Logic and Logical Philosophy
Volume 7 (1999), 91–99

Carlos A. Oller

PARACONSISTENCY AND ANALYTICITY

Abstract. William Parry conceived in the early thirties a theory of entailment, the theory of analytic implication, intended to give a formal expression to the idea that the content of the conclusion of a valid argument must be included in the content of its premises. This paper introduces a system of analytic, paraconsistent and quasi-classical propositional logic that does not validate the paradoxes of Parry's analytic implication. The interpretation of the expressions of this logic will be given in terms of a four-valued semantics, and its proof theory will be provided by a system of signed semantic tableaux that incorporates the techniques developed to improve the efficiency of the tableaux method for many-valued logics.

1. Introduction

It is often pointed out that inconsistency in information is the norm and not the exception [8]. One of the motivations for paraconsistent logics, i.e. logics that do not have *ex contradictione quodlibet* as a valid rule, is to provide a framework for reasoning from inconsistent databases. An adequate paraconsistent logic will, in the presence of a contradiction, perform a kind of damage control: it will allow us to derive all the reasonable conclusions until we can detect and eradicate the inconsistency (if that is desirable).

Several systems of paraconsistent logic have been presented in the literature which involve a weakening of classical logic. Of course one might want the resulting system not to be too weak, which would make it not very useful for applications. A couple of attractive paraconsistent logics that fulfill this desideratum have been recently presented in the literature, namely Joke Meheus' AN system [10] and the quasi-classical logic developed by Ph. Besnard and A. Hunter [3]. Unlike other paraconsistent logics, AN and quasi-classical logic keep Disjunctive Syllogism, Double Negation, and the interdefinition relations between the connectives expressed by the De Morgan rules. In the case of quasi-classical logic, on which I will concentrate for comparisons, paraconsistency is achieved by restricting the proof theory so that compositional proof rules like Addition cannot be followed by decompositional rules like Resolution.

Quasi-classical logic is not analytic, i.e. a valid argument may have propositional letters in the conclusion not appearing in the premises. Nevertheless, there are applications for which analyticity of the inference procedure is a desirable feature. For example, the importance of using an analytic logic when we query a database is clear from the following example. Suppose that a certain database consists of the following sentence: *If a patient has symptom A, then (s)he has disease D.* If we query this database regarding the disease suffered by patient P who has symptom A, we would consider the answer: *Patient P has disease D or disease D'* misleading, if not incorrect.

It is interesting to point out that analyticity is not often mentioned among the motivations for paraconsistent logic, even though analyticity and paraconsistency seem to be closely related. In fact, if a consequence relation is analytic in the sense defined in the previous paragraph it is also paraconsistent, even though the converse does not hold.

William Parry [11, 12] conceived in the early thirties a theory of entailment, the theory of analytic implication (AI), intended to avoid the paradoxes of strict implication and to give a formal expression to the intuition that a sentence X entails a sentence Y only if the content of X contains that of Y. First-degree entailments of AI propositional calculus are, according to Parry's characterization, the formulas "X entails Y" such that $X \supset Y$ is a truth-functional tautology, and Y contains no propositional letter not contained in X. First-degree AI formulas can be regarded as expressing a consequence relation between classical formulas.

The notion of analytic consequence can be characterized in terms of the Dunn-Epstein's [5, 6] semantics. Let $\mathcal{L}$ be a classical propositional language, v a Boolean valuation of the set of all wffs of $\mathcal{L}$, and s a function that assigns to each wff of $\mathcal{L}$ a subset of a non-empty set $\mathbb{S}$, which can be taken as a set of bits of content, according to the following conditions:

S1. $s(\neg X) = s(X)$.

S2. $s(X \vee Y) = s(X \wedge Y) = s(X \supset Y) = s(X) \cup s(Y)$.

S3. $s(\Gamma)$ is the union of all the $s(X_i)$ such that $X_i \in \Gamma$.

A set of sentences Γ analytically entails a proposition X (in symbols, $\Gamma \vDash_a X$) iff X is true in every Boolean valuation v in which all the members of Γ are true, and given any content-assignment s, the content of X is included in the content of Γ. The class of theorems, i.e. the set of expressions $\vDash_a X$ is according to this definition, empty. Clearly, this notion of entailment rules out the Lewis' principles as valid entailments; also, the rule of Addition is not a valid entailment in Parry's system, whereas Disjunctive Syllogism is. However, the following counterintuitive formulas which will be referred to as "the paradoxes of Parry's analytic implication" must be accepted as valid entailments:

$$X \wedge \neg X \wedge Y \vDash_a \neg Y \qquad\qquad X \wedge \neg X \wedge Y \vDash_a Y \wedge \neg Y$$

So, even if this notion of entailment avoids the paradoxes of strict implication, it still validates some paradoxical theses.

In this paper a system of analytic quasi-classical logic will be presented. This system is paraconsistent and analytic, i.e. no argument with propositional letters occurring in the conclusion but not in the premises is valid in it. It shares with Besnard and Hunter's quasi-classical logic the property of keeping the usual interdefinition relations between connectives while achieving paraconsistency by a modification in the characterization of logical consequence. Furthermore, the paradoxes of Parry's analytic implication are not valid in analytic quasi-classical logic.

The interpretation of the expressions of this logic will be given in terms of a four-valued semantics, an approach to paraconsistency that has been defended, for example, in [2, 1]. The proof theory for analytic quasi-classical logic will be provided by a system of signed semantic tableaux that incorporates the techniques developed to improve the efficiency of the tableaux method for many-valued logics [9].

2. Semantics for AL

In this section the definitions of basic semantic concepts (truth value, designated truth value, valuation, satisfiability, model, logical consequence) for classical propositional logic are generalized for many-valued logics: a *truth value* is one of the members of the set $\mathbb{T} = \{\mathtt{b}, \mathtt{t}, \mathtt{f}, \mathtt{n}\}$. A *designated truth value* is one of the members of the set $\mathbb{D} = \{\mathtt{b}, \mathtt{t}\}$, and a *non-designated truth value* is one of the members of the set $\mathbb{ND} = \{\mathtt{f}, \mathtt{n}\}$. $\mathtt{b}$ (both), $\mathtt{t}$ (true), and $\mathtt{f}$ (false) will be called *definite truth values*, because they indicate that we have some definite information — or even too much information, in the case of $\mathfrak{b}$ — about a formula; $\mathfrak{n}$ (neither) indicates the absolute absence of information. A valuation v is a function from the set $\mathbb{F}$ of formulas into the set $\mathbb{T}$ of truth values. The value of X under v is called the *truth value of* X *under* v. An *AL-valuation* assigns values to the formulas of AL according to the following truth tables:

x	$\neg x$
$\mathfrak{b}$	$\mathfrak{b}$
$\mathfrak{t}$	$\mathfrak{f}$
$\mathfrak{f}$	$\mathfrak{t}$
$\mathfrak{n}$	$\mathfrak{n}$

$\vee$	$\mathfrak{b}$	$\mathfrak{t}$	$\mathfrak{f}$	$\mathfrak{n}$
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{n}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{n}$
$\mathfrak{f}$	$\mathfrak{b}$	$\mathfrak{t}$	$\mathfrak{f}$	$\mathfrak{n}$
$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{n}$

$\wedge$	$\mathfrak{b}$	$\mathfrak{t}$	$\mathfrak{f}$	$\mathfrak{n}$
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{f}$	$\mathfrak{n}$
$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{t}$	$\mathfrak{f}$	$\mathfrak{n}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{n}$
$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{n}$

The truth table for $\rightarrow$ can be built taking into account the the definition $X \rightarrow Y =_{\mathrm{df}} \neg X \vee Y$.

A set Γ of formulas is *satisfiable* if there is a AL-valuation v such that for any $X \in \Gamma$: $v(X) \in \mathbb{D}$. In this case v is an $AL-model$ for Γ. A formula X is satisfiable iff the set $\{X\}$ is satisfiable. A formula X is a tautology if $\{X\}$ is satisfiable for any valuation v; it is easy to prove that the set of ACQ-tautologies is empty. A formula X is a logical consequence of the set of formulas Γ iff in every valuation where Γ is satisfiable, X is satisfiable.

One of the main differences between Belnap's and AL tables is that no compound formula is assigned a definite truth value, i.e. $\mathfrak{b}$, $\mathfrak{t}$, or $\mathfrak{f}$, unless all its components are assigned one. Intuitively, this means that we cannot assert a compound sentence unless we have definite information about all its components. This constraint is necessary to comply with the requirement of analyticity imposed on the AL consequence relation: if the conclusion has propositional letters not occurring in the premises there will always be an AL-valuation under which the conclusion is assigned the truth value $\mathfrak{n}$ and therefore, given the definition above, it will not be an AL-consequence of the premises.

It is easy to check that Parry's paradoxes of analytic implication are not valid AL-entailments: if X is assigned the truth value $\mathfrak{b}$ and Y the truth value $\mathfrak{t}$, then the premises of both $X \wedge \neg X \wedge Y \vDash_a \neg Y$ and $X \wedge \neg X \wedge Y \vDash_a Y \wedge \neg Y$ are assigned the truth value $\mathfrak{t}$ while their conclusions are assigned $\mathfrak{f}$.

3. A tableau system for AL

In this section a tableau system for AL will be presented. The following definitions provide an adaptation of the basic concepts of the tableau method for multiple-valued logics: a *signed tableau* is a n-ary tree of nodes such that each node consists of a signed formula, i.e. a formula preceded by a sign. A *branch* θ *of a signed tableau* τ, is an acyclic path from the root of τ to its leaves. For each pair of sign and connective there is a *rule* that determines what nodes may be adjoined to the branches of a tableau. The α*-rules* are those rules that do not cause the tableau to branch and the β*-rules* are those rules that cause the tableau to branch. A formula to which an α-rule can be applied is an α (formula) and a formula to which a β-rule can be applied is a β (formula). A tableau τ_2 is an *immediate extension* of a tableau τ_1 iff it is obtained from τ_1 by one application of an α-rule or a β-rule. α_1 and α_2 are the nodes added to a tableau as a result of the application of an α-rule and a β-sequence is one of the sequences of formulas that may be added to a branch of a tableau as a result of the application of a β-rule. A *branch* θ *of a signed tableau* τ *is closed* iff it contains two signed formulas $\sigma_1 X$ and $\sigma_2 X$, where σ_1 and σ_2 are opposite signs. A *branch* θ *of a signed tableau* τ *is open* iff it is not closed. A *branch* θ *of a signed tableau* τ *is complete* iff for every α which occurs in θ the α_i occur in θ and for every β which occurs in θ at least one of the β-sequences occur in θ. A *tableau* τ *is completed* iff every branch of it is either closed or completed. A *signed tableau* τ *is closed* iff every branch of τ is closed and *open* otherwise. There exists a *proof of an argument* $X_1, X_2, \ldots, X_n \vdash Y$ iff for each designated sign σ and each non-designated sign $\overline{\sigma}$ there is a closed tableau for $\{\sigma X_1, \sigma X_2, \ldots, \sigma X_n, \overline{\sigma} Y\}$.

The tableau method for classical bivalent logic can be described as a systematic search for a counterexample, i.e. a valuation under which the premises are true and the conclusion false; if no such counterexample exists the argument is valid. In the case of many-valued logics we can have more than one designated truth value and/or more than one non-designated truth value, e.g. in AL there are two designated truth values and two non-designated truth values. Therefore, showing the validity of an AL argument involves building four, instead of just one, tableaux: each of these four tableaux shows that the premises cannot take one of the designated truth values and the conclusion one of the non-designated truth values.

Nevertheless, a method was developed for increasing the efficiency of tableau proof systems for many-valued logics that has been applied to several of these logics [4, 9]. The basic idea behind this method is to increase the expressivity of the signs in order to reduce the number of tableaux needed to show validity of an argument. Instead of having a one-to-one association between truth values and signs we can use truth-value sets as signs. For example, instead of having a sign associated with the truth value t and another one associated with the truth value b, the two designated truth values of AL, we can fuse both of them in the sign (T/B), which has the intended interpretation of t or b.

The following definitions apply this method to the AL tableau system: the *set of signs for* AL is the set $\Sigma = \{\mathsf{T}, \mathsf{F}, \mathsf{B}, \mathsf{N}, (\mathsf{T}/\mathsf{B}), (\mathsf{F}/\mathsf{N})\}$. A *sign* is one of the members of this set. (In fact, if the complex signs (B/F) and (B/F/T) were added then the rules below would be simpler. For perspicuity, I keep the number of complex signs to a minimum). The *set of designated signs for AL is the set* $\Sigma_{\mathrm{d}} = \{(\mathsf{T}/\mathsf{B})\}$. The *set of non-designated signs for* AL is the set $\Sigma_{\mathrm{nd}} = \{(\mathsf{F}/\mathsf{N})\}$. The following table defines the notion of *opposite signs* (two signs in the same row are opposite signs):

T	F
T	B
T	N
T	(F/N)
F	B
F	N
F	(T/B)
B	N
B	(F/N)
(T/B)	N
(T/B)	(F/N)

The following are the tableau rules for the AL logic:

RULES FOR ¬

$$\frac{\mathsf{T}\neg X}{\mathsf{F}X} \qquad \frac{\mathsf{F}\neg X}{\mathsf{T}X} \qquad \frac{\mathsf{B}\neg X}{\mathsf{B}X} \qquad \frac{\mathsf{N}\neg X}{\mathsf{N}X} \qquad \frac{(\mathsf{T}/\mathsf{B})\neg X}{\mathsf{F}X \quad \mathsf{B}X} \qquad \frac{(\mathsf{F}/\mathsf{N})\neg X}{\mathsf{T}X \quad \mathsf{N}X}$$

RULES FOR $\wedge$

$$\frac{\mathsf{T}X \wedge Y}{\begin{array}{c}\mathsf{T}X\\ \mathsf{T}Y\end{array}} \qquad \frac{\mathsf{F}X \wedge Y}{\begin{array}{ccc}\mathsf{F}X & (\mathsf{T/B})X & \mathsf{F}X\\ \mathsf{F}Y & \mathsf{F}Y & (\mathsf{T/B})Y\end{array}} \qquad \frac{\mathsf{B}X \wedge Y}{\begin{array}{ccc}\mathsf{B}X & \mathsf{B}X & \mathsf{B}Y\\ \mathsf{B}Y & \mathsf{T}Y & \mathsf{T}X\end{array}}$$

$$\frac{\mathsf{N}X \wedge Y}{\begin{array}{cc}\mathsf{N}X & \mathsf{N}Y\end{array}} \qquad \frac{(\mathsf{T/B})X \wedge Y}{\begin{array}{c}(\mathsf{T/B})X\\ (\mathsf{T/B})Y\end{array}} \qquad \frac{(\mathsf{F/N})X \wedge Y}{\begin{array}{cc}(\mathsf{F/N})X & (\mathsf{F/N})Y\end{array}}$$

RULES FOR $\vee$

$$\frac{\mathsf{T}X \vee Y}{\begin{array}{cccc}\mathsf{T}X & (\mathsf{T/B})X & \mathsf{F}X & \mathsf{T}X\\ (\mathsf{T/B})Y & \mathsf{T}Y & \mathsf{T}Y & \mathsf{F}Y\end{array}} \qquad \frac{\mathsf{F}X \vee Y}{\begin{array}{c}\mathsf{F}X\\ \mathsf{F}Y\end{array}}$$

$$\frac{\mathsf{B}X \vee Y}{\begin{array}{ccc}\mathsf{B}X & \mathsf{F}X & \mathsf{B}X\\ \mathsf{B}Y & \mathsf{B}Y & \mathsf{F}Y\end{array}} \qquad \frac{\mathsf{N}X \vee Y}{\begin{array}{cc}\mathsf{N}X & \mathsf{N}Y\end{array}} \qquad \frac{(\mathsf{F/N})X \vee Y}{\begin{array}{ccc}\mathsf{N}X & \mathsf{N}Y & \begin{array}{c}\mathsf{F}X\\ \mathsf{F}Y\end{array}\end{array}}$$

$$\frac{(\mathsf{T/B})X \vee Y}{\begin{array}{ccccc}(\mathsf{T/B})X & \mathsf{T}X & \mathsf{F}X & (\mathsf{T/B})X & \mathsf{F}X\\ (\mathsf{T/B})Y & \mathsf{F}Y & \mathsf{T}Y & \mathsf{F}Y & (\mathsf{T/B})Y\end{array}}$$

4. Soundness and completeness

The standard proofs of soundness and completeness for classical bivalent propositional logic [13] can be straightforwardly adapted to give their counterparts for AL. In fact, the point of Hähnle's paper [9] was to provide a general soundness and completeness result for truth value set-signed logics.

5. Comparison of AL with AI and QC

The table below compares some properties of the AL, Analytic Implication, and Besnard and Hunter's Quasi-Classical Logic consequence relations. In this table '$\vdash$' denotes the inference relation of classical logic, and in '$\vdash_x$' the 'x' can take the values 'AL', 'QC', or 'AI'.

Property	AL	QC	AI
Supraclassicality: if $\Gamma \vdash X$, then $\Gamma \vdash_x X$	no	no	no
Reflexivity: $\Gamma \cup \{X\} \vdash_x X$	yes	yes	yes
Monotonicity: if $\Gamma \vdash_x Y$, then $\Gamma \cup \{X\} \vdash_x Y$	yes	yes	yes
Cut: If $\Gamma \vdash_x X$ and $\Gamma \cup \{X\} \vdash_x Y$, then $\Gamma \vdash_x Y$	yes	no	yes
Left Logical Equivalence: if $\Gamma \cup \{X\} \vdash_x Z$ and $\vdash Y \leftrightarrow X$, then $\Gamma \cup \{Y\} \vdash_x Z$	no	no	no
Right Weakening: if $\Gamma \vdash_x Y$ and $\vdash Y \rightarrow X$, then $\Gamma \vdash_x X$	no	no	no
And: if $\Gamma \vdash_x X$ and $\Gamma \vdash_x Y$, then $\Gamma \vdash_x X \wedge Y$	yes	yes	yes
Or: if $\Gamma \cup \{X\} \vdash_x Z$ and $\Gamma \cup \{Y\} \vdash_x Z$, then $\Gamma \cup \{X \vee Y\} \vdash_x Z$	yes	yes	yes
Consistency Preservation: if $\Gamma \vdash_x \bot$, then $\Gamma \vdash \bot$	yes	yes	yes
Conditionalization: if $\Gamma \cup \{X\} \vdash_x Y$, then $\Gamma \vdash_x X \rightarrow Y$	no	no	no
Deduction: if $\Gamma \vdash_x X \rightarrow Y$, then $\Gamma \cup \{X\} \vdash_x Y$	no	no	yes

Even though these three consequence relations share most of these properties, AL and AI are better behaved relations, from the point of view of deductive consequence relations, because Cut holds for both of them and fails for QC.

References

[1] A. R. Anderson, N. D. Belnap Jr., J. M. Dunn, *Entailment. The Logic of Relevance and Necessity, vol. II*, Princeton University Press, Princeton, 1992.

[2] N. D. Belnap Jr., A Useful Four-valued Logic, in J. M. Dunn and G. Epstein (eds.), *Modern Uses of Multiple-valued Logic*, Reidel, Dordrecht, pp. 8–37, 1977. A more recent version of this paper can be found in [1], pp. 506–541.

[3] Ph. Besnard and A. Hunter, Quasi-classical logic. non-trivializable classical reasoning from inconsistent information, in C. Froidevaux and J. Kohlas (eds.), *Symbolic and Quantitative Approaches to Uncertainty (ECSQARU'95)*, Springer, LNCS 946, Heidelberg, pp. 44–51, 1995.

[4] A. Bloesch, *Tableau Style Proof Systems for Many-Valued Logics*, Technical Report No 94–18, Software Verification Research Centre, Department of Computer Science, The University of Queensland, Queensland (Aus.), 1994.

[5] J. M. Dunn, A Modification of Parry's Analytic Implication, *Notre Dame Journal of Formal Logic*, 13 (1972), pp. 195-205.

[6] R. L. Epstein, *The Semantic Foundations of Logic. Volume 1: Propositional Logics*, Kluwer Academic Publishers, Dordrecht, 1990.

[7] K. Fine, Analytic implication, *Notre Dame Journal of Formal Logic*, 27 (1986), pp. 169–179.

[8] D. Gabbay and A. Hunter, Making Inconsistency Respectable 1: A Logical Framework for Inconsistency in Reasoning, in Ph. Jorrand and J. Kelemen (eds.), *Fundamentals of Artificial Intelligence*, Springer, LNCS 535, Heidelberg, pp. 19–32, 1991.

[9] R. Hähnle, Towards an Efficient Tableau Proof Procedure for Multiple-Valued Logics, in *Proceedings Workshop Computer Science*, Springer, LNCS 533, Heidelberg, pp. 248–260, 1991.

[10] J. Meheus, An extremely rich paraconsistent logic and the adaptive logic based on it, Universiteit Gent, Centre for Logic and Philosophy of Science, Preprint 65, Gent, 1998.

[11] W. T. Parry, Ein Axiomensystem für eine neue Art von Implikation (analytische Implikation), *Ergebnisse eines mathematischen Kolloquiums*, 4 (1933), pp. 5–6.

[12] W. T. Parry, Analytic Implication: Its History, Justification and Varieties, in J. Norman and R. Sylvan (eds.), *Directions in Relevant Logic*, Kluwer, Boston, pp. 101–118, 1989.

[13] R. Smullyan, *First-Order Logic*, Springer, New York, 1968.

Carlos A. Oller
Facultad de Filosofia y Letras
Universidad de Buenos Aires
Casilla de Correo 1015
Correo Central
C1000WAK – Buenos Aires, Argentina
coller@ciudad.com.ar