Guido Vanackere

PREFERENCES AS INCONSISTENCY-RESOLVERS:

The Inconsistency-adaptive Logic PRL

Abstract. In this paper I generalize the new approach to nonmonotonic reasoning that was presented in [6]. This generalization results in the inconsistency-adaptive logic **PRL** (**PR** stands for preference-based reliability strategy). I give proof theory, semantics, mention interesting properties, and comment on the reconstruction and amelioration of other nonmonotonic logics and mechanisms.

1. Introduction

Applying a paraconsistent logic to an inconsistent set of premises avoids triviality but results, in general, in a too poor consequence-set. An inconsistency-adaptive logic derives all 'rich' consequences where no inconsistency occurs and isolates inconsistencies. However, an inconsistency-adaptive logic does not resolve inconsistenties; the consequence-set will still contain inconsistencies. When we meet inconsistencies, in everyday life, we are interested in weeding out one half of each inconsistency. A straightforward way to do this, exists in tracking the sources where the inconsistent information derives from, and dropping the conclusion derived from the least reliable source.

The idea of annotating the premises according to their sources, is taken from [7], but the machinery of **PRL** is different. The annotated premise A^k is a weakening of the 'normal' premise A and is equated with $\sim^k \neg A$, in which,

for $k = 1, 2, \ldots$: $\sim^k$ is a paraconsistent negation and $\neg$ is classical negation. I use letters as symbols for numbers, for the simple reason that this makes it easy to interpret the annotations as preferences. "$\sim^k \neg A$" can be read as: "we give preference k to the fact that we have no reason to reject A".

In this paper I do not question "Where do preferences on (the sources of) the premises come from?". In fact, **PRL** is an excellent logic even if the annotations are not interpreted as preferences. When preferences are known, the **PRL**-consequence-set is as strong and specific as we want it to be.

I give a quick overview of the results of the **PRL**-mechanism. Let Γ^A be obtained from Γ by annotating all members of Γ. Where **XL** is a logic, $Cn_{\mathbf{XL}}(\Gamma)$ is the set of **XL**-consequences of Γ, and $Cn^0_{\mathbf{XL}}(\Gamma^A)$ the set of **XL**-consequences of Γ^A in which no annotations occur.[1] **CL** is Classical Logic. If Γ is consistent, then $Cn^0_{\mathbf{PRL}}(\Gamma^A) = Cn_{\mathbf{CL}}(\Gamma)$. If Γ is inconsistent, **PRL** reveals the conflicting premises. At this point, there are two possibilities: (1) It is not possible to interpret the annotations as preferences, *i.e.* it is not possible to introduce preferences on conflicting premises. In this case **PRL** 'deletes' both halves of each inconsistency. Let $\Gamma^C \subseteq \Gamma$ be the set of conflicting premises. Then $Cn^0_{\mathbf{PRL}}(\Gamma^A) = Cn_{\mathbf{CL}}(\Gamma - \Gamma^C)$. Obviously $Cn_{\mathbf{CL}}(\Gamma)$ is trivial. (2) It is possible to introduce preferences on the conflicting premises. This is a non-logical step that can be made beforehand or at any stage of a proof. It is an advantage of **PRL** that we do not need to know the preferences of all premises. **PRL** reveals those premises on which the introduction of preferences will lead to the resolution of inconsistencies. If preferences of the conflicting premises are known, **PRL** deletes that half of each inconsistency that derives from the least preferred premises. Let $\Gamma^P \subset \Gamma^C$ be the set of conflicting premises that are not least preferred. In this case $Cn^0_{\mathbf{PRL}}(\Gamma^A) = Cn_{\mathbf{CL}}(\Gamma - (\Gamma^C - \Gamma^P))$. Obviously $Cn_{\mathbf{CL}}(\Gamma - \Gamma^C) \subseteq Cn_{\mathbf{CL}}(\Gamma - (\Gamma^C - \Gamma^P))$.

In section 2, I give a brief characterisation of inconsistency-adaptives logics in general. In section 3, I give proof theory, semantics and metatheory of the lower limit logic **pPRL**. In section 4, I do the same for **PRL**. In section 5, I comment on the influence of the formulation of the premises. In the 6th and last section, I try to show that **PRL** is a better nonmonotic tool than the (non-modal versions of) default logic. I also mention the possibility to reconstruct Rescher-mechanisms by means of **PRL**.

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[1] Hence, $Cn^0_{\mathbf{XL}}(\Gamma^A)$ does not contain formulas of the form A^k, $\sim^k \neg A$, or complex formulas containing such subformulas. All members of $Cn^0_{\mathbf{XL}}(\Gamma^A)$ are 'normal' classically formalized formulas.

2. Inconsistency-adaptive logics in general

Let **IAL** be any inconsistency-adaptive logic. **IAL** oscillates between a paraconsistent lower limit logic (**PL**) and a 'rich' upper limit logic (*e.g.* **CL**). Where no inconsistency occurs, **IAL** behaves like the upper limit logic; where an inconsistency occurs, **IAL** behaves like the lower limit logic. In [3] Diderik Batens argues that, in general, a weaker lower limit logic goes along with a richer **IAL**.

The question "how to get grip on "where an inconsistency occurs?" is formally answered by means of *the minimal DEK-consequences* (minimal disjunctions of (where necessary existentially quantified) contradictions that are **PL**-derivable from the premises). In **IAL** some derivations are made on condition of the consistent behaviour of some involved (sub)formulas. The minimal DEK-consequences of the premises indicate which conditions are overruled. For instance, if $\Gamma \vdash_{\mathbf{PL}} (A \mathbin{\&} \sim A) \vee (B \mathbin{\&} \sim B) \vee (C \mathbin{\&} \sim C)$, $\Gamma \nvdash_{\mathbf{PL}} (A \mathbin{\&} \sim A) \vee (B \mathbin{\&} \sim B)$, $\Gamma \nvdash_{\mathbf{PL}} (A \mathbin{\&} \sim A) \vee (C \mathbin{\&} \sim C)$, and $\Gamma \nvdash_{\mathbf{PL}} (B \mathbin{\&} \sim B) \vee (C \mathbin{\&} \sim C)$, then $(A \mathbin{\&} \sim A) \vee (B \mathbin{\&} \sim B) \vee (C \mathbin{\&} \sim C)$ is a minimal DEK-consequence of Γ, and hence at least one of the formulas A, B, C behaves inconsistently. Another example: suppose q is derived from $\Gamma \cup \{\sim p, p \vee q\}$, on condition of the consistent behaviour of p. This condition is overruled if, *e.g.*, $\Gamma \cup \{\sim p, p \vee q\} \vdash_{\mathbf{PL}} p \mathbin{\&} \sim p$. In this case $p \mathbin{\&} \sim p$ is a minimal DEK-consequence of $\Gamma \cup \{\sim p, p \vee q\}$ and thus p behaves inconsistenly. Therefore $\Gamma \cup \{\sim p, p \vee q\} \nvdash_{\mathbf{IAL}} q$.

There are two well-known strategies to select unreliable formulas among the factors of minimal DEK-consequences:[2]

1. *Reliability.* The consequence-set of Γ is the set of formulas that are true in all models of Γ in which the only true contradictions come from the factors of the minimal DEK-consequences of Γ. The factors of minimal DEK-consequences are the only unreliable formulas.

Example. If p is derived from Γ on condition of the consistent behaviour of q, and $(q \mathbin{\&} \sim q) \vee (r \mathbin{\&} \sim r)$ is a minimal DEK-consequence of Γ, then both q and r are unreliable and hence p is not finally derivable from Γ.

2. *Minimizing abnormality.* The consequence-set of Γ is the set of all formulas that are true in all minimally abnormal models of Γ, i.e. in all models of Γ that verify a set of inconsistencies that is not a superset of a set of inconsistencies verified by another model of Γ.

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[2] In [5], Kristof De Clercq develops two more strategies.

Example. If p is derived from Γ on condition of the consistent behaviour of q, and p is also derived from Γ on condition of the consistent behaviour of r, and $(q \,\&\, {\sim}\, q) \vee (r \,\&\, {\sim}\, r)$ is a minimal DEK-consequence of Γ, then one type of minimally abnormal models verify $q \,\&\, {\sim}\, q$ and falsify $r \,\&\, {\sim}\, r$ whereas the other verify $r \,\&\, {\sim}\, r$ and falsify $q \,\&\, {\sim}\, q$. Hence all minimally abnormal models satisfy a condition on which p is derivable. Therefore p is finally derivable from Γ.

The reliability strategy has an easier proof-mechanism, but leads, in specific cases, to a poorer consequence-set than the minimal abnormality strategy. The latter difference however vanishes when preferences are taken into account. **PRL** is constructed by means of the reliability strategy.

3. The underlying paraconsistent logic pPRL

3.1. Proof theory

The language of **pPRL** is the language of classical predicate logic with identity, extended with an infinite set of paraconsistent negations ${\sim}^1$, ${\sim}^2$, ... The logic is obtained by adding to **CL** (with $\neg$ as the classical negation) the axiom schema $(\mathrm{A}{\sim}^k)$.

$$(\mathrm{A}\,{\sim}^k) \qquad\qquad A \vee {\sim}^k A \qquad (\text{for } k = 1,\, 2,\, \dots)$$

Premises receive a preference $k \geq 1$ or an annotation that can be interpreted as a symbol for a preference (notation A^k). A higher number corresponds to a lower preference. Theorems receive preference 0, the highest preference. Formally, the preferences are handled by equating A^0 with A, and A^k with ${\sim}^k \neg A$. Substitution of identity is restricted as follows

$$(\mathrm{A} = 2) \qquad\qquad \alpha = \beta \;\supset\; (A \supset B),$$

where B is obtained by replacing in A an occurrence of α that occurs outside the scope of a paraconsistent negation by β.

In view of the weak interpretation of the premises, **pPRL** is an extremely poor logic. In a way of speaking it is not even possible to derive the premises. More exactly: $A^k \nvdash_{\mathbf{pPRL}} A$. Moreover, $Cn^0_{\mathbf{pPRL}}(\Gamma^A) = Cn_{\mathbf{CL}}\emptyset$. If the statement, defended in [3] holds — a poorer lower limit logic goes along with a richer adaptive logic — **pPRL** must be an excellent lower limit logic.

In view of the axioms $(\mathrm{A}{\sim}^k)$ and $(\mathrm{A} = 2)$, **pPRL** does not spread inconsistencies: no inconsistency entails another. **pPRL** shares this property with

CLuN, the lower limit logic of the inconsistency-adaptive logics **ACLuN1** and **ACLuN2** (see [2]).

Let $!^k A$ stand for $A \mathbin{\&} \sim^k A$ and let $\sharp^k A$ stand for $\neg A \mathbin{\&} \sim^k \neg A$ (hence $!^k \neg A \equiv \sharp^k A$). The following theorem expresses a property of **pPRL** that is important in view of the construction of the adaptive logic **PRL**. (The easy proof is left to reader.)

THEOREM 1 (Conditional Derivation CD).

$$\Gamma^A \vdash_{\mathbf{pPRL}} \sim^k \neg A \quad \textit{iff} \quad \Gamma^A \vdash_{\mathbf{pPRL}} \sharp^k A \vee A.$$

CD can be used as a derivation rule in **pPRL**-proofs. This rule allows to derive the 'normal' interpretation A of the premise A^k in disjunction with $\sharp^k A$. In the adaptive logic **PRL**, $\sharp^k A$ is considered as false, unless and until proven otherwise. Therefore, it is right to talk about "conditional derivation": the normal interpretation of the premises is derivable *on condition of the $\sim^k$-consistent behaviour of the classical negation of the premises.*

3.2. Semantics

Let $\mathcal{S}$ be the set of sentential letters, $\mathcal{P}^r$ the set of predicative letters of rank r, $\mathcal{C}$ and $\mathcal{V}$ the set of letters for individual constants and variables respectively, $\mathcal{F}$ the set of all formulas (open and closed), and $\mathcal{N} = \{\sim^k A \mid A \in \mathcal{F}, k \geq 1\}$.

A **pPRL**-model is a couple $M = \langle D, v \rangle$ in which D is a set and v is an assignment-function defined by:

S1.1 $v : \mathcal{S} \longrightarrow \{0, 1\}$
S1.2 $v : \mathcal{C} \cup \mathcal{V} \longrightarrow D$ is such that $D = \{v(\alpha) \mid \alpha \in \mathcal{C} \cup \mathcal{V}\}$
S1.3 $v : \mathcal{P}^r \longrightarrow P(D^r)$ (the power set of the r-th Cartesian product of D)
S1.4 $v : \mathcal{N} \longrightarrow \{0, 1\}$

The valuation-function v_M determined by the model M is defined as follows:

S2.1 $v_M : \mathcal{F} \longrightarrow \{0, 1\}$
S2.2   where $A \in \mathcal{S}$, $v_M(A) = v(A)$
S2.3 $v_M(\pi^r \alpha_1 ... \alpha_r) = 1$ iff $\langle v(\alpha_1), \ldots, v(\alpha_r) \rangle \in v(\pi^r)$
S2.4 $v_M(\alpha = \beta) = 1$ iff $v(\alpha) = v(\beta)$
S2.5 $v_M(\neg A) = 1$ iff $v_M(A) = 0$
S2.6 $v_M(\sim^k A) = 1$ iff $v_M(A) = 0$ or $v(\sim^k A) = 1$
S2.7 $v_M(A \supset B) = 1$ iff $v_M(A) = 0$ or $v_M(B) = 1$
S2.8 $v_M(A \mathbin{\&} B) = 1$ iff $v_M(A) = 1$ and $v_M(B) = 1$
S2.9 $v_M(A \vee B) = 1$ iff $v_M(A) = 1$ or $v_M(B) = 1$

S2.10 $v_M(A \equiv B) = 1$ iff $v_M(A) = v_M(B)$
S2.11 $v_M((\forall\alpha)A(\alpha)) = 1$ iff $v_M(A(\beta)) = 1$ for all $\beta \in \mathcal{C} \cup \mathcal{V}$
S2.12 $v_M((\exists\alpha)A(\alpha)) = 1$ iff $v_M(A(\beta)) = 1$ for at least one $\beta \in \mathcal{C} \cup \mathcal{V}$

Truth in a model, semantic consequence and validity are defined as usual.

3.3. Metatheory

The proofs of the soundness and completeness theorems are analogous to those for **pHL2** (see [6]), and hence left to reader.

THEOREM 2. $\Gamma^A \vdash_{\textbf{pPRL}} A$ iff $\Gamma^A \vDash_{\textbf{pPRL}} A$.

4. PRL

4.1. Intuitive approach

In order to get grip on the machinery of **PRL**, it is interesting to take a look at some **pPRL**-proofs.

Proof 1.

1.	$(\forall x)Px^k$	–	PREM
2.	$(Pa \supset p)^m$	–	PREM
3.	$\sim^k \neg(\forall x)Px$	(1)	DEF
4.	$\sim^m \neg(Pa \supset p)$	(2)	DEF
5.	$\sharp^k(\forall x)Px \vee (\forall x)Px$	(3)	CD
6.	$\sharp^m(Pa \supset p) \vee (Pa \supset p)$	(4)	CD
7.	$\sharp^k(\forall x)Px \vee Pa$	(5)	UI
8.	$\sharp^k(\forall x)Px \vee \sharp^m(Pa \supset p) \vee p$	(6,7)	MP[3]

Proof 2.

1.	$(\forall x)(Bx \supset Fx)^k$	–	PREM
2.	$(\forall x)(Px \supset \neg Fx)^m$	–	PREM
3.	$(\forall x)(Px \supset Bx)^n$	–	PREM

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[3] This application of Modus Ponens is derivable: From $A \vee (B \supset C)$ and $D \vee B$ we can derive by addition $A \vee D \vee (B \supset C)$ and $A \vee D \vee B$. From these formulas we can derive $A \vee D \vee ((B \supset C) \,\&\, B)$ and from this follows $A \vee D \vee C$. This kind of sped up derivations is generally valid in **pPRL**:

  $\sharp C_1 \vee \cdots \vee \sharp C_m \vee ((A_1 \,\&\, .. \,\&\, A_n) \supset B), \sharp D_1 \vee \cdots \vee \sharp D_k \vee (A_1 \,\&\, .. \,\&\, A_n))/$
  $\sharp C_1 \vee \cdots \vee \sharp C_m \vee \sharp D_1 \vee \cdots \vee \sharp D_k \vee B$

The application of the instantiation rule UI in line (7) is an example of such a sped up derivation.

4.	Pa^r	–	PREM
5.	$\sim^k \neg(\forall x)(Bx \supset Fx)$	(1)	DEF
6.	$\sim^m \neg(\forall x)(Px \supset \neg Fx)$	(2)	DEF
7.	$\sim^n \neg(\forall x)(Px \supset Bx)$	(3)	DEF
8.	$\sim^r \neg Pa$	(4)	DEF
9.	$\sharp^k(\forall x)(Bx \supset Fx) \vee (\forall x)(Bx \supset Fx)$	(5)	CD
10.	$\sharp^m(\forall x)(Px \supset \neg Fx) \vee (\forall x)(Px \supset \neg Fx)$	(6)	CD
11.	$\sharp^n(\forall x)(Px \supset Bx) \vee (\forall x)(Px \supset Bx)$	(7)	CD
12.	$\sharp^r Pa \vee Pa$	(8)	CD
13	$\sharp^k(\forall x)(Bx \supset Fx) \vee (Ba \supset Fa)$	(9)	UI
14	$\sharp^m(\forall x)(Px \supset \neg Fx) \vee (Pa \supset \neg Fa)$	(10)	UI
15	$\sharp^n(\forall x)(Px \supset Bx) \vee (Pa \supset Ba)$	(11)	UI
16.	$\sharp^r Pa \vee \sharp^m(\forall x)(Px \supset \neg Fx) \vee \neg Fa$	(12,14)	MP
17.	$\sharp^r Pa \vee \sharp^n(\forall x)(Px \supset Bx) \vee Ba$	(11,12)	MP
18.	$\sharp^r Pa \vee \sharp^n(\forall x)(Px \supset Bx) \vee \sharp^k(\forall x)(Bx \supset Fx) \vee Fa$		
		(13,17)	MP
19.	$\sharp^r Pa \vee \sharp^m(\forall x)(Px \supset \neg Fx) \vee \sharp^n(\forall x)(Px \supset Bx) \vee$		
	$\sharp^k(\forall x)(Bx \supset Fx) \vee (Fa \,\&\, \neg Fa)$	(16,18)	CONJ
20.	$\neg(Fa \,\&\, \neg Fa)$	–	THEOREM
21.	$\sharp^r Pa \vee \sharp^m(\forall x)(Px \supset \neg Fx) \vee \sharp^n(\forall x)(Px \supset Bx) \vee \sharp^k(\forall x)(Bx \supset Fx)$		
		(19,20)	DS

Whether the annotations are interpreted as preferences or not, has no influence on **pPRL**-derivations. The difference shows up when we construct the adaptive logic **PRL** on **pPRL**. I illustrate the machinery of **PRL**, in comparison with **CL**. Unless and until an inconsistency or a disjunction of inconsistencies is derived from the premises, **PRL** interprets a set of premises consistently. Hence, if no disjunction of inconsistencies is derived, all disjuncts of the form $\sharp^k C$ are false, and hence, for instance in proof 1, p is derivable from $\sharp^k(\forall x)Px \vee \sharp^m(Pa \supset p) \vee p$.

If we apply **CL** or **PRL** — even without introducing preferences — to the first set of premises, we can derive p.

If we apply **CL** to the second set of premises, we derive triviality (both Fa and $\neg Fa$ are derivable). If we apply **PRL** to the second set of premises, without interpreting the annotations as preferences, neither Fa nor $\neg Fa$ are derivable (and hence we avoid triviality). If we apply **PRL** to the second set of premises, and we introduce preferences, then, (1) if $k > m, r$ or $n > m, r$, then $\neg Fa$ is derivable, but Fa not; (2) if $m > k, n, r$, then Fa is

derivable, but $\neg Fa$ not, and (3) if $r \geq k, m, n$, then neither Fa nor $\neg Fa$ are derivable.[4]

The formula in line (21) reveals the conflicting premises. If no premise is preferred, all premises mentioned in the formula in line (18) are considered as unreliable. If preferences are known, only the least preferred premise (it is, the premise annotated with the highest number) is unreliable. Indeed, **PRL** interprets a set of premises as consistent as possible. If a set of premises is inconsistent, a disjunction of inconsistencies is derivable. The minimal disjunctions of inconstencies (see exact definition below) indicate the unreliable premises. The introduction of preferences results in a smaller set of unreliable premises and hence in a richer consequence-set.

Where $\sharp^k A$ is a formula in which the variables $\alpha_1, \ldots, \alpha_m$ $(m \geq 0)$ occur free, let $\exists \sharp^k A$ be $(\exists \alpha_1) \ldots (\exists \alpha_m)(\sharp^k A)$. Let $\mathsf{DENK}\{A_1^{k_1}, \ldots, A_n^{k_n}\}$ refer to $\exists \sharp^{k_1} A_1 \vee \cdots \vee \exists \sharp^{k_n} A_n$, a disjunction of (where necessary) existentially quantified contradictions of classically negated premises.[5] $A_1^{k_1}, \ldots, A_n^{k_n}$ are the *factors* of the DENK-formula $\mathsf{DENK}\{A_1^{k_1}, \ldots, A_n^{k_n}\}$.

THEOREM 3. $\Gamma \vDash_{\mathbf{CL}} A$ *iff there are* $C_1^{k_1}, \ldots, C_n^{k_n} \in \mathcal{F}$ $(0 \leq n)$ *such that* $\Gamma^A \vDash_{\mathbf{pPRL}} \mathsf{DENK}\{C_1^{k_1}, \ldots, C_n^{k_n}\} \vee A$.

The proof is analogous to the proof of Theorems 6 and 7 in [6].

4.2. Proof theory

The idea of the proof theory of **PRL** is that we apply all rules derivable in **pPRL** unconditionally, whereas we equal every formula of the form $\sim^k \neg A$ with A on condition that $\neg A$ behaves $\sim^k$-consistently.

Every line of a **PRL**-proof gets a fifth element in which we write the formulas on the consistent behaviour of which we rely to derive the formula in the second element by the rule mentioned in the fourth element from the formulas of the lines enumerated in the third element.

DEFINITION. *A occurs unconditionally* at some line of a proof iff the fifth element of that line is empty.

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[4] In contrast with **PRL**, **HL2** (see [6]), assumes that 'descriptions of facts' do not come from unreliable sources. With **PRL** it is also possible to consider Pa as unreliable.

If B stands for "is a bird", F for "can fly", and P for "is a penguin", it is natural to take $r, n < m < k$, and hence penguin a cannot fly. Also: birds that are not penguins can fly.

[5] In view of the fact that the contradictions are contradictions of classically negated formulas, I speak of DENK instead of DEK (confer [2]). In section 5 it will become clear why it is necessary to assume that some disjunctions of contradictions have to be existentially quantified.

DEFINITION. *A behaves $\sim^k$-consistently at a stage of a proof* iff $\sharp^k A$ does not occur unconditionally in the proof at that stage.

DEFINITION. *The $\sim^{k_1}$-consistent behaviour of A_1 is connected to* the $\sim^{k_2}$-consistent behaviour of A_2, ..., and the $\sim^{k_n}$-consistent behaviour of A_n *at a stage of a proof* iff $\mathsf{DENK}\{A_1^{k_1}, \ldots, A_n^{k_n}\}$ occurs unconditionally in the proof at that stage whereas $\mathsf{DENK}\{A_2^{k_2}, \ldots, A_n^{k_n}\}$ does not occur unconditionally in it.

DEFINITION. Where A^k is a premise, the number k is the *preference of A*.

DEFINITION. If A occurs as the second element of a line in a proof, the *line preference of A* is the lowest preference (i.e. the highest number) of the formulas in the fifth element of that line. If the fifth element is empty, the line preference of A is 0.

DEFINITION. The *derivation preference of A at a stage of a proof*, is the highest line preference of A at that stage of the proof. If A does not occur as the second element of a line at that stage, the derivation preference is maximally low.

DEFINITION. *A is reliable with respect to its preference at a stage of a proof*, iff the preference of A is higher than the derivation preference of $\sharp^k A$ at that stage of the proof.[6]

DEFINITION. *A is reliable at a stage of a proof* iff A behaves $\sim^k$-consistently (for all k) at that stage and its consistent behaviour is not connected to the consistent behaviour of other formulas, *or* A is reliable with respect to its preference at that stage of the proof.

Given these definitions, proofs in **PRL** are governed by an unconditional rule, a conditional rule and a marking rule. The application of a rule produces the next stage.

RU All derivation rules of **pPRL** are unconditionally valid in any **PRL**-proof. The fifth element of a new line is the union of the fifth elements of the lines mentioned in its third element.

RC From a line (i) with $\mathsf{DENK}\{C_1^{k_1}, \ldots, C_m^{k_m}\} \vee A$ as second line, derive a new line with A as second element, (i) as third element, RC as fourth element and the union of $\{C_1^{k_1}, \ldots, C_m^{k_m}\}$ and of the fifth element of line (i) as fifth element.

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[6] If the derivation preference of $\sharp^k A$ is maximally low, A is reliable with respect to its preference.

RM If C is not (anymore) reliable, then mark all lines the fifth element of which contains C.[7]

At any stage of the proof, it is obligatory to apply RM and permitted to apply RU and RC. A marked line does not belong to the proof. If the fifth element of a line is empty, the formula in its second element is **pPRL**-derivable from the premises and cannot be marked later. If the fifth element is not empty, its formula is provisionally derived. Unless it can also be derived at a line the fifth element of which is empty, it is not a **pPRL**-consequence. The unconditional occurrence of DENK-formulas in the proof determines which formulas are not anymore reliable.

As usual proofs may be sped up by derived rules. In general, I will use "conditional" rules as short cut rules for a combination of typical **pPRL**-rules, RC, and a classical derivation rule. "CUI", for instance, is the short cut rule for (1) $A^k/\sim^k\neg A$, (2) CD, (3) RC, and (4) UI.

A wff may be derived at some stage of a proof, while the line in which it occurs may be marked at a later stage of the proof. Therefore we need to distinguish between provisional and *final* consequences.

DEFINITION. An extension of a **PRL**-proof is *intelligent* iff it has the following property: if both DENK(Σ) and DENK($\Sigma \cup \Pi$) occur unconditionally in the extension, then the former precedes the latter.

DEFINITION. A is *finally derived* at some line in an **PRL**-proof iff it is the second element of that line and the line will not be marked in any intelligent extension of the proof.

DEFINITION. $\Gamma^A \vdash_{\textbf{PRL}} A$, A is finally **PRL**-derivable from Γ^A, iff there is a **PRL**-proof from Γ^A in which A is finally derived.

It is possible to prove that $Cn_{\textbf{PRL}}(\Gamma^A)$ may be characterized without refering to the dynamics of the proofs. The characterization refers to **pPRL** only. The central point is that it depends only on **pPRL**-derivability (which is monotonic) whether a wff is reliable in an intelligent extension of the proof.

LEMMA 1. *If in an* **PRL**-*proof from* Γ^A, A *occurs as the second element and* $\{C_1^{k_1}, \ldots, C_m^{k_m}\}$ $(0 \leq m)$ *occurs as the fifth element of a line, then* $\Gamma^A \vdash_{\textbf{pPRL}} A \vee \mathsf{DENK}\{C_1^{k_1}, \ldots, C_m^{k_m}\}$.[8]

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[7] As new lines in a proof take over the fifth elements of the lines they are derived from by means of RC or an application of RU, it is obvious that lines derived from marked lines have to be marked too.

[8] The proof of Lemma 1 is completely analogous to the proof of Lemma 2 in [2] and Lemma 2 in [6].

This means that whenever A is derived in a **PRL**-proof on condition of the consistent behaviour of $\{C_1^{k_1}, \ldots, C_m^{k_m}\}$, then $A \vee \mathsf{DENK}\{C_1^{k_1}, \ldots, C_m^{k_m}\}$ is **PRL**-derivable unconditionally.

DEFINITION. A *DENK-consequence* of Γ^A is a DENK-formula which is **pPRL**-derivable from Γ^A.

DEFINITION. $\mathsf{DENK}(\Delta)$ is a *minimal DEK-consequence* of Γ^A iff it is a DENK-consequence of Γ^A, and for no $\Phi \subset \Delta$, $\mathsf{DENK}(\Phi)$ is a DENK-consequence of Γ^A.

THEOREM 4. $\Gamma^A \vdash_{\mathbf{PRL}} A$, *iff there are* $C_1^{k_1}, \ldots, C_m^{k_m} \in \mathcal{F}$ $(0 \leq m)$ *such that* $\Gamma^A \vdash_{\mathbf{pPRL}} A \vee \mathsf{DENK}\{C_1^{k_1}, \ldots, C_m^{k_m}\}$, *and none of* $C_1^{k_1}, \ldots, C_m^{k_m}$ *is a factor of a minimal* DEK-*consequence of* Γ *unless it is reliable with respect to its preference.*[9]

It follows from Theorem 4 that whenever A occurs as the second element of a line of a **PRL**-proof with $\{C_1^{k_1}, \ldots, C_m^{k_m}\}$ as fifth element, a new line can be added with $A \vee \mathsf{DENK}\{C_1^{k_1}, \ldots, C_m^{k_m}\}$ as second element and an empty fifth element, and vice versa. If a classical contradiction is derived in a line with $\{C_1^{k_1}, \ldots, C_m^{k_m}\}$ as fifth element, a new line can be added with $\mathsf{DENK}\{C_1^{k_1}, \ldots, C_m^{k_m}\}$ as second element and an empty fifth element. In **PRL**-proofs the latter derivation rule is called IDENK (introduction of a DENK-formula).

The following Theorem expresses an important feature of **PRL**:

THEOREM 5. *If* $\Gamma^A \vdash_{\mathbf{PRL}} A$, *then it is possible to extend any proof from* Γ^A *into a proof in which A is finally derived from* Γ^A.[10]

I now list some derivable marking rules in **PRL**.

MR1. If $\mathsf{DENK}\{C_1^{k_1}, \ldots, C_m^{k_m}\}$ occurs unconditionally as the second element of a line of a **PRL**-proof, then mark all lines the fifth element of which contains an unreliable factor of that DENK-formula.

MR2. If the derivation preference of A (resp. $\neg A$) at line (i) is lower than the derivation preference of $\neg A$ (resp. A) at any line of the proof, then mark line (i).

THEOREM 6. *The marking rules* MR1 *and* MR2 *are derivable in* **PRL**.[11]

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[9] The proof of Theorem 4 is completely analogous to the proof of Theorem 15 in [6].

[10] The proof of Theorem 9 is completely analogous to the proof of Theorem 9 in [6].

[11] The proofs for MR1 and MR2 are respectively analogous to the proofs of Theorems 10 and 15 in [6]

4.3. Semantics

The **PRL**-semantics is obtained from the **pPRL**-semantics by defining, for each Γ^A, a subset of the **PRL**-models of Γ^A. Any Γ^A defines a set of semantically unreliable formulas. The **PRL**-models of Γ^A are those **pPRL**-models of Γ^A in which only unreliable formulas behave inconsistently. The set of unreliable formulas with respect to Γ^A is a subset of the factors of the minimal DENK-consequences of Γ.

DEFINITION. If $\Gamma^A \vDash_{\mathbf{PRL}} \sim^n B$ (whereas $\Gamma^A \nvDash_{\mathbf{PRL}} \sim^m B$ ($1 \leq m < n$), then $\mathsf{CP}_{\Gamma^A}(B) = n$. If $\Sigma = \{B_1^{k_1}, \ldots, B_n^{k_n}\}$ ($n \geq 1$), then $\mathsf{CP}_{\Gamma^A}(\mathsf{DENK}(\Sigma))$ is the maximum of $k_1, \ldots, k_n$.

DEFINITION. If B^k is a factor of the minimal DENK-consequence $\mathsf{DENK}(\Sigma)$ of Γ^A, and $\mathsf{CP}_{\Gamma^A}(B) = \mathsf{CP}_{\Gamma^A}(\Sigma)$, then B^k is **PRL**-unreliable with respect to Γ^A. $\mathcal{U}(\Gamma^A)$ is the set of all formulas that are **PRL**-unreliable with respect to Γ^A.

DEFINITION. Where M is a **pPRL**-model, $\mathsf{ab}(M) = \{A \mid v_M(\exists \sharp^k A) = 1\}$.

DEFINITION. M is a **PRL**-model of Γ^A iff it is a **pPRL**-model of Γ^A and $\mathsf{ab}(M) \subseteq \mathcal{U}(\Gamma^A)$.

DEFINITION. $\Gamma^A \vDash_{\mathbf{PRL}} A$ iff A is true in all **PRL**-models of Γ^A.

4.4. Metatheory

The proofs of the soundness and completeness theorems for **PRL** are analogous to those for **HL2** (see [6]).

THEOREM 7. $\Gamma^A \vDash_{\mathbf{PRL}} A$ iff $\Gamma^A \vdash_{\mathbf{PRL}} A$

4.5. Example

Proof 3.[12]

1.	p^j	–	PREM	–
2.	q^k	–	PREM	–
3.	r^l	–	PREM	–
4.	$(p \supset \neg q)^m$	–	PREM	–

---

[12] CSUB (conditional substitution) is the short cut rule for the combination of $A^k \vdash \sim^k \neg A$, CD and RC.

5.	$(r \supset \neg q)^n$	–	PREM	–
6.	p	(1)	CSUB	p^j
7.	q	(2)	CSUB	q^k
8.	r	(3)	CSUB	r^l
9.	$p \supset \neg q$	(4)	CSUB	$(p \supset \neg q)^m$
10.	$r \supset \neg q$	(5)	CSUB	$(r \supset \neg q)^n$
11.	$\neg q$	(6,9)	MP	$p^j, (p \supset \neg q)^m$
12.	$\neg q$	(8,10)	MP	$r^l, (r \supset \neg q)^n$
13.	$\neg p$	(7,9)	MT	$q^k, (p \supset \neg q)^m$
14.	$\neg r$	(7,10)	MT	$q^k, (r \supset \neg q)^n$
15.	$\sharp^j p \vee \sharp^k q \vee \sharp^m (p \supset \neg q)$	(7,11)	IDENK	–
16.	$\sharp^k q \vee \sharp^l r \vee \sharp^n (r \supset \neg q)$	(7,12)	IDENK	–

If preferences are not known p, q, r, $p \supset \neg q$ and $r \supset \neg q$ are unreliable, and none of them is **PRL**-derivable. If, for instance, $k > j, l, m, n$, then q is the only unreliable formula, and p, r, $p \supset \neg q$, $r \supset \neg q$, $\neg q \in Cn^0_{\mathbf{PRL}}\{p^j, q^k, r^l, (p \supset \neg q)^m, (r \supset \neg q)^n\}$. If $j > l > k, m, n$, then p and r are the only unreliable formulas, and q, $\neg p$, $\neg r$, $p \supset \neg q$, $r \supset \neg q \in Cn^0_{\mathbf{PRL}}\{p^j, q^k, r^l, (p \supset \neg q)^m, (r \supset \neg q)^n\}$.

5. About the premises

PRL-consequences are to a certain degree dependent of the formulation of the premises. However, if a premise is replaced by an *equivalent premise*, the consequence-set remains the same.

Example: $\Gamma^A \cup \{(p \supset q)^k\}$ and $\Gamma^A \cup \{(\neg p \vee q)^k\}$ have the same PRL-consequence-set.

When premises that have the same annotation (because they originate from the same source), are put together in one conjunction, the **PRL**-consequence-set changes drastically.

Example.

Premises	Consequences		
	$k < l$	$k > l$	$k = l$
$A^k, B^k, \neg A^l$	A, B	$\neg A, B$	B
$(A \& B)^k, \neg A^l$	A, B	$\neg A$	–

Hence, if one is interested in tracking (and deleting) unreliable sources, one has to put all premises from one source in one conjunction. This results

in a poorer consequence-set. If one is interested in tracking (and deleting) individual unreliable premises, one has to analyse the conjunctions.

If universally quantified formulas are premises, we can treat them in two ways. The first way is the way they are treated in the previous part of this paper. The second way is to consider them as (possibly infinite) lists of instances. Consider $(\forall x)A(x)^k$ as the short for $A(a)^k$, $A(b)^k$, $A(c)^k$, ... This results in the following instantiation rule:

$$(\forall x)A(x)^k \vdash_{\textbf{(p)PRL}} \sim^k \neg A(a).$$

In this case **PRL** includes **HL2**. Thus, it is not only possible to track unreliable universally quantified formulas, but also to track unreliable instances. This way, **PRL** avoids that a universally quantified formula becomes unreliable as soon as one exception occurs.

The following proof illustrates the richness of this version of **PRL**.

Proof 4.

1.	$(\forall x)(Ix \supset Wx)^k$	–	PREM	–
2.	$(\forall x)(Fx \supset Wx)^l$	–	PREM	–
3.	$(Ia \lor Fa)^m$	–	PREM	–
4.	$(Fb \,\&\, \neg Wb)^n$	–	PREM	–
5.	$Ia \supset Wa$	(1)	CUI	$(Ia \supset Wa)^k$
6.	$Fa \supset Wa$	(2)	CUI	$(Fa \supset Wa)^l$
7.	$Fb \supset Wb$	(2)	CUI	$(Fb \supset Wb)^l$
8.	$Ia \lor Fa$	(3)	CSUB	$(Ia \lor Fa)^m$
9.	Fb	(4)	CSIM	$(Fb \,\&\, \neg Wb)^n$
10.	$\neg Wb$	(4)	CSIM	$(Fb \,\&\, \neg Wb)^n$
11.	Wa	(5,6,8)	DIL	$(Ia \supset Wa)^k, (Fa \supset Wa)^l,$ $(Ia \lor Fa)^m$
12.	Wb	(9,7)	MP	$(Fb \,\&\, \neg Wb)^n, (Fb \supset Wb)^l$

If I stands for "is Italian", F stands for "is French", W for "likes wine", and if premise (4) does not get a lower preference than premise (3), we might expect that $n \leq m < k = l$. Hence, in view of MR2, line (12) has to be marked, and hence, $\neg Wb$ and Wa are finally derivable, whereas Wb is not.

With this interpretation of universally quantified formulas, some disjuncts of DENK-formulas can be *existentially quantified*. Whenever a formula $A(a)$ is derived from $(\forall x)A(x)^k$ by means of CUI, whereas a does not occur in the premises, the fifth element of the line in which $A(a)$ is

derived will be the open formula $A(x)^k$. Hence it is possible that $A(x)^k$ is a factor of a DENK-formula $\mathsf{DENK}(\{A(x)^k\} \cup \Sigma)$, which is the short for $(\exists x)\sharp^k A(x) \vee \mathsf{DENK}(\Sigma)$.

6. PRL and other nonmonotonic mechanisms

6.1. Default logic

The definitions in this section, as well as the premises of Proofs 2 and 4 (above) are taken from [4].

A default theory is a pair (D, W). W is a set of first order formulas representing the facts which are known to be true with certainty. D is a set of defaults of the form

$$\frac{A : B_1, \ldots, B_n}{C}$$

where A, B_i and C are classical formulas. The default has the intuitive reading: if A is provable and for all i ($1 \leq i \leq n$), $\neg B_i$ is not provable, then derive C. Defaults with free variables are schemata representing all of their closed instances ([4] p. 41.).

In **PRL** 'defaults' can simply be formulated as formulas of the form $(A \supset C)^k$ and $(\forall x)(A(x) \supset C(x))^k$. Universally quantified formulas are considered as (possibly infinite) lists of instances. *The (possibly infinite) list of formulas $B_1^{k_1}$, $B_2^{k_2}$, ... does not need to be known in advance!* Whenever A (respectively $A(a)$) is derived with a derivation preference l, and hence C (resp. $C(a)$) can be derived with a derivation preference n which is the maximum of k and l, and $\neg C$ (respectively $\neg C(a)$) is derived with a line preference m, the **PRL**-machinery guarantees that (1) C (resp. $C(a)$) is derivable iff $n < m$, (2) $\neg C$ (resp. $\neg C(a)$) is derivable iff $m < n$, and (3) neither C (resp. $C(a)$) nor $\neg C$ (resp. $\neg C(a)$) are derivable iff $n = m$. In any of these cases, **PRL** reveals the involved formulas B_i. If we are interested in reconstructing a default theory within **PRL**, the preference of the formulas B_i should not be higher than or equal to the preference of $A \supset C$. The members of W, the facts, can all get a preference 1 (this means that they can only be 'deleted' by theorems). The possibility to introduce an infinite number of preferences, allows to express a refined ordering on the **PRL**-equivalents of defaults.

Such a reconstruction is nothing but a special case of **PRL**. **PRL** allows even for the contradiction of facts. Actually, not all facts are true with certainty in all contexts. With the **PRL**-reinterpretation of the premises, it

is not required to assume that W is consistent. Generally, the machinery of **PRL** comes to the following 'default':

> C is derived with derivation preference k: $\neg C$ is not derived with line preference $l \leq k$ / $C \in Cn^0_{\mathbf{PRL}}(\Gamma^A)$.

The original version of default logic does not satisfy several "intended" properties. It is for instance not possible to derive *Likes wine* from (1) *Italian*: *Likes Wine/Likes Wine*, (2) *French:Likes Wine/Likes Wine* and (3) *Italian* $\vee$ *French*. As illustrated in Proof 4 (above) **PRL** derives this intended consequence.

At the other hand, default logic derives *Usable(Left arm)* & *Usable(Right arm)* from (1) *true*: *Usable(x)* & $\neg$*Broken(x)/Usable(x)* and (2) *Broken(Left arm)* $\vee$ Broken(*Right arm*). In the following proof it is shown that **PRL** deals perfectly with these premises.

Proof 4.

1.	$(\forall x)(Ux)^3$	–	PREM	–
2.	$(\forall x)(Bx \supset \neg Ux)^2$	–	PREM	–
3.	$(Bl \vee Br)^1$	–	PREM	–
4.	Ul	(1)	CUI	Ul^3
5.	Ur	(1)	CUI	Ur^3
6.	$Bl \supset \neg Ul$	(2)	CUI	$(Bl \supset \neg Ul)^2$
7.	$Br \supset \neg Ur$	(2)	CUI	$(Br \supset \neg Ur)^2$
8.	$Bl \vee Br$	(3)	CSUB	$(Bl \vee Br)^1$
9.	$\neg Ul \vee \neg Ur$	(6,7,8)	DIL	$(Bl \supset \neg Ul)^2, (Br \supset \neg Ur)^2,$ $(Bl \vee Br)^1$
10.	$\neg Ur$	(4,9)	DS	$(Bl \supset \neg Ul)^2, (Br \supset \neg Ur)^2,$ $(Bl \vee Br)^1, Ul^3$
11.	$\neg Ul$	(5,9)	DS	$(Bl \supset \neg Ul)^2, (Br \supset \neg Ur)^2,$ $(Bl \vee Br)^1, Ur^3$
12.	$\sharp^2(Bl \supset \neg Ul) \vee \sharp^2(Br \supset \neg Ur)^2 \vee \sharp^1(Bl \vee Br) \vee \sharp^3 Ul$ (5,10) IDENK –				
13.	$\sharp^2(Bl \supset \neg Ul) \vee \sharp^2(Br \supset \neg Ur)^2 \vee \sharp^1(Bl \vee Br) \vee \sharp^3 Ur$ (4,11) IDENK –				

In view of MR1 and the minimal DENK-consequences derived in lines (12) and (13) both Ul^3 and Ur^3 are unreliable, and hence lines (4), (5), (10) and (11) have to be marked. This means that none of Ul, Ur, $\neg Ul$, $\neg Ur$ are finally derivable. However $\neg Ul \vee \neg Ur$ is a **PRL**-consequence of the premises.

6.2. Rescher's strong and weak consequences

Diderik Batens showed me, that if we apply the minimal abnormality strategy instead of the reliability strategy, (thus obtaining the logic **PRL2**), $Cn^0_{\mathbf{PRL2}}(\Gamma^A)$ is the set of Rescher's strong consequences of Γ. Indeed, A is a **PRL2**-consequence of Γ^A iff A is true in all minimally abnormal models of Γ^A, whereas these minimally abnormal models verify the maximally consistent subsets of Γ^A. If A is true in some minimally abnormal models of Γ^A, then A is a weak consequence of Γ^A. I refer to Diderik Batens article in the Proceedings of the JS-Symposium (Toruń, Poland, 1998) for the details.

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GUIDO VANACKERE
Center for Logic and Philosophy of Science
University of Ghent, Belgium
Guido.Vanackere@rug.ac.be