$ then $X \vdash_{\mathbf{ACLuN2}} A$. For the exhaustive formulations of semantics and syntax of both logics, see for example [3].

We say that a formula of the form $(C_1 \wedge \sim C_1) \vee \dots \vee (C_m \wedge \sim C_m)$ is a minimally inconsistent disjunction in a given proof if it appears in the proof unconditionally and no formula of the form $(C_{i_1} \wedge \sim C_{i_1}) \vee \dots \vee (C_{i_k} \wedge \sim C_{i_k})$ where $k < m$, appears in the given proof unconditionally.

A formula of the form $(C_1 \wedge \sim C_1) \vee \dots \vee (C_m \wedge \sim C_m)$ is a minimally inconsistent consequence of a given set X iff it is a consequence of X in the sense of the logic **CLuN** and no formula of the form $(C_{i_1} \wedge \sim C_{i_1}) \vee \dots \vee (C_{i_k} \wedge \sim C_{i_k})$ where $k < m$, is a consequence of X . Here are the rules:

RU If $\vdash_{\mathbf{CLuN}} (A_1 \wedge \dots \wedge A_n) \rightarrow B$ and A_1, \dots, A_n appears in the proof, B can be added. The assumption under which B is derived is the sum of assumptions under which formulas A_1, \dots, A_n appeared in the proof.

RC If $\vdash_{\mathbf{CLuN}} \left((C_1 \wedge \sim C_1) \vee \dots \vee (C_m \wedge \sim C_m) \right) \vee \left((A_1 \wedge \dots \wedge A_n) \rightarrow B \right)$, and A_1, \dots, A_n appears in the proof, then B can be added provided that no formula of the form $C_i \wedge \sim C_i$ for all $1 \leq i \leq n$ occurs as the disjunct of a minimally inconsistent disjunction in that proof. The appropriate assumption is the sum of assumptions under which formulas A_1, \dots, A_n appeared in the proof plus the set $\{C_1, \dots, C_m\}$.



RD It is obligatory that if A has been inferred under an assumption of consistent behavior of a set containing a formula B , and the formula $B \wedge \sim B$ became a disjunct of a minimally inconsistent disjunction, then A has to be deleted from the proof.

Using the above meta-rules one can formulate the notion of a proof of a given formula on the basis of a set of premises X . But since RD can cause the deletion of a step in a proof, the notion of final derivability has to be introduced. We say that A is finally derived on the basis of premises the set X iff there is a proof of A on the basis of that set, in which A appears under some assumption Δ , and even if A were deleted in some extension of the proof, it could be further extended in such a way that A is derived under Δ once more.

For our purposes the notion of a model is needed. By **CLuN**-model we mean any valuation v classical with respect to positive connectives, while in the case of negation for any A the following is fulfilled: $v(A) = 1$ or $v(\sim A) = 1$. In the standard way we semantically define the consequence relation for **CLuN**.

A **CLuN**-model v is the **ACLuN1** model of a given set X iff it satisfies X and the only formulas of the form $B \wedge \sim B$ valid under v are disjuncts of minimally inconsistent semantical consequences of the set X .

Since for the logic **CLuN** the completeness result is valid, the word ‘semantical’ can be dropped in the last paragraph.

A **CLuN**-model v is the **ACLuN2** model of a given set X iff it satisfies X and it is a minimally inconsistent **CLuN**-model of X , i.e., there is no **CLuN**-model of X , which would have validated fewer inconsistencies.

2. Jaśkowski’s Logic D_2

Jaśkowski’s logic D_2 is a propositional logic defined with the help of the modal logic **S5**.

A formula A is a theorem of the system D_2 iff it is built in the standard way with the help of ‘ \leftrightarrow ’, ‘ \rightarrow ’, ‘ \wedge ’, ‘ \vee ’ and ‘ \sim ’, and the formula $\diamond A'$, which arises by the substitution of $C \wedge \diamond D$, $\diamond C \rightarrow D$, $(\diamond C \rightarrow D) \wedge \diamond(\diamond D \rightarrow C)$ respectively for subformulas $C \wedge D$, $C \rightarrow D$ and $C \leftrightarrow D$ in $\diamond A$, is the theorem of **S5**.¹

The notion of the theorem of D_2 can be expressed more accurately. Let us use the following definition of Jaśkowski’s transformation:

¹ The Appendix contains some basic notions from the modal logic pp. 22–27.



DEFINITION 1. By a Jaśkowski's transformation we mean the function $(\cdot)^d: \text{For} \longrightarrow \text{For}_m$ from the set of all propositional formulas into the set of all modal propositional formulas, defined by induction for any $A \in \text{For}$:

1. If A is a propositional variable, then $A^d = A$
2. (a) A is of the form $B \vee C$, we put $A^d = B^d \vee C^d$
 (b) A is of the form $B \wedge C$, we put $A^d = B^d \wedge \Diamond C^d$
 (c) A is of the form $B \rightarrow C$, we put $A^d = \Diamond B^d \rightarrow C^d$
 (d) A is of the form $B \leftrightarrow C$, we put $A^d = (\Diamond B^d \rightarrow C^d) \wedge \Diamond(\Diamond C^d \rightarrow B^d)$
 (e) A is of the form $\sim B$, we put $A^d = \sim(B^d)$.

Jaśkowski's logic defined with the help of M-fragment of the modal logic \mathbf{P} we call set:

$$\mathbf{P}^J = \{A \in \text{For} : \Diamond A^d \in \mathbf{P}\}$$

i.e., for $A \in \text{For}$: $A \in \mathbf{P}^J$ iff $\Diamond A^d \in \mathbf{P}$.

By the above definition we have:

$$\mathbf{D}_2 = \mathbf{S5}^J.$$

Connectives of conjunction, implication and equivalence can be treated as abbreviations of some modal formulas. In this sense we can call them "discussive" and denote them with the additional symbol ' $_d$ ':

$$\begin{aligned} A \wedge_d B &:= (A \wedge B)^d \\ A \rightarrow_d B &:= (A \rightarrow B)^d \\ A \leftrightarrow_d B &:= (A \leftrightarrow B)^d \end{aligned}$$

Now we can define the consequence relation by saying that from A_1, \dots, A_n a formula A is derivable in the sense of \mathbf{D}_2 iff on the basis of $\Diamond A_1^d, \dots, \Diamond A_n^d$ a formula $\Diamond A^d$ is $\mathbf{S5}$ -provable.²

PROPOSITION 2. *The Logic \mathbf{D}_2 is closed under the Modus Ponens rule:*

$$\text{(MP)} \quad \frac{A \rightarrow B \quad A}{B}$$

² It is obvious that in the case of the presented definition the addition of Gödel's rule makes no difference. See the definition 18 on page 23.



PROOF. It is enough to see that the following rule is provable in **S5**:

$$\frac{\diamond(\diamond A \rightarrow B) \quad \diamond A}{\diamond B}$$

Let us assume that in a given proof $\diamond(\diamond A \rightarrow B)$ and $\diamond A$ appears. By Lemma 23 from p. 24 one can derive $\diamond A \rightarrow \diamond B$, therefore by the rule of Modus Ponens (being used in **S5**) we get $\diamond B$. \square

3. Jaśkowski's logic and inconsistency adaptive logics

In the present section we will compare Jaśkowski' logic and inconsistency adaptive logics **ACLuN1** and **ACLuN2**. To achieve this aim connectives of implication, conjunction and equivalence will be treated as discussive ones. Firstly, let us notice that each axiom of the logic **CLuN** is a theorem of the logic **D₂**: *a fortiori* the full positive logic is contained in **D₂**. Corollary 4 was *implicite* expressed already by Jaśkowski (see [11] and [12]). We present its full proof here.

THEOREM 3. *Each of the following schema represents a theorem of **D₂**.*

- Ax1. $A \rightarrow (B \rightarrow A)$
- Ax2. $((A \rightarrow B) \rightarrow A) \rightarrow A$
- Ax3. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
- Ax4. $A \wedge B \rightarrow A$
- Ax5. $A \wedge B \rightarrow B$
- Ax6. $A \rightarrow (B \rightarrow A \wedge B)$
- Ax7. $A \rightarrow A \vee B$
- Ax8. $B \rightarrow A \vee B$
- Ax9. $(A \rightarrow B) \rightarrow ((C \rightarrow B) \rightarrow (A \vee C \rightarrow B))$
- Ax10. $(A \leftrightarrow B) \rightarrow (A \rightarrow B)$
- Ax11. $(A \leftrightarrow B) \rightarrow (B \rightarrow A)$
- Ax12. $(A \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow (A \leftrightarrow B))$
- Ax13. $(A \rightarrow \sim A) \rightarrow \sim A$



PROOF. Using the introduced definitions of discussive connectives and the definition of Jaśkowski's logic it is enough to show that modal versions (denoted by 'Ax_d') of considered axioms are theorems of **S5**.

Ad Ax_d1. On the basis of **S5** a formula $\diamond(\diamond A \rightarrow (\diamond B \rightarrow A))$ is by Lemma 23 and extensionality equivalent to $\diamond A \rightarrow (\diamond B \rightarrow \diamond A)$. The latter is a substitution of $A \rightarrow (B \rightarrow A)$, so belongs to **S5**.

Ad Ax_d2. By Lemma 23 the following equivalences hold in **S5**: $\diamond[\diamond(\diamond(\diamond A \rightarrow B) \rightarrow A) \rightarrow A]$, $\diamond(\diamond(\diamond A \rightarrow B) \rightarrow A) \rightarrow \diamond A$, $(\diamond(\diamond A \rightarrow B) \rightarrow \diamond A) \rightarrow \diamond A$, $((\diamond A \rightarrow \diamond B) \rightarrow \diamond A) \rightarrow \diamond A$. The latter formula is a substitution of the law of Peirce. Therefore $Ax_2 \in \mathbf{D}_2$.

Ad Ax_d3. The axiom $\diamond[\diamond(\diamond A \rightarrow (\diamond B \rightarrow C)) \rightarrow (\diamond(\diamond A \rightarrow B) \rightarrow (\diamond A \rightarrow C))]$ belongs to **S5**, since by Lemma 23 it is equivalent to the formula $(\diamond A \rightarrow (\diamond B \rightarrow \diamond C)) \rightarrow ((\diamond A \rightarrow \diamond B) \rightarrow (\diamond A \rightarrow \diamond C))$. The latter is a substitution of Frege's law.

Ad Ax_d4. $\vdash_{\mathbf{S5}} \diamond(\diamond(A \wedge \diamond B) \rightarrow A)$ by lemmas 23 and 24, this formula is equivalent to the law of absorption for conjunction.

Ad Ax_d5. $\vdash_{\mathbf{S5}} \diamond(\diamond(A \wedge \diamond B) \rightarrow B)$. The proof proceeds in an analogue way.

Ad Ax_d6. $\vdash_{\mathbf{S5}} \diamond(\diamond A \rightarrow (\diamond B \rightarrow A \wedge \diamond B))$. The proof goes similarly since the formula under consideration is equivalent to the law of adjunction.

Ad Ax_d7. $\vdash_{\mathbf{S5}} \diamond(\diamond A \rightarrow A \vee B)$ This follows directly from the law of absorption for disjunction, Lemma 23 and the monotonicity rule.

Ad Ax_d8. $\vdash_{\mathbf{S5}} \diamond(\diamond B \rightarrow A \vee B)$ as in the previous case.

Ad Ax_d9. $\vdash_{\mathbf{S5}} \diamond(\diamond(\diamond A \rightarrow B) \rightarrow (\diamond(\diamond C \rightarrow B) \rightarrow (\diamond(A \vee C) \rightarrow B)))$. By Lemma 23 this formula is equivalent to $(\diamond A \rightarrow \diamond B) \rightarrow ((\diamond C \rightarrow \diamond B) \rightarrow (\diamond(A \vee C) \rightarrow \diamond B))$, while by regularity (t3 on p. 23) it is equivalent to the substitution of: $(A \rightarrow B) \rightarrow ((C \rightarrow B) \rightarrow (A \vee C \rightarrow B))$.

Ad Ax_d10. $\vdash_{\mathbf{S5}} \diamond[\diamond((\diamond A \rightarrow B) \wedge \diamond(\diamond B \rightarrow A)) \rightarrow (\diamond A \rightarrow B)]$. By lemmas 23 and 24 this formula is equivalent to: $((\diamond A \rightarrow \diamond B) \wedge (\diamond B \rightarrow \diamond A)) \rightarrow (\diamond A \rightarrow \diamond B)$, which is a substitution of the law of the absorption for conjunction.

Ad Ax_d11. $\vdash_{\mathbf{S5}} \diamond[\diamond((\diamond A \rightarrow B) \wedge \diamond(\diamond B \rightarrow A)) \rightarrow (\diamond B \rightarrow A)]$. The reason is the same as above.

Ad Ax_d12. $\vdash_{\mathbf{S5}} \diamond[\diamond(\diamond A \rightarrow B) \rightarrow (\diamond(\diamond B \rightarrow A) \rightarrow ((\diamond A \rightarrow B) \wedge \diamond(\diamond A \rightarrow B)))]$. This formula is equivalent to the substitution of the law of adjunction: $(\diamond A \rightarrow \diamond B) \rightarrow [(\diamond B \rightarrow \diamond A) \rightarrow ((\diamond A \rightarrow \diamond B) \wedge (\diamond A \rightarrow \diamond B))]$.

Ad Ax_d13. $\vdash_{\mathbf{S5}} \diamond(\diamond(\diamond A \rightarrow \sim A) \rightarrow \sim A)$. By Lemma 23 and condition (1) from page 22 of inter-definability of '□' and '◇' it is equivalent to $(\diamond A \rightarrow$



$\sim \Box A) \rightarrow \sim \Box A$, and by the disjunctive syllogism it is equivalent to the schema $(\sim \Diamond A \vee \sim \Box A) \rightarrow \sim \Box A$. The last one is a theorem of the logic **S5** by the contraposition of the axiom $D: \sim \Diamond A \rightarrow \sim \Box A$. (MP) and the substitution of $(A \rightarrow B) \rightarrow (A \vee B \rightarrow B)$. \square

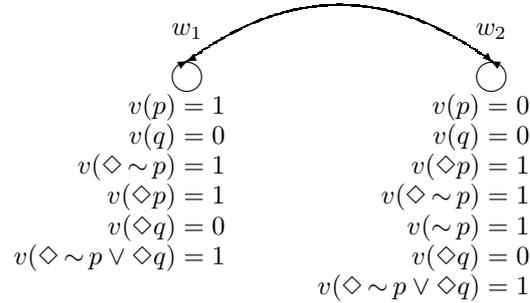
As a consequence of the last theorem and Proposition 2 we obtain:

COROLLARY 4. *The logic **CLuN** is contained in **D₂**.*

However, the analogous observation does not hold for adaptive logics built on **CLuN** as the lower limit logic:

PROPOSITION 5. *The adaptive logics **ACLuN1** and **ACLuN2** are crossing with **D₂**.*

PROOF. Since for any set of formulas X and a formula A holds $X \vdash_{CLuN} A \Rightarrow X \vdash_{ACLuN1} A \Rightarrow X \vdash_{CL} A$, $X \vdash_{CLuN} A \Rightarrow X \vdash_{ACLuN2} A \Rightarrow X \vdash_{CL} A$ (see [2, 3]) and simultaneously, if X is a consistent set, then $X \vdash_{ACLuN1} A \Leftrightarrow X \vdash_{ACLuN2} A \Leftrightarrow X \vdash_{CL} A$, therefore $\{p, \sim p \vee q\} \vdash_{ACLuN1} q$ and $\{p, \sim p \vee q\} \vdash_{ACLuN2} q$. It is easy to see, that $\{p, \sim p \vee q\} \not\vdash_{D_2} q$. Indeed $\{\Diamond p, \Diamond \sim p \vee \Diamond q\} \not\vdash_{S5} \Diamond q$:



On the other hand, one can observe that $\{p, \sim p\} \not\vdash_{ACLuN2} \sim(p \wedge \sim p)$. Indeed, minimally inconsistent **CLuN**-models of our premises do not validate the formula $\sim(p \wedge \sim p)$ (the formula $\{p \wedge \sim p\}$ is a **CLuN** consequence of the set of our premises and there is no **CLuN**-model of our premises which would validate fewer inconsistencies). Since in general $X \vdash_{ACLuN1} A \Rightarrow X \vdash_{ACLuN2}$, the same observation holds for the **ACLuN1** consequence: $\{p, \sim p\} \not\vdash_{ACLuN1} \sim(p \wedge \sim p)$. Of course $\{p, \sim p\} \vdash_{D_2} \sim(p \wedge \sim p)$. In fact, $\Diamond \sim(p \wedge \Diamond \sim p)$ is equivalent to formula $\Diamond(p \rightarrow \Box p)$, which belongs to **S5**.

Let us notice that $\{p \wedge \sim p, p \vee \sim(p \wedge \sim p)\} \not\vdash_{ACLuN1} \sim(p \wedge \sim p)$. Since on the basis of the premises one can unconditionally derive $p \wedge \sim p$, while only assuming consistent behavior of p one can derive $\sim(p \wedge \sim p)$. \square



LEMMA 6. Let $\mathbf{KD}^*\mathbf{T}^*$ is the normal modal logic arising by extension of the logic \mathbf{K} with the axioms:

$$(T^*) \quad \Box\Box A \rightarrow \Box A, \text{ equivalently } \Box(\Box A \rightarrow A)$$

$$(D^*) \quad \Box A \rightarrow \Box A, \text{ equivalently } \Box(\Box A \rightarrow A)$$

In $\mathbf{KD}^*\mathbf{T}^*$ the following formulas are provable:

$$(D) \quad \Box A \rightarrow \Box A$$

$$(DN1) \quad \Box(\Box(A \vee B) \rightarrow (\Box A \vee \Box B))$$

$$(DN2) \quad \Box[\Box(\Box A \rightarrow B) \rightarrow (\Box(\Box C \rightarrow B) \rightarrow (\Box(A \vee C) \rightarrow B))]$$

$$(DN3) \quad \Box A \rightarrow \Box(\Box A \wedge A)$$

$$(DN4) \quad \Box\Box(\Box A \vee A) \rightarrow \Box A$$

PROOF. *Ad* (D)

1. $(\Box A \rightarrow A) \rightarrow (A \rightarrow A)$ (MP), the law of identity, and
 $(A \rightarrow A) \rightarrow ((\Box A \rightarrow A) \rightarrow (A \rightarrow A))$
2. $\Box(\Box A \rightarrow A) \rightarrow \Box(A \rightarrow A)$ 1 and the monotonicity rule
3. $\Box(A \rightarrow A)$ (MP), 2 and (D*)

Ad (DN1)

1. $\Box\Box(A \vee B) \rightarrow \Box\Box A \vee \Box\Box B$ **t3**, monotonicity, and **t4**
2. $\Box\Box A \vee \Box\Box B \rightarrow \Box\Box B \vee \Box\Box A$ commutativity of ' \vee '
3. $\Box\Box(A \vee B) \rightarrow \Box\Box B \vee \Box\Box A$ the law of syllogism, 1 and 2
4. $\Box\Box\Box(A \vee B) \rightarrow \Box\Box\Box B \vee \Box\Box\Box A$ 3, monotonicity rule and **t4**
5. $\Box\Box\Box B \vee \Box\Box\Box A \rightarrow \Box\Box\Box A \vee \Box\Box\Box B$ commutativity of ' \vee '
6. $\Box\Box\Box(A \vee B) \rightarrow \Box\Box\Box A \vee \Box\Box\Box B$ the law of syllogism, 4 and 5
7. $\Box\Box\Box B \rightarrow \Box\Box\Box\Box B$ the substitution to the version of (T*): $A/(\Box\Box B)$
8. $\Box\Box\Box\Box B \rightarrow \Box\Box\Box B$ the axiom (T*) and monotonicity rules
9. $\Box\Box\Box B \rightarrow \Box\Box\Box B$ the law of syllogism, 7 and 8
10. $\Box\Box\Box A \vee \Box\Box\Box B \rightarrow \Box\Box\Box A \vee \Box\Box\Box B$
the substitution of $(A \rightarrow B) \rightarrow ((C \vee A) \rightarrow (C \vee B))$, (MP) and 9
11. $\Box\Box\Box(A \vee B) \rightarrow \Box\Box\Box A \vee \Box\Box\Box B$ the law of syllogism, 6 and 10
12. $\Box\Box\Box A \vee \Box\Box\Box B \rightarrow \Box(\Box\Box A \vee \Box\Box B)$ **t3**
13. $\Box\Box\Box(A \vee B) \rightarrow \Box(\Box\Box A \vee \Box\Box B)$ the law of syllogism, 11 and 12
14. $\Box(\Box\Box(A \vee B) \rightarrow (\Box\Box A \vee \Box\Box B))$ 13 and **t5**



Ad (DN2)

1. $(\Box\Diamond A \rightarrow \Diamond B) \rightarrow \{(\Box\Diamond C \rightarrow \Diamond B) \rightarrow [(\Box\Diamond(A \vee C) \rightarrow (\Box\Diamond A \vee \Box\Diamond C)) \rightarrow (\Box\Diamond(A \vee C) \rightarrow \Diamond B)]\}$
the substitution of
 $(A \rightarrow B) \rightarrow \{(C \rightarrow B) \rightarrow ((D \rightarrow A \vee C) \rightarrow (D \rightarrow B))\}$
2. $\Box\Diamond(\Diamond A \rightarrow B) \rightarrow \Diamond(\Diamond A \rightarrow B)$ the axiom (D*): $A/(\Diamond A \rightarrow B)$
3. $\Box(\Box\Diamond A \rightarrow \Diamond B) \rightarrow (\Box\Diamond A \rightarrow \Diamond B)$ 2 and t5
4. $\Box\Diamond(\Diamond C \rightarrow B) \rightarrow \Diamond(\Diamond C \rightarrow B)$ the axiom (D*): $A/(\Diamond C \rightarrow B)$
5. $\Box(\Box\Diamond C \rightarrow \Diamond B) \rightarrow (\Box\Diamond C \rightarrow \Diamond B)$ 4 and t5
6. $\Box(\Box\Diamond A \rightarrow \Diamond B) \rightarrow \{\Box(\Box\Diamond C \rightarrow \Diamond B) \rightarrow [(\Box\Diamond(A \vee C) \rightarrow (\Box\Diamond A \vee \Box\Diamond C)) \rightarrow (\Box\Diamond(A \vee C) \rightarrow \Diamond B)]\}$
the commutativity rule, the law of syllogism, 5, 3 and 1
7. $(\Box\Diamond(A \vee C) \rightarrow (\Box\Diamond A \vee \Box\Diamond C)) \rightarrow \{ \Box(\Box\Diamond A \rightarrow \Diamond B) \rightarrow [\Box(\Box\Diamond C \rightarrow \Diamond B) \rightarrow (\Box\Diamond(A \vee C) \rightarrow \Diamond B)] \}$
the commutativity rule and 6
8. $\Diamond(\Box\Diamond(A \vee C) \rightarrow (\Box\Diamond A \vee \Box\Diamond C)) \rightarrow \Diamond\{ \Box(\Box\Diamond A \rightarrow \Diamond B) \rightarrow [\Box(\Box\Diamond C \rightarrow \Diamond B) \rightarrow (\Box\Diamond(A \vee C) \rightarrow \Diamond B)] \}$
monotonicity rule and 7
9. $\Diamond\{ \Box(\Box\Diamond A \rightarrow \Diamond B) \rightarrow [\Box(\Box\Diamond C \rightarrow \Diamond B) \rightarrow (\Box\Diamond(A \vee C) \rightarrow \Diamond B)] \}$
(MP), 8 and (DN1)
10. $\Diamond[\Box\Diamond(\Diamond A \rightarrow B) \rightarrow [\Box\Diamond(\Diamond C \rightarrow B) \rightarrow \Diamond(\Diamond(A \vee C) \rightarrow B)]]$
9, laws of extensionality and t5
11. $\Diamond\Diamond[\Diamond(\Diamond A \rightarrow B) \rightarrow [\Diamond(\Diamond C \rightarrow B) \rightarrow (\Diamond(A \vee C) \rightarrow B)]]$
10, extensionality and t5
12. $\Box\Diamond\Diamond[\Diamond(\Diamond A \rightarrow B) \rightarrow [\Diamond(\Diamond C \rightarrow B) \rightarrow (\Diamond(A \vee C) \rightarrow B)]]$
11 and (RG)
13. $\Diamond[\Diamond(\Diamond A \rightarrow B) \rightarrow [\Diamond(\Diamond C \rightarrow B) \rightarrow (\Diamond(A \vee C) \rightarrow B)]]$
the axiom (T*), 12 and (MP)

Ad (DN3) We will prove an auxiliary theorem:

$$(DN3') \quad \Box\Diamond(\Diamond A \vee A) \rightarrow \Diamond A$$

1. $\Diamond[\Diamond(\Diamond\Diamond A \rightarrow A) \rightarrow (\Diamond(\Diamond A \rightarrow A) \rightarrow (\Diamond(\Diamond A \vee A) \rightarrow A))]$
the substitution of (DN2): $A/\Diamond A$, B/A and C/A
2. $\Box(\Box\Diamond\Diamond A \rightarrow \Diamond A) \rightarrow [\Box(\Box\Diamond A \rightarrow \Diamond A) \rightarrow (\Box\Diamond(\Diamond A \vee A) \rightarrow \Diamond A)]$
1, extensionality and t5
3. $\Box(\Box\Diamond\Diamond A \rightarrow \Diamond A)$ (T*) and (RG)
4. $\Box(\Box\Diamond A \rightarrow \Diamond A)$ (D*) and (RG)
5. $\Box\Diamond(\Diamond A \vee A) \rightarrow \Diamond A$ (MP), 2, 3 and 4



Using the rule of contraposition to (DN3'), where $\sim A$ is substituted for A we get: $\sim \diamond \sim A \rightarrow \sim \square \diamond (\diamond \sim A \vee \sim A)$. (DN3) is arising by the condition (1), *via* de Morgan's law:

6. $\square \diamond (\diamond \neg A \vee \neg A) \rightarrow \diamond \neg A$ contraposition of the substitution of (DN3')
7. $\square A \rightarrow \diamond \square (\square A \wedge A)$ 6, contraposition, de Morgan's law and (1)

Ad (DN4)

1. $(\square A \vee A) \rightarrow (\diamond A \vee A)$ the axiom D and addition of a new disjunct to arguments of an implication
2. $\square \diamond (\square A \vee A) \rightarrow \square \diamond (\diamond A \vee A)$ 1 and the monotonicity rules
3. $\square \diamond (\diamond A \vee A) \rightarrow \diamond A$ (DN3')
4. $\square \diamond (\square A \vee A) \rightarrow \diamond A$ the law of syllogism, 2 and 3 \square

THEOREM 7. *The minimal normal logic, which contains axioms $\diamond(\text{Ax1})^d$ – $\diamond(\text{Ax13})^d$, closed with respect to the rule:*

$$\frac{\diamond A \quad \diamond(\diamond A \rightarrow B)}{\diamond B}$$

is the logic $\mathbf{KD}^\mathbf{T}^*$, i.e., $\mathbf{KD}^*\mathbf{T}^* = \mathbf{J}(\mathbf{CLuN})$*

PROOF. Firstly, we will show that each of axioms Ax_d1 – Ax_d13 is provable in $\mathbf{KD}^*\mathbf{T}^*$.

Ad Ax_d1 .

1. $\square \diamond A \rightarrow \diamond A$ (D*)
2. $\square \diamond B \rightarrow (\square \diamond A \rightarrow \diamond A)$ 1 and the classical logic
3. $\square \diamond A \rightarrow (\square \diamond B \rightarrow \diamond A)$ 2 and the commutativity law
4. $\diamond(\diamond A \rightarrow (\diamond B \rightarrow A))$ 3, t5 and extensionality rule

Ad Ax_d2 . Let us start with the observation, that in the considered logic the following theorem is provable:

$$\text{(DN4')} \quad \square \diamond (\square \diamond A \vee A) \rightarrow \diamond A$$

Indeed:

1. $\square \diamond A \rightarrow \diamond A$ (D*)
2. $\square \diamond A \vee A \rightarrow \diamond A \vee A$ 1 and the addition of a new disjunct to arguments of an implication
3. $\square \diamond (\square \diamond A \vee A) \rightarrow \square \diamond (\diamond A \vee A)$ 2, monotonicity rules
4. $\square \diamond (\diamond A \vee A) \rightarrow \diamond A$ (DN3')
5. $\square \diamond (\square \diamond A \vee A) \rightarrow \diamond A$ 3, 4 and the law of syllogism



Using (DN4') we prove Ax_d2 .

1. $((\Box\Diamond A \wedge \sim\Diamond B) \vee A) \rightarrow (\Box\Diamond A \vee A)$ the law of absorption and the addition of a new disjunct to arguments of an implication
2. $\Box\Diamond((\Box\Diamond A \wedge \sim\Diamond B) \vee A) \rightarrow \Box\Diamond(\Box\Diamond A \vee A)$ 1 and monotonicity rules
3. $\Box\Diamond((\Box\Diamond A \wedge \sim\Diamond B) \vee A) \rightarrow \Diamond A$ 2, (DN4') and the law of the syllogism
4. $\Diamond[\Diamond((\Box\Diamond A \wedge \sim\Diamond B) \vee A) \rightarrow A]$ 3 and **t5**
5. $\Diamond[\Diamond(\sim(\Box\Diamond A \wedge \sim\Diamond B) \rightarrow A) \rightarrow A]$ 4, extensionality and the disjunctive syllogism
6. $\Diamond[\Diamond((\Box\Diamond A \rightarrow \Diamond B) \rightarrow A) \rightarrow A]$ 5, extensionality and the law of negation of ' \rightarrow '
7. $\Diamond[\Diamond(\Diamond(\Diamond A \rightarrow B) \rightarrow A) \rightarrow A]$ 6 and **t5**

Ad Ax_d3 .

1. $(\Box\Diamond A \wedge \Box\sim C) \rightarrow \Box(\Diamond A \wedge \sim C)$ the regularity condition **t2**
2. $(\Box(\Box\Diamond A \rightarrow \Diamond B) \wedge (\Box\Diamond A \wedge \Box\sim C)) \rightarrow (\Box(\Box\Diamond A \rightarrow \Diamond B) \wedge \Box(\Diamond A \wedge \sim C))$ 1 and the the addition of a new conjunct to arguments of an implication
3. $(\Box(\Box\Diamond A \rightarrow \Diamond B) \wedge \Box(\Diamond A \wedge \sim C)) \rightarrow \Box((\Box\Diamond A \rightarrow \Diamond B) \wedge (\Diamond A \wedge \sim C))$ **t2**
4. $(\Box(\Box\Diamond A \rightarrow \Diamond B) \wedge (\Box\Diamond A \wedge \Box\sim C)) \rightarrow \rightarrow \Box((\Box\Diamond A \rightarrow \Diamond B) \wedge (\Diamond A \wedge \sim C))$ the law of syllogism, 2. and 3.
5. $\Box((\Box\Diamond A \rightarrow \Diamond B) \wedge (\Diamond A \wedge \sim C)) \rightarrow \Diamond\Box[\Box((\Box\Diamond A \rightarrow \Diamond B) \wedge (\Diamond A \wedge \sim C)) \wedge ((\Box\Diamond A \rightarrow \Diamond B) \wedge (\Diamond A \wedge \sim C))]$ (DN3): $A/[(\Box\Diamond A \rightarrow \Diamond B) \wedge (\Diamond A \wedge \sim C)]$
6. $\Box((\Box\Diamond A \rightarrow \Diamond B) \wedge (\Diamond A \wedge \sim C)) \rightarrow \Box\Diamond A$ the absorption law and the monotonicity rule
7. $[\Box((\Box\Diamond A \rightarrow \Diamond B) \wedge (\Diamond A \wedge \sim C)) \wedge ((\Box\Diamond A \rightarrow \Diamond B) \wedge (\Diamond A \wedge \sim C))] \rightarrow \rightarrow [\Box\Diamond A \wedge ((\Box\Diamond A \rightarrow \Diamond B) \wedge (\Diamond A \wedge \sim C))]$ 6 and the the addition of a new conjunct to arguments of an implication
8. $\Diamond\Box[\Box((\Box\Diamond A \rightarrow \Diamond B) \wedge (\Diamond A \wedge \sim C)) \wedge ((\Box\Diamond A \rightarrow \Diamond B) \wedge (\Diamond A \wedge \sim C))] \rightarrow \rightarrow \Diamond\Box[\Box\Diamond A \wedge ((\Box\Diamond A \rightarrow \Diamond B) \wedge (\Diamond A \wedge \sim C))]$ 7 and the monotonicity rules
9. $\Box\Diamond A \wedge (\Box\Diamond A \rightarrow \Diamond B) \rightarrow \Diamond B$ the substitution of the principle of detachment
10. $[\Box\Diamond A \wedge ((\Box\Diamond A \rightarrow \Diamond B) \wedge (\Diamond A \wedge \sim C))] \rightarrow (\Diamond A \wedge (\Diamond B \wedge \sim C))$ 9 and the the addition of a new conjunct to arguments of an implication, and laws of associativity and commutativity of conjunction
11. $\Diamond\Box[\Box\Diamond A \wedge ((\Box\Diamond A \rightarrow \Diamond B) \wedge (\Diamond A \wedge \sim C))] \rightarrow \Diamond\Box[\Diamond A \wedge (\Diamond B \wedge \sim C)]$



- 10 and the monotonicity rules
12. $\diamond\Box[\diamond A \wedge (\diamond B \wedge \sim C)] \rightarrow \diamond[\Box\diamond A \wedge (\Box\diamond B \wedge \Box\sim C)]$
extensionality and regularity
13. $(\Box(\Box\diamond A \rightarrow \diamond B) \wedge (\Box\diamond A \wedge \Box\sim C)) \rightarrow \diamond(\Box\diamond A \wedge (\Box\diamond B \wedge \Box\sim C))$
the law of syllogism 4, 5, 8, 11 and 12
14. $\sim\diamond(\Box\diamond A \wedge (\Box\diamond B \wedge \Box\sim C)) \rightarrow \sim[\Box(\Box\diamond A \rightarrow \diamond B) \wedge (\Box\diamond A \wedge \Box\sim C)]$
13, contraposition
15. $\Box[\Box\diamond A \rightarrow (\Box\diamond B \rightarrow \diamond C)] \rightarrow [\Box(\Box\diamond A \rightarrow \diamond B) \rightarrow (\Box\diamond A \rightarrow \diamond C)]$
14, the law of negation of '→'
and condition (1)
16. $\Box[\Box\diamond A \rightarrow \diamond(\diamond B \rightarrow C)] \rightarrow [\Box\diamond(\Box\diamond A \rightarrow B) \rightarrow \diamond(\Box\diamond A \rightarrow C)]$
15 and t5
17. $\diamond[\diamond(\Box\diamond A \rightarrow (\diamond B \rightarrow C)) \rightarrow (\diamond(\Box\diamond A \rightarrow B) \rightarrow (\Box\diamond A \rightarrow C))]$
16 and t5

Ad Ax_d4.

1. $A \wedge \diamond B \rightarrow A$ the absorption law
2. $\Box\diamond(A \wedge \diamond B) \rightarrow \Box\diamond A$ 1 and monotonicity rules
3. $\Box\diamond A \rightarrow \diamond A$ (D*)
4. $\Box\diamond(A \wedge \diamond B) \rightarrow \diamond A$ the law of syllogism, 2 and 3
5. $\diamond(\Box\diamond(A \wedge \diamond B) \rightarrow A)$ 4 and t5

Ad Ax_d5.

1. $A \wedge \diamond B \rightarrow \diamond B$ the absorption law
2. $\Box\diamond(A \wedge \diamond B) \rightarrow \Box\diamond\diamond B$ 1 and monotonicity rules
3. $\Box\diamond\diamond B \rightarrow \diamond B$ (T*)
4. $\Box\diamond(A \wedge \diamond B) \rightarrow \diamond B$ the law of syllogism, 2 and 3
5. $\diamond(\Box\diamond(A \wedge \diamond B) \rightarrow B)$ 4 and t5

Ad Ax_d6.

1. $A \rightarrow (\diamond B \rightarrow A \wedge \diamond B)$ the law of adjunction
2. $\diamond A \rightarrow \diamond(\diamond B \rightarrow A \wedge \diamond B)$ 1 and the monotonicity rule
3. $\Box\diamond A \rightarrow \diamond A$ (D*)
4. $\Box\diamond A \rightarrow \diamond(\diamond B \rightarrow A \wedge \diamond B)$ the law of syllogism, 3 and 2
5. $\diamond(\Box\diamond A \rightarrow (\diamond B \rightarrow A \wedge \diamond B))$ 4 and t5

Ad Ax_d7.

1. $\diamond A \rightarrow \diamond(A \vee B)$ the absorption law and the monotonicity rule



2. $\Box \Diamond A \rightarrow \Diamond A$ (D*)
3. $\Box \Diamond A \rightarrow \Diamond(A \vee B)$ the law of syllogism, 2 and 1
4. $\Diamond(\Diamond A \rightarrow A \vee B)$ 3 and t5

Ad Ax_d8. I.e., $\Diamond(\Diamond B \rightarrow A \vee B)$. The proof is analogous.

Ad Ax_d9. I.e., $\Diamond[\Diamond(\Diamond A \rightarrow B) \rightarrow (\Diamond(\Diamond C \rightarrow B) \rightarrow (\Diamond(A \vee C) \rightarrow B))]$. This is just the formula (DN2) from Lemma DN1.

Ad Ax_d10.

1. $\Diamond[\Diamond((\Diamond A \rightarrow B) \wedge \Diamond(\Diamond B \rightarrow A)) \rightarrow (\Diamond A \rightarrow B)]$
Ax_d4, A/ $(\Diamond A \rightarrow B)$, B/ $(\Diamond B \rightarrow A)$

Ad Ax_d11.

1. $\Diamond[\Diamond((\Diamond A \rightarrow B) \wedge \Diamond(\Diamond B \rightarrow A)) \rightarrow (\Diamond B \rightarrow A)]$
Ax_d5, A/ $(\Diamond A \rightarrow B)$, B/ $(\Diamond B \rightarrow A)$

Ad Ax_d12.

1. $\Diamond[\Diamond(\Diamond A \rightarrow B) \rightarrow [\Diamond(\Diamond B \rightarrow A) \rightarrow ((\Diamond A \rightarrow B) \wedge \Diamond(\Diamond B \rightarrow A))]]$
Ax_d6, A/ $(\Diamond A \rightarrow B)$, B/ $(\Diamond B \rightarrow A)$

Ad Ax_d13.

1. $\Box \Diamond(\Box \sim A \vee \sim A) \rightarrow \Diamond \sim A$ (DN4): A/ $\sim A$
2. $\Box \Diamond(\sim \Box \sim A \rightarrow \sim A) \rightarrow \Diamond \sim A$ the disjunctive syllogism
and the extensionality
3. $\Diamond(\Diamond(\Diamond A \rightarrow \sim A) \rightarrow \sim A)$ t5 and the condition (1)

Let us assume that $\Diamond A \in \mathbf{KD}^*\mathbf{T}^*$ and $\Diamond(\Diamond A \rightarrow A) \in \mathbf{KD}^*\mathbf{T}^*$. By (RG) and t5 we have that $\Box \Diamond A \in \mathbf{KD}^*\mathbf{T}^*$ and $\Box \Diamond A \rightarrow \Diamond A \in \mathbf{KD}^*\mathbf{T}^*$. So, by (MP) we have $\Diamond A \in \mathbf{KD}^*\mathbf{T}^*$.

In the opposite direction, it is easy to see that axioms (D*) and (T*) cannot be omitted by postulated minimality — indeed these axioms are just appropriate substitutions of some the axioms Ax_d1–Ax_d13:

Ad T*.

1. $\Diamond(\Diamond(\Diamond A \wedge \Diamond A) \rightarrow A)$ the substitution of Ax_d5: A/ $\Diamond A$ and B/A
2. $\Diamond(\Diamond \Diamond A \rightarrow A)$ 1, idempotency and extensionality
3. $\Box \Diamond \Diamond A \rightarrow \Diamond A$ 2 and t5

Ad D*.

1. $\Diamond(\Diamond A \rightarrow (\Diamond A \rightarrow A))$ the substitution of Ax_d1: B/A



PROOF. We show that using given axiomatization one can prove specific axioms and the rule of the logic $\mathbf{S5}_M$. Let us notice that:

Ad (MT)

1. $\sim \diamond \square (\square A \rightarrow A) \rightarrow \square \diamond (\square A \wedge \sim A)$
the law of negation of ' \rightarrow ' and the condition (1)
2. $\diamond (\square A \wedge \sim A) \rightarrow \diamond \square A \wedge \diamond \sim A$ t6: $A/\square A, B/\sim A$
3. $\square \diamond (\square A \wedge \sim A) \rightarrow \square (\diamond \square A \wedge \diamond \sim A)$ 2 and the monotonicity rule
4. $\square (\diamond \square A \wedge \diamond \sim A) \rightarrow \sim \diamond (\diamond \square A \rightarrow \square A)$ the law of negation of ' \rightarrow '
and condition (1)
5. $\sim \diamond \square (\square A \rightarrow A) \rightarrow \sim \diamond (\diamond \square A \rightarrow \square A)$ the law of syllogism
and 1, 3 and 4
6. $\diamond (\diamond \square A \rightarrow \square A) \rightarrow \diamond \square (\square A \rightarrow A)$ the contraposition of 5
7. $\square \diamond \square A \rightarrow \diamond \square A$ (D*): $A/\square A$
8. $(\square \diamond \square A \rightarrow \diamond \square A) \rightarrow \diamond (\diamond \square A \rightarrow \square A)$ t5
9. $\diamond \square (\square A \rightarrow A)$ 2 \times (MP), 7, 8 and 6

Ad (M5)

1. $\sim \diamond \diamond \square (\diamond \square A \rightarrow \square A) \rightarrow \square \square \diamond (\diamond \square A \wedge \sim \square A)$ the law of negation
of ' \rightarrow ' and condition (1)
2. $\square \square \diamond (\diamond \square A \wedge \sim \square A) \rightarrow (\square \square \diamond \diamond \square A \wedge \sim \diamond \diamond \square \square A)$ t6: $A/(\diamond \square A),$
 $B/(\sim \square A)$, the monotonicity rule and condition (1)
3. $(\square \square \diamond \diamond \square A \wedge \sim \diamond \diamond \square \square A) \rightarrow \sim (\square \square \diamond \diamond \square A \rightarrow \diamond \diamond \square \square A)$
the law of negation of ' \rightarrow '
4. $\sim \diamond \diamond \square (\diamond \square A \rightarrow \square A) \rightarrow \sim (\square \square \diamond \diamond \square A \rightarrow \diamond \diamond \square \square A)$ the law
of syllogism, 1, 2 and 3
5. $(\sim \diamond \diamond \square (\diamond \square A \rightarrow \square A) \rightarrow \sim (\square \square \diamond \diamond \square A \rightarrow \diamond \diamond \square \square A)) \rightarrow$
 $\rightarrow ((\square \square \diamond \diamond \square A \rightarrow \diamond \diamond \square \square A) \rightarrow \diamond \diamond \square (\diamond \square A \rightarrow \square A))$
the law of contraposition
6. $(\square \square \diamond \diamond \square A \rightarrow \diamond \diamond \square \square A) \rightarrow \diamond \diamond \square (\diamond \square A \rightarrow \square A)$ (MP), 4 and 5
7. $\square \diamond \diamond \square A \rightarrow \diamond \square A$ (T*): $A/\square A$
8. $\square \square \diamond \diamond \square A \rightarrow \square \diamond \square A$ 7 and the monotonicity rule
9. $\square \diamond \square A \rightarrow \diamond \square A$ D*: $A/\square A$
10. $\square \square \diamond \diamond \square A \rightarrow \diamond \square A$ the law of syllogism, 8 and 9
11. $\square A \rightarrow \diamond \square \square A$ the equivalent version of (T*)
12. $\diamond \square A \rightarrow \diamond \diamond \square \square A$ 11 and monotonicity rule
13. $\square \square \diamond \diamond \square A \rightarrow \diamond \diamond \square \square A$ the law of syllogism, 10 and 12
14. $\diamond \diamond \square (\diamond \square A \rightarrow \square A)$ (MP), 13 and 6
15. $\square \diamond \diamond \square (\diamond \square A \rightarrow \square A)$ 14 and Gödel's rule



16. $\Box\Diamond\Box(\Diamond\Box A \rightarrow \Box A) \rightarrow \Diamond\Box(\Diamond\Box A \rightarrow \Box A)$ (T*): $A/(\Box(\Diamond\Box A \rightarrow \Box A))$
 17. $\Diamond\Box(\Diamond\Box A \rightarrow \Box A)$ (MP), 15 and 16

It is easy to see that the rule (RT*) is provable:

1. $\Diamond\Diamond A$ assumption
 2. $\Box\Diamond\Diamond A$ Gödel's rule
 3. $\Box\Diamond\Diamond A \rightarrow \Diamond A$ (T*)
 4. $\Diamond A$ (MP), 2 and 3

In the opposite direction we show that the formulas (T*) and (D*) belong to **S5_M**. Indeed we have:

1. $\Box(\Box\Diamond A \rightarrow \Diamond A) \rightarrow \Diamond(\Box\Diamond A \rightarrow \Diamond A)$ (D): $A/(\Box\Diamond A \rightarrow \Diamond A)$
 2. $\Diamond\Box(\Box\Diamond A \rightarrow \Diamond A) \rightarrow \Diamond\Diamond(\Box\Diamond A \rightarrow \Diamond A)$ the monotonicity rule and 1
 3. $\Diamond\Box(\Box\Diamond A \rightarrow \Diamond A)$ (MT): $A/\Box A$
 4. $\Diamond\Diamond(\Box\Diamond A \rightarrow \Diamond A)$ (MP), 2 and 3
 5. $\Diamond(\Box\Diamond A \rightarrow \Diamond A)$ (RT*) and 4
 6. $\Diamond(\Box\Diamond A \rightarrow \Diamond A) \rightarrow \Diamond\Diamond(\Diamond A \rightarrow A)$ t5 and monotonicity rule
 7. $\Diamond\Diamond(\Diamond A \rightarrow A)$ (MP), 5 and 6
 8. $\Diamond(\Diamond A \rightarrow A)$ (RT*) and 7
 9. $\Diamond(\Diamond A \rightarrow A) \rightarrow (\Box\Diamond A \rightarrow \Diamond A)$ t5
 10. $\Box\Diamond A \rightarrow \Diamond A$ (MP), 8 and 9

In the proof of the schema (T*) we use the schema (D*).

1. $\Box\Diamond\Diamond A \rightarrow \Diamond\Diamond A$ (D*): $A/\Box A$
 2. $\Box\Box\Diamond\Diamond A \rightarrow \Box\Diamond\Diamond A$ the monotonicity rule and 1
 3. $\Box\Box\Diamond\Diamond A \rightarrow \Diamond\Diamond A$ the law of syllogism, 2 and 1
 4. $(\Box\Box\Diamond\Diamond A \rightarrow \Diamond\Diamond A) \rightarrow \Diamond(\Box\Diamond\Diamond A \rightarrow \Diamond A)$ t5
 5. $\Diamond(\Box\Diamond\Diamond A \rightarrow \Diamond A) \rightarrow \Diamond\Diamond(\Diamond\Diamond A \rightarrow A)$ t5 and the monotonicity rule
 6. $\Diamond\Diamond(\Diamond\Diamond A \rightarrow A)$ $2 \times$ (MP), 3, 4 and 5
 7. $\Diamond(\Diamond\Diamond A \rightarrow A)$ the rule (RT*) and 6
 8. $\Diamond(\Diamond\Diamond A \rightarrow A) \rightarrow (\Box\Diamond\Diamond A \rightarrow \Diamond A)$ t5
 9. $\Box\Diamond\Diamond A \rightarrow \Diamond A$ (MP), 7 and 8 \square

PROOF OF THEOREM 12. Let $\Diamond A \in \mathbf{S5}$. There exists a **S5**-proof: $D_1, \dots, D_n = \Diamond A$. Let us consider a sequence of formulas $\Diamond\Box D_1, \dots, \Diamond\Box D_n = \Diamond\Box\Diamond A$. We show the construction of the **S5_M**-proof of the formula $\Diamond\Box\Diamond A$. The proof goes by induction on $1 \leq i \leq n$. By the induction hypothesis, for each $k < i$ there exists in the sense of the logic **S5_M** a proof of the formula



$\diamond\Box D_k$. We show how to construct an analogous sequence for $\diamond\Box D_i$. If in the given sequence the formula D_i is either the axiom (T) or (5), then in both cases $\diamond\Box D_i$ is also provable as an axiom of the logic $\mathbf{S5}_M$. If on the other hand D_i is either the axiom (K) or a classical theorem, then $\diamond\Box D_i$ can be easily also proved in the sense of $\mathbf{S5}_M$: we use Gödel's rule twice with D_i obtaining in this way the antecedent of the substitution of the axiom (D): $\Box\Box D_i \rightarrow \diamond\Box D_i$. The consequent of this substitution is the required formula. If, however, D_i is of the form $\Box D_j$, for some $j < i$, then by the induction hypothesis for j there exists in $\mathbf{S5}_M$ a proof of formula $\diamond\Box D_j$. By Lemma 8 and the monotonicity rule the given be the induction hypothesis sequence we can extend with steps $k + 1 - k + 4$:

- | | | |
|------|---|--|
| k. | $\vdash_{\mathbf{S5}_M} \diamond\Box D_j$ | by the induction hypothesis |
| k+1. | $\Box D_n \rightarrow \diamond\Box\Box D_j$ | the version of T^* : $A/(D_j)$ – Lemma 8 |
| k+2. | $\diamond\Box D_j \rightarrow \diamond\diamond\Box\Box D_j$ | k+1 and the monotonicity rule |
| k+3. | $\Box\diamond\diamond\Box\Box D_j$ | (MP), k, k+2 and (RG) |
| k+4. | $\diamond\Box\Box D_j$ | (T^*): $A/(\Box\Box D_j)$, (MP) and k+3 |

Let us consider the case that in the initial proof D_i arises by the rule (MP), i.e., $\exists_{m,j < i} D_m = D_j \rightarrow D_i$, therefore by the induction hypothesis we have $\vdash_{\mathbf{S5}_M} \diamond\Box D_j$ and $\vdash_{\mathbf{S5}_M} \diamond\Box(D_j \rightarrow D_i)$. We associate the appropriate sequences, adding the following steps:

- | | | |
|---------|---|---|
| l. | $\vdash_{\mathbf{S5}_M} \diamond\Box D_j$ | by the induction hypothesis |
| l+k. | $\vdash_{\mathbf{S5}_M} \diamond\Box(D_j \rightarrow D_i)$ | by the induction hypothesis |
| l+k+1. | $\Box(D_j \rightarrow D_i) \rightarrow \diamond\Box\Box(D_j \rightarrow D_i)$ | the version of (T^*): $A/(D_j \rightarrow D_i)$ |
| l+k+2. | $\diamond\Box(D_j \rightarrow D_i) \rightarrow \diamond\diamond\Box\Box(D_j \rightarrow D_i)$ | l+k+1 and the monotonicity rule |
| l+k+3. | $\diamond\diamond\Box\Box(D_j \rightarrow D_i)$ | (MP), l+k and l+k+2 |
| l+k+4. | $\Box\Box\diamond\Box D_j \rightarrow \diamond\diamond\diamond\Box D_i$ | the axiom (K) and $t1$, $t5$ |
| l+k+5. | $\Box\Box\diamond\Box D_j$ | $2 \times$ (RG) and l. |
| l+k+6. | $\diamond\diamond\diamond\Box D_i$ | (MP) and l+k+4, l+k+5 |
| l+k+7. | $\Box\diamond\diamond\diamond\Box D_i$ | (RG) and l+k+6 |
| l+k+8. | $\Box\diamond\diamond\diamond\Box D_i \rightarrow \diamond\diamond\Box D_i$ | the axiom (T^*): $A/(\diamond\Box D_i)$ |
| l+k+9. | $\diamond\diamond\Box D_i$ | (MP), l+k+7 and l+k+8 |
| l+k+10. | $\Box\diamond\diamond\Box D_i$ | (RG) and l+k+9 |
| l+k+11. | $\Box\diamond\diamond\Box D_i \rightarrow \diamond\Box D_i$ | the axiom (T^*): $A/(\Box D_i)$ |
| l+k+12. | $\diamond\Box D_i$ | (MP), l+k+10 and l+k+11 |



which ends the inductive proof. So, for $i = n$ we have $\vdash_{\mathbf{S5}_M} \diamond \square \diamond A$. By the axiom (D*), the provable in $\mathbf{S5}_M$ rule of monotonicity and (MP) we have: $\vdash_{\mathbf{S5}_M} \diamond \diamond A$, while by Gödel's rule, with the help of the axiom (T*) we conclude that $\vdash_{\mathbf{S5}_M} \diamond A$. Since all our axioms belongs to the M -fragment of $\mathbf{S5}$, so the postulated minimality $\mathbf{S5}_M$ is stated. \square

3.1. Semantics of the logic $\mathbf{S5}_M$

Now we give conditions for frames which establish the completeness result for the logic under consideration. We'll use Theorem 13 and the following observations semantically characterizing logic $\mathbf{KD}^*\mathbf{T}^*$.

THEOREM 14. *A formula is valid in all frames satisfying the condition*

$$(*) \quad \forall_w \exists_{\bar{w}} (wR\bar{w} \wedge \forall_{w'} (\bar{w}Rw' \rightarrow wRw'))$$

iff it is provable in the logic \mathbf{K} extended with (D), i.e., in $\mathbf{K}[D^*]$ ($= \mathbf{D}^*$).*

PROOF. “ \Leftarrow ” Using the standard procedure, *via* Lemma 34 from p. 26 it is enough to show that the axiom (D*) is valid in each frame satisfying the given condition. We assume to the contrary, that there is a frame satisfying the condition (*), where the formula (D*) is not valid, so there is a world w and some valuation v , that $w \not\models_v (D^*)$, i.e., $w \models_v \square \diamond p$ and $w \not\models_v \diamond p$. By the definition of the notion of the truth at a world for ‘ \diamond ’ we have $\bar{w} \not\models_v p$ for all worlds \bar{w} , where $wR\bar{w}$; in particular, we have $w' \not\models_v p$ for the world w' postulated in the condition (*). By the assumption and the definition of truth for ‘ \square ’, we see that $w' \models_v \diamond p$, thus there is w'' , that $w'Rw''$ and $w'' \models_v p$, however by (*) we obtain wRw'' , i.e., $w \models_v \diamond p$ which is a contradiction.

“ \Rightarrow ” Let us consider the canonical model of the logic \mathbf{D}^* .

Firstly we show that for each world w , the set $\{A : \square A \in w\} \cup \{\square A : \square A \in w\}$ is consistent with respect to \mathbf{D}^* . Assume otherwise, i.e., there are formulas $\square A_1, \dots, \square A_n, \square A'_1, \dots, \square A'_m \in w$ that $\vdash_{\mathbf{D}^*} \neg(A_1 \wedge \dots \wedge A_n \wedge \square A'_1 \wedge \dots \wedge \square A'_m)$. By classical logic we have $\vdash_{\mathbf{D}^*} A_1 \wedge \dots \wedge A_n \wedge A'_1 \wedge \dots \wedge A'_m \rightarrow A_1 \wedge \dots \wedge A_n$, and on the basis of the logic \mathbf{K} using **t2** and obvious induction we get $\vdash_{\mathbf{D}^*} \square(A_1 \wedge \dots \wedge A_n \wedge A'_1 \wedge \dots \wedge A'_m) \rightarrow (\square A_1 \wedge \dots \wedge \square A_n \wedge \square A'_1 \wedge \dots \wedge \square A'_m)$, while using the absorption law for conjunction we obtain $\vdash_{\mathbf{D}^*} \square(A_1 \wedge \dots \wedge A_n \wedge A'_1 \wedge \dots \wedge A'_m) \rightarrow (\square A'_1 \wedge \dots \wedge \square A'_m)$. Let us denote $A_1 \wedge \dots \wedge A_n \wedge A'_1 \wedge \dots \wedge A'_m$ as A . Our observations can be written as follows: $\vdash_{\mathbf{D}^*} A \rightarrow A_1 \wedge \dots \wedge A_n$ and $\vdash_{\mathbf{D}^*} \square A \rightarrow (\square A'_1 \wedge \dots \wedge \square A'_m)$.

Apparently $\square A \in w$; by the law of adjunction of implications we have: $\vdash_{\mathbf{D}^*} A \wedge \square A \rightarrow A_1 \wedge \dots \wedge A_n \wedge (\square A'_1 \wedge \dots \wedge \square A'_m)$, and *via* the law of



contraposition $\vdash_{\mathbf{D}^*} \neg[A_1 \wedge \dots \wedge A_n \wedge (\Box A'_1 \wedge \dots \wedge \Box A'_m)] \rightarrow \neg(A \wedge \Box A)$. Therefore, by the assumption and Modus Ponens, we have: $\vdash_{\mathbf{D}^*} \neg(A \wedge \Box A)$, equivalently $\vdash_{\mathbf{D}^*} \Box A \rightarrow \neg A$. By Gödel's rule and **t1** we conclude $\vdash_{\mathbf{D}^*} \Diamond \Box A \rightarrow \neg \Box A$. Using Lemma 9 and the law of syllogism we get $\vdash_{\mathbf{D}^*} \Box A \rightarrow \neg \Box A$, i.e., $\neg \Box A \in w$, therefore $\Box A \notin w$. However $\Box A \in w$, which is a contradiction.

We are ready to prove that the accessibility relation R of the canonical model fulfills the condition $(*)$. Since the set $\mathcal{W} = \{A : \Box A \in w\} \cup \{\Box A : \Box A \in w\}$ is consistent with respect to \mathbf{D}^* , there is thus a maximally consistent set w' containing \mathcal{W} , by the definition of the accessibility relation in the canonical model we state that wRw' . Indeed, if $\Box A \in w$, then $A \in \mathcal{W} \subseteq w'$. Let w'' be any set that $w'Rw''$. We prove that wRw'' . Let us assume that $\Box B \in w$. By the definition of w' clearly $\Box B \in w'$, but since $w'Rw''$, so $B \in w''$.

Thus the canonical model of the logic \mathbf{D}^* belongs to the class of models fulfilling $(*)$, then if some formula is valid in all frames satisfying the condition $(*)$, it is also valid in the canonical model, but by Lemma 36 each formula valid in the canonical model, is provable in \mathbf{D}^* . \square

THEOREM 15. *A formula is valid in all frames fulfilling the condition*

$$(\otimes) \quad \forall_w \exists_{\bar{w}} (wR\bar{w} \wedge \forall_{w'} \forall_{w''} (\bar{w}Rw' \wedge w'Rw'' \rightarrow wRw''))$$

iff it is provable in the logic \mathbf{K} with the axiom T^ , i.e., in $\mathbf{K}[T^*]$ ($= \mathbf{T}^*$).*

PROOF. “ \Leftarrow ” We prove the the axiom T^* is valid in each frame fulfilling the given condition.

Assume to the contrary that there exists a frame fulfilling the condition (\otimes) , in which the formula T^* is refuted. Then there is a world w and valuation v , that $w \not\models_v T^*$, i.e., $w \models_v \Box \Diamond \Diamond p$ and $w \not\models_v \Diamond p$. The last condition *via* the definition of truth says that $\underline{w} \not\models_v p$ for all possible worlds \underline{w} , such that $wR\underline{w}$, i.e., we also have $\bar{w} \not\models_v p$ for \bar{w} , which is postulated by (\otimes) . By the assumption $w \models_v \Box \Diamond \Diamond p$ and the conditions of truth for ‘ \Box ’ for ‘ \Diamond ’ there are worlds w' and w'' , that $\bar{w}Rw'$, $w'Rw''$ and $w'' \models_v p$, then by the condition (\otimes) we get wRw'' , i.e., $w \models_v \Diamond p$ which is a contradiction.

“ \Rightarrow ” We follow the proof of the previous theorem. Let us consider the canonical model of the logic \mathbf{T}^* . We prove the canonical frame satisfies (\otimes) . Let us start with the observation that for each possible world w , the set of formulas $\{A : \Box A \in w\} \cup \{\Box \Box A : \Box A \in w\}$ is \mathbf{T}^* -consistent. Assume otherwise, i.e., there are formulas $\Box A_1, \dots, \Box A_n, \Box A'_1, \dots, \Box A'_m \in w$ that



$\vdash_{\mathbf{T}^*} \neg(A_1 \wedge \dots \wedge A_n \wedge \Box\Box A'_1 \wedge \dots \wedge \Box\Box A'_m)$. By the law of absorption we have $\vdash_{\mathbf{T}^*} A_1 \wedge \dots \wedge A_n \wedge A'_1 \wedge \dots \wedge A'_m \rightarrow A_1 \wedge \dots \wedge A_n$, while on the basis of \mathbf{K} , and *via* **t2** we get $\Box\Box(A_1 \wedge \dots \wedge A_n \wedge A'_1 \wedge \dots \wedge A'_m) \rightarrow (\Box\Box A_1 \wedge \dots \wedge \Box\Box A_n \wedge \Box\Box A'_1 \wedge \dots \wedge \Box\Box A'_m)$, thus once more using the law of absorption for conjunction we see that $\vdash_{\mathbf{K}} \Box\Box(A_1 \wedge \dots \wedge A_n \wedge A'_1 \wedge \dots \wedge A'_m) \rightarrow (\Box\Box A'_1 \wedge \dots \wedge \Box\Box A'_m)$. If we denote the formula $A_1 \wedge \dots \wedge A_n \wedge A'_1 \wedge \dots \wedge A'_m$ by A , then the above observations can be written: $\vdash_{\mathbf{T}^*} A \rightarrow A_1 \wedge \dots \wedge A_n$ and $\vdash_{\mathbf{T}^*} \Box\Box A \rightarrow (\Box\Box A'_1 \wedge \dots \wedge \Box\Box A'_m)$. Clearly $\Box A \in w$, indeed since $\Box A_1, \dots, \Box A_n, \Box A'_1, \dots, \Box A'_m \in w$, thus *via* the law of adjunction also $\Box A_1 \wedge \dots \wedge \Box A_n \wedge \Box A'_1 \wedge \dots \wedge \Box A'_m \in w$. In \mathbf{K} we have $\Box A_1 \wedge \dots \wedge \Box A_n \wedge \Box A'_1 \wedge \dots \wedge \Box A'_m \rightarrow \Box(A_1 \wedge \dots \wedge A_n \wedge A'_1 \wedge \dots \wedge A'_m)$, so $\Box(A_1 \wedge \dots \wedge A_n \wedge A'_1 \wedge \dots \wedge A'_m) \in w$; and using the law of addition of implications we state: $A \wedge \Box\Box A \rightarrow A_1 \wedge \dots \wedge A_n \wedge (\Box\Box A'_1 \wedge \dots \wedge \Box\Box A'_m)$, by the contraposition law and the assumption we have: $\vdash_{\mathbf{T}^*} \neg(A \wedge \Box\Box A)$, equivalently $\vdash_{\mathbf{T}^*} \Box\Box A \rightarrow \neg A$. By Gödel's rule and **t1** we deduce that $\vdash_{\mathbf{T}^*} \Diamond\Box\Box A \rightarrow \neg\Box A$. By Lemma 8 we have $\vdash_{\mathbf{T}^*} \Box A \rightarrow \Diamond\Box\Box A$, and by the law of syllogism we get $\vdash_{\mathbf{T}^*} \Box A \rightarrow \neg\Box A$, i.e., $\neg\Box A \in w$, contrary to the earlier observation.

We show that canonical frame of the logic \mathbf{T}^* satisfies the condition (\otimes) . The set $\{A : \Box A \in w\} \cup \{\Box\Box A : \Box A \in w\}$ is consistent with respect to \mathbf{T}^* , so it is contained in a maximally consistent set. Let us denote it by \bar{w} . By the definition of the accessibility relation in the canonical frame we get $wR\bar{w}$. Let w', w'' be any possible world such that $\bar{w}Rw'$ and $w'Rw''$. We show that wRw'' . Let us assume that $\Box B \in w$, by the definition of the world \bar{w} we see that $\Box\Box B \in \bar{w}$, by *via* the assumption about w' and w'' we get $\Box B \in w'$ and $B \in w''$, since B was any formula, we have wRw'' . The rest of the proof follows in the standard way, as in the previous theorem. \square

The above semantical conditions express a kind of weaker version of the condition of transitivity of the relation R . Therefore obviously these theorems can be generalized:

THEOREM 16. *A formula is valid in all frames fulfilling the condition*

$$\forall_w \exists_{\bar{w}} (wR\bar{w} \wedge \forall_{w_1} \forall_{w_2} (\bar{w}Rw_1 \wedge w_1Rw_2 \wedge \dots \wedge w_{n-1}Rw_n \rightarrow wRw_n))$$

iff it is provable in logic \mathbf{K} with the axiom $\Box \underbrace{\Diamond \dots \Diamond}_n A \rightarrow \Diamond A$.

PROOF. It is analogous. \square

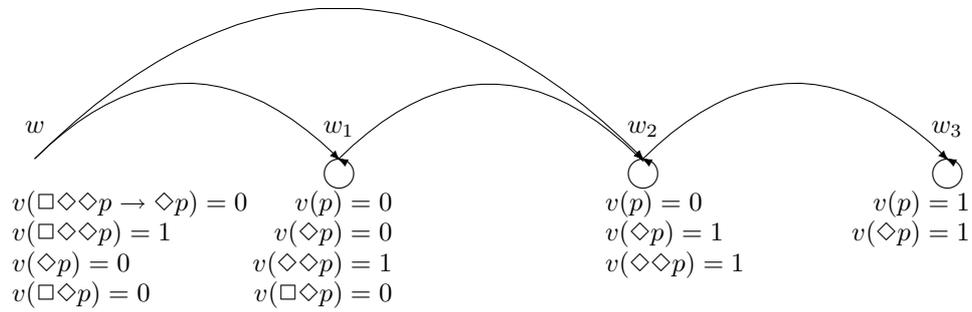
There follows an easily-provable corollary:



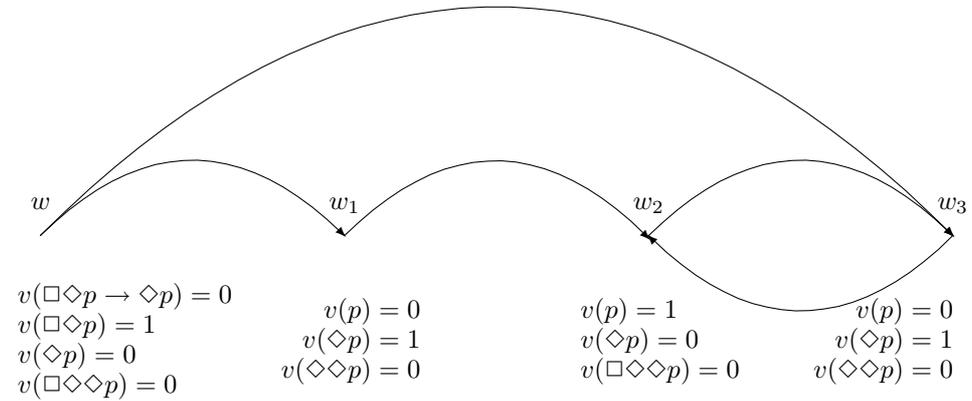
COROLLARY 17. *Formulas (T*) and (D*) are independent on the basis of \mathbf{K} .*

PROOF. We point out a model whose frame fulfills (*) but does not fulfill the condition (⊗).

$W := \{w, w_1, w_2, w_3\}$, the relation R between possible worlds is indicated with arrows, which ends in the second argument of an ordered pair. One can easily see that in the world w formula $\Box\Diamond\Diamond p \rightarrow \Diamond p$ is not satisfied, while formula D^* is true in the model.



Now let us consider a model which for which the condition (⊗) is satisfied, and the condition (*) is not satisfied. Let $W = \{w, w_1, w_2, w_3\}$. In each world the formula (T*) is satisfied, while the formula D^* is not satisfied in the world w .



□



Appendix: Preliminaries in modal logic

Syntax

By a *logic* we mean any set of formulas, which is closed under the substitution rule and the consequence relation.

We recall standard notions concerning *modal logics*. The language of a modal logic arises by enriching the standard propositional language with the unary logical constant ' \diamond ' (so called *possibility*) or ' \square ' (*necessity*), which are inter-definable by the following condition:

$$(1) \quad \diamond A \leftrightarrow \sim \square \sim A.$$

We say that ' \diamond ' and ' \square ' are dual connectives.

By the logic **K** we mean the smallest modal logic containing the classical logic, the Kripke axiom

$$(K) \quad \square(A \rightarrow B) \rightarrow (\square A \rightarrow \square B),$$

closed under rules of Modus Ponens (MP) and the Gödel's rule:

$$(RG) \quad \frac{A}{\square A}$$

A *normal modal logic* is a set of formulas which contains **K** and is closed under the same as **K** rules.

By the logic **D** we mean the minimal normal logic which contains the axiom³:

$$(D) \quad \square A \rightarrow \diamond A$$

The logic **T** is the normal modal logic obtained by adding to **K**, the axiom:

$$(T) \quad \square A \rightarrow A \text{ or equivalently } A \rightarrow \diamond A$$

The logic **S4** is the normal modal logic defined by adding to **T**, the axiom

$$(4) \quad \diamond \diamond A \rightarrow \diamond A \text{ or equivalently } \square A \rightarrow \square \square A.$$

The logic **S5** is the normal modal logic defined by adding to **T**, the axiom

$$(5) \quad \diamond \square A \rightarrow \square A \text{ or equivalently } \diamond A \rightarrow \square \diamond A.$$

³ We say that this normal logic is defined by adding to **K** the axiom (D*).



The logic **Triv** is the normal modal logic defined by adding to **D**, the axiom

$$\text{(Triv)} \quad A \rightarrow \Box A \text{ or equivalently } \Diamond A \rightarrow A.$$

For each of those logics one can define a consequence relation allowing (MP) to be the only rule of inference. For example we have:

DEFINITION 18. We say that a given formula A is **S5**-provable on the basis of X (notation: $X \vdash_{\mathbf{S5}} A$) iff there is a sequence of formulas $C_1, \dots, C_n = A$, where for each $1 \leq i \leq n$: $C_i \in X$ either C_i is a theorem of **S5**, or arises by (MP) from earlier formulas in the that sequence,⁴

Let us recall some well-known theorems and rules provable in normal modal logics which will be needed in the sequel.

LEMMA 19. *The following rule are provable in all normal logics*

1. *The monotonicity rule:* $\frac{A \rightarrow B}{\Box A \rightarrow \Box B}$,
2. *The dual form of the monotonicity rule:* $\frac{A \rightarrow B}{\Diamond A \rightarrow \Diamond B}$,
3. *The extensionality:* $\frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$,
4. *The dual form of extensionality:* $\frac{A \leftrightarrow B}{\Diamond A \leftrightarrow \Diamond B}$.

LEMMA 20. *The following formulas are theorems in all normal logics*

- (t1) $\Box(A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$,
- (t2) *Regularity:* $\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$,
- (t3) *The dual form of the regularity:* $\Diamond(A \vee B) \leftrightarrow (\Diamond A \vee \Diamond B)$
- (t4) $\Box(A \vee B) \rightarrow (\Box A \vee \Box B)$, $\Box(A \vee B) \rightarrow (\Diamond A \vee \Box B)$,
- (t5) $\Diamond(A \rightarrow B) \leftrightarrow (\Box A \rightarrow \Diamond B)$,
- (t6) $\Diamond(A \wedge B) \rightarrow (\Diamond A \wedge \Diamond B)$,
- (t7) $\Diamond A \wedge \Box B \rightarrow \Diamond(A \wedge B)$,
- (t8) $\Box A \leftrightarrow \neg \Diamond \neg A$

\perp can be defined as the abbreviation of: $p \wedge \sim p$ (or any other contr-tautology).

⁴ Notice that we do not use Gödel's rule here.



LEMMA 21. $\vdash_{\mathbf{K}} \diamond \perp \leftrightarrow \perp$

PROOF.

1. $\perp \rightarrow \diamond \perp$ by the classical logic $\perp \rightarrow A$
2. $\sim \perp$ by the classical logic
3. $\Box \sim \perp$ 2 and (RG)
4. $\sim \perp \rightarrow \Box \sim \perp$ 3 and weakening
5. $\sim \Box \sim \perp \rightarrow \sim \sim \perp$ 4 and the law of contraposition
6. $\diamond \perp \rightarrow \perp$ 5, (1) and the law of double negation \square

LEMMA 22. $\vdash_{\mathbf{S5}} \diamond \diamond A \rightarrow \diamond A$, i.e., (S5) \vdash (4).

PROOF.

1. $\diamond A \rightarrow \Box \diamond A$ the equivalent version of 5
2. $\diamond \diamond A \rightarrow \diamond \Box \diamond A$ Lemma 2
3. $\diamond \Box \diamond A \rightarrow \Box \diamond A$ the substitution of the axiom (5): $A/\diamond A$
4. $\Box \diamond A \rightarrow \diamond A$ the instance of (T): $A/\diamond A$
5. $\diamond \diamond A \rightarrow \diamond A$ the law of syllogism and 2, 3, 4 \square

LEMMA 23. $\vdash_{\mathbf{S5}} \diamond(\diamond A \rightarrow B) \leftrightarrow (\diamond A \rightarrow \diamond B)$

PROOF. follows from two above theorems: $\vdash_{\mathbf{K}} \diamond(\diamond A \rightarrow B) \leftrightarrow (\Box \diamond A \rightarrow \diamond B)$ and $\vdash_{\mathbf{S5}} (\Box \diamond A \rightarrow \diamond B) \leftrightarrow (\diamond A \rightarrow \diamond B)$. \square

LEMMA 24. $\vdash_{\mathbf{S5}} \diamond(A \wedge \diamond B) \leftrightarrow (\diamond A \wedge \diamond B)$.

PROOF. “ \Rightarrow ” By **t6** we have $\vdash_{\mathbf{K}} \diamond(A \wedge \diamond B) \rightarrow \diamond A \wedge \diamond \diamond B$. And by Lemma 22 we obtain $\vdash_{\mathbf{S5}} \diamond A \wedge \diamond \diamond B \rightarrow \diamond A \wedge \diamond B$.

“ \Leftarrow ” Using classical logic and *S5* we have $\vdash_{\mathbf{S5}} (\diamond A \wedge \diamond B) \rightarrow (\diamond A \wedge \Box \diamond B)$, and substituting $\diamond B$ for B in the formula **t7** we see that $\vdash_{\mathbf{K}} \diamond A \wedge \Box \diamond B \rightarrow \diamond(A \wedge \diamond B)$. \square

Let us introduce the following notation:

For any modal logic \mathbf{P} , we define the set $\diamond \mathbf{P} := \{\diamond A : \diamond A \in \mathbf{P}\}$ which is called the *M*-fragment of the logic \mathbf{P} .

For any modal logic \mathbf{P} , let $M(\mathbf{P}) := \{A \in \text{For}_m : \diamond A \in \mathbf{P}\}$, where For_m is the set of all modal formulas. The set $M(\mathbf{P})$ is called *M*-analogon of the logic \mathbf{P} .



Semantics

In this section we present the Kripke semantics for modal logics. Once more we start with some standard definitions, which will be used later.

DEFINITION 25. A *relational frame* (in shortage a *frame*) is a pair $\langle W, R \rangle$ consisting of nonempty set W , and a binary relation R on W . Elements of the set W we call the *points of the frame*, while R is the *accessibility relation*.

DEFINITION 26. A *valuation* is any function $v: \text{Var} \rightarrow 2^W$.

DEFINITION 27. A *model* is a triple $\langle W, R, v \rangle$, where $\langle W, R \rangle$ is a frame and v is a valuation. We say that $\langle W, R, v \rangle$ is based on the frame $\langle W, R \rangle$.

DEFINITION 28. A formula A is *true* in the point $w \in W$ under the valuation v (notation: $w \models_v A$) iff

1. if A is a propositional letter p , then:

$$w \models_v p \Leftrightarrow w \in v(p).$$

2. if A is of the form $\neg B$, for some formula B , then:

$$w \models_v \neg B \Leftrightarrow \text{it is not the case that } w \models_v B \text{ (as abbreviation we use } w \not\models_v B \text{)}.$$

3. if A is of the form $B \wedge C$, for some formulas B and C , then

$$w \models_v B \wedge C \Leftrightarrow w \models_v B \text{ and } w \models_v C.$$

4. if A is of the form $B \vee C$, for some formulas B and C , then

$$w \models_v B \vee C \Leftrightarrow w \models_v B \text{ or } w \models_v C.$$

5. if A is of the form $B \rightarrow C$, for some formulas B and C , then

$$w \models_v B \rightarrow C \Leftrightarrow w \not\models_v B \text{ or } w \models_v C.$$

6. if A is of the form $B \leftrightarrow C$, for some formulas B and C , then

$$w \models_v B \leftrightarrow C \Leftrightarrow (w \models_v B \text{ and } w \models_v C) \text{ or } (w \not\models_v B \text{ and } w \not\models_v C).$$

7. if A is of the form $\diamond B$, for a formula B , then

$$w \models_v \diamond B \Leftrightarrow \exists w' \in W (wRw' \wedge w' \models_v B).$$

8. if A is of the form $\Box B$, for a formula B , then

$$w \models_v \Box B \Leftrightarrow \forall w' \in W (wRw' \Rightarrow w' \models_v B).$$



DEFINITION 29. A formula A is *true* in a model $M = \langle W, R, v \rangle$ (notation $M \models A$) iff $w \models_v A$ for each $w \in W$.

DEFINITION 30. A formula A is *valid* in frame $\langle W, R \rangle$ iff it is true in all models based on $\langle W, R \rangle$.

DEFINITION 31. 1. Logic \mathbf{P} is complete with respect to the class of frames \mathcal{C} iff (a formula A is a theorem of \mathbf{P} iff A is valid in each frame from \mathcal{C} . In that case we say that frames of the class \mathcal{C} are \mathbf{P} -frames.

2. Any frame in which all theorems of a given modal logic \mathbf{P} are valid we call *a frame for \mathbf{P}* .

DEFINITION 32. A is a $\mathbf{S5}$ -consequence of the set X (notation: $X \models_{\mathbf{S5}} A$) iff for any model $\langle W, R, v \rangle$ with equivalence accessibility-relation and for any $w \in W$ if $w \models_v X$ then $w \models_v A$.

LEMMA 33. 1. $\mathbf{S5}$ -frames are frames with the equivalence accessibility-relation.

2. $X \vdash_{\mathbf{S5}} A$ iff $X \models_{\mathbf{S5}} A$.

Let us recall the classical:

LEMMA 34. (a) *The axiom K is valid in any frame.*

(b) *All classical tautologies are valid in each frame.*

(c) *The set of all formulas valid in a given frame is closed under Gödel's rule, (MP) and substitution.*

DEFINITION 35. The set X of formulas is *inconsistent with respect to the modal logic \mathbf{P}* (or shortly \mathbf{P} -inconsistent) iff there are formulas $A_1, \dots, A_n \in X$, such that $\vdash_{\mathbf{P}} \neg(A_1 \wedge \dots \wedge A_n)$. The set X is *consistent with respect to the modal logic \mathbf{P}* (or shortly \mathbf{P} -consistent) iff it not \mathbf{P} -inconsistent.

We have also:

LEMMA 36. *All maximally consistent sets with respect to the modal logic \mathbf{P} contain \mathbf{P} and are closed under (MP).*

Finally let us recall the definition of a compact logic:

DEFINITION 37. A logic \mathbf{P} is *compact* iff for every \mathbf{P} -consistent set of formulas X , there is a world w in some model M based on a frame for \mathbf{P} , such that all the formulas in X are true in that world in M .



Canonical models

DEFINITION 38. Let \mathbf{P} be any modal logic.

1. Let W be a class of all maximally consistent sets with respect to \mathbf{P} , and R be a binary relation on W defined as follows: $wRw' \Leftrightarrow \forall A(\Box A \in w \Rightarrow A \in w')$. The canonical frame of the logic \mathbf{P} is the pair $\langle W, R \rangle$.
2. A canonical model of the logic \mathbf{P} is the model $\langle W, R, v \rangle$, where $\langle W, R \rangle$ is the canonical frame of the logic \mathbf{P} and the following condition is satisfied for any variable p_i :

$$v(p_i) = \{w \in W : p_i \in w\}.$$

i.e., $w \models_v p_i \Leftrightarrow p_i \in w$.

We have:

LEMMA 39. Let $\langle W, R, v \rangle$ be a canonical model of the logic \mathbf{P} .

- (a) For each formula A and $w \in W$ the following holds: $w \models_v A \Leftrightarrow A \in w$.
- (b) A formula is true in $\langle W, R, v \rangle$ iff it is a theorem of \mathbf{P} .

Let us stress that if the notion of truth in the canonical frame instead of truth in the canonical model were used, the point (b) of the above theorem would not hold in general. This is why the important notion of canonical logic is often formulated:

DEFINITION 40. A logic \mathbf{P} is *canonical* iff all theorems of \mathbf{P} are valid in the canonical frame of \mathbf{P} , i.e., if the canonical frame of \mathbf{P} is a frame for \mathbf{P} .

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