Logic and **L**ogical **P**hilosophy
Volume 9 (2001), 35–46

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A NEW AXIOMATIZATION OF JAŚKOWSKI'S DISCUSSIVE LOGIC

Abstract. In 1995 N. C. A. da Costa and F. Doria proposed the modal-type elegant axiomatization of Jaśkowski's discussive logic D_2. Yet his own problem which was formulated in 1975 in a following way: Is it possible to formulate natural and simple axiomatization for D_2, employing classical disjunction and conjunction along with discussive implication and conjunction as the only primitive connectives? — still seems left open. The matter of fact is there are some axiomatizations of D_2 proposed, e.g., by T. Furmanowski (1975), J. Kotas and N. C. A. da Costa (1979), G. Achtelik, L. Dubikajtus, E. Dudek and J. Konior (1981), satisfying da Costa's conditions, but they are rather looking very complicated and unnatural. An attempt is made to solve da Costa's problem. The new axiomatization of D_2 is proposed essentially based on da Costa's-Doria axiomatization from 1995.

In his papers [Jaśkowski 1948] and [Jaśkowski 1949] Polish logician Stanisław Jaśkowski first built the system of paraconsistent logics. He called her a discursive logics because his intention was to describe logics of the *discursive systems* which cannot be said to include theses that express opinion in agreement with one another. Thus in discursive (or discussive) systems some proposition and its negation can be both true but this does not lead to its trivialization.

Jaśkowski himself did not proposed any axiomatization of discursive logics and his consideration was essentially based on the interplay of classical logic and modal system S5 enriched with the three additional connectives of discussive implication, discussive conjunction and discussive equivalence. He defines these connective in the following way:

discussive implication: $\alpha \to \beta \;\equiv\; \Diamond\alpha \supset \beta$,

discussive conjunction: $\alpha \wedge \beta \;\equiv\; \Diamond\alpha \,\&\, \beta$,

discussive equivalence: $\alpha \leftrightarrow \beta \;\equiv\; (\alpha \to \beta) \wedge (\beta \to \alpha)$.

Hereafter we use for convenience the classical connectives $\vee$, $\&$, $\supset$, $\neg$ while Jaśkowski himself used a Polish notation.

Jaśkowski s system D_2 would be described as the smallest set of formulas α satisfying the following conditions:

1. in α only the signs of sentential variables and the signs '$\to$', '$\wedge$', '$\leftrightarrow$', '$\vee$', '$\&$' and '$\neg$' occur;
2. the expression $\Diamond\alpha_{\mathrm{L}}$, where α_{L} is an expression obtained from α by eliminating in accordance with the definitions above the signs '$\to$', '$\wedge$', '$\leftrightarrow$', is a thesis of S5-system (cf. [Kotas 1971, p. 82]).

Following this course of modal-classic interplay da Costa and Dubikajtis in 1968 first gave an infinite axiom set for D_2 [da Costa, Dubikajtis 1968].

The next axiomatization of D_2 was proposed by Kotas in 1974. It would be described in the following way [Kotas 1974].

Let **A** be the set consisting of the following formulas and the rules:

(A_1) $\Box(p \supset (\neg p \supset q))$

(A_2) $\Box((p \supset q) \supset ((q \supset r) \supset (p \supset r)))$

(A_3) $\Box((\neg p \supset p) \supset p)$

(A_4) $\Box(\Box p \supset p)$

(A_5) $\Box(\Box(p \supset q) \supset (\Box p \supset \Box q))$

(A_6) $\Box(\neg\Box p \supset \Box\neg\Box p)$

(R_1) substitution rule

(R_2) $$\frac{\Box\alpha \qquad \Box(\alpha \supset \beta)}{\Box\beta}$$

(R_3) $$\frac{\Box\alpha}{\Box\Box\alpha}$$

(R_4) $$\frac{\Box\alpha}{\alpha}$$

(R_5) $$\frac{\Box\neg\Box\neg\alpha}{\alpha}$$

Let us also introduce the following definition:

$$p \Rightarrow q \stackrel{\text{def}}{=} \neg((\neg r \vee r) \wedge \neg(\neg p \vee q))$$

and two translations: $\mathfrak{i}_1$ from D_2 to S5

(i) $\mathfrak{i}_1(\alpha) = \alpha$, when α is a propositional variable;
(ii) $\mathfrak{i}_1(\neg\alpha) = \neg\mathfrak{i}_1(\alpha)$;
(iii) $\mathfrak{i}_1(\alpha \vee \beta) = \neg\mathfrak{i}_1(\alpha) \supset \mathfrak{i}_1(\beta)$;
(iv) $\mathfrak{i}_1(\alpha \wedge \beta) = \neg(\neg\mathfrak{i}_1(\alpha) \vee \Box\neg\mathfrak{i}_1(\beta)$;
(v) $\mathfrak{i}_1(\alpha \rightarrow \beta) = \neg\Box\neg\mathfrak{i}_1(\alpha) \supset \mathfrak{i}_1(\beta)$;

and $\mathfrak{i}_2$ from S5 to D_2

(vi) $\mathfrak{i}_2(\alpha) = \alpha$, when α is a propositional variable;
(vii) $\mathfrak{i}_2(\neg\alpha) = \neg\mathfrak{i}_2(\alpha)$;
(viii) $\mathfrak{i}_2(\alpha \supset \beta) = \neg\mathfrak{i}_2(\alpha) \vee \mathfrak{i}_2(\beta)$;
(ix) $\mathfrak{i}_2(\Box\alpha) = \neg((\neg p \vee p) \wedge \neg\mathfrak{i}_2(\alpha))$.

The formulas $\mathfrak{i}_2(A_k)$, $k = 1, \ldots, 6$, $\mathfrak{i}_2\mathfrak{i}_1(Fpq) \Rightarrow Fpq$, $Fpq \Rightarrow \mathfrak{i}_2\mathfrak{i}_1(Fpq)$, where instead of the symbol F, symbols $\wedge$, $\rightarrow$, $\vee$ should be put in turn, and the rules $(R_k, \mathfrak{i}_2)$, $k = 1, \ldots, 5$, connected through the translation $\mathfrak{i}_2$ with the rules (R_k), $k = 1, \ldots, 5$, constitute the complete axiom system of D_2.

In 1975 da Costa proposed the following two axiomatizations of D_2.
First version [da Costa 1975, p. 9]:

1. $\Box\alpha$, whenever α is a tautology
2. $\Box(\Box(\alpha \supset \beta) \supset \Box(\Box\alpha \supset \Box\beta))$
3. $\Box(\Box\alpha \supset \alpha)$
4. $\Box(\alpha \supset \Box\Diamond\alpha)$
5. $\Box(\Box(\alpha \supset \Box\Box\alpha)$
6. $\dfrac{\alpha \qquad \Box(\alpha \supset \beta)}{\beta}$ (strict detachment)
7. $\dfrac{\Diamond\alpha}{\alpha}$ (depossibilization)

Second version [da Costa 1975, p. 10]:

1. $\Box\alpha$, whenever α is a tautology

2. $$\frac{\alpha \quad \Box(\alpha \supset \beta)}{\beta}$$ (strict detachment)

3. $$\frac{\Box(\alpha \supset \beta)}{\Box(\Box\alpha \supset \beta)}$$

4. $$\frac{\Diamond\alpha}{\alpha}$$ (depossibilization)

5. $$\frac{\Box(\alpha \supset \beta)}{\Box(\alpha \supset \Box\beta)}$$ whenever α is fully modalized.

The first non-modal axiomatization of D_2 was proposed by Furmanowski (he never published this result by himself and we know it only from the presentation in [Kotas 1975, p. 166–167]):

A1. $\neg(\alpha \supset (\neg\alpha \supset \beta)) \to 0$

A2. $(\alpha \supset \beta) \supset ((\beta \supset \gamma) \supset (\alpha \supset \gamma)) \to 0$

A3. $\neg((\neg\alpha \supset \beta) \supset \alpha) \to 0$

A4. $\neg((\neg\alpha \supset \beta) \supset \alpha) \to \beta$

A5. $\neg((\neg(\alpha \supset \beta) \to 0) \to ((\neg\alpha \to 0) \supset (\neg\beta \supset\!\to 0))) \to 0$

A6. $\neg(\neg\neg(\neg\alpha \supset 0) \vee \neg\neg(\neg\alpha \to 0)) \to 0$

A7. $(\neg(\alpha \supset \beta) \to \gamma) \to ((\neg\alpha \to \gamma) \to (\neg\beta \supset\!\to \gamma))$

A8. $(\neg\alpha \to 0) \to \alpha$

A9. $(\alpha \to \beta) \to (\neg(\alpha \to \beta) \to \beta)$

A10. $\neg(\neg\neg\alpha \to \beta) \to \alpha$

where α, β, γ are arbitrary formulas, $0 \equiv \neg(\neg\alpha \vee \alpha)$.

One more non-modal axiomatization was built by da Costa and Dubikajtis in 1977. According to [da Costa, Dubikajtis 1977] the set of axioms of D_2 would be decomposed into two parts:

Axioms without negation:

A01. $p \to (q \to p)$

A02. $[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$

A03. $[(p \to q) \to p)] \to p$

A04. $(p \wedge q) \to p$

A05. $(p \wedge q) \to q$

A06. $p \to [q \to (p \wedge q)]$

A07. $p \to (p \vee q)$

A08. $q \to (p \vee q)$

A09. $(p \to r) \to [(q \to r) \to ((p \vee q) \to r)]$

Axioms with negation:

A1. $p \to \neg\neg p$

A2. $\neg\neg p \to p$

A3. $\neg(\neg p \vee p) \to q$

A4. $\neg(p \vee q) \to \neg(q \vee p)$

A5. $\neg(p \vee q) \to (\neg p \wedge \neg q)$

A6. $\neg(\neg\neg p \vee q) \to \neg(p \vee q)$

A7. $[\neg(p \vee q) \to r] \to [(\neg p \to q) \vee r]$

A8. $\neg[(p \vee q) \vee r] \to \neg[p \vee (q \vee r)]$

A9. $\neg[(p \to q) \vee r] \to [p \wedge \neg(q \vee r)]$

A10. $\neg[(p \wedge q) \vee r] \to [p \to \neg(q \vee r)]$

A11. $\neg[\neg(p \vee q) \vee r] \to [\neg(\neg p \vee r) \vee \neg(\neg q \vee r)]$

A12. $\neg[\neg(p \to q) \vee r] \to [p \to \neg(\neg q \vee r)]$

A13. $\neg[\neg(p \wedge q) \vee r] \to [p \wedge \neg(\neg q \vee r)]$

Also the following rules of inference were adopted:

substitution rule in the classical form,

derivation rule for discussive implication: $\dfrac{\alpha \qquad \alpha \to \beta}{\beta}$

In 1981 Achtelik, Dubikajtis, Dudek and Konior proposed (answering the problem in [da Costa, Dubikajtis 1977]) to replace axioms A1–A13 by the following axioms [Achtelik *et al* 1981, p. 4]:

N1. $p \to \neg\neg p$

N2. $\neg(\neg p \vee p) \to q$

N3. $\neg(p \vee q) \to \neg p$

N4. $\neg(p \vee q) \to \neg(q \vee p)$

N5. $\neg((p \vee q) \vee r) \to \neg(p \vee (q \vee r))$

N6. $\neg(\neg(p \vee q) \vee r) \to \neg(\neg(\neg p \vee q) \vee \neg(\neg q \vee r))$

N7. $\neg((p \wedge q) \vee r) \rightarrow (p \rightarrow \neg(q \vee r))$

N8. $\neg(\neg(p \wedge q) \vee r) \rightarrow \neg(\neg q \vee r)$

N9. $\neg((p \rightarrow q) \vee r) \rightarrow (p \wedge \neg(q \vee r))$

N10. $\neg(\neg(p \rightarrow q) \vee r) \rightarrow (p \rightarrow \neg(\neg q \vee r))$

N11. $\neg(\neg(p \vee q) \rightarrow r) \rightarrow ((\neg p \rightarrow q) \vee r)$

Fairly unusual on this background appears a new formulation of discussive logic given by Kotas and da Costa in 1979. In their calculus SD_2 they use the following abbreviation:

$\neg * A$ for $\neg(A \wedge (\alpha \vee \neg\alpha))$ (strong negation)

The deductive structure of SD_2 is established in the following manner [Kotas, da Costa 1979, pp. 431–432]:

1. Primitive rules of inference:

R1. $\dfrac{A \rightarrow B \qquad A}{B}$

R2. $\dfrac{A \wedge B}{A} \qquad \dfrac{A \wedge B}{B}$

R3. $\dfrac{A \qquad B}{A \wedge B}$

R4. $\dfrac{A \vee B \qquad \neg^* B}{A}$

R5. $\dfrac{\neg(A \vee \neg A)}{B}$

R6. $\dfrac{\neg A \wedge \neg^* B}{\neg(A \vee B)}$

R7. $\dfrac{\neg(A \vee B)}{\neg A \wedge \neg B}$

R8. $\dfrac{\neg^*(A \vee B)}{\neg^* A \wedge \neg^* B}$

R9. $\dfrac{\neg(A \wedge B)}{\neg A \vee \neg B}$

R10. $\dfrac{\neg^*(A \rightarrow B)}{A \wedge \neg^* B}$

R11. $\dfrac{\neg(A \rightarrow B)}{A \wedge \neg B}$

R12. $\dfrac{\neg((A \wedge B) \vee C) \qquad A}{\neg(B \vee C)}$

R13. $\dfrac{\neg(\neg(A \to B) \vee C) \qquad A}{\neg(\neg B \vee C)}$

R14. $\dfrac{\neg((A \wedge B) \vee C) \qquad A}{\neg(\neg A \vee C) \vee \neg(\neg B \vee C)}$

R15. $\dfrac{\neg(\neg(A \wedge B) \vee C)}{\neg(\neg B \vee C)}$

R16. $\dfrac{\neg(A \vee B)}{\neg(B \vee A)}$

R17. $\dfrac{\neg((A \vee B) \vee C)}{\neg(A \vee (B \vee C))}$

R18. $\dfrac{\neg((A \to B) \vee C)}{\neg(A \to (B \vee C))}$

R19. $\dfrac{\neg(\neg\neg A \vee B)}{\neg(A \vee B)}$

Ayda I. Arruda proved that R9, R10 and R19 are dependent.

2. Rules for the construction of proofs:

Every formula may be considered a having the following form:

$$(*) \qquad A_1 \to (A_2 \to (A_3 \to \cdots \to (A_{n-1} \to A_n) \dots)),$$

because if '$\to$' is not the principal symbol of a formula A, then A may be considered of the form $(*)$ for $n = 1$.

2.1. A direct proof of $(*)$ is formed as follows:

- A_1, A_2, A_3, ..., A_{n-1} are written in the first $n-1$ lines as suppositions of the proof;
- formulas may be added as new lines of the proof according to the following rules:
  - formula obtained from previous ones in the proof by rules R1–R19:
  - formulas already proved;
- the proof is finished when we obtain A_n.

2.2. An indirect proof of $(*)$ is formed as follows:

- we write A_1, A_2, A_3, ..., A_{n-1} in the first $n-1$ lines as suppositions of the proof;

- $\neg^* A_n$ is also written in the next line as a supposition of the indirect proof (rule ip);
- we may add formulas as new lines of the indirect proof precisely as in the case of direct proof (i and ii above);
- the proof is finished when we obtain two lines containing any formulas of the form A and $\neg^* A$, or $\neg\neg^* A$ and $\neg^* A$, or A and $\neg^*\neg\neg A$.

A formula is said to be a theorem (or thesis) if there exists a finished proof of it.

Finally, in 1995 da Costa proposed the following elegant axiomatization of Jaśkowski's discussive logic D_2:

1. $\Box\alpha$, if α is an axiom of S5
2. $\dfrac{\Box\alpha \qquad \Box(\alpha \supset \beta)}{\Box\beta}$
3. $\dfrac{\Box\alpha}{\alpha}$
4. $\dfrac{\Diamond\alpha}{\alpha}$
5. $\dfrac{\Box\alpha}{\Box\Box\alpha}$

Yet his own problem which was formulated in 1975 in a following way [da Costa 1975, p. 14]:

> Is it possible to formulate **natural** and **simple** axiomatization for D_2, employing $\to$, $\wedge$, $\vee$ and $\neg$ as the only primitive connectives?

still seems left open. Da Costa himself never mentioned later this problem notwithstanding the new attempts to axiomatize D_2. The matter of fact is that axiomatizations of D_2 above proposed by Furmanowski (1975), Achtelik, Dubikajtus, Dudek and Konior (1979–1980), Kotas and da Costa (1979), satisfying da Costa's conditions (i.e., employing '$\to$','$\wedge$', '$\vee$' and '$\neg$' as the only primitive connective), are rather looking very complicated and unnatural. In any case this process still goes on and it allows us to contribute the problem.

We shall attempt to solve da Costa's problem on the base of modal da Costa-Doria axiomatization from 1995. The new axiomatization of D_2 is based on the following definition:

$$\begin{aligned} \alpha \to \beta &\equiv \neg(\alpha \wedge \neg\beta), \\ \Diamond\alpha &\equiv (\alpha \to \alpha) \to \alpha, \\ \alpha^\circ &\equiv \neg(\neg\alpha \wedge \alpha), \\ \Box\alpha &\equiv \alpha^\circ \wedge \alpha \equiv \alpha^+. \end{aligned}$$

Note that the definition of α° bears a formal resemblance with $\alpha^\circ \equiv \neg(\alpha \,\&\, \neg\alpha)$ of da Costa's C_n systems.

In effect, our definitions means that we introduce two translations: $\mathtt{i}$ from S5 to D_2

$$\begin{aligned} \mathtt{i}(\alpha) &= \alpha, \text{ when } \alpha \text{ is a propositional variable,} \\ \mathtt{i}(\Diamond\alpha) &= (\mathtt{i}(\alpha) \to \mathtt{i}(\alpha)) \to \mathtt{i}(\alpha), \\ \mathtt{i}(\Box\alpha) &= (\mathtt{i}(\alpha))^+, \\ \mathtt{i}(\alpha \vee q) &= \mathtt{i}(\alpha) \vee \mathtt{i}(\beta), \\ \mathtt{i}(\neg\alpha) &= \neg\mathtt{i}(\alpha). \end{aligned}$$

$\mathtt{j}$ from D_2 to S5

$$\begin{aligned} \mathtt{j}(\alpha) &= \alpha, \text{ when } \alpha \text{ is a propositional variable,} \\ \mathtt{j}(\alpha \to \beta) &= \Diamond\mathtt{j}(\alpha) \supset \mathtt{j}(\beta), \\ \mathtt{j}(\alpha \wedge \beta) &= \Diamond\mathtt{j}(\alpha) \,\&\, \mathtt{j}(\beta), \\ \mathtt{j}(\alpha \vee \beta) &= \mathtt{j}(\alpha) \vee \mathtt{j}(\beta), \\ \mathtt{j}(\neg\alpha) &= \neg\mathtt{j}(\alpha). \end{aligned}$$

The idea of the axiomatization becomes clear: we replace all '$\Box p$', '$\Diamond p$' in axiomatics above with the respective formulas from our definition. The result is the following theorem:

THEOREM. *The logic* D_2 *may be axiomatized by means of the following axioms and derivation rules:*

1. *(Axioms of* PC*)*$^+$
2. $((\alpha \supset \beta)^+ \supset (\alpha^+ \supset \beta^+))^+$
3. $(\alpha^+ \supset \alpha)^+$
4. $((\alpha \to \alpha) \to \alpha) \supset ((\alpha \to \alpha) \to \alpha)^+$

5. $\dfrac{\alpha^+ \qquad (\alpha \supset \beta)^+}{\beta^+}$

6. $\dfrac{\alpha^+}{\alpha}$

7. $\dfrac{(\alpha \to \alpha) \to \alpha}{\alpha}$

8. $\dfrac{\alpha^+}{\alpha^{++}}$

9. $\dfrac{\alpha \supset \beta}{\alpha \to \beta}$

PROOF. It easily can be seen that under definitions $\Box\alpha \equiv \alpha^+$ and $\Diamond\alpha \equiv (\alpha \to \alpha) \to \alpha$ our axiomatization contains that of da Costa-Doria. The other way round, under this definitions da Costa-Doria axiomatization contains our axiomatization too, the only new different case with the translation of the rule 9 is obvious.

Now let us observe that $(\alpha \supset \beta)^{++} \vdash_{D_2} 1 \to (\alpha \supset \beta)$. This means that the direct translation of the strict implication $\Box(\alpha \supset \beta)$ provides us with the more power strict implication than that of Kotas for D_2 (cf. [Kotas 1974, p. 197]). Taking our observation into account in order to establish the equivalence of the systems D_2 (in our formulation) and M-S5 (the set of all formulas of the system S5 which after being preceded by the sign '$\Box$' become theses of S5) we, following Kotas' method in [Kotas 1974], need to show:

1. If $\alpha \in D_2$, then $\mathtt{j}(\alpha) \in$ M-S5;
2. if $\beta \in$ M-S5, then $\mathtt{i}(\beta) \in D_2$;
3. $(\alpha \supset \mathtt{ij}(\alpha))^+ \in D_2$ and $(\mathtt{ij}(\alpha) \supset \alpha)^+ \in D_2$;
4. $\Box(\alpha \supset \mathtt{ji}(\alpha)) \in$ M-S5 and $\Box(\mathtt{ji}(\alpha) \supset \alpha) \in$ M-S5.

Case 1: let $\alpha \in D_2$ or $\Diamond\alpha\Box\mathrm{L} \in$ S5 ($\Diamond\alpha_{\mathrm{L}}$ is an expression obtained from α by eliminating in accordance with the definitions of $\mathtt{j}$-translation the signs '$\to$', '$\wedge$', '$\leftrightarrow$'). In order to prove that $\mathtt{j}(\alpha) \in$ M-S5 we shall prove that for any formula γ of the system we have

$$\Diamond\gamma_{\mathrm{L}} \supset \Diamond\mathtt{j}(\gamma) \in \mathrm{S5}. \tag{1}$$

It is obvious for γ be a propositional variable. It is easy to prove that if for formulas γ_1 and γ_2 (1) holds, then for $\neg\gamma_1$, $\gamma_1 \vee \gamma_2$, $\gamma_1 \wedge \gamma_2$ (1) holds too. Hence, by hypothesis that $\Diamond\alpha_{\mathrm{L}} \in$ S5, we have $\Diamond\mathtt{j}(\alpha) \in$ S5 or $\mathtt{j}(\alpha) \in$ M-S5.

Case 2: it is obvious in case of propositional variable, negation and disjunction. In case of $\Diamond\beta$ if $\mathtt{i}(\beta) \in \mathrm{D}_2$; we have $\mathtt{i}(\Diamond\beta) = (\mathtt{i}(\beta) \to \mathtt{i}(\beta)) \to \mathtt{i}(\beta) \in \mathrm{D}_2$. For $\Box\beta$ if $\mathtt{i}(\beta) \in \mathrm{D}_2$; we have $\mathtt{i}(\Box\beta) = (\beta)^+ \in \mathrm{D}_2$.

Case 3: it is obvious that for propositional variable α 3. holds. It is easy to prove that if for α_1 and α_2 3. holds, then also for $\neg\alpha_1$, $\alpha_1 \vee \alpha_2$, $\alpha_1 \wedge \alpha_2$ and $\alpha_1 \to \alpha_2$. Thus for any formula α 3. holds.

Case 4: the proof is analogous to the case 3. □

References

[Achtelik *et al* 1981] Achtelik, G., L. Dubikajtis, E. Dudek, and J. Konior, "On independence of axioms of Jaśkowski s propositional calculus", Reports on Mathematical Logic 11, 3–11.

[da Costa, Dubikajtis 1968] da Costa, N. C. A., and L. Dubikajtis, "Sur la logique discoursive de Jaśkowski", Bulletin Acad. Polonaise des Sciences Math., Astr. et Phys. 16, 551–557.

[da Costa 1975] da Costa, N. C. A., "Remarks on Jaśkowski's Discussive Logic", Reports on Mathematical Logic 4, 7–15.

[da Costa, Dubikajtis 1977] da Costa, N. C. A., and L. Dubikajtis, "On Jaśkowski's Discussive Logic", pages 37–56 in *Non-Classical Logic. Model Theory and Computability*, A. I. Arruda, N. C. A. da Costa and R. Chuaqui (eds.), North-Holland.

[da Costa, Doria 1995] da Costa, N. C. A., and F. Doria, "On Jaśkowski's Discussive Logic", Studia Logica 54, 1, 33–60.

[Jaśkowski 1948] Jaśkowski, S., "Rachunek zdań dla systemów dedukcyjnych sprzecznych", Studia Soc. Sci. Torunensis, sectio A, vol. I, no. 5. English translation [Jaśkowski 1969] and [Jaśkowski 1999a].

[Jaśkowski 1949] Jaśkowski, S., "O koniunkcji dedukcyjnej w rachunku zdań dla systemæw dedukcyjnych sprzecznych", Studia Soc. Sci. Torunensis, sectio A, vol. I, no. 8 (1949). English translation [Jaśkowski 1999b].

[Jaśkowski 1969] Jaśkowski, S., "Propositional calculus for contradictory deductive systems", Studia Logica XXIV, 143–157.

[Jaśkowski 1999a] Jaśkowski, S., "Propositional calculus for inconsistent deductive systems", Logic and Logical Philosophy 7, 35–56.

[Jaśkowski 1999b] Jaśkowski, S., "On the discussive conjunction in the propositional calculus for inconsistent deductive systems", Logic and Logical Philosophy 7, 57–59.

[Kotas 1971] Kotas, J., "On the algebra of classes of formulae of Jaśkowski s Disscussive System", Studia Logica XXVII, 81–92.

[Kotas 1974] Kotas, J., "The axiomatization of S. Jaśkowski s Discussive System", Studia Logica XXXIII, 2, 195–200.

[Kotas 1975] Kotas, J., "Discussive sentential calculus of Jaśkowski", Studia Logica XXXIV, 2, 149–168.

[Kotas, da Costa 1979] Kotas, J., and N. C. A. da Costa, "A new formulation of Discussive Logic", Studia Logica XXXVIII, 4, 429–445.

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