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ON THE PRACTICAL VALUE OF HERBRAND DISJUNCTIONS

Abstract. Herbrand disjunctions are a means for reducing the problem of whether a first-oder formula is valid in an open theory $\mathcal{T}$ or not to the problem whether an open formula, one of the so called Herbrand disjunctions, is $\mathcal{T}$-valid or not. Nevertheless, the set of Herbrand disjunctions, which has to be examined, is undecidable in general. Fore this reason the practical value of Herbrand disjunctions has been estimated negatively (cf. [30]).

Relying on completeness proofs which are based on the algebraization technique presented in [30], but taking a more optimistic view, we show how Herbrand disjunctions become the base of a method for building in theories into automatic theorem provers [26]. Surveying newer results we discuss how to treat heterogeneous theories [29] as well as practical implications of different normal form transformations.

1. Introduction

It was one of the key points of David Hilbert's program to find a uniform method which allows for a given mathematical theory $\mathcal{T}$ and an arbitrary given formula F to decide whether F is valid in $\mathcal{T}$. Jacques Herbrand [18] described a method which allows for any open first-order theory $\mathcal{T}$ and for any formula F to enumerate a set of open formulas, the so-called Herbrand disjunctions, such that F is $\mathcal{T}$-valid if and only if for some Herbrand disjunction H holds

$$\mathcal{T} \models H. \tag{1}$$

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This way the validity problem $\mathcal{T} \models F$ for an arbitrary formula F has been reduced to the decidable validity problem $\mathcal{T} \models H$ for an open formula H. This result may have been seen as a fruitful step towards the ultimate goal formulated by Hilbert. Unfortunately, Gödel's incompleteness theorem made clear that, in general, Herbrand's result may be improved at most by finding better enumerations of the set of Herbrand disjunctions. This led logicians to a pessimistic estimation of the practical value of Herbrand's result. For example, in [30] just can be read: "However, this method has no practical meaning ...".

The development of automated theorem proving showed that this view point is over-pessimistic. Though, in general, the set of Herbrand disjunctions for a given formula is (only) enumerable, there is the hope that for certain interesting formulas in a reasonable time may be found Herbrand disjunctions H such that $\mathcal{T} \models H$. Nevertheless, substantial work has been necessary in order to improve both the representation of Herbrand disjunctions and the search for an Herbrand disjunction satisfying (1).

In particular means for treating multiple copies of subformulas (so called amplifications) and for a more goal oriented search for instances of (sub-) formulas had to be developed. The matings [1] or matrix method [5, 7] are the most direct, however computationally improved, algorithmic realizations of Herbrand disjunctions. Further techniques like theory unification as well as the more general form of unifying theory connections had to be developed and integrated with the matings or matrix method (cf. [3, 26]) in order to treat theories.

In the present paper we will illustrate those refinements of the method of Herbrand disjunctions by a case study concerning translations of Jaśkowski's discussive logic D_2 [19]. We discuss a translation working in two steps. A first translation step returns a modal formula F' such that $\models_d F$ if and only if $\models_{S4} F'$ for a given D_2-formula F. This first translation has been described in [19]. According to [10, 13] formula F' may be translated into a first-order formula F'' such that $\models_{S4} F'$ if and only if $\mathcal{T} \models F''$ for an appropriately chosen first-order theory $\mathcal{T}$. The theory $\mathcal{T}$, which has to be considered, is open. Thus, the target logic of our translation meets the requirements of Herbrand's theorem.

The sub-area of automated reasoning which is investigating methods for proving theorems under theories is called hybrid reasoning. A hybrid reasoning system is usually constructed from cooperating foreground reasoner and background reasoner. The foreground reasoner takes care of the general logical structure of a formula to be proved. The background reasoner is consulted whenever the meaning of a built-in theory must be considered. In [3, 26] has been formed a general framework for building in theories. For a newer presentation and a more complete bibliography see [4]. Different to [3, 26], which rely on syntactical representations of theories, [9] consideres also theories given semntically, i.e. by classes of models.

While in [3, 9, 27] have been considered homogeneous built-in theories, in the example descussed here, we have to take care of the internal structure of the built-in theory. For the target logic of the translation of multi-modal logics the built-in theory is combined from two sub-theories, one being a definite theory without equality, and the other being an equational theory. Reasoning within this equational sub-theory may ba reduced to associative unification with unit. Analoguously, also reasoning in the second sub-theory has a restricted form. Moreover, both sub-theories of the built-in theories are related to different parts of formulas.

Another reason for considering hybrid theories are normal form transformations. Usually, before any proof search starts, formulas are transformed to some normal form because the simpler the formulas to be proved are the simpler the proof calculus may be designed. Let us consider the following sequence of normal form transformations. After transformation to negation normal form anti-prenexing for treating quantifiers is used in order to decrease, if possible, the range of quantifiers, which are to be eliminated later by Skolemization. Next the structure preserving transformation to definitional normal form should be used. The value of those transformations has been discussed and analysed in [6, 14, 15]. Skolemization introduces new function symbols and eventually new axioms. The definitional normal form transformation introduces new predicate symbols of new axioms. This means, that instead of solving the validity problem $\mathcal{T} \models F$ a transformed problem $\mathcal{T}' \models F'$ with some, in a certain sense, less complex formula F' and an, in general extended, built-in theory $\mathcal{T}'$ will be attacked. Theory $\mathcal{T}'$ therefore may be treated as a hybrid theory. As a final remark in favor of the advised normal form transformations let us mention that human readable proof presentations may be generated easier if structure preserving normal form transformations has been used.

This paper is organized as follows. In Section 2 will be introduced necessary general notions. Section 3.1 illustrates the use of Herbrand disjunctions. The translation of formulas of a paraconsistent logic to modal logic S4 and further to first-order logic and the resulting target calculus will be discussed in Section 3.2. Section 4 is devoted to the presentation of a generic approach to building in theories into theorem provers. The application of the general approach to reasoning under hybrid theories will be presented in Section 4.4. Implementation issues are briefly described in Section 5.

Related work. The algebraic translation of multi-modal logic has been developed in [2, 10, 13]. An alternative translation of modal logic to first-order logic, the relational one, has been described by Alan Frisch and Richard Scherl as an instance of constraint reasoning [16]. Similar to the algebraic translation as described in [13] is the functional translation due to Hans Jürgen Ohlbach [25]. A general approach to building in theories into theorem provers via theory connections has been proposed by Wolfgang Bibel. In [6] this approach has been illustrated by the treatment of equality by so-called eq-connections. The extension of resolution to theory resolution is due to Mark Stickel. Many improvements of resolution have been shown as special kinds of theory resolution in [32]. For the lifting to the full first-order calculus see [3] or [26]. Another approach, considering theories given by classes of models, has been presented by Hans-Jürgen Bürckert [9]. Our approach carries over to that case if one considers a complete set for theory connections of each model of the considered class. The case of constraint reasoning may be seen also as a special case of reasoning in a hybrid theory with one theory being the empty theory.

2. Preliminaries

In order to keep the paper self-contained we recall basic notions concerning logic in general and theory reasoning in particular. We assume that the reader is familiar with the basic notions of first-order logic in clause form (cf. [21]). We consider clauses as disjunctions of conjunctions literals and we will ask for the validity of those formulas in a theory. Though our presentation is formulated for clause logic it may be carried over to full first-order logic. A clause with at most (exactly) one positive literal will be called a *Horn (definite) clause*. A definite clause consisting only of equational literals will be called a *conditional equation*. A *clause* is represented as a multi-set of literals. A *matrix* is a multi-set of clauses. Multi-Sets will be denoted as sequences of their elements. A set of copies of clauses of a matrix M will be called an *amplification* of M (see [22] for a more general definition of this notion). Clauses will be abbreviated also by Γ, C, D etc. Γ_1, Γ_2 denotes the union $\Gamma_1 \cup \Gamma_2$, whereas Γ, L denotes $\Gamma \cup \{L\}$ etc. A clause $L_1, \dots, L_n$ means the conjunction $(L_1 \wedge \cdots \wedge L_n)$ of its elements. The meaning of a matrix $C_1, \dots, C_n$ is the disjunction $C_1 \vee \cdots \vee C_n$.

This paper will focus on a family of proof procedures that generate goal driven a set of instances of clauses such that its validity in a given theory may be proved by checking a simple sufficient criterion. In order to formulate this criterion we first of all need the notions of a path and of a spanning theory mating. A *(partial) path (in) through* a matrix M is a multi-set containing (at most) exactly one literal from each clause of M. Paths will be abbreviated also by p or q. A set of partial paths in a matrix M is called a *mating* in M. A partial path u in a matrix M is *spanning* a path p through M if $u \subseteq p$. A mating U in a matrix M is *spanning* if for every path p through M exists an element $u \in U$ which is spanning p. If L is a positive literal then $\bar{L}$ denotes the literal $\neg L$. If L has the form $\neg K$ then $\bar{L}$ denotes the literal K. If p is the path $L_1, \ldots, L_n$ then $\bar{p}$ denotes the clause $\bar{L_1}, \ldots, \bar{L_n}$. And, vice versa, if Γ is the clause $L_1, \ldots, L_n$ then $\bar{p}$ denotes the path $\bar{L_1}, \ldots, \bar{L_n}$. The set of variables occurring in a term t, literal L, clause Γ or path p will be denoted by $Var(t)$, $Var(L)$, $Var(\Gamma)$ or $Var(p)$ respectively.

A *substitution* is a mapping from the set of variables into the set of terms which is almost everywhere equal to the identity. The *domain* of a substitution σ is the set $D(\sigma) = \{X \mid \sigma(X) \neq X\}$. The set of *variables introduced by* σ is the set $I(\sigma) = \bigcup_{x \in D(\sigma)} Var(\sigma(X))$. If the variables $X_1, \ldots, X_n$ are the elements of the domain of a substitution σ and the terms $t_1, \ldots, t_n$ are the corresponding values then σ will be denoted by $\{X_1 \mapsto t_1, \ldots, X_n \mapsto t_n\}$. A substitution σ may be extended canonically to a mapping from the set of terms into the set of terms. This extension will be denoted by σ too. For a set of variables V and substitutions σ and ρ we write $\sigma =_V \rho$ if for every element $X \in V$ holds $\sigma(X) = \rho(X)$. In the previous equation the lower index V may be omitted if V is the set of all variables. The *composition* $\sigma\theta$ of substitutions σ and θ is the substitution which assigns to every variable X the term $\theta(\sigma(X))$. A substitution σ is called *idempotent* if $\sigma = \sigma\sigma$. A substitution σ is idempotent iff $D(\sigma) \cap I(\sigma) = \emptyset$. If M is the multi-set of clauses $C_1, \ldots, C_n$ then $M' = C'_1, \ldots, C'_k$ is a *sub-matrix* of M iff there is a sequence of pairwise disjoint indices $i_1, \ldots, i_k$ s.t. C'_l is a sub-multi-set of C_{i_l} for each l with $1 \leq l \leq k$. A set of matrices which is closed w.r.t. the application of substitutions, forming amplifications and sub-matrices will be called a *query language*. For a path $p = L_1, \ldots, L_n$ and a query language $\mathcal{Q}$ we will write $p \in \mathcal{Q}$ in order to abbreviate $\{\{L_1\}, \ldots, \{L_n\}\} \in \mathcal{Q}$.

Let $\mathcal{T}$ be an open, i.e. quantifier-free, theory. A $\mathcal{T}$-model is an interpretation satisfying $\mathcal{T}$. A query (a clause, a path, a literal) S is *$\mathcal{T}$-satisfiable* if there is a $\mathcal{T}$-model satisfying S. It is *$\mathcal{T}$-unsatisfiable* else. A query (a clause, a path, a literal) S is *$\mathcal{T}$-valid* if every $\mathcal{T}$-model satisfies S. Let $\mathcal{E}$

be the theory of equality, i.e. the clause set consisting of clauses expressing reflexivity, symmetry, transitivity and functional and predicative substitutivity. Let $\mathcal{T}$ be an arbitrary theory. Then the set of predicate (function) symbols occurring in the formulas of a theory $\mathcal{T}$ be denoted by $\mathcal{P}(\mathcal{T})$ ($\mathcal{F}(\mathcal{Q})$ respectively).

3. Two examples illustrating Herband disjunctions

In this section we discuss two examples in order to illustrate the use of Herbrand disjunctions and possibilities of refinements.

The first example is more devoted to the introduction of Herbrand disjunctions.

The second example is taken from the paraconsistent logic D2. We use this examples in order to illustrate automated reasoning in this logic. For this purpose we use a two-step translation of this logic into first-order logic. The first step is Jaśkowski's [19] translation into the modal logic S4. The second step is the so called algebraic translation [13]. The algebraic translation of modal logics introduces new semantical items, possible worlds, following Kripke's approach to the definition of semantics for modal logics [20]. Moreover, so called transitions — semantical items of a further kind — are introduced. Transitions allow to pass from one world to another. The features of specific modal logics are expressed in terms of the target logic by first-order theories. Those theories consist of certain sub-theories. Thus, they may be considered as hybrid theories [28, 29]. The form of those theories allows to use general techniques, which have been developed for reasoning under first-order theories, for theorem proving in non-classical logics as well.

3.1. Theorem proving with Herbrand disjunctions

Given a theory $\mathcal{T}_1$ consisting of one axiom

$$\forall U \forall V \forall W (f(U,V) \wedge f(V,W) \rightarrow g(U,W)) \tag{2}$$

we try to prove the following theorem F

$$\forall X \exists s f(X,s) \rightarrow \forall d \exists Y g(d,Y) \tag{3}$$

by use of Herbrand disjunctions. In a preparatory step formula (3) is transformed to negation normal form and then to prenex form (4). Dropping leading universal quantifiers we obtain formula (5). Each of the sequences of formulas (5), ..., (8) has to be understood as disjunctions — so called Herbrand disjunctions.

$$(4) \qquad \forall d \exists X \exists Y \forall s(\neg f(X,s) \vee g(d,Y))$$

$$(5) \qquad \exists X \exists Y \forall s(\neg f(X,s) \vee g(d,Y))$$

$$(6) \qquad \neg f(d,s) \vee g(d,Y),\ \exists X \exists Y \forall s'(\neg f(X,s') \vee g(d,Y))$$

$$(7) \quad \neg f(d,s) \vee g(d,Y),\ \neg f(s,s') \vee g(d,Y),\ \exists X \exists Y \forall s''(\neg f(X,s'') \vee g(d,Y))$$

$$(8) \quad \begin{array}{l} \underline{\neg f(d,s)} \vee g(d,Y),\ \underline{\neg f(s,s')} \vee g(d,Y),\ \neg f(X,s'') \vee \underline{g(d,s')}, \\ \exists X \exists Y \forall s(\neg f(X,s''') \vee g(d,Y)) \end{array}$$

In particular, each of the Herbrand disjunctions (6), ..., (8) is the so-called direct derivative of its predecessor. Each direct derivative is obtained from its predecessor by

1. choosing a quantified disjunct of the Herbrand disjunction,
2. then instantiating the existentially quantified variables of a maximal quantifier prefix of the form $\exists X_1 \dots \exists X_n \forall s_1 \dots \forall s_m$ by terms,
3. afterwards dropping the universal quantifiers of that quantifier prefix, and, finally,
4. adding a new version of the disjunct which has been obtained by bound renaming of universal quantifiers.

Thus, each direct derivative is equivalent to its predecessor. The sequence (5), ..., (8) represents a proof for (3). Indeed, the disjunction of underlined formulas in (8) is an obvious consequence of axiom (2). Therefore the universal closure $\overline{\delta}$ of the Herbrand disjunction δ in (8) is valid under $\mathcal{T}_1$. Consequently, (3) is valid under $\mathcal{T}_1$.

In principle the method of Herbrand disjunctions could be used as a systematic method for searching proofs in predicate logic. Nevertheless, the method suffers from sever redundancies. In the following those redundancies and appropriate remedies will be discussed. The equations (9), ..., (12) present a refined version of the derivation (4), ..., (8) which has been discussed before.

The first redundancy is introduced already in the first transformation step. For the efficiency of the proof search it is important that quantifiers have a possibly small range (cf. [14]). Therefore anti-prenexing should be used instead of the transformation to prenex form. Formula (9) which is obtained from formula (3) by transformation to negation normal form is already in anti-prenex form. The next step, Skolemization, gives formula (10). Let us remark that, in order to keep the notation comparable with the approach of Herbrand disjunctions, we use the positive (or affirmative) representation in opposition to the negative (or refutational) representation which is widely used in automated theorem proving. Therefore each universal quantifier is substituted by a Skolem-function depending on each variable bound by an existential quantifier having the considered universal quantifier in its scope.[1] If the obtained formula is, in opposition to Formula (10), not yet in disjunctive normal form, a further transformation is necessary. The disjunctive normal form is a disjunction of existentially closed conjunctions. Transforming to this form it is important not to use the simple method relying just on applying the de Morgan's laws. This trivial method destroys the formula structure and may blow up the formula size exponentially. Both effects are undesired because of their disastrous implications for both proof search and presentation (cf. [15]).

$$\exists X \forall s \neg f(X,s) \vee \forall d \exists Y g(d,Y) \tag{9}$$

$$\exists X \neg f(X, s(X)) \vee \exists Y g(d,Y) \tag{10}$$

$$\neg f(X, s(X)) \vee g(d,Y) \tag{11}$$

$$(\neg f(X, s(X)) \vee \neg f(X', s(X') \vee g(d,Y)) \left\{ \begin{array}{c} X \mapsto d, \\ X' \mapsto s(d), \\ Y \mapsto s(s(d)) \end{array} \right\} \tag{12}$$

Finally, existential quantifiers may be dropped (11). Formula (11) is not $\mathcal{T}_1$-valid. But an analysis of the structure of $\mathcal{T}_1$ shows that two instances of the first disjunct of Formula (10) might be helpful for completing the proof. Indeed, a proof may be found after adding the instance $\neg f(X, s(X))$ and $\neg f(X', s(X'))$ and applying the substitution $\{X \mapsto d, X' \mapsto s(d), Y \mapsto s(s(d))\}$. This way, essentially, the same proof argument has found as before in (12). Now let us compare both approaches. First of all, let us draw the reader's attention to the fact that the proof search in the second approach consists of only one step, i.e. trying new clause instances and computing a unifier in (12). None of the three preparatory steps does contain any search. The search itself is guided by the search for sets of literals $\{\neg f(X, s(X)), \neg f(X', s(X')), g(d,Y)\}$, a so called theory-connection[2], having the following two properties:

[1] By some abuse of notation we denote the introduced Skolem function by the same identifier as the variable it substitutes.

[2] Here a $\mathcal{T}_1$-connection.

1. any two of their elements are connected by a disjunction and
2. their disjunction is valid after applying an appropriate substitution.

Moreover, this substitution may be computed in the present case by solving the unification problem $\{X = d, X' = s(X), Y = s(X')\}$. For a wide class of theories both the general form of the theory-connections and the unification problems may be determined in a generic way.

3.2. Automated reasoning in para-consistent logic

In the present section we show how to translate the para-consistent logic D_2 into a first-order logic. The first step of this translation uses just the definition of the discussive connectives as defined by Jaśkowski and his scholars in terms of the modal system S4. The second step is the so-called algebraic translation of the modal system S4 into first-order logic. For a source of para-consistent logic the reader is referred to [11] and for a detailed presentation of the algebraic translation of modal logics to [13]. Here we can illustrate only basic features of the target logic of this translation. Let us consider Formula (13) which is a tautology of Jaśkowski's discussive logic D_2.

$$(((p \rightarrow_{\mathrm{d}} q) \wedge (p \rightarrow_{\mathrm{d}} (\neg q))) \rightarrow_{\mathrm{d}} (\neg p)) \tag{13}$$

According to the translation given by Jaśkowski [19], in order to prove this sentence in D_2, one has to prove that the modal formula (14) is valid in the modal system S4. In order to be able to treat also multi-modal systems we made a minor change, i.e. adding an index 'a' to the modalities.

$$\Diamond_a(\Diamond_a((\Diamond_a p \rightarrow q) \wedge (\Diamond_a p \rightarrow (\neg q))) \rightarrow (\neg p)) \tag{14}$$

Formula (14) may be transformed to negation normal form (15).

$$\Diamond_a(\Box_a((\Diamond_a p \wedge \neg q) \vee (\Diamond_a p \wedge q)) \vee \neg p) \tag{15}$$

Now, let us consider the translation of the modal system S4 into a first-order theory. Modality $\Box_a$ will be characterized by the axiom schemes (16) and (17) within this theory. Let us remark that Φ denotes an arbitrary formula.

$$\Box_a \Phi \rightarrow \Box_a \Box_a \Phi \tag{16}$$

$$\Box_a \Phi \rightarrow \Phi \tag{17}$$

The translation of formulas will be illustrated by the translation of Formula (15) into Formula (18). In order to ease the comparison of the multi-modal formula (18) and the first-order sentence in (18) the latter one has been written in a way emphasizing the logical structure. Modal operators, $\Box_a$ and $\Diamond_a$, have been translated by restricted quantifiers, $\forall_{\alpha_1:k(a,\alpha_1)}$ and $\forall_{\alpha_1:k(a,\alpha_1)}$ respectively. Moreover each predicate symbol has obtained an additional argument, $\varepsilon!\alpha_1!\alpha_2$ for example.

$$
\begin{array}{l}
\exists_{\gamma:k(a,\gamma)}(\\
\quad \forall_{\alpha:k(a,\alpha)}(\\
\quad\quad (\exists_{\beta:k(a,\beta)} p(\varepsilon!\gamma!\alpha!\beta) \wedge \neg q(\varepsilon!\gamma!\alpha!\beta)) \\
\quad\quad \vee \\
\quad\quad (\exists_{\delta:k(a,\delta)} p(\varepsilon!\gamma!\alpha!\delta) \wedge q(\varepsilon!\gamma!\alpha!\delta)) \\
\quad) \\
\quad \vee \\
\quad \neg p(\varepsilon!\gamma) \\
)
\end{array} \tag{18}
$$

First of all let us discuss the role of this additional argument. It represents a possible world, which has been coded by a term. The term $\varepsilon!\alpha_1$ represents a world, which is accessible from the initial world ε via the transition α_1. Formally this has been expressed by the infix operator !, which takes two arguments, a world (here ε) and a possible transition (here α_1), and returns a world accessible from the given world via that transition. The operator ! associates to the left, therefore brackets will be omitted wherever possible. Transitions can be combined by the associative binary operator $*$. Moreover, there is a distinguished transition, which is denoted by 1. The operations $*$, ! and 1 form a monoid operating on the set of worlds, i.e. we have the equational theory $\mathcal{T}$ consisting of the axioms (19), ..., (23) introduced below.

$$w!1 = w \tag{19}$$

$$w!(\alpha_1 * \alpha_2) = (w!\alpha_1)!\alpha_2 \tag{20}$$

$$(\alpha_1 * \alpha_2) * \alpha_3 = \alpha_1 * (\alpha_2 * \alpha_3) \tag{21}$$

$$1 * \alpha = \alpha \tag{22}$$

$$\alpha * 1 = \alpha \tag{23}$$

Now let us consider the restricted quantifiers, which have been introduced by the translation, in more detail. The restricted quantifier $\forall_{\alpha_1:k(a,\alpha_1)}$ is the translation of the modal operator $\Box_a$. The sort information $\alpha_1 : k(a, \alpha_1)$ given by the restricted quantification of variable α_1 just says that this variable is related to the interpretation of the modality $\Box_a$. The term $\varepsilon!\alpha_1!\alpha_2!\alpha_3$ represents a world which may be accessed from world ε making use of three

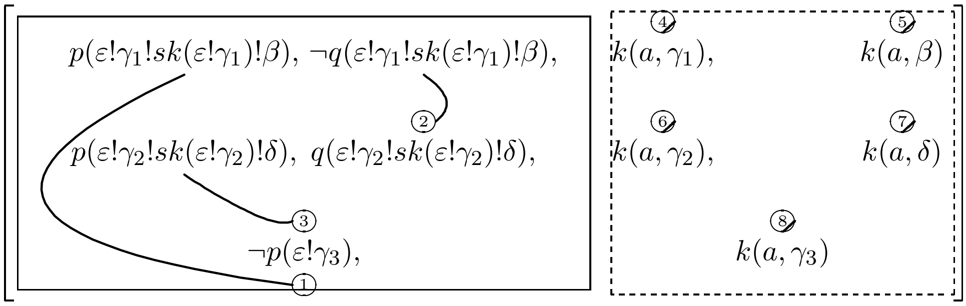

Figure 1. The matrix form of the translated D_2-formula

transitions α_1, α_2 and α_3, one after the other. The properties of the modality will be expressed by the clauses (26) and (27) given below. After this preparation we are ready to consider the translation of Formula (18) into clause normal form (24).

$$
\begin{aligned}
&\exists\gamma\ \neg k(a, sk(\varepsilon!\gamma)) \wedge k(a,\gamma)\\
&\vee\\
&\exists\gamma\ \exists\ \beta\ p(\varepsilon!\gamma!sk(\varepsilon!\gamma)!\beta) \wedge \neg q(\varepsilon!\gamma!sk(\varepsilon!\gamma)!\beta) \wedge k(a,\beta) \wedge k(a,\gamma)\\
(24)\qquad &\vee\\
&\exists\gamma\ \exists\delta\ p(\varepsilon!\gamma!sk(\varepsilon!\gamma)!\delta) \wedge q(\varepsilon!\gamma!sk(\varepsilon!\gamma)!\delta) \wedge k(a,\delta) \wedge k(a,\gamma)\\
&\vee\\
&\exists\gamma\ \neg p(\varepsilon!\gamma) \wedge k(a,\gamma)
\end{aligned}
$$

In the first disjunct of formulas (24) occurs only the predicate symbol k. Therefore we decide to consider the negation of this clause as part of the built-in theory (25) ... (27) rather than this clause as a part of the formula to be proven. In Figure 1 we write the instantiated part of a Herbrand disjunction of formula (24) as a matrix with the literals of each clause forming a row. Each clause is interpreted as the conjunction of its literals, whereas the clauses are connected by disjunction. The 3 clauses in Figure 1 correspond to the three main sub-formulas of Formula (18). The matrix in Figure 1 has to be proved under the union of the theories $\mathcal{T}$, consisting of Formulas (19), ..., (23), and $\Re$, consisting of formulas (25), ..., (27) as axioms. Let us recall, that the properties (16), ..., (17) of the modal system S4, which characterizes the modality $\Box_a$, are expressed by clauses (26), ..., (27) in terms of the target logic. Clause (25) characterizes the properties of the

Instantiated $\Re$-connection	Used axioms
$k(a,1)$, $k(a,1)$, $k(a,1)$, $k(a,1)$	(27)
$k(a, 1 * sk(\varepsilon!1) * 1)$	(25), (26), (27)

Table 1. Solving constraints of the matrix in Figure 1

Skolem function sk which had to be introduced for the universal quantifier $\forall_{\alpha:k(a,\alpha)}$. For details see [13].

(25) $$\neg k(a,\alpha) \vee\ k(a, sk(\varepsilon!\alpha))$$

(26) $$\neg k(a,\alpha_1) \vee\ \neg k(a,\alpha_2) \vee\ k(a, \alpha_1 * \alpha_2)$$

(27) $$k(a,1)$$

The proof task in the target logic is to show that the matrix in Figure 1 is valid in the union of the theories $\mathcal{T}$ and $\Re$. The last mentioned matrix has the following syntactic properties.

1. Equality does not occur as a predicate symbol in the matrix.
2. Sort literals (i.g. $k(a,\beta)$) are positive.

From the first observation we deduce that all theory connections within the boxed part of the matrix in Figure 1 are binary connections of the form $p(\underline{t}), \neg\, p(\underline{s})$ where the tuples of terms $\underline{t}$ and $\underline{s}$ are component-wise $\mathcal{T}$-unifiable. The second syntactic property and the form of theory $\Re$ make sure that sort literals occurring in the matrix in Figure 1 may be elements only of unary $\Re$-connections.

Now we can discuss the remaining details of Figure 1. Three $\mathcal{T}$-connections are indicated by arcs in the boxed part. They may be simultaneously $\mathcal{T}$-unified by the substitution

(28) $$\{\gamma_1, \gamma_2, \beta, \delta \mapsto 1, \gamma_3 \mapsto 1 * sk(\varepsilon!1) * 1\}$$

It is easy to verify that every sort literal in the dashed boxed sub-matrix in Figure 1 is an $\Re$-connection. Substitution (28) is also a simultaneous $\Re$-unifier for these $\Re$-connections. Table 1 gives for each of those $\Re$-connections the axioms which have to be used for proving this statement. The mentioned theory connections may be found subsequently by theory inference steps of the form given in Example 4.5. An appropriate calculus will be introduced in Section 4.3. The reader may have observed that none of the equational axioms (19), ..., (23) has been mentioned in Table 1. Indeed, the following proposition holds.

Proposition 3.1. *Suppose that the equation $t = s$ is valid in the equational theory $\mathcal{T}$. Then for the literal $k(a, s)$ (and analogously for $k(b, s)$) holds that $k(a, s)$ is $\Re$-valid iff $k(a, t)$ is $\Re$-valid.*

From this observation follows that for proving the $\Re$-validity of a literal $k(a, t)$ we don't need to apply equational axioms. Speaking more operationally, when inferencing within the constraint theory $\Re$ there is no need to apply $\mathcal{T}$-unification but only syntactical unification. This is a useful feature of the target logic of multi-modal logic. Assumption (1) of Proposition 4.4 is related to this feature.

4. A generic approach to theory reasoning

In the present section we introduce a formal framework for constructing complete total theory reasoning calculi for open, i.e. quantifier free, theories. A complete theory reasoning calculus for an open theory needs the following key capabilities: (1) finding theory connections, (2) computing unifiers for theory connections, and (3) managing amplifications and representations of sets of paths which are not spanned by a currently found theory mating. The ingredients for constructing a complete theory reasoning calculus — a complete set of theory connections (Definition 4.3) with a solvable unification problem (Definition 4.4) and a calculus managing amplifications of matrices and keeping track of unsolved goals — will be introduced in the subsections 4.1, 4.2 and 4.3 respectively. Implementation issues are discussed in Section 5. In the present section we formalize what it means to have for a given theory "enough" theory connections in order to prove all theory valid matrices which belong to a given query language. We formulate a Herbrand theorem by use of this notion (cf. Subsection 4.1). The notion of a complete set of unifiers for a theory connection generalizes the notion of complete set of theory unifiers of a pair of terms.

4.1. Complete sets of theory connections

In order to formulate sufficient conditions for the completeness of a theory reasoning calculus we introduce the notion of a set of theory connections which is complete with respect to a given query language. For an open theory first-order $\mathcal{T}$, given as a set of clauses, we formalize (see Definition 4.3), what it means, to have "enough" theory connections in order to prove all theory valid matrices, which belong to a given query language.

DEFINITION 4.1 ($\mathcal{T}$-complementary, $\mathcal{T}$-unifier). A path u is called *$\mathcal{T}$-complementary* if and only if the universal closure of the disjunction of the elements of u, $\bar{\forall}\left(\bigvee_{L\in u} L\right)$, is $\mathcal{T}$-valid. A substitution σ is a *$\mathcal{T}$-unifier* of u if and only if $\sigma(u)$ is $\mathcal{T}$-complementary.

Remark 4.1. The $\mathcal{T}$-complementarity of a path u has been defined via the $\mathcal{T}$-validity of the universal closure of the disjunction of the elements of u according to the positive representation which has been chosen in the present paper. In the negative representation $\mathcal{T}$-complementarity of a path u we would have been defined via the $\mathcal{T}$-unsatisfiability of the conjunction of the elements of u. The remaining notions and results may be defined independently on the chosen (positive or negative) representation. □

DEFINITION 4.2 (Connection, $\mathcal{T}$-Connection)). Let $\mathcal{T}$ be a theory, M a matrix, $\mathcal{U}$ a set of multi-sets of literals and $\mathcal{Q}$ a query language. Any partial path u in M will be called a *$\mathcal{T}$-connection* in M if there exists a $\mathcal{T}$-unifier for u. If $\mathcal{T}$ is the empty theory then the prefix $\mathcal{T}$ may be omitted.

DEFINITION 4.3 (Complete set of theory connections). Let $\mathcal{T}$ be a theory, M a matrix, $\mathcal{U}$ a set of $\mathcal{T}$-connections and $\mathcal{Q}$ a query language.

1. Any set of $\mathcal{T}$-connections in a matrix M, which are elements of $\mathcal{U}$, is called a *$\mathcal{U}$-mating* in M.
2. A decidable set $\mathcal{U}$ of $\mathcal{T}$-connections which is closed w.r.t. application of substitutions will be called *$\mathcal{T}$-complete w.r.t. $\mathcal{Q}$* if
   (a) for each $\mathcal{T}$-complementary ground path $p \in \mathcal{Q}$ exists $u \in \mathcal{U}$ such that $u \subseteq p$ and
   (b) for each $\mathcal{T}$-complementary ground path of the form $\sigma(u) \in \mathcal{U}$ such that $u \in \mathcal{Q}$ holds $u \in \mathcal{U}$.

Example 4.1. In the simplified version of equational reasoning, discussed in Section 3.2, the equality symbol does not occur in the query language. The following characterization may be specialized, setting $n = 1$, to the case studied in Section 3.2. In terms of Definition 4.3 the set $\mathcal{U}_{\mathcal{T}}$ of connections of the form $p(t_1, \ldots, t_n), \neg p(s_1, \ldots, s_n)$ for simultaneously pairwise $\mathcal{T}$-unifiable terms t_i and s_i is complete w.r.t. to the query language $\mathcal{Q}_{\mathcal{T}}$. □

Example 4.2. Let us now consider the query language $\mathcal{Q}_{\Re}$ discussed in Subsection 3.2. It contains only positive clauses with a single predicate symbol k and the function symbols as in Example 4.1. The theory $\Re$ is formed from the definite clauses (25) and (27) given in Subsection 3.2. As an example consider the positive clause $k(a, \gamma_2), k(a, \delta)$ which occurs as a fragment of the second clause in Figure 1. Each literal of this clause becomes $\Re$-valid after applying the substitution $\{\gamma_2 \mapsto 1, \delta \mapsto 1\}$. Since $\Re$ is definite all $\Re$-connections in queries from $\mathcal{Q}_{\Re}$ are units. Thus, the set $\mathcal{U}_{\Re}$ of positive literals with predicate symbol k having a $\Re$-unifier is a set of $\Re$-connections complete with respect to the query language $\mathcal{Q}_{\Re}$. □

The less literals a connection consists of the more paths it may span. Therefore, we are interested to find theory connections which are minimal with respect to set-theoretical inclusion. Every extra literal may cause that additional sub-goals have to be solved. The following proposition makes sure that a complete set of theory connections contains also all minimal connections.

PROPOSITION 4.1 (Properties of complete sets of theory connections).
Let the set of $\mathcal{T}$-connections $\mathcal{U}$ be $\mathcal{T}$-complete with respect to the query language $\mathcal{Q}$. Let u be a path such that $u \in \mathcal{Q}$. If u is minimal $\mathcal{T}$-complementary then $u \in \mathcal{U}$.

Having a complete set of theory connections a Herbrand theorem may be proved. The following version of Herbrand's theorem applies to the discussed examples 4.1 and 4.2.

THEOREM 4.1 (Herbrand's theorem). *Let $\mathcal{T}$ be an open theory, $\mathcal{Q}$ a query language, $\mathcal{U}$ a set of $\mathcal{T}$-connections which is complete w.r.t. to $\mathcal{Q}$. Then for every $\mathcal{T}$-valid matrix $M \in \mathcal{Q}$ there exists an amplification M' of M, a $\mathcal{U}$-mating U which is spanning in M' and a substitution σ such that $\sigma(u)$ is $\mathcal{T}$-complementary for each $u \in U$.*

4.2. The unification problem for sets of theory connections

The Herbrand theorem gives neither a hint how to find the substitution σ nor how to decide the existence of σ. In order to obtain a proof calculus for a given complete set of $\mathcal{T}$-connections $\mathcal{U}$ we also need to be able to compute or to represent for every $u \in \mathcal{U}$ all substitutions σ such that $\sigma(u)$ is $\mathcal{T}$-valid. This will be formulated in the following definition.

DEFINITION 4.4 (more general $\mathcal{T}$-unifier, $\mathcal{T}$-unification problem in $\mathcal{U}$).
Let $\mathcal{U}$ be a set of multi-sets of literals.

(1) Let ϱ and σ be $\mathcal{T}$-unifiers of a path $u \in \mathcal{U}$ such that $D(\varrho), D(\sigma) \subseteq Var(u)$. Then ϱ is called *more general than* σ if there exists η such that $\varrho\eta =_{Var(u)} \sigma$. This will be denoted by $\varrho \leq \sigma$.
(2) A set S of $\mathcal{T}$-unifiers of a multi-set $u \in \mathcal{U}$ will be called *complete* if for each $\mathcal{T}$-unifier σ of u exists a substitution $\varrho \in S$ such that $\varrho \leq \sigma$.
(3) We say that the *$\mathcal{T}$-unification problem in $\mathcal{U}$ is solvable* if
    (a) for every $u \in \mathcal{U}$ there exists an enumerable complete set S_u of $\mathcal{T}$-unifiers for u and
    (b) for a given $u \in \mathcal{U}$ it is decidable whether $S_u \neq \emptyset$.
(4) A substitution σ is called a *simultaneous $\mathcal{T}$-unifier* of a set U of multi-sets of literals if and only if $\sigma(u)$ is $\mathcal{T}$-complementary for every $u \in U$.

Example 4.3. Let $\mathcal{U}_{\mathcal{T}}^{upp}$ be the subset of $\mathcal{T}$-connections defined in Example 4.1 obeying the following, so called, unique prefix property (cf. [12]). A formula or a term has the unique prefix property if the binary symbol $*$ does not occur and for each variable α introduced for a modal operator holds that it occurs always in the same left context. Formulas obtained by the algebraic translation have this property. This corresponds to the fact that each variable introduced for a modal operator occurs always in the same modal context. For those restricted $\mathcal{T}$-unification problems exists an efficient unification algorithm [12]. □

In a connection calculus we have to find a *simultaneous $\mathcal{T}$-unifier* of a spanning mating of $\mathcal{T}$-connections incrementally. The solvability of the unification problem in a set of theory connections $\mathcal{U}$ implies the solvability of the simultaneous unification problem in $\mathcal{U}$.

PROPOSITION 4.2. *Suppose that the $\mathcal{T}$-unification problem is solvable for the set of $\mathcal{T}$-connections $\mathcal{U}$ and that S_u denotes the complete set of $\mathcal{T}$-unifiers for each $u \in \mathcal{U}$. Then every simultaneous $\mathcal{T}$-unifier θ of a set of $\mathcal{T}$-connections $U \subseteq \mathcal{U}$ can be approximated incrementally. Indeed, for each enumeration $u_1, \ldots, u_n$ of the elements of U may be constructed sequences $\{\sigma_i\}_{i=1}^n$, $\{\eta_i\}_{i=0}^n$, $\{\varrho_i\}_{i=0}^n$ such that*

(1) *$\eta_0 = \theta$ and $\varrho_0 = \{\ \}$ and*
(2) *for every $i, 1 \leq i \leq n$*
    (a) $\sigma_i \in S_{\varrho_{i-1}(u_i)}$,

(b) $\sigma_i \eta_i = \eta_{i-1}$ and
(c) $\varrho_i = \varrho_{i-1}\sigma_i$.

(3) $\varrho_n \eta_n = \theta$ and

Example 4.4. Let θ be the simultaneous $\mathcal{T}$-unifier (28) of the 8 connections (three binary and 5 unary) indicated in Fig. 1. That 8 connections may be found in 8 deduction steps. Those inferences determine the unifiers $\sigma_1, \ldots, \sigma_8$ which are subsequent approximations of the simultaneous $\mathcal{T}$-unifier of the three connections. We have $\theta = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8$ with

$$\begin{array}{lll} & \sigma_1 = \{\gamma_3 \mapsto \gamma_1 * sk(\varepsilon!\gamma_1) * \beta\} & \sigma_4 = \{\gamma_2 \mapsto 1\} \\ (29) & \sigma_2 = \{\gamma_1 \mapsto \gamma_2, \beta \mapsto \delta\} & \sigma_5 = \{\delta \mapsto 1\} \\ & \sigma_3 = \sigma_6 = \sigma_7 = \sigma_8 = \{\} & \end{array}$$

□

4.3. The pool calculus with built-in theory

In this section we introduce a generalization of the pool calculus [24] towards theory reasoning. For an amplification M' of the matrix M to be proved a *pool* of so-called *hooks* represents the set of paths through M' which are not spanned by the set of theory connections which have been found up to the current proof state. Each hook, denoted by $(p \perp \Gamma)$, and consisting of a partial path p in M' and a partial clause Γ in M', represents all paths through M' continuing p via some literal of Γ. Figure 2 shows a three-step derivation under the built-in theory $\mathcal{T}$ given by the equations (19), ..., (23). In each deduction step a $\mathcal{T}$-connection is detected and a most general $\mathcal{T}$-unifier has to be computed. The $\mathcal{T}$-connections are drawn as arcs. The final proof state is the rightmost in the second row in Figure 2. The mating U formed by those $\mathcal{T}$-connections is spanning that matrix.

Let us consider the derivation in Figure 2 in more detail. A diagonal arrow pointing to the *current goal* appears in every but the lower rightmost matrix. The *current path* p is given by the set of boxed literals. In each of the inference steps a $\mathcal{T}$-connection is found which contains the current goal L. In the last inference the found connection is subset of $p \cup \{L\}$. This so-called *reduction step* does not generate additional goals. This is not the case in the first two inference steps, so-called *extension steps.* The first extension step solves the initial goal $\neg p(\varepsilon!\gamma_3)$ with the substitution σ_1 but opens a new goal $\neg q(\varepsilon!\gamma_1!sk(\varepsilon!\gamma_1)!\beta)$ in the second clause. This goal is solved by extension step 2 extending the current substitution to substitution σ_2. Extension step 2 in turn opens the goal $p(\varepsilon!\gamma_2!sk(\varepsilon!\gamma_2)!\delta)$ in the third clause. The latter is solved by the last inference.

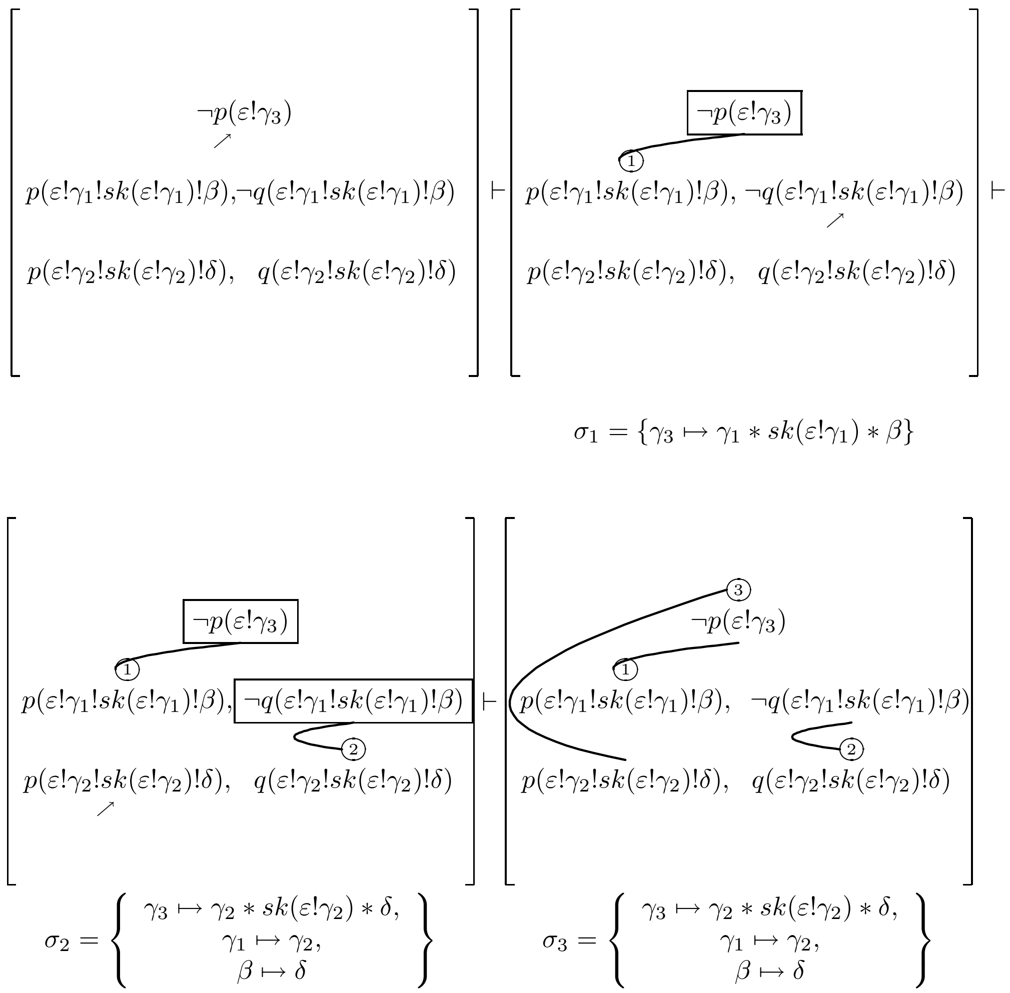

Figure 2. A Sample Deduction

Now we look at the general case. In order to describe a state of the derivation we have to represent the set of complete paths which still have to be considered. For this purpose we use *hooks* $(p \perp \Gamma)$. A hook represents all those paths which continue the path fragment p via one of the literals of the clause fragment Γ. We use the sign $\perp$ in order to indicate that the partial path p and the clause Γ are in a sense orthogonal. The elements of Γ are called *goals*. The current goal of the hook has to be solved by applying an inference rule. Inference rules describe how new hooks are generated from a given hook. A derivation starts with an *initial hook* which has the form $(\emptyset \perp \Gamma)$ where Γ contains all literals of a given goal clause. Hooks of the form $(p \perp \emptyset)$ are said to be *solved*. Solved hooks need not to be considered any more. A derivation is complete if no more unsolved hooks are left. For more details see also [24], [27] or [4].

Definition 4.5 (Pools, hooks). A *hook* for a matrix M is a pair (p, Γ) where p is a partial path in an amplification M' of M and Γ is a sub-clause of a clause $\Gamma' \in M'$ such that $p \cap \Gamma = \emptyset$. The hook (p, Γ) will be denoted by $(p \perp \Gamma)$. The partial path p is called the *current path*. The elements of Γ are called *goals*. The *set of paths represented by the hook* $(p \perp \Gamma)$ is the set $Paths_{(p \perp \Gamma)} = \{p' \mid \exists L(p \cup \{L\} \subset p', p' \cap \Gamma = \{L\}), p' \text{ is a path through } M'\}$. A hook $(p \perp \emptyset)$ will be called a *solved hook*, and a hook of the form $(\emptyset \perp \Gamma)$ is called an *initial hook*.

An inference step chooses a hook, removes it from the pool, and eventually produces some new hooks. The rules of a calculus describe how to construct new hooks from a chosen hook.

Definition 4.6 ($\mathcal{T}$-connection inference). Let $\mathcal{U}$ be a complete set of $\mathcal{T}$-connections and M a matrix. A *$\mathcal{T}$-connection inference* is an inference rule of the form

$$\frac{(p \perp \Gamma_0, L_0) \qquad \Gamma_1 \cup \{L_1\}, \; \ldots \; , \Gamma_n \cup \{L_n\}}{(p \perp \Gamma_0), \; (p, L_0 \perp \Gamma_1), \; \ldots, \; (p, L_0, \ldots, L_{n-1} \perp \Gamma_n)} \quad \sigma$$

where (1) $(p \perp \Gamma_0, L_0)$ is a hook, called the *chosen hook*, (2) if $0 < n$ then the clauses $\Gamma_1 \cup \{L_1\}, \ldots, \Gamma_n \cup \{L_n\}$ are copies of clauses from M, called the *extension clauses*, (3) σ is a substitution, (4) the hooks $(p \perp \Gamma_0)$, $(p, L_0 \perp \Gamma_1), \ldots, (p, L_0, \ldots, L_{n-1} \perp \Gamma_n)$ are called *new hooks* and (5) there exists a sub-path q of p such that $u \in \mathcal{U}$ and $\sigma(u)$ is $\mathcal{T}$-complementary for the partial path $u = q \cup \{L_0, \ldots, L_n\}$. A $\mathcal{T}$-connection inference is called an *extension step* if $n \neq 0$ and a *reduction step* else.

Example 4.5. Let us return to the sample derivation in Figure 2. In that example an equational theory $\mathcal{T}$ has been assumed which contains the equation $(\varepsilon!\alpha)!\beta = \varepsilon!(\alpha * \beta)$. Let $\mathcal{U}$ be the set of all unordered pairs of literals $\{p(t_1, \ldots, t_n), \neg p(t'_1, \ldots, t'_n)\}$ such that for each i with $1 \leq i \leq n$ the terms t_i and t'_i are $\mathcal{T}$-unifiable. A theory extension is an inference rule of the form

$$\frac{(p \perp L_0, \Gamma_0) \qquad L_1, \Gamma_1}{(p \perp \Gamma_0),\ (p, L_0 \perp \Gamma_1)}\ \sigma$$

where (1) L_1, Γ_1 is a copy of a clause from M, called the *extension clause* and (2) for $u = \{L_0, L_1\}$ holds $u \in \mathcal{U}$ and $\sigma(u)$ is $\mathcal{T}$-complementary. A theory reduction rule has the form

$$\frac{(p \perp L_0, \Gamma_0)}{(p \perp \Gamma_0)}\ \sigma$$

where for some literal $L_1 \in p$ and $u = \{L_0, L_1\}$ holds $u \in \mathcal{U}$ and $\sigma(u)$ is theory complementary. □

DEFINITION 4.7 (Rule application). A rule

$$\frac{h \qquad \Gamma_1, \ldots, \Gamma_n}{H}\ \sigma$$

may be applied to a pool P if $h \in P$. The new pool is obtained from P by removing h, then adjoining those hooks from H which are not solved and finally applying the substitution σ to the resulting pool. The clause copies used in an inference within a derivation must have always a set of new variables, i.e. those not occurring already in the pool. Moreover if $u \in \mathcal{U}$ is the $\mathcal{T}$-connection chosen in the considered rule application then the variables from $Var(\sigma) \setminus Var(u)$ must not occur in P.

An *initial pool* in a derivation consists of a single initial hook. Now a *derivation* may be defined as a sequence of rule applications which starts from an initial pool. A derivation is called *ground* if the unifier in every $\mathcal{T}$-connection step is empty. A derivation is *successful* if its last element is the empty pool. The calculus is sound, because in every state of a derivation the pool represents all paths, such that there still have to be found theory connections spanning them.

PROPOSITION 4.3 (Soundness). *The theory connection calculus is sound.*

The completeness proof consists of the steps Herbrand theorem, ground completeness and lifting lemma. The Herbrand theorem (4.1) and the lifting lemma rely on the completeness of a given set of theory connections $\mathcal{U}$ and the solvability of the theory unification problem in $\mathcal{U}$. The proof of the ground completeness relies on the properties of minimal spanning matings. The following result may be found already in [27].

Theorem 4.2 (General Completeness theorem). *Suppose that for a theory $\mathcal{T}$ and a query language $\mathcal{Q}$ there is given a decidable set $\mathcal{U}$ of $\mathcal{T}$-connections which is $\mathcal{T}$-complete w.r.t. $\mathcal{Q}$ and the $\mathcal{T}$-unification problem in $\mathcal{U}$ is solvable. Then for every $\mathcal{T}$-valid query from $\mathcal{Q}$ exists a clause $\Gamma \in \mathcal{Q}$ and a successful derivation starting from the initial pool $\{(\perp \Gamma)\}$ such that in each inference according to Definition 4.6 for the chosen connection u holds $u \in \mathcal{U}$ and the chosen $\mathcal{T}$-unifier σ is an element of the complete set of $\mathcal{T}$-unifiers S_u for u.*

4.4. Hybrid theories

In Section 3 we have discussed the translation of the paraconsistent logic D2 to a first-order theory via the modal logic S4. This translation justifies the treatment of the background reasoner of a hybrid reasoner as a hybrid system itself. Let us now forge precise notions which enable us to construct a $\mathcal{T} \cup \Re$-reasoner from a $\mathcal{T}$-reasoner and a $\Re$-reasoner. A formula will be considered as consisting of a $\mathcal{T}$-layer and an $\Re$-layer. The intended $\mathcal{T} \cup \Re$-reasoner should try to find a $\mathcal{U}_{\mathcal{T}}$-connection if the current goal is in the $\mathcal{T}$-layer and a $\mathcal{U}_{\Re}$-connection if the current goal is in the $\Re$-layer. We formulate sufficient conditions such that $\mathcal{U}_{\mathcal{T}} \cup \mathcal{U}_{\mathcal{R}}$ is a complete set of $\mathcal{T} \cup \Re$-connections for $\mathcal{Q}$ if so are $\mathcal{U}_{\mathcal{T}}$ for $\mathcal{Q}_{\mathcal{T}}$ and $\mathcal{U}_{\mathcal{R}}$ for $\mathcal{Q}_{\Re}$. If, moreover, the theory unification problems in both $\mathcal{U}_{\mathcal{T}}$ and $\mathcal{U}_{\mathcal{R}}$ do not interfere, we just can use the unification algorithms for the connections belonging to one of both layers without change.

Definition 4.8. Let a theory be given by its sub-theories $\mathcal{T}$ and $\Re$ which are formulated within the signatures Σ and Δ respectively. Then we say that $\mathcal{T}$ and $\Re$ *form a hybrid theory* in the union $\Sigma \cup \Delta$ of both signatures.

Definition 4.9. Let the theories $\mathcal{T}$ and $\Re$ form a hybrid theory in the union $\Sigma \cup \Delta$ of their signatures and let $\mathcal{Q}$ be a query language formulated in a signature which contains $\Sigma \cup \Delta$.

Every clause C in a matrix $M \in \mathcal{Q}$ contains then two sub-clauses $C_{\mathcal{T}}$ and $C_{\Re}$ consisting of literals L expressed in signature Σ (respectively L'

expressed in signature Δ). The set of nonempty sub-clauses $C_{\mathcal{T}}$ of M will be called the $\mathcal{T}$-layer of M. Analogously will be defined the $\Re$-layer of M. By $\mathcal{Q}_{\mathcal{T}}$ (analogously $\mathcal{Q}_{\Re}$) will be denoted the set of all matrices being the $\mathcal{T}$-layer (respectively the $\Re$-layer) of a query from $\mathcal{Q}$. $\mathcal{Q}_{\mathcal{T}}$ (analogously $\mathcal{Q}_{\Re}$) will be called the *$\mathcal{T}$-layer* (respectively the $\Re$-layer) of $\mathcal{Q}$. If for a matrix $M \in \mathcal{Q}$ every of its clauses is the union of its $\mathcal{T}$- and $\Re$-layers then M will be called *covered by its $\mathcal{T}$- and $\Re$-layers*. If every matrix $M \in \mathcal{Q}$ is covered by its $\mathcal{T}$- and $\Re$-layers then query language $\mathcal{Q}$ is said to be *covered by its $\mathcal{T}$- and $\Re$-layers*.

Example 4.6. In the example in Figure 1 the signatures Σ of the $\mathcal{T}$-layer and Δ of the $\Re$-layer share the function symbols !, ε, $*$, sk and a. Σ contains p and q as the only predicate symbols, Δ contains k and the equality symbol =. The target language of the algebraic translation of multi-modal logic is covered by its $\mathcal{T}$- and $\Re$-layers. Since the sets of predicate symbols of the $\mathcal{T}$-layer and the $\Re$-layer are disjoint, for each literal L the sets of $\mathcal{T}$- and of $\Re$-connections L might belong to are disjoint. □

DEFINITION 4.10. Let $\mathcal{T}$ and $\Re$ form a hybrid theory in the union $\Sigma \cup \Delta$ of signatures. Let $\mathcal{Q}$ be a query language formulated in a signature containing both signatures Σ and Δ. Moreover, let $\mathcal{U}_{\mathcal{T}}$ and $\mathcal{U}_{\Re}$ be sets of $\mathcal{T}$-connections and of $\Re$-connections. We say that $\mathcal{U}_{\mathcal{T}}$ and $\mathcal{U}_{\Re}$ are *separated w.r.t.* $\mathcal{Q}$ if and only if there does not exist connections $u \in \mathcal{U}_{\mathcal{T}}$ and $u' \in \mathcal{U}_{\Re}$ with $\emptyset \neq u \cap u'$.

The following proposition 4.4 gives sufficient criteria for the theory completeness of the union of sets of theory connections that are theory complete with respect to the constituent sub-theories of a hybrid theory. The case of the target logic of the multi-modal logic will be covered by Proposition 4.4.

DEFINITION 4.11. Let M be a set of instances of clauses and U a mating in M. For every literal L in M we define the *set R_L of clauses reachable from L via U* as the least set being closed with respect to the following condition: If there exists a connection $u \in U$ such that one of the literals of u is L or a literal in a clause being element of R_L then also any clause containing a literal of u different from L belongs to R_L.

PROPOSITION 4.4. *Let theories $\mathcal{T}$ and $\Re$ be expressed in the signatures Σ and Δ respectively form a hybrid theory such that $\mathcal{T} \cup \Re$ is consistent. The query language $\mathcal{Q}$ is formulated in the union $\Sigma \cup \Delta$ of signatures. Moreover suppose that:*

(1) *The sets of $\mathcal{T}$-connections $\mathcal{U}_{\mathcal{T}}$ and of $\Re$-connections $\mathcal{U}_{\Re}$ are complete w.r.t. $\mathcal{Q}_{\mathcal{T}}$ and $\mathcal{Q}_{\Re}$ respectively.*

(2) *In $\mathcal{Q}$ equality literals occur only negative.*

(3) *In both theories positive equality literals may occur only within conditional equations.*

(4) *The sets of predicate symbols occurring in $\mathcal{T} \cup \mathcal{Q}_{\mathcal{T}}$ and $\Re \cup \mathcal{Q}_{\Re}$ are disjoint.*

(5) *If equality occurs in $\mathcal{T} \cup \Re$ then let $\mathcal{T}_1$ be that of the sub-theories $\mathcal{T}$ and $\Re$ that does not contain equality and $\mathcal{U}_1$ be the set of theory connections for that sub-theory. Moreover let $\mathcal{E}$ be the set of equational axioms in $\mathcal{T} \cup \Re$. For every $u \in \mathcal{U}_1$ and substitution σ holds $\mathcal{E} \cup \mathcal{T}_1 \models \sigma(\bigvee \bar{u})$ if and only if $\mathcal{T}_1 \models \sigma(\bigvee \bar{u})$.*

Then the sets of $\mathcal{T}$-connections $\mathcal{U}_{\mathcal{T}}$ and $\Re$-connections $\mathcal{U}_{\Re}$ are separated with respect to $\mathcal{Q}$ and $\mathcal{U}_{\mathcal{T}} \cup \mathcal{U}_{\Re}$ is $\mathcal{T}, \Re$-complete with respect to $\mathcal{Q}$.

Proof. Let us suppose that theories $\mathcal{T}$ and $\Re$, signatures Σ and Δ, query language $\mathcal{Q}$ and the sets of $\mathcal{T}$-connections $\mathcal{U}_{\mathcal{T}}$ and of $\Re$-connections $\mathcal{U}_{\Re}$ satisfy the assumptions of the proposition. In order to show that $\mathcal{U}_{\mathcal{T}}$ and $\mathcal{U}_{\Re}$ are separated with respect to $\mathcal{Q}$ it is sufficient to observe that the sets of predicate symbols occurring in $\mathcal{T} \cup \mathcal{Q}_{\mathcal{T}}$ and $\Re \cup \mathcal{Q}_{\Re}$ are disjoint. In order to show that $\mathcal{U}_{\mathcal{T}} \cup \mathcal{U}_{\Re}$ is $\mathcal{T}, \Re$-complete with respect to $\mathcal{Q}$ we show first of all that $\mathcal{U}_{\mathcal{T}} \cup \mathcal{U}_{\Re}$ has property (2.1) formulated in Definition 4.3. Let p be a $\mathcal{T}, \Re$-complementary ground path. We have to show that there exists a sub-path such that $u \in \mathcal{U}_{\mathcal{T}} \cup \mathcal{U}_{\Re}$. We consider p as a set of unit clauses. By the compactness theorem for first-order logic there exists a finite set M of instances of clauses of $\mathcal{T}$ and of $\Re$ and a minimal mating U spanning $M \cup p$. Let u be the multi-set of all literals of p which are element of a connection in U. Then u is not empty because of the consistency of $\mathcal{T} \cup \Re$. Because the sets of predicate symbols occurring in $\mathcal{T} \cup \mathcal{Q}_{\mathcal{T}}$ and $\Re \cup \mathcal{Q}_{\Re}$ are disjoint either for every connection $u' \in U$ holds $u \in \mathcal{Q}_{\Sigma}$ or for every connection $u' \in U$ holds $u \in \mathcal{Q}_{\Delta}$. Therefore, u is either element of $\mathcal{Q}_{\mathcal{T}}$ or of $\mathcal{Q}_{\Re}$. If $u \in \mathcal{Q}_{\mathcal{T}}$ (the case $u \in \mathcal{Q}_{\Re}$ may be treated analogously) then there exists $u'' \in \mathcal{U}_{\mathcal{T}}$ such that $u'' \subseteq u$, and therefore $u'' \subseteq p$, because $\mathcal{U}_{\mathcal{T}}$ is $\mathcal{T}$-complete with respect to $\mathcal{Q}_{\mathcal{T}}$. Both $\mathcal{U}_{\mathcal{T}}$ and $\mathcal{U}_{\Re}$ satisfy condition (2.2) of Definition 4.3. Therefore also $\mathcal{U}_{\mathcal{T}} \cup \mathcal{U}_{\Re}$ has this property. □

Example 4.7. Let us observe that in a matrix, which belongs to the target language of the algebraic translation of multi-modal logic, theory connections either are in the non-sort part, i.e. those discussed in Example 4.1, or in the sort part, i.e. those discussed in Example 4.2. This is obvious because both parts of the hybrid theory are expressed by use of disjoint sub-sets of predicate symbols and equality does not occur in the query language. Therefore, in order to obtain a complete set of theory connections for the hybrid theory consisting of $\mathcal{T}$ and $\Re$ it is sufficient to take just the union of the complete sets of theory connections $\mathcal{U}_{\mathcal{T}}$ and $\mathcal{U}_{\Re}$. □

Now we discuss briefly the unification problem in sets of hybrid theory connections. We restrict our attention to the case that for given theories $\mathcal{T}$ and $\Re$ a complete set of theory connections is given by the union of sets of theory connections that are complete with respect to the respective theories. What we have in mind is that unification of a theory connection u is either $\mathcal{T}$-unification if u is a $\mathcal{T}$-connection or $\Re$-unification otherwise. This leads to the notion of non-interfering unification problems.

DEFINITION 4.12. Let $\mathcal{U}_{\Re}$ and $\mathcal{U}_{\mathcal{T}}$ be sets of theory connections for the components of a hybrid theory $\mathcal{T}, \Re$. We say that *the unification problems in $\mathcal{U}_{\Re}$ and $\mathcal{U}_{\mathcal{T}}$ do not interfere* if and only if

(1) For every $u \in \mathcal{U}_{\mathcal{T}}$ and for every substitution σ holds: σ is a $\mathcal{T}$-unifier of u if and only if σ is $\mathcal{T}, \Re$-unifier of u, and

(2) for every $u \in \mathcal{U}_{\Re}$ and for every substitution σ holds: σ is a $\Re$-unifier of u if and only if σ is $\mathcal{T}, \Re$-unifier of u.

Let $\mathcal{U}_{\mathcal{T}} \cup \mathcal{U}_{\Re}$ be the set of theory connections discussed in Section 3.2 for the target logic of the algebraic translation of multi-modal logic. Then the unification problems in $\mathcal{U}_{\mathcal{T}}$ and $\mathcal{U}_{\Re}$ do not interfere.

PROPOSITION 4.5. *Let theories $\mathcal{T}$ and $\Re$, which are expressed in the signatures Σ and Δ respectively, form a hybrid theory, such that $\mathcal{T} \cup \Re$ is consistent. The query language $\mathcal{Q}$ is formulated in the union $\Sigma \cup \Delta$ of signatures. Moreover suppose that the assumptions (1)–(5) of Proposition 4.4 are satisfied. Then the unification problems in $\mathcal{U}_{\mathcal{T}}$ and $\mathcal{U}_{\Re}$ do not interfere.*

PROOF. In the non-trivial direction of the equivalence to be proved we have to show that every $\mathcal{T} \cup \Re$-unifier of a $\mathcal{T}$-connection $u \in \mathcal{U}_{\mathcal{T}}$ is a $\mathcal{T}$-unifier of u and that every $\mathcal{T} \cup \Re$-unifier of a $\Re$-connection $u \in \mathcal{U}_{\Re}$ is a $\Re$-unifier of u. The latter claim is satisfied because $\mathcal{T}$ and $\Re$ have no common predicate symbols and $\Re$ does not contain the equality sign. The former claim follows from assumption (5). □

Now a completeness theorem for hybrid theories may be proved.

Theorem 4.3 (Completeness theorem for hybrid theories). *Let* $\mathcal{Q}$ *be a query language expressed in a signature containing* Σ *and* Δ*. Moreover, let* $\mathcal{Q}_{\Re}$ *and* $\mathcal{Q}_{\mathcal{T}}$ *be the* $\Re$*-layer and* $\mathcal{T}$*-layer of* $\mathcal{Q}$ *respectively. Let* $\mathcal{U}_{\Re}$ *and* $\mathcal{U}_{\mathcal{T}}$ *be complete sets of* $\Re$*-connections and* $\mathcal{T}$*-connections which satisfy the assumptions of Proposition 4.4. Then for every* $\mathcal{T},\Re$*-valid query* $M \in \mathcal{Q}$ *exists a clause* $\Gamma \in M$ *and a successful derivation starting from the initial pool* $\{(\ \perp\ \Gamma)\}$ *such that in each inference according to Definition 4.6 for the chosen connection* u *holds either* $u \in U_{\Re}$ *or* $u \in \mathcal{U}_{\mathcal{T}}$ *and for the chosen theory unifier* $\sigma \in S_u$*, with* S_u *being the set of* $\mathcal{T}$*-unifiers or, respectively,* $\Re$*-unifiers.*

Proof. Due to Proposition 4.4 the set of $\mathcal{T},\Re$-connections $\mathcal{U}_{\Re} \cup \mathcal{U}_{\mathcal{T}}$ is $\mathcal{T},\Re$-complete w.r.t. query language $\mathcal{Q}$. Due to Proposition 4.5 the unification problem in $\mathcal{U}_{\Re} \cup \mathcal{U}_{\mathcal{T}}$ is solvable and applying the $\mathcal{T}$-unification procedure to $\mathcal{U}_{\mathcal{T}}$-connections and the $\Re$-unification procedure to $\mathcal{U}_{\Re}$-connections provides a solution to the $\mathcal{U}_{\Re} \cup \mathcal{U}_{\mathcal{T}}$-unification problem. Thus the assumptions of Theorem 4.2 are satisfied and the calculus for the hybrid theory is complete. □

Let $\mathcal{U}_{\mathcal{T}}$ and $\mathcal{U}_{\Re}$ be the set of theory connections discussed in Section 3.2 for the target logic of the algebraic translation of multi-modal logic. Then we obtain a complete calculi instantiating the theory pool calculus (cf. Section 4.3) as a corollary of Theorem 4.3.

5. Concerning implementation

A prover for multi-modal logic has been implemented by a joint effort of research groups in Leipzig and Caen. We used the calculi description interface CaPrI of the PTTP-prover ProCom [23]. The algebraic translation of Francoise Debart and Patrice Enjalbert from multi-modal logic to a language of constrained clauses has been implemented by Zoltán Rigó [31]. The translation generates a constraint theory that provides information about the interaction between modalities, the properties of the occurring modalities and the dependencies introduced by Skolemization. For reasoning in the non-constraint part of a matrix being element of the target language an A1-unification algorithm due to Francoise Debart and Patrice Enjalbert [12] is used. The algorithm has been tuned for this application. The used implementation is due to Gilbert Boyreau [8]. ProCom and his interface has been implemented by Gerd Neugebauer. He also integrated constraint reasoning into ProCom.

6. Conclusion

In the present paper we examined how to develop applicable proof procedures from the rather theoretical concept of Herbrand disjunctions. The presented approach has been illustrated by target logics obtained from a certain translation of the paraconsistent logic D2 into first-order theories. The first step of the translation is Jaśkowski's [19] translation into the modal logic S4. The second step is the so called algebraic translation [13]. To the target of this translation we applied a general framework which allows to build in theories into provers which are based on the connection method. For this purpose we introduced the notion of a hybrid theory. We obtained a completeness result for a connection method based calculus dealing with hybrid theories. A brief overview about an implementation has been given.

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