Logic and Logical Philosophy
Volume 9 (2001), 5–24

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PARAINCONSISTENCY, or inconsistency tamed, investigated and exploited

Introduction

1. Any educated person knows, or at least should know[1], that most cases of incoherences, impossibilities and — in a theoretical framework — inconsistencies are rather suspicious members of a domain.

In particular, being inconsistent is a rather bad property of a theory. But why?

2. Our aim in the paper is, firstly, to discuss several answers to the question, and secondly, and more importantly to provide a proper frames to explain and to exploit inconsistencies. The framework which will force inconsistencies to work in a positive way, i.e., to enlarge and to deep our understanding of problems involved.

Two domains of inconsistencies

3. Let me first distinguish two main domains of inconsistencies:

The theoretical domain, where inconsistencies occur in the realm of reasoning. The chief problems involved here are therefore logical and cognitive ones.

[1] With exceptions of Hegel, Hegelians, etc.

THE OBJECTIVE DOMAIN, where inconsistencies occur (or seem to occur) in the realm of what is given (including psychological phenomena), or in the broader realm of what is real. Sometimes we consider even incoherences from the borderline of the ontological space of all possibilities, including impossibilities of several types. In consequence, such items considered here occur in a common extension of these two domains, i.e., in the realm of what is possible plus the realm of what is impossible.

The problem of theoretical inconsistency is, as we noted above, a logical and cognitive one, whereas in the case of objective inconsistencies (or rather — incoherences) it is either the problem of common sense and science (for what is given and what is real) or the problem of ontology (for the realm of possibilities — the ontological space *sensu stricto*, plus the realm of impossibilities, which form together the ontological space *sensu largo*).

General preliminaries

4. Let me start with differentiating inconsistencies from incoherences, both objective and theoretical.

4.1. An objective incoherence is any internal tension or contradiction in the object under consideration. They are quite common in many domains. In particular, in the domain of mind (take, for example, emotional ambivalence), in the domain of beliefs (Tertulian's *dictum credo*, *quia absurdum*, which is a quite reasonable position for irrational believers), in the domain of physics (take, for example, simultaneous attraction and repulsion).

Usually, objective inconsistencies are not so easy to find. Sometimes their discovery is indeed a great achievement.

4.2. Theoretical incoherences usually are global (the incoherence of a system, etc.), whereas theoretical inconsistencies primarily are local (the inconsistency of two opposite statements) and only secondarily are they global (of a theory, etc.). As we will see, in many cases it depends on which basic logic is chosen. Let me add that they frequently occur during discussions.

4.3 Some additional remarks concerning distinctions between incoherences and impossibilities in a modal framework will be given later on.

Metalogical preliminaries

5. Return to the initial observation that nearly everybody knows that inconsistency is a rather unpleasant (most of students of logic simply think — bad!) property of a theory. Let us ask again: Why?

5.1. From an epistemological (or cognitive) point of view both serious inconsistencies and/or incoherences are bad, for 1) they limit or even stop our (illusion of?) understanding of a subject hosting them, 2) they can shake a system of our knowledge or beliefs, and *last but not least*, 3) they can stupefy us by producing false impressions or the illusion of understanding.

5.2. To see this more carefully let us distinguish between *paradoxes*, which are claims, observations or results incoherent with our basic assumptions, opinions, beliefs, intuitions, etc.; and *antinomies*, which are statements (results) inconsistent with claims of a given theory.

Some of them are explicit, some implicit. The question of their explication is important and does not invite a quick reply.

Antinomies need a theory, hence logic; whereas paradoxes not necessarily. Usually, theoretical paradoxes are antinomies; sometimes also conversely.

6. To be more strict, let me distinguish now between *inconsistency* and *overflowing*.[2] To be strict enough we need a bit of metalogic.

A *logical calculus* (or simply *calculus*) is a consequence operator C on a given language. Its *logics* are all its *systems* (or *theories*), i.e., sets of formulas closed on C, which, in addition, are closed on *substitution*.

Suppose that the language is such that we can distinguish between: *An inconsistency sign*, say **inc**: $\mathbf{inc} \in C(X)$ iff $C(X)$ is inconsistent, and *an overflowing sign*, say $\bot$: $\bot \in C(X)$ iff $C(X) = \mathrm{FOR}$.

For a calculus in a language with a "proper" negation, a common inconsistency sign is, for suitable A, the formula $A \wedge \neg A$.

Overflowing is, for sure[3], quite bad, for it demolishes the theoretical usefulness of the overflowed $C(X)$. By accepting it we cannot distinguish between theorem and non-theorem, between truth and falsity.

7. Let me add that the role of *negation* in the inconsistency sign is not essential. First of all, *the negation* used in it must indeed be negation, what occurs, for example, in intuitionistic or classical like logics. For many nonclassical logics *negation* is, however, negation by name only.

On the other hand, quite often the inconsistency sign is a *positive* formula, as in the case of classical arithmetic "$0 = 1$".

8. Now, it is easy to explain why inconsistencies are dreadful from the point of view of classical — like logics.

[2] In the original Jaśkowski's terminology. Some people like to talk about *explosions*.

[3] At least for people not too crazy. Postmodern "thinkers" are with their "deconstruction" of the limit between Truth and Falsity — and not only in this case — exceptions.

The classical, so called *Scotian*, calculus consists in following claims:

(i) The signs of overflowing and inconsistency are logically equivalent: $C(\bot) = C(\mathbf{inc})$,

(ii) Both of them are equivalent (more exactly – equal) to the standard classical sign of inconsistency: $A \wedge \neg A$, using real, boolean or pseudo-boolean negation. Hence

(iii) $C(A \wedge \neg A) = \mathrm{FOR}$.

Thus for classical-like logics the problem of paraconsistency[4] reduces to the question of validity of the Duns Scotus rule: $A \wedge \neg A/B$. A calculus is paraconsistent, if the rule is invalid for some A, whereas it is not such in the contrary case.

It is such, for a given calculus C in a language with the true negation and with $A \wedge \neg A$ being its sign of inconsistency

(1) *For any theory T, that T is overflowed is equivalent to that T is inconsistent iff T is closed on Duns Scotus rule* RDS*: $A \wedge \neg A/B$.*

As a matter of fact it is quite reasonable to take a stronger position. *Strong paraconsistency* for a given calculus C is defined by a stronger condition: For all A, $C(A \wedge \neg A) \neq \mathrm{FOR}$.

9. Paraconsistent logics are non-Scotian ones. Hence, we accept that:

(i) their signs of inconsistency are not equivalent to suitable signs of overflowing (if any),

(ii) $A \wedge \neg A$ is not necessary their common sign of overflowing, but in most cases it is its sign of inconsistency.

Hence, in a paraconsistent logic

(2) *For at least one A, $C(A \wedge \neg A) \neq \mathrm{FOR}$.*

Paraconsistent logics are thereby logics for studying inconsistencies expressed in the classical way.[5]

10. Jaśkowski's problem (1948, [2], [3]): Find an interesting and rich[6] (i.e., close to classical logic as far as reasonable) as well as well-motivated paraconsistent logic.

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[4] Standard name introduced in the late 1970s by F. M. Quesada.

[5] This can be, and indeed was, generalized to some non-classical cases.

[6] But not too rich!

In his seminal works Jaśkowski was explicitly motivated by Jan Łukasiewicz's 1910 great book [5], being the starting work for Polish Logic, and also by the philosophy of Veihinger *als ob* popular at the beginning of the last century.

In the papers cited above Jaśkowski, not only stated the problem but also provided an answer — the first solution to it, which is probably still the best.[7]

11. The richness of a given paraconsistent logic can be estimated in a standard way. The problem of being "interesting" is more subtle and somehow pragmatic. One point at least is however clear. An interesting paraconsistent logic should not only block inconsistency overflowing but also supply us with an understanding of how inconsistency emerges in a given domain and give an insight into its machinery, thus opening the way to the better understanding of the domain itself.

Parainconsistency

12. The title of the present paper immediately gives the basic idea and the program I would like to present and defend in my talk. It offers also something like a definition of the notion of *parainconsistency.*

Notice first that the popular name "paraconsistent logic" is, in a sense, misleading. It suggests that such logics are consistent in a special, weak sense. But, as we known, it is just exactly the reverse. They are simply inconsistent, but unlike the classical logic they are able to work with inconsistencies.

Thus, the situation is quite similar to the misleading naming of the "law of consistency", which in many textbooks is named the "law of inconsistency".

13. Let me also add a few opening words on the status of classical-like logics in our subject. In the case of domains with inconsistencies they are useless for investigating them, for the classical calculus suggests that the only effect of inconsistency is overflowing, i.e., trivialization of an appropriate theory.

Even worse, the classical logic states that all inconsistencies are equivalent, despite our well-founded experiences and intuitions.

14. The basic paraconsistent logics are *para*inconsistent, for they are inconsistent in a special — *para* — sense. They are inconsistent, for they contain at least one inconsistent formula, but they tolerate it and can be made to work in pursuit of better understanding of a domain under discussion. This mean that they are invented not only to block or eliminate

[7] *Pace* G. Priest.

inconsistencies, but rather *for* the study of inconsistencies (in Portuguese: *para* inconsistencies, because Portuguese *para* means English *for*).

We observe that Jaśkowski's logic as well as many paraconsistent logics built up in the last 55 years are parainconsistent in the above sense.

15. Parainconsistent logics are therefore logics for the investigation of serious and real inconsistencies. We must find a framework for understanding them, and in consequence to tame and to exploit them.

16. Our feelings concerning such delicate questions depend, as you will see, upon quite different attitudes to the inconsistencies we can chose.

A few remarks on inconsistencies

17. Let me distinguish, first of all, several types of inconsistencies: We move from stupid claims (like "I am here and also I am not"), which are not even jokes and should be thoroughly ignored; through several nice logical jokes (like "Have you lost your rush-mats"), to *serious* puzzles, paradoxes and antinomies.

They have appeared several times and are still appearing, sometimes challenging the very foundations of our knowledge. From a theoretical point of view only serious inconsistencies deserve our attention and research.

18. To organize our procedure let me list here a few paradigmatic cases of serious inconsistencies:

The Paradox of the Liar: It is not a logical joke or puzzle, but a real paradox for the classical, Platonic theory of truth and falsity. The theory says that, where A is an indicative proposition: *A is true if just what it claims is*, and *A is false if just what it claims is not* is based on our most natural and convincing intuitions. Therefore the famous and very ingenious counterexample against *bivalence* invented by Eubulides of Megara really shook the background of our common-sense knowledge. The paradox was also very productive in forcing us to a much deeper analysis of the notions of truth and falsity.

Zeno's paradoxes of motion (or in general — change): They are true and challenging inconsistencies in some theories of motion and still a source of interesting investigations as well as one of the starting points for the modern theory of dynamics.

The case of infinitesimals in Mathematical Analysis gives the false impression that the Calculus is grounded on inconsistent statements. The solution of the problem took nearly 200 years (Cauchy, Weierstrass) and another one hundred years to see that the classical approach of Newton and Leibniz was not as bad as many people had believed after Berkeley's critique (consider the case of Nonstandard Analysis of Abraham Robinson). Notice here a very characteristic reaction of mathematicians: they were still developing the Calculus by means of classical logic, bracketing the paradox.

Quite a similar story has happened in the case of set-theoretical paradoxes shown by the pragmatic reaction of Cantor. Many people used to say, and still do say, quite naively that Cantor's theory was "naive". Not so naive, however. Just after discovering the first paradoxes Cantor changed his previous view that basic mathematics is free of inconsistent items and distinguished between *consistent* multiplicities (sets) and *inconsistent* ones, being in such a way a pioneer of von Neumann's type of set theory (which was next developed by Bernays and Gödel) as well as of a *modal* approach to set theory, which is still waiting for further investigation.

We recall also several paradoxes of Quantum Mechanics, which are still the subject of hard work; as well as Bohr's theory of atoms, and the *delta* function of Dirac (which is, in fact, not a function defined in the standard way, but — as was discovered next — a special distribution).

19. In what way do people react after such serious inconsistencies' emerge? We can distinguish at least three positions — two extreme and one mediatory:

- Inconsistency *enemies* (sometimes even *inquisitors*): Any inconsistency is an offense to reason, a sort of high treason. It is a sign of logical disease (A. Tarski) and should be treated in a proper way.
- Inconsistency *believers* (sometimes *lovers*): inconsistencies really exist and play a most important, positive role. Among believers we find quite a lot of Hegelians.[8]
- Inconsistency *investigators*: They do not accept inconsistencies, but they treat them as a challenge, both logical and essential. "Contradiction is not a failure, it is an opportunity" (A. N. Whitehead).

[8] The position has rather a long tradition, starting with the sophists, Nicolas of Cusa, Hegel and Hegelians of several types (including the dialectic philosophers). In our time the position is defended by several Australian philosophers, including the late Richard Routley (laser Sylvan), Chris Mortensen, and, under the name of *dialethism*, by Graham Priest.

I have been for several years rather suspicious on it. Now, when I am fighting with cancer of the colon, I came to the opinion that most (or even any) case of cancer is an inconsistency occurring in the world, which should be taken as the paradigmatic case by any true dialectical theory. I am therefore preparing myself to become a Hegelian after death.

An opportunity to extend our understanding, is offered, obviously, only by serious and challenging inconsistencies. Which inconsistencies are such, however?

Notice, that between investigators are many Leibnizians. In what follows, I will take a moderate investigator's position.

20. Global versus local effects. Overflowing (or explosion).

For a given theory T a paradox (or inconsistency) occurring in it is *global*, if it affects the full content of the theory; *local* — if it affects only its proper part.

From a point of view of classical-like logics, overflowing is global. From a material, or essential, point of view it, however, must not be such. Therefore, in the case of an important theory having a serious inconsistency destroying it, we usually stay with the theory, searching for a remedy and trying to extend our understanding.

In most cases we try to revise an inconsistent theory. As a first step we try to weaken the theory, hoping to extend it in a reasonable way in subsequent steps.

To this end, we try first to *localize* the paradox, i.e., to find its place and essential sources of it, in the hope of showing that the inconsistency under investigation is local. Quite often the theory is then revised, by *enlarging* its domain with new elements (like extending the standard domain of rational numbers by irrational numbers after discovering them) or by the *differentiation* of several kinds of objects in the domain caused by inconsistencies (like distinction between classes and sets introduced by Cantor after discovering the set-theoretical paradoxes).

21. For a successful localization it is important to stop the overflowing effect (i.e., classical explosion). Parainconsistent logics then show their usefulness.

Notice that moderate investigators wish to use parainconsistent logics as auxiliary devices, in addition to classical logic, which, at least in a metamathematical setting, remains the basic practical logic. To paraphrase Whitehead: "One God, one country, one logic" (but **with several auxiliary logics**).[9]

[9] And one wife with several female assistants as well. One logic here means the classical one. Paraincosistent logics are in most cases auxiliary ones.

Modal, or Polish, approaches to parainconsistency

Definition and short history

22. A modal approach to parainconsistency, for a classical-like language with calculi and logics expressed in it, is undertaken by the introduction of an auxiliary modal language with suitable calculi and logics used for paraphrasing the inconsistency problem in a way that provides it with a new reading and way of understanding it.

The approach was introduced in 1948 by Stanisław Jaśkowski. His motivation will be discussed in one of the forthcoming sections.

The approach was rediscovered, or rather reintroduced, by several American scholars in late 1970s in quite a similar way but differently motivated. It started to play an important role in the development of non-monotonic logics. In the last few years it has again been investigated, though without a deep understanding what is behind it.

23. The basic historical data are as follows. Let me start with some remarks concerning the land of its birth.

In Poland, its prehistory is the famous book of Jan Łukasiewicz [5], 1910. This book, written by one of the two fathers of Polish logic, started by challenging the very traditional, Aristotelian Principle of Noncontradiction:

> since the contradiction of the statement cannot be true at the same time of the same thing, it is obvious that contraries cannot apply at the same time to the same thing. [. . .] Therefore if it is impossible at the same time to affirm and deny a thing truly, it is also impossible for contraries to apply to a thing at the same time [. . .].[10]

After distinguishing metaphysical, logical and psychological versions of the Principle, Łukasiewicz argued that none of them is well — motivated and thereby should be considered as doubtful. This consideration together with parallel investigation of the question of determinism and sentences speaking about the future led Łukasiewicz in the next decade to introduce the family of his many — valued logics.

As regards the Consistency Principle Łukasiewicz considered it as ethical Principle[11], being foundation of logical rationalism. Łukasiewicz position started a very vivid discussion and has never been forgotten amongst Polish

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[10] Aristotle, *Metaphysics*, IV, vi, 10–11. Translated by Hugh Tredennick.

[11] Notice a great similarity between Łukasiewicz's position and the well-known view of Ludwig Wittgenstein from the 1930s.

scholars. Obviously, it is also the chief starting point for Jaśkowski's investigation.

As regards other scholars: Stanisław Leśniewski had defended the Principle from the very beginning (the first paper on the subject was published in 1912). It was taken next as his own position by his pupil Alfred Tarski. A somewhat similar position was taken independently in the same time by Leon Chwistek (1912).

On the other hand, the breakdown step, though whilst keeping the spirit of Łukasiewicz with respect to parainconsistent logics, was taken in 1948 by Stanisław Jaśkowski [2], [3]. His work was continued from the 1960s to the 1980s by his pupils Lech Dubikajtis and Jerzy Kotas. In particular, in the early sixties Lech Dubikajtis started the close co-operation of Polish scholars with the father of Brazilian paraconsistent logics, Newton da Costa. In the seventies the Torunian group became especially active, which included Jerzy Kotas and his pupils (Tomasz Furmanowski, Wiesław Dziobiak, Jerzy Błaszczuk, and Max Urchs). Some generalization of the full research project was in 1975 given by Jerzy Perzanowski [8].

Outside the main stream let me mention: In the eighties — Witold Łukaszewicz & non-monotonic logicians from Warsaw, and working in the spirit of Meinong, Jacek Paśniczek from Lublin.

24. American scholars started the modal investigation of inconsistency with David Kaplan's post-Carnapian approach to axiomatization of the ontology of Wittgenstein's *Tractatus* which is itself modal in spirit indeed. This was led out in the famous doctoral dissertation of Kaplan [4] from the early 1960s. In his dissertation Kaplan introduced the modal theory (or rather the family of such theories) named **S13** which is defined as the extension of Lewis' modal logic **S5** by a pair of contingency statements $\{\Diamond A, \Diamond \neg A\}$.

The Kaplan's approach was implicitly touched upon by Steve Thomason in his famous preprint on dialectical logics from the early 1970s, which was finally published in Reports on Mathematical Logic as [11] in the 1990s.

Next, in the 1980s Nicholas Rescher and Robert Brandom published an influential book [1], whose basic idea is simply a repetition of Jaśkowski's main idea re-expressed in the apparatus of relational semantics for modal logics.

At almost the same time, M. McDermott published his important paper [6] on modal approach to non-monotonic logic, which is also close in its basic ideas to those of Jaśkowski.

I believe that most of these similarities are accidental and independent. There is simply one truth which shows its face on several occassions. An exact and careful comparison of these approaches are, to the best of my knowledge, still waiting for discussion.

25. Finally, let me add that an important approach to paraconsistency invented by relevance logicians, should also be considered to be implicitly modal due to the implicit connection of relevance to modalities.

Preliminaries on modal logic

26. Before coming to a discussion of Jaśkowski's approach we need to recall some basic data on modal logics.

26.0. Modal language. Is defined in a standard way from the set of propositional variables by means of the following functors: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\Diamond$, $\Box$.

26.1. Two Aristotelian definitions of contingency.

$\mathrm{Ct}^* A$	$:= \Diamond A \wedge \Diamond \neg A$	symmetric, or two-sided
$\mathrm{Ct}\, A$	$:= A \wedge \Diamond \neg A$	non-symmetric, or one-sided

26.2 Rules and axioms.

Rules

MP	$\dfrac{A \qquad A \rightarrow B}{B}$	Modus Ponens
RE	$\dfrac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$	rule of extensionality, or congruence
RM	$\dfrac{A \rightarrow B}{\Box A \rightarrow \Box B}$	rule of monotonicity
RAM	$\dfrac{A \rightarrow B}{\Box B \rightarrow \Box A}$	Rule of antimonotonicity
RD	$\dfrac{A}{\Diamond A}$	Aristotle's rule of possibilization
RG	$\dfrac{A}{\Box A}$	Gödel's rule of necessitation
RJ	$\dfrac{\Diamond A}{A}$	Jaśkowski's rule of depossibilization

Both Gödel's rule of necessitation as well as Jaśkowski's rule of depossibilization are rather unusual from a woman (or man) usual point of view. We are living in a world full of contingencies, whereas both rules taken in their full generality say something contrary: in the first case contingencies become necessary truths, whereas — in the second case — contingencies are facts.

Notice that Gödel's rule RG is closely connected with relational semantics in its Dana Scott's standard version; whereas Jaśkowski's rule RJ is important for the axiomatization of Jaśkowki's logic.

Axioms

K	$\Box(A \to B) \to (\Box A \to \Box B)$	Kripke's axiom of regularity
D	$\Box A \to \Diamond A$	Aristotle's basic axiom
T	$\Box A \to A$	von Wright's basic axiom
R	$\Diamond A \to \Box A$, or $\neg\,\mathrm{Ct}^*\,A$	Leibniz's axiom of metaphysical rationalism
4	$\Box A \to \Box\Box A$	Lewis' axiom of transitivity
5	$\Diamond\Box A \to \Box A$	Lewis' axiom of euclidity
B	$\Diamond\Box A \to A$	Kripke's *Brouwerian* axiom of symmetry
GL	$\Box(\Box A \to A) \to \Box A$	Gödel-Löb's axiom of provability

26.3. Calculi. Logics and their classes.

Calculi

C	– based on classical tautologies expressed in modal language and MP
Ce	– extension of C by RE
Cm	– extension of C by RM
Cam	– extension of C by RAM
Cg	– extension of C by RG
Cn	– extension of C by RG and K

Logics

CL	$:=$ C(∅)	the classical logic in ML
K	$:=$ Cn(∅)	Kripke's minimal normal logic
D	$:=$ Cn(D)	Deontic standard logic of von Wright
T	$:=$ Cn(T)	Gödel-Feys-von Wright's logic of common sense universe
BK	$:=$ Cn(**T**, B)	*Brouwerian* logic of Kripke

$\mathbf{S4} := \mathrm{Cn}(\mathbf{T}, 4)$		Lewis' fourth modal logic
$\mathbf{S5} := \mathrm{Cn}(\mathbf{T}, 5)$		Lewis's fifth modal logic
$\mathbf{R} := \mathrm{Cn}(\mathsf{R})$		Leibniz's logic of metaphysical rationalism
$\mathbf{GL} := \mathrm{Cn}(\mathsf{GL})$		Gödel-Löb's logic of provability
$\mathbf{GL^s} := \mathrm{C}(\mathbf{GL}, \mathsf{T})$		Gödel-Solovay's logic of provability

Crucial logics

which are crucial for a topography of modal logics, cf. [9].

$\mathbf{TR} := \mathrm{C}(\Box A \leftrightarrow A)$	the modal *identity* logic
$\mathbf{NEG} := \mathrm{C}(\Box A \leftrightarrow \neg A)$	the modal *negation* logic
$\mathbf{VER} := \mathrm{C}(\Box A)$	the modal *verum* logic
$\mathbf{FALS} := \mathrm{C}(\Diamond A) = \mathrm{C}(\mathrm{Ct}^* A)$	the modal *falsum* logic

Notice that **VER** is the logic of very extreme rationalism (for it claims that everything is necessary); whereas **FALS** is the Cartesian logic of total contingentialism (due to its claim that everything is possible, and hence that everything is contingent).

Classes of logics

$\mathsf{MOD} := \{\boldsymbol{P} : \boldsymbol{P}$ is a logic of C$\}$	the class of all *modal* logics
$\mathsf{CON} := \{\boldsymbol{P} : \boldsymbol{P}$ is a logic of Ce$\}$	the class of all *congruential*, or *classical* modal logics
$\mathsf{MON} := \{\boldsymbol{P} : \boldsymbol{P}$ is a logic of Cm$\}$	the class of all *monotonic* modal logics
$\mathsf{AMON} := \{\boldsymbol{P} : \boldsymbol{P}$ is a logic of Cam$\}$	the class of all *antimonotonic* modal logics
$\mathsf{NR} := \{\boldsymbol{P} : \boldsymbol{P}$ is a logic of Cn$\}$	Krike's class of all *normal* modal logics
$\mathsf{QNR} := \{\boldsymbol{P} : \boldsymbol{P}$ is a modal overlogic of $\mathbf{K}\}$	the class of all *quasinormal* modal logics

26.4. Homogenous counterparts (cf. [8]). Let $\boldsymbol{P}$ be a modal logic.

$\mathrm{M}(\boldsymbol{P}) := \{A : \boldsymbol{P} \vdash \Diamond A\}$	$\Diamond$-counterpart
$\mathrm{L}(\boldsymbol{P}) := \{A : \boldsymbol{P} \vdash \Box A\}$	$\Box$-counterpart
$\mathrm{Ct}(\boldsymbol{P}) := \{A : \boldsymbol{P} \vdash \mathrm{Ct}\, A\}$	Ct-counterpart
$\mathrm{Ct}^*(\boldsymbol{P}) := \{A : \boldsymbol{P} \vdash \mathrm{Ct}^*\, A\}$	Ct^*-counterpart

27. Investigation of $\mathrm{M}(\boldsymbol{P})$ — $\Diamond$-counterpart of $\boldsymbol{P}$ plays quite fundamental role in the construction of Jaśkowskian logics.

Stanisław Jaśkowski's seminal contribution, cf. [2], [3]

28. Consider a classical-like language L with the usual connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$; and its modal counterpart ML with $\Box$ and its dual $\Diamond$ added. In the future the language ML will be treated as metalanguage of L.

Let $\boldsymbol{P}$ be a modal logic. To define its *parainconsistent counterpart* consider Jaśkowski's transformation

$$\mathrm{J}\colon \mathrm{L} \Rightarrow \mathrm{ML}$$

defined by putting, with two further characteristic conditions:

$$\begin{aligned} p^{\mathrm{J}} &:= p, \\ (\neg A)^{\mathrm{J}} &:= \neg A^{\mathrm{J}}, \\ (A \vee B)^{\mathrm{J}} &:= A^{\mathrm{J}} \vee B^{\mathrm{J}}, \\ (A \wedge B)^{\mathrm{J}} &:= A^{\mathrm{J}} \wedge \Diamond B^{\mathrm{J}} \\ (A \rightarrow B)^{\mathrm{J}} &:= \Diamond A^{\mathrm{J}} \rightarrow B^{\mathrm{J}}. \end{aligned}$$

It is sometimes reasonable to use a more regular, symmetric, condition for $\wedge$:

$$(A \wedge B)^{\mathrm{J}^*} := \Diamond A^{\mathrm{J}^*} \wedge \Diamond B^{\mathrm{J}^*}.$$

29. Define now Jaśkowski's logic based upon $\boldsymbol{P}$:

$$\mathsf{J}(\boldsymbol{P}) := \{A \in \mathrm{FOR} : \boldsymbol{P} \vdash \Diamond A^{\mathrm{J}}\}.$$

In particular, Jaśkowski's original logic $\mathbf{D_2} := \mathsf{J}(\mathbf{S5})$.

30. Jaśkowski in fact provided two motivations for his construction. The first, a methodological consideration based on Veihinger philosophy *als ob*, says that — at least in science — everything should be treated in a pure hypothetical way.

As for the second, what happens in serious discussions is taken into account. In such discussions, at least at their beginning, I should take my opponent seriously, which means that his claim B should be considered at least as possible: $\Diamond B$. Therefore, if I am comparing my position A with the position of my opponent B, I should use a description which leads immediately to the idea of discussive conjunction: $A \wedge_{\mathrm{d}} B := A \wedge \Diamond B$. Similarly, if

I am considering the consequences of the position of my opponent, expressed by A, then the discussive implication is represented as an entailment between my presentation of my opponent's view $\Diamond A$ and its consequence B. In such a way, we come to the idea of discussive implication: $A \rightarrow_{\mathrm{d}} B := \Diamond A \rightarrow B$.

Jaśkowski's logic proved its usefulness and power. Observe, however, lack of an ontological motivation for it. Let me therefore ask: What does it add to our understanding of parainconsistency? What is behind its construction? How many similar logics can be constructed?

I will return to these fundamental questions in the forthcoming, last parts of the paper.

Several basic facts and observations

31. Let me establish first a connection between M — counterpart of a given logic and Jaśkowski-type logics based on it.

(3) If $\mathrm{M}(\boldsymbol{P}) = \mathrm{M}(\boldsymbol{Q})$ then $\mathsf{J}(\boldsymbol{P}) = \mathsf{J}(\boldsymbol{Q})$.

Therefore Jaśkowski's logics based on $\boldsymbol{P}$ and $\boldsymbol{Q}$ are fully defined by their M-counterpart.

In his doctoral dissertation [7] of 2002 Marek Nasieniewski observed that under a special proviso the reverse is also true.

(4) Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be normal logics. Then $\mathsf{J}(P) = \mathsf{J}(Q)$ implies that $\mathrm{M}(\boldsymbol{P}) = \mathrm{M}(\boldsymbol{Q})$.

32. Now let me discuss Jaśkowski — type logics based on the crucial logics:

(5) $\mathrm{M}(\mathbf{TR}) = \mathbf{TR} = \mathrm{L}(\mathbf{TR})$, hence $\mathsf{J}(\mathbf{TR}) = \mathbf{CL}$. Also $\mathrm{Ct}(\mathbf{TR}) = \varnothing = \mathrm{Ct}^*(\mathbf{TR})$.

On the other hand

(6) $\mathrm{M}(\mathbf{FALS}) = \mathrm{FOR}$, hence $\mathsf{J}(\mathbf{FALS}) = \mathrm{FOR}$! But $\mathrm{L}(\mathbf{FALS}) = \varnothing$.

(7) $\mathrm{M}(\mathbf{VER}) = \varnothing$, hence $\mathsf{J}(\mathbf{VER}) = \varnothing$. But $\mathrm{L}(\mathbf{VER}) = \mathrm{FOR}$.

(8) $\mathrm{M}(\mathbf{NEG}) = \mathbf{NEG}^{\neg} = \mathrm{L}(\mathbf{NEG})$, hence $\mathsf{J}(\mathbf{NEG}) = \mathbf{CL}^{\neg}$;
where $\mathbf{NEG}^{\neg}$ and $\mathbf{CL}^{\neg}$ are respectively families of all negations of theorems of $\mathbf{NEG}$ and $\mathbf{CL}$.

33. By a generalization of a criterion formulated for normal logics by Perzanowski [8] we can prove a rather general and quite useful condition:

(9) Let $\boldsymbol{P}$ be monotonic or antimonotonic. $\mathrm{M}(\boldsymbol{P}) = \varnothing$ iff $\boldsymbol{P} \leq \mathbf{VER}$; and $\mathrm{L}(\boldsymbol{P}) = \varnothing$ iff $\boldsymbol{P} \leq \mathbf{FALS}$.

The first part of it says that in a quite big class of logics we cannot fruitfully repeat Jaśkowski's construction. Hence

(10) $\mathrm{M}(\mathbf{GL}) = \varnothing$. But

(11) $\mathrm{C}(\mathrm{M}(\mathbf{GL^s})) = \mathbf{VER}$.

The last observation suggests that a very promising Jaśkowskian logic is generated by $\mathbf{GL^s}$, Gödel-Solovay's logic of provability.

34. The research of parainconsistent Jaśkowskian logics also leads to an investigation of Ct^*-counterparts.

(12) If $\boldsymbol{P}$ is included in $\mathbf{TR}$, or in $\mathbf{VER}$, or in $\mathbf{NEG}$, then $\mathrm{Ct}^*(\boldsymbol{P}) = \varnothing$.

But

(13) $\mathrm{Ct}^*(\mathbf{FALS}) = \mathrm{FOR}$. Also

(14) If $\boldsymbol{P} \vdash \mathsf{R}$, then for any $\boldsymbol{Q}$ including $\boldsymbol{P}$, $\mathrm{Ct}^*(\boldsymbol{Q}) = \varnothing$.

35. Let me mention also, that in the case of C-logics **FALS** plays quite a distinguished role in comparing impossibility and standard inconsistency. Namely

(15) $\boldsymbol{P} \vdash \neg \Diamond A \leftrightarrow A \wedge \neg A$ iff $\boldsymbol{P} = \mathbf{FALS}$.

In this case impossibility is equivalent to inconsistency only in the logic **FALS**!

Polish parainconsistency revisited and generalized

36. Now we are ready to discuss the basic questions of §§ 5 and 30.

36.1. First of all, ML with logics expressed in it should be treated as the *modal metalanguage* for the *classical* language L with its classical-like logics and theories, including parainconsistent ones (in particular Jaśkowski's discussive logic) built upon it.

Therefore, Jaśkowski's transformation defined in § 28 $\mathrm{J}\colon \mathrm{L} \Rightarrow \mathrm{ML}$ is, in fact, an embedding of the classical language L of the calculus $\mathbf{D_2}$ into its modal metalanguage ML.

36.2. What happens with inconsistencies through this transformation? It is easy to see that in the case of L-inconsistency $A \wedge \neg A$ its J-image is $\mathrm{Ct}(A^{\mathrm{J}})$, i.e., the claim that A^{J} is Ct-contingent; whereas when we use the more regular version J^*, then J^*-image of the inconsistency under J^* is $Ct^*(A^{J^*})$, i.e., the claim that A^{J^*} is Ct^*-contingent. In both cases inconsistencies are transformed into contingencies!

37. To see the point, let me recall the quite fundamental classification of concepts (in our case — of propositions), given by Leibniz, into *necessary* ones, which are finitely analytical and are the immediate subject of the principle of consistency, and *contingent* propositions, which usually are empirical and the subject of the principle of sufficient reason.

Seen trough Jaśkowskian eyes, inconsistencies are thereby implicit and special contingencies! Which ones, however?

38. Let me recall also the second deep insight of Leibniz: contingencies are also analytical, but they are calculable in infinite number of steps. Therefore, in a finite world of our everyday experience we can only approximate contingent truths.

39. Indeed, there exist quite a lot of natural contingencies in the world, like my being and speaking here in Gent, during the First World Congress on Paraconsistent Logics. We should therefore be careful and ask which contingencies (if any) encode inconsistencies? They, for sure, must be a sort of *theoretical contingencies.*

40. To shed light on this very problem let me compare now Jaśkowski's approach with the celebrated provability interpretation of modalities by Gödel.

As we know, it compares the classical language of elementary arithmetic extended by Gödelian provability operators Prov( ) and Cons( ), which are purely arithmetical formulas, with the modal language ML defined by the conditions:

$$\begin{aligned} \Diamond A &:= \mathrm{Cons}(\ulcorner A \urcorner), \\ \Box A &:= \mathrm{Prov}(\ulcorner A \urcorner). \end{aligned}$$

41. Observe that Gödel's interpretation of modalities goes in the reverse direction:

$$\mathrm{G}\colon \mathrm{ML} \Rightarrow \mathrm{L}.$$

Let us now ask for a Gödelian interpretation of contingencies. Clearly $\mathrm{G}(\Diamond A \wedge \Diamond \neg A)$ is equal to $\mathrm{Cons}(\ulcorner A \urcorner) \wedge \mathrm{Cons}(\ulcorner \neg A \urcorner)$, with respect to a given, and rich enough theory. Hence contingency in the provability interpretation means simply — on the metalevel — *metalogical independence*!

42. To derive a further lesson let us compose both interpretations. Let T be an inconsistent theory in L.

Through Jaśkowski's interpretation we reach several contingencies in its modal metatheory MT. The picture of T-inconsistency by J becomes MT-contingency, which, in turn, by Gödel's interpretation (if applicable) encodes that some statement is independent with respect to some *hidden kernel* of the theory T, say Ker(T).

43. Just as in the case of Russell's paradoxical statement for Cantorian set theory **CST**, which was shown to be independent with respect **CST**-kernel defined to be **ZF** part of Cantorian (and von Neumann's and Gödel-Bernays') extension of **ST**:

$$\mathrm{Ker}(\mathbf{CST}) = \mathbf{ZF}.$$

44. Let us take another example. $\mathbf{S13_p}$ as we remember is Kaplan's Wittgensteinian system written more carefully (with its parameter p indicated). Now it is easy to see that

$$\mathrm{Ker}(\mathbf{S13_p}) = \mathbf{S5}.$$

45. The best place to test this connection is in the logic of arithmetical truths $\mathbf{GL^s}$, which, as we noticed before, is interested in both interpretations.

Schema for the revision of (not only) inconsistent theories

46. By its depth both Jaśkowski's interpretation and Gödel's interpretation are conjugate and based on the same fundamental idea: the connection of inconsistencies with theoretical contingencies (or logically independent statements).

They also suggests that the construction of J(T) can be used to search for a hidden kernel Ker(T) of (caused by inconsistencies) theory T.

47. The idea is simple. If you have difficulties with a theory T please go to its kernel and next add what is possible in a reasonable direction.

The most important question now is therefore to define in a proper way the kernel of the theory. Which way is proper, however?

48. We have several options. Let T be an inconsistent theory in the language L with its conjugate theory MT in the modal metalanguage ML.

Then Jaśkowski's receipt is the following one:

$$\mathrm{Ker}_{\mathrm{M}T}(T) \;:=\; \mathbf{J}(T),\ \text{i.e., } \{A : \Diamond A^{\mathrm{J}} \in MT\}.$$

In particular, $\mathbf{D_2} = \mathrm{Ker}_{\mathbf{S5}}(T)$, where T is an inconsistent theory based on classical, or intuitionistic, logic.

On the other hand, Gödel's receipt is the following one:

$$\mathrm{Ker}_{\boldsymbol{P}A}(T) \ := \ \{A : A^{\mathrm{G}} \in \boldsymbol{P}A\}.$$

It was shown by Solovay that

(16) $\mathrm{Ker}_{\boldsymbol{P}A}(T) = \mathbf{GL}$.

49. Finally, let us state probably the most natural candidates, for a given transformation *, to play the role of the kernel of a given theory T with respect to a fixed logic $\boldsymbol{P}$.

Strong version: $\mathrm{Ker}_{\boldsymbol{P}}(T) := \{A : \Box A^* \in \boldsymbol{P}\}$;
Weak (but broader) version: $\mathrm{Ker}_{\boldsymbol{P}}(T) := \{A : \Diamond A^* \in \boldsymbol{P}\}$.

Conclusion

50. As we can see, the success of Jaśkowski's approach is not an accident. Behind it is to be found a combination of the most powerful and fruitful ideas of modal philosophy and logic.

51. To finish the paper, let me summarize its essentials via the diagram:

$$\begin{tikzcd}
\text{Language L} & \text{Modal Metalanguage ML} \\
\text{Theory } T \arrow[r, shift left, "\mathrm{J}"] & \text{Its metatheory } \mathrm{M}T \arrow[l, shift left, "\mathrm{G}"] \arrow[dl, "\mathbf{J(M\boldsymbol{T})}"] \\
\mathrm{Ker}(T) \arrow[u] &
\end{tikzcd}$$

52. To resume, Jaśkowski's logic $\mathbf{D_2}$ can be understood as the kernel of a suitable theory through **S5** according to the above scheme.

Acknowledgments. The author wishes to thank many scholars who hear several versions of this paper, starting with members of the late Professor Stig Kanger's group in Uppsala in April 1980. The lecture was repeated in Kraków twice, in Leipzig, in Heidelberg, in Erlangen, in Kraków again, in João Pessoa, in Toruń twice, in Gent in 1997 as the invited lecture at the First World Congress on Paraconsistency, and also in Poznań.

Support obtained from Polish Committee for Scientific Research (KBN) by means of grant 1 H01 006 18 and indirectly by the Flemish Minister responsible for Science and Technology (Contract Bill 2002/41) is also thankfully acknowledged.

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