Logic and Logical Philosophy
Volume 7 (1999), 63–80

Newton C. A. da Costa
Roque da C. Caiero*

k-TRANSFORMS IN CLASSICAL AND PARACONSISTENT LOGICS

Abstract. We study some metamathematical properties of various classical and paraconsistent logical systems. In particular, we discuss the concept of a k-transform of a formula and consider some of its applications.

Keywords: k-transform, paraconsistent logic, predicate calculus.

1. Introduction

This is essentially an expository paper in which we treat various known classical and paraconsistent logical systems and study some of their metamathematical properties. The investigation of these properties is limited to the syntactical level and only finitary methods are employed.

In most cases a paraconsistent system has no well defined (and formalized) semantics as its *starting point*. In effect, in order to build such a semantics, in the strict sense of the word, we clearly need a previous theory to function as its underlying basis. In this connection classical logic and extant set theory are, as a matter of principle, excluded.

So, three ways are open to construct the semantics of a paraconsistent system **S**: 1) We treat **S** syntactically, with the help of informal hints and

* The second author is supported by a grant from the Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Section of Philosophy.

more or less definite intuitions, its semantics remaining informal, at least to begin with; 2) We take profit of a previously developed paraconsistent system to construct a semantics for **S**; 3) We may employ, say, a set theory whose underlying logic is **S**, to elaborate semantics for some its parts (for instance, the elementary logic of **S**).

In what follows, we proceed syntactically. It is implicit that finitary syntactical means (in the sense of [15]) are allowed in any study of the syntactical counterpart of a logical system, paraconsistent or not.

2. Hierarchies of paraconsistent systems

The basic condition that a paraconsistent logic $\mathcal{S}$ must satisfy is that, from any two contradictory statements, one can not deduce, according to the rules of $\mathcal{S}$, any statement whatever.

The first author introduced, several years ago (*cf.* [9], [10] and [11]), a hierarchy of paraconsistent propositional calculi, and two corresponding hierarchies of paraconsistent first-order predicate calculi and of paraconsistent first-order predicate calculi with equality. These systems, described below, were not intended as *true* logics, governing the actual world, but as theoretical systems showing the possibility in principle of strong systems of paraconsistent logic. They served as foundations for the construction of strong paraconsistent set theories and paraconsistent mathematics (see, for examples, [11]). The applications of such systems to actual, concrete problems constitute a matter of fact and of experience, that can not be solved *a priori*.

2.1. The hierarchy of propositional calculi $\mathcal{C}_n$, $1 \leqslant n \leqslant \omega$

2.1.1. The classical propositional calculus $\mathcal{C}_0$

The primitive symbols of the language **L** are the following: 1) A denumerably infinite family of propositional variables; 2) Connectives: $\neg$, $\vee$, $\rightarrow$ and $\wedge$ ($\leftrightarrow$, equivalence, is defined as usual); 3) Auxiliary symbols: parentheses. With these symbols, we define formula and other syntactical concepts as usual.

Let φ, ψ and χ be formulas. We present briefly, in Hilbert-Bernays style (see [15]), the propositional postulates (axiom schemes and primitive deduction rules) of the $\mathcal{C}_0$ as follows:

$\rightarrow_1$) $\quad \varphi \rightarrow (\psi \rightarrow \varphi)$

$\rightarrow_2$) $\quad (\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow \chi)$

$\rightarrow_3$) $\quad \varphi, (\varphi \rightarrow \psi) \,/\, \psi$

$\wedge_1$) $(\varphi \wedge \psi) \to \varphi$

$\wedge_2$) $(\varphi \wedge \psi) \to \psi$

$\wedge_3$) $\varphi \to (\psi \to (\varphi \wedge \psi))$

$\vee_1$) $\varphi \to (\varphi \vee \psi)$

$\vee_2$) $\psi \to (\varphi \vee \psi)$

$\vee_3$) $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$

$\neg_1$) $(\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi))$

$\neg_2$) $\neg\varphi \to (\varphi \to \psi)$

$\neg_3$) $\varphi \vee \neg\varphi$

The notions of proof, theorem, etc. of $\mathcal{C}_0$ and the paraconsistent calculi introduced below are the standard ones. We shall use abbreviations like $\to_{1,2,3}$: it denotes the set of postulates $\to_1$, $\to_2$ and $\to_3$.

2.1.2. The paraconsistent propositional calculus $\mathcal{C}_1$

The language of $\mathcal{C}_1$ is the same as that of $\mathcal{C}_0$, that is **L**.

Definition 1. $\varphi^\circ \equiv_{\mathrm{Def}} \neg(\varphi \wedge \neg\varphi)$.

Definition 2. $\neg^*\varphi \equiv_{\mathrm{Def}} \neg\varphi \wedge \varphi^\circ$.

The postulates of the paraconsistent calculus $\mathcal{C}_1$ are those from $\to_1$ to $\vee_3$ above, plus the following:

$\neg'_1$) $\psi^\circ \to ((\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi)))$

$\neg'_2$) $(\varphi^\circ \wedge \psi^\circ) \to ((\varphi \to \psi)^\circ \wedge (\varphi \wedge \psi)^\circ \wedge (\varphi \vee \psi)^\circ)$

$\neg'_3$) $\neg\neg\varphi \to \varphi$

$\neg'_4$) $\varphi \vee \neg\varphi$

Remark. The negation $\neg^*$ has all properties of classical negation. Though $\mathcal{C}_1$ is obviously weaker than $\mathcal{C}_0$, in certain sense the later is contained in the former. The abbreviation φ° means that, intuitively, φ is a "good" formula; the "good" formulas satisfy all postulates of $\mathcal{C}_1$.

2.1.3. The paraconsistent propositional calculi $\mathcal{C}_n$, $1 < n < \omega$

The language of $\mathcal{C}_n$, $1 < n < \omega$, is that of $\mathcal{C}_0$. Let us define:

Definition 3. $\varphi^n \equiv_{\text{Def}} \varphi^{\circ\circ\cdots\circ}$ (with $\circ$ n times) $n \geqslant 1$.

Definition 4. $\varphi^{(n)} \equiv_{\text{Def}} \varphi^{(n-1)} \wedge \varphi^n$ for $n > 1$ and $\varphi^{(1)} \equiv_{\text{Def}} \varphi^1$.

Definition 5. $\neg^{(n)}\varphi \equiv_{\text{Def}} \neg\varphi \wedge \varphi^{(n)}$, $n > 1$.

We note that in $\mathcal{C}_n$, $1 \leqslant n < \omega$, the negation $\neg^{(n)}$ has all properties of the classical negation.

The calculi $\mathcal{C}_n$, $1 < n < \omega$, are characterized by the postulates $\rightarrow_{1,2,3}$, $\wedge_{1,2,3}$, $\vee_{1,2,3}$ and the following axiom schemes:

$\neg^n_1$) $\quad \psi^{(n)} \rightarrow ((\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi)))$

$\neg^n_2$) $\quad (\varphi^{(n)} \wedge \psi^{(n)}) \rightarrow ((\varphi \rightarrow \psi)^{(n)} \wedge (\varphi \wedge \psi)^{(n)} \wedge (\varphi \vee \psi)^{(n)})$

$\neg^n_3$) $\quad \neg\neg\varphi \rightarrow \varphi$

$\neg^n_4$) $\quad \varphi \vee \neg\varphi$

2.1.4. The paraconsistent propositional calculus $\mathcal{C}_\omega$

$\mathcal{C}_\omega$ has the language **L** and its postulates are $\rightarrow_{1,2,3}$, $\wedge_{1,2,3}$, $\vee_{1,2,3}$, plus the following:

$\neg^\omega_1$) $\quad \neg\neg\varphi \rightarrow \varphi$

$\neg^\omega_2$) $\quad \varphi \vee \neg\varphi$

$\mathcal{C}_n$, $0 \leqslant n \leqslant \omega$, are consistent (see [9] and [10]). In $\mathcal{C}_n$, $1 \leqslant n \leqslant \omega$, the principle of contradiction, *i.e.* $\neg(\varphi \wedge \neg\varphi)$, is not a valid schema, and from two contradictory formulas, φ and $\neg\varphi$, it is not in general possible to deduce an arbitrary formula.

2.2. The predicate calculi without equality $\mathcal{C}^*_n$, $0 \leqslant n \leqslant \omega$

The language of $\mathcal{C}^*_n$, $0 \leqslant n \leqslant \omega$, is defined as in [6] and [9], where the propositional language is modified to be transformed into the language of first-order logic without equality.

2.2.1. The predicate calculus $\mathcal{C}_n^*$, $0 < n < \omega$

The postulates of $\mathcal{C}_n^*$ are those of $\mathcal{C}_n$, $0 < n < \omega$, to which we add the following (x is a variable, $\varphi(x)$ is a formula, ψ is a formula which does not contain x free, and d is a term is free for x in $\varphi(x)$):

$\forall_1$) $\forall x\varphi(x) \to \varphi(d)$

$\forall_2$) $\psi \to \varphi(x) \,/\, \psi \to \forall x\varphi(x)$

$\forall_3$) $\forall x(\varphi(x))^{(n)} \to (\forall x\varphi(x))^{(n)}$

$\exists_1$) $\varphi(d) \to \exists x\varphi(x)$

$\exists_2$) $\varphi(x) \to \psi \,/\, \exists x\varphi(x) \to \psi$

$\exists_3$) $\forall x(\varphi(x))^{(n)} \to (\exists x\varphi(x))^{(n)}$

K) $\varphi \leftrightarrow \psi$, where φ and ψ are congruent formulas, or one is obtained from the other by the suppression of vacuous quantifiers (see [15], p. 153).

2.2.2. The predicate calculus $\mathcal{C}_0^*$

The postulates of $\mathcal{C}_0^*$ are those of $\mathcal{C}_0$ plus $\forall_{1,2}$ and $\exists_{1,2}$.

2.2.3. The predicate calculus $\mathcal{C}_\omega^*$

$\mathcal{C}_\omega^*$ has the following postulates: those of $\mathcal{C}_\omega$ plus $\forall_{1,2}$ and $\exists_{1,2}$ and K above.

2.3. The predicate calculi $\mathcal{C}_n^=$, $0 \leqslant n \leqslant \omega$

The hierarchy of predicate calculi with equality $\mathcal{C}_n^=$, $0 \leqslant n \leqslant \omega$, is obtained from $\mathcal{C}_n^*$, $0 \leqslant n \leqslant \omega$, by the addition of the symbol '=' of equality and the corresponding usual axiom schemes:

$=_1$) $\forall x(x = x)$

$=_2$) $x = y \to (\varphi(x) \leftrightarrow \varphi(y))$,
where $\varphi(z)$ is a formula, and x and y are distinct variables free for z in $\varphi(z)$.

In the next three definitions, **S** is a formal system whose underlying logic contains one of the propositional calculi $\mathcal{C}_n$, $0 \leqslant n \leqslant \omega$.

DEFINITION 6. **S** is said to be *trivial* (or *overcomplete*) if for any sentence φ of its language $\vdash \varphi$ in **S**; otherwise, **S** is said to be *non trivial*.

DEFINITION 7. **S** is called *inconsistent* if there exists a formula φ such that $\vdash \varphi$ and $\vdash \neg\varphi$ in **S**; otherwise, **S** is called *consistent*.

DEFINITION 8. **S** is called *finitely trivializable* if and only if there exists a formula φ such that adjoining φ to the system as a new axiom, the resulting formal system is trivial.

$\mathcal{C}_n$, $\mathcal{C}_n^*$ and $\mathcal{C}_n^=$, $0 \leqslant n < \omega$, are finitely trivializable, but $\mathcal{C}_\omega$, $\mathcal{C}_\omega^*$ and $\mathcal{C}_\omega^=$ are not. For details on the paraconsistent logics defined see, for example,[9] and [11] and the works cited there.

3. Some type theories

We now present a hierarchy of type theories (or higher-order logics) related to the calculi described above.

To begin with, we define the notion of a type symbol.

DEFINITION 9. *Type* is a syntactical notion, characterized as follows:

t1. τ is a type;

t2. if t_1, t_2, …, t_m, $0 < m < \omega$, are types, then the $\langle t_1, t_2, \ldots, t_m \rangle$ is a type;

t3. the only types are those given by t1 and t2.

DEFINITION 10. The *height of a type t*, denoted by $h(t)$, is defined as follows:

h1. if $t = \tau$, then $h(t) = 0$;

h2. if $t = \langle t_1, t_2, \ldots, t_m \rangle$, $0 < m < \omega$, then
$h(t) = \max\{h(t_1), h(t_2), \ldots, h(t_m)\} + 1$.

3.1. The type system $\mathcal{T}_0$

$\mathcal{T}_0$ is a classical simple theory of types. It has the following primitive symbols: 1) The connectives, quantifiers and auxiliary symbols of $\mathcal{C}_0^=$, to which we add the comma; 2) Given any type t, a denumerably infinite family of variables of type t; 3) If t is a type, a family of constant symbols of this type. Variables and constants of type different of the type τ are called predicate symbols. We easily introduce the concepts of atomic formula, formula, closed formula (or sentence), etc.

The postulates of $\mathcal{T}_0$ are those of $\mathcal{C}_0^=$, with obvious adaptations.

Two remarks are in order: 1) The type stratification is here conceived especially as a device to hinder some circular definitions (in the case of classical logic they contribute to the elimination of some known paradoxes); 2) Extensionality is not assumed in general because we interpret $\mathcal{T}_0$ as an intentional type theory. However, extensional predicates are not ruled out: they are particular predicates and their theory can be encompassed by $\mathcal{T}_0$.

3.2. The system $\mathcal{T}_n$, $1 \leqslant n \leqslant \omega$

The system $\mathcal{T}_n$, $1 \leqslant n \leqslant \omega$, is derived from $\mathcal{T}_0$, as $\mathcal{C}_n^=$ is obtained from $\mathcal{C}_0^=$, with clear modifications.

4. k-transforms

In this section, we study the concepts of k-transform and of P-k-transform (see [3], [7] and [10]), and apply them to solve some metamathematical problems connected with the logical systems of the preceding sections.

4.1. k-transforms in classical logic

Our symbolism and terminology are borrowed from Kleene's book (see [15]), with obvious changes (*cf.* [3, 10]). In particular, $\mathcal{C}_0^*$ with k individuals, denoted by $\mathcal{C}_0^k$ and called predicate calculus with k-individuals, is $\mathcal{C}_0^*$ plus the new individual constants $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$, $\ldots$, $\mathbf{k}$, $k \geqslant 1$. We introduce the notion of a variable which occurs in a formula associated with a proper predicate symbol, as follows:

DEFINITION 11. Let P be a proper predicate symbol of arity m ($m > 0$), that occurs in the formula φ; the *variable x occurs in φ associated with P* if there exist in φ occurrences of P of the form $P(x_1, x_2, ..., x_m)$, where, for some i, x_i is x, with $0 < i \leqslant m$.

The notion of k-transform of a formula of $\mathcal{C}_0^k$ is defined in [14] and [15].

The classical predicate calculus with k individuals and q proper predicate letters, denoted by $\mathcal{C}_0^{k,q}$, is $\mathcal{C}_0^*$ with k individuals and only q proper predicate symbols (or letters), *i.e.*, predicate symbols whose arity is greater than zero. The calculus $\mathcal{C}_0^{k,0}$, *i.e.*, with zero proper predicate symbols, is $\mathcal{C}_0$ with a suitable definition of atomic formula.

The P-k-transforms of a formula of $\mathcal{C}_0^k$ is introduced by the following definition (one starts with a formula φ and proceeds step by step to the P-k-transforms of the subformulas of φ):

Definition 12. Let P be a predicate symbol of arity m of $\mathcal{C}_0^k$. The *P-k-transforms of the formula* φ of $\mathcal{C}_0^k$ are formulas obtained as follows:

pk1. In φ there are no free occurrences of variables associated with P:

- pk1.1) If φ is of the form $\neg\alpha$, then the P-k-transform of φ is $\neg(\alpha^{\maltese})$, where $\alpha^{\maltese}$ is the P-k-transform of α;
- pk1.2) If φ is $\alpha \to \beta$, then the P-k-transform of φ is $(\alpha^{\maltese}) \to (\beta^{\maltese})$, where $\alpha^{\maltese}$ and $\beta^{\maltese}$ are the P-k-transforms of α and β respectively. Analogously, we define the P-k-transforms of $\alpha \vee \beta$ and $\alpha \wedge \beta$;
- pk1.3) Let φ be $\forall x\alpha(x)$, then:
  - (a) If there are no occurrences of x in $\alpha(x)$ both free and associated with P, the P-k-transform of φ is $\forall x(\alpha^{\maltese}(x))$, $\alpha^{\maltese}(x)$ being the P-k-transform of $\alpha(x)$;
  - (b) Otherwise, the P-k-transform of φ is the conjunction $\alpha^{\maltese}(\mathbf{1}) \wedge \alpha^{\maltese}(\mathbf{2}) \wedge \cdots \wedge \alpha^{\maltese}(\mathbf{k})$, where $\alpha^{\maltese}(\ell)$ is the P-k-transform of $\alpha(\ell)$, $0 < \ell \leqslant k$.
- pk1.4) If φ is $\exists x\alpha(x)$, using disjunction instead of conjunction, we define the P-k-transform of $\exists x\alpha(x)$;
- pk1.5) If φ is a predicate symbol different of P, with any terms attached to it, or if φ is P, but there are no variables in φ associated with P, then the P-k-transform of φ is φ;

pk2. In φ there are free occurrences of variables associated with P: let φ be $\alpha(x_1, x_2, ..., x_m)$ where x_1, x_2, $\ldots$, x_m are free variables associated with P. The P-k-transforms of φ are the P-k-transforms of all formulas $\alpha(p_1, p_2, ..., p_m)$, where p_1, p_2, $\ldots$, p_m is any arrangement with repetition of rang m, of the k expressions $\mathbf{1}$, $\mathbf{2}$, $\ldots$, $\mathbf{k}$.

In order to simplify the exposition, we suppose that all vacuous quantifiers of φ are suppressed, that no variable occurs free and bound in φ, and that any variable in φ is linked to only one occurrence of a quantifier.

Let P_1, P_2, $\ldots$, P_h be h predicate symbols. One may generalize the concept of P-k-transform ($k \geqslant 0$) and define the concept of a P_1-P_2-...-P_h-k-transform of a given formula φ. When $h = 0$, the P_1-P_2-...-P_h-k-transform of φ is φ. If all predicate symbols of φ belong to $\{P_1, P_2, \ldots, P_h\}$, then the P_1-P_2-...-P_h-k-transforms of φ are the standard k-transforms of Hilbert and Bernays (see [3] and [15]). When k is zero, the P_1-P_2-...-P_h-k-transform of φ is also φ.

LEMMA 1. *If* $\vdash \varphi$ *in* $\mathcal{C}_0^*$ *(or in* $\mathcal{C}_0^k$*), then the* P-k*-transforms of* φ *are provable in* $\mathcal{C}_0^k$.

PROOF. The postulates of $\mathcal{C}_0^*$ (or of $\mathcal{C}_0^k$, $k \geqslant 1$) are $\rightarrow_{1,2,3}$, $\wedge_{1,2,3}$, $\vee_{1,2,3}$, $\neg_{1,2,3}$ and $\forall_{1,2}$ and $\exists_{1,2}$ above. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be a (formal) proof of φ in $\mathcal{C}_0^*$ (or in $\mathcal{C}_0^k$). We shall prove, by induction, that the P-k-transforms of any formula α_i, $1 \leqslant i \leqslant m$, in the proof of φ, are provable in $\mathcal{C}_0^k$.

If $i = 1$, α_i is a propositional axiom or a quantificational axiom. In the first case, it is clear that the P-k-transforms of α_i are theorems of $\mathcal{C}_0^k$. In the second, α_i is of one of the forms $\forall_1$ or $\exists_1$.

In the case of $\forall_1$, α_i is of the form $\forall x\alpha(x;d) \rightarrow \alpha(d;d)$. Therefore, we have:

1. There are no free occurrences of x associated with P in $\alpha(x;d)$. If $\alpha(d;d)$ does not have free occurrences of d associated with P, the P-k-transforms of α_i are of the form $\forall x(\alpha(x;d)^{\maltese}) \rightarrow \alpha(d;d)^{\maltese}$, where $\alpha(x;d)^{\maltese}$ and $\alpha(d;d)^{\maltese}$ are the P-k-transforms of $\alpha(x;d)$ and $\alpha(d;d)$ respectively. If there are free occurrrences of d associated with P in $\alpha(d;d)$, then the P-k-transforms of α_i are of the form $\forall x(\alpha(x;\ell)^{\maltese}) \rightarrow \alpha(\ell;\ell)^{\maltese}$, in which ℓ is a numeral and $\alpha(x;\ell)^{\maltese}$ and $\alpha(\ell;\ell)^{\maltese}$ are the P-k-transforms of $\alpha(x;\ell)$ and $\alpha(\ell;\ell)$.
2. $\alpha(x;d)$ contains free occurrences of x associated with P. Therefore, the P-k-transforms of α_i are expressions like $\alpha(\mathbf{1};\ell)^{\maltese} \wedge \alpha(\mathbf{2};\ell)^{\maltese} \wedge \cdots \wedge \alpha(\mathbf{k};\ell)^{\maltese} \rightarrow \alpha(\ell;\ell)^{\maltese}$, whose meanings are clear.

In both hypotheses, the P-k-transforms of α_i are provable in $\mathcal{C}_0^k$.

We treat postulate $\exists_1$ similarly.

Now, let us suppose that $i > 1$ and that all P-k-transforms of the formulas $\alpha_1, \alpha_2, \ldots, \alpha_{i-1}$ are provable in $\mathcal{C}_0^k$. We have to show that the same is true of α_i.

When α_i is an axiom, its P-k-transforms are provable. Hence, it suffices to consider the cases in which α_i is an immediate consequence of a preceding formula by one of the rules $\forall_2$ or $\exists_2$, or is an immediate consequence of two previous formulas by rule $\rightarrow_3$.

1. Rule $\forall_2$: α_i is, then, of the form $\chi \rightarrow \forall x\beta(x)$, in which χ does not contain x free, and there exists α_j, $j < i$, such that α_j is $\chi \rightarrow \beta(x)$. When x does not have free occurrences associated with P in $\beta(x)$, the P-k-transforms of α_j and α_i are of the forms $\chi^{\maltese} \rightarrow \beta(x)^{\maltese}$ and $\chi^{\maltese} \rightarrow \forall x(\beta(x)^{\maltese})$. Otherwise, α_j and α_i have P-k-transforms of forms $\chi^{\maltese} \rightarrow \beta(\ell)^{\maltese}$, $\ell = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{k}$,

and $\chi^{\maltese} \to \beta(\mathbf{1})^{\maltese} \wedge \beta(\mathbf{2})^{\maltese} \wedge \cdots \wedge \beta(\mathbf{k})^{\maltese}$. In both alternatives, from the P-k-transforms of α_j one can deduce any P-k-transforms of α_i.

2. Rule $\exists_2$ is handled analogously.
3. Rule $\to_3$: α_i is an immediate consequence of two formulas β and $\beta \to \alpha_i$, appearing in the formal deduction before α_i. Any P-k-transforms of β, denoted by $\beta^{\maltese}$, is provable in $\mathcal{C}_0^k$, as well as any P-k-transforms of $\beta \to \alpha_i$, denoted by $\beta^{\maltese} \to \alpha_i^{\maltese}$. Hence, $\alpha_i^{\maltese}$, *i.e.*, any P-k-transforms of α_i is also provable.

So, the proof of *Lemma 1* is complete. □

Lemma 2. *If $\Gamma \vdash \varphi$ in $\mathcal{C}_0^*$ (or in $\mathcal{C}_0^k$), Δ is a set of the P-k-transforms of the formulas of Γ and β is any P-k-transform of φ, then $\Delta \vdash \beta$ in $\mathcal{C}_0^k$.*

Proof. By an extension of the proof of *Lemma 1*. □

Lemma 3. *Any (formal) theorem φ of $\mathcal{C}_0^*$ (or of $\mathcal{C}_0^k$) is such that any P_1-P_2-...-P_h-k-transform of φ is provable in $\mathcal{C}_0^k$.*

Proof. Consequence of *Lemma 1*. □

Lemma 4. *If $\Gamma \vdash \varphi$ in $\mathcal{C}_0^*$ (or in $\mathcal{C}_0^k$), Δ is the set of P_1-P_2-...-P_h-k-transforms of the formulas of Γ, and β is any P_1-P_2-...-P_h-k-transform of φ, then $\Delta \vdash \beta$ in $\mathcal{C}_0^k$.*

Proof. Application of *Lemma 3*. □

On the above lemmas, see [3], [7] and [8].

Theorem 1. *Let $\Gamma \cup \{\varphi\}$ be a set of formulas of $\mathcal{C}_0^*$ with q proper (i.e., arity equal to or greater than 1) predicate symbols, with or without k individuals, such that $\Gamma \vdash \varphi$ in this calculus. Then, any P_1-P_2-...-P_h-k-transform of φ can be deduced from the P_1-P_2-...-P_h-k-transforms of the formulas of Γ in $\mathcal{C}_0^*$ with $q-h$ proper predicate symbols, $h \leqslant q$, and k individuals.*

Proof. By *Lemma 4*, in the (formal) deduction of any P_1-P_2-...-P_h-k-transform of φ from the P_1-P_2-...-P_h-k-transforms of the formulas of Γ, the predicate symbols to which are attached only numerals, can be treated as atomic formulas of arity zero, that is, as propositional letters. □

Theorem 2 (Hilbert and Bernays). *If $\Gamma \vdash \varphi$ in $\mathcal{C}_0^*$, then any k-transform of φ can be deduced from the k-transforms of the formulas of Γ in $\mathcal{C}_0$.*

PROOF. Let us suppose that $P_1, P_2, \dots, P_h$ are all proper predicate symbols which occur in the formulas $\Gamma \cup \{\varphi\}$. By the preceding proposition, the P_1-P_2-...-P_h-k-transforms of φ, *i.e.*, its k-transforms, can be deduced from the P_1-P_2-...-P_h-k-transforms of the formulas of Γ in the predicate calculus with zero proper predicate symbols, that is, $\mathcal{C}_0$. □

In the next two theorems, we employ the terminology of Church's book (see [5]).

THEOREM 3. *Every valid formula of the predicate calculus $\mathcal{C}_0^*$ has a thesis of $\mathcal{C}_0$ as* associated formula of the propositional calculus *(cf. [5], p. 180).*

PROOF. Immediate consequence of the preceding theorem. □

THEOREM 4. *The classical predicate calculus of first order is consistent with reference to the transformation of α into $\neg\alpha$, is absolutely consistent, and is consistent in the sense of Post (see [5], p. 108). But it is not complete with reference the transformation of α into $\neg\alpha$, is not absolutely complete, and is not complete in the sense of Post (cf. [5], p. 110).*

PROOF. Apply the above theorems with appropriate adaptations. □

THEOREM 5. *For any valid formula of the classical predicate calculus in which there are no quantifiers, there exists a proof composed only of formulas without quantifiers.*

PROOF. Corollary to the preceding results. □

THEOREM 6. *Every valid formula of the predicate calculus $\mathcal{C}_0^*$ without quantifiers is an instance of a propositional tautology.*

THEOREM 7. *If the formula φ does not contain the symbol of equality and $\vdash \varphi$ in $\mathcal{C}_0^=$, then $\vdash \varphi$ in $\mathcal{C}_0^k$.*

PROOF. When $\vdash \varphi$ in $\mathcal{C}_0^=$, then φ is a consequence, in $\mathcal{C}_0^*$, of $\forall x(x = x)$ and of a finite number of formulas of the form $x = y \to (\alpha(x) \leftrightarrow \alpha(y))$. Therefore, the =-1-transform of φ, that is φ, is a consequence of the =-1-transforms of formulas of these forms. In other words, φ would be deductible from the formula $\mathbf{1} = \mathbf{1}$, as it is easy to verify. Since in the resulting deduction the formula $\mathbf{1} = \mathbf{1}$ behaves like a predicate symbol of arity zero, we have that $\vdash \varphi$ in $\mathcal{C}_0^k$, taking into account that we may replace $\mathbf{1} = \mathbf{1}$ by a closed formula which is propositionally valid. □

Theorem 8. *Let $\Gamma \cup \{\varphi\}$ be a set of formulas of $\mathcal{C}_0^*$. Then, $\Gamma \vdash \varphi$ in $\mathcal{C}_0^k$ if and only if $\Gamma \vdash \varphi$ in $\mathcal{C}_0^=$.*

Proof. Corollary to *Theorem 7.* □

It is easy to see that our exposition remains valid when function symbols are added to the predicate calculi here studied.

Open formula and *quasi*-tautology are defined as in Shoenfield's book (*cf.* [17], p. 49).

Using the previous theorems, it is not difficult to prove the following:

Theorem 9 (Hilbert and Ackermann). *Let Γ be a set of open formulas of the classical predicate calculus, with or without, equality; then Γ is inconsistent if and only if there is a quasi-tautology which is a disjunction of negations of instances of formulas of Γ.*

As in Shoenfield's book, using *Theorem 9* we can prove, by finitary means, the consistency of Robinson's arithmetic system, as well as Herbrand's theorem. Connections of the present exposition with the two ε-theorems of Hilbert and Bernays ([14] and [16]) will be left to future papers.

Some of our results are derivable through a different notion of transform.

Let P be a predicate symbol of arity $m > 0$ and i be a number less than m. If, in the formula φ, we supress, in each occurrence of P, the argument of place i, then P is transformed into a predicate symbol of arity $m - 1$. The formula so obtained from φ is called the $(P; i)$-transform of φ (we suppose that in φ all vacuous quantifiers are eliminated). In general, we define the $(P; i_1, i_2, ..., i_n)$-transform of φ, $n \leqslant m$, and P_1, P_2, …, P_u being u predicate symbols of arities m_1, m_2, …, m_u respectively, we also define the $(P_1; i_1^1, i_2^1; ..., i_{n_1}^1)$, $(P_2; i_1^2, i_2^2; ..., i_{n_2}^2)$, …, $(P_u; i_1^u, i_2^u; ..., i_{n_h}^u)$-transform of a formula φ.

We have, for example:

Theorem 10. *$\Gamma \vdash \varphi$ in $\mathcal{C}_0^=$ implies that the $(P; i_1, i_2, ..., i_n)$-transform of φ, $n \leqslant m$, is provable in $\mathcal{C}_0^*$ from the $(P; i_1, i_2, ..., i_n)$-transforms of the formulas of Γ. If $n = m$, this deduction can be made in $\mathcal{C}_0$.*

$\mathcal{T}_0$ is the classical type theory, already defined, which possesses predicate symbols of all types and heights. And $\mathcal{T}_{0,u}$, $0 < u < \omega$, is the portion of $\mathcal{T}_0$ that contains only predicate symbols of height equal to or less than u, and quantifications associated with variables of height strictly smaller than u. $\mathcal{T}_{0,0}$ is $\mathcal{C}_0$ and $\mathcal{T}_{0,1}$ is $\mathcal{C}_0^=$.

There are various methods of defining different concepts of transforms of a given formula in $\mathcal{T}_0$ (or in $\mathcal{T}_{0,u}$). For example, supposing that to each type t there corresponds a set $\delta(t)$ of constants of this type, $\{d_1, d_2, \ldots, d_{\overline{\delta}(t)}\}$, in order to get the $\delta(t)$-transforms of φ, we proceed as follows:

1. Firstly, if the free variables of φ are $x_1, x_2, \ldots, x_n$ of types $t_1, t_2, \ldots, t_n$, we replace $x_1, x_2, \ldots, x_n$ by $c_1, c_2, \ldots, c_n$, where $c_i \in \delta(t_i)$. Every such replacement gives rise to one transform;
2. Afterwards, we replace any part of φ of the form $\forall x\alpha(x)$, x being of type t, by $\alpha(d_1)^{✠} \wedge \alpha(d_2)^{✠} \wedge \cdots \wedge \alpha(d_{\overline{\delta}(t)})^{✠}$, where $\alpha(d_i)^{✠}$ represents the transform of $\alpha(d_i)$, $1 \leqslant i \leqslant \overline{\delta}(t_i)$;
3. Finally, parts of the form $\exists x\alpha(x)$ are replaced by analogous disjunctions.

One has the following theorems:

THEOREM 11. *In $\mathcal{T}_0$ without equality: $\Gamma \vdash \varphi$ implies that any $\delta(t)$-transform of φ is derivable from the $\delta(t)$-transforms of the formulas of Γ in $\mathcal{T}_{0,0}$ (i.e., in $\mathcal{C}_0$).*

THEOREM 12. *The function $\delta(t)$ is restricted to $\mathcal{T}_{0,n}$ (without equality). Then $\Gamma \vdash \varphi$ implies that any $\delta(t)$-transform of φ can be deduced from the $\delta(t)$-transforms of the formulas of Γ in $\mathcal{T}_{0,0}$.*

With the help of an extension of the notion of P-k-transform, one can prove the proposition below:

THEOREM 13. *If $\Gamma \cup \{\varphi\}$ is a set of formulas of $\mathcal{T}_0$ (or $\mathcal{T}_{0,n}$, $n > 0$) without equality, then $\Gamma \vdash \varphi$ in $\mathcal{T}_0$ (or $\mathcal{T}_{0,n}$) with equality if only if $\Gamma \vdash \varphi$ in $\mathcal{T}_0$ (or $\mathcal{T}_{0,n}$) without equality.*

4.2. *k*-transforms in paraconsistent logic

Evidently, most of the results of the preceding section are valid for non classical logics. Thus, for example, we have the next propositions about the intuitionistic and minimal predicate calculi.

The intuitionistic predicate calculus can be characterized by the postulates $\rightarrow_{1,2,3}$, $\wedge_{1,2,3}$, $\vee_{1,2,3}$, $\neg_{1,2}$, $\forall_{1,2}$ and $\exists_{1,2}$; a postulate list for the minimal predicate calculus is composed by the postulates $\rightarrow_{1,2,3}$, $\wedge_{1,2,3}$, $\vee_{1,2,3}$, $\neg_1$, $\forall_{1,2}$ and $\exists_{1,2}$.

Theorem 14. *Let $\Gamma \cup \{\varphi\}$ be a set of formulas of the predicate calculus. Then, $\Gamma \vdash \varphi$ in the intuitionistic predicate calculus (in the minimal predicate calculus) if and only if $\Gamma \vdash \varphi$ in the intuitionistic predicate calculus with equality (in the minimal predicate calculus with equality).*

Theorem 15. *If formula φ does not contain the symbol of equality, then $\vdash \varphi$ in the intuitionistc (minimal) predicate calculus if and only if $\vdash \varphi$ in the intuitionistic (minimal) predicate calculus with equality.*

Theorem 16. *$\vdash \varphi$ in the intuitionistic (minimal) predicate calculus implies that any k-transform of φ is provable in the intuitionistic (minimal) predicate calculus with k individuals.*

Theorem 17. *Let φ be a formula in which there are no occurrences of the symbols of negation and of equality. Then $\vdash \varphi$ in the intuitionistic (minimal) predicate calculus if and only if $\vdash \varphi$ in the positive intuitionistic (minimal) predicate calculus.*

In $\mathcal{C}_n$, $1 \leqslant n \leqslant \omega$, since this calculus is paraconsistent, we have that, for example, the schemes $(\varphi \wedge \neg\varphi) \rightarrow \beta$ and $\varphi \rightarrow (\neg\varphi \rightarrow \beta)$ are not provable (*cf.* [9] and [10]). The problem, therefore, is to show that the calculi $\mathcal{C}_n^*$, $\mathcal{C}_n^=$ and $\mathcal{T}_n$, $1 \leqslant n \leqslant \omega$, have the same property. With the results of the preceding section we are able to prove that this is so, as well as many other results concerning paraconsistent logic. The remaining theorems of the present section are only stated, since their proofs are simple variations of the proofs of the corresponding classical results.

Theorem 18. *Theorem 1 is valid for $\mathcal{C}_n^*$, $1 \leqslant n \leqslant \omega$.*

Corollary 18.1. *The following schemes are not valid in $\mathcal{C}_n^*$, $1 \leqslant n \leqslant \omega$: (1) $(\varphi\wedge\neg\varphi) \rightarrow \beta$; (2) $\varphi \rightarrow (\neg\varphi \rightarrow \beta)$; (3) $\neg\varphi \rightarrow (\varphi \rightarrow \beta)$; (4) $(\varphi \leftrightarrow \neg\varphi) \rightarrow \beta$; (5) $\neg(\varphi \wedge \neg\varphi)$; (6) $(\varphi \leftrightarrow \beta) \rightarrow (\neg\varphi \leftrightarrow \neg\beta)$.*

Corollary 18.2. *The following schemes are not provable in $\mathcal{C}_n^*$, $1 \leqslant n < \omega$: (1) $\forall x\varphi(x) \leftrightarrow \neg\exists x\neg\varphi(x)$; (2) $\exists x\varphi(x) \leftrightarrow \neg\forall x\neg\varphi(x)$.*

Theorem 19. *Let $\Gamma \cup \{\varphi\}$ be a colection of formulas of $\mathcal{C}_n^*$, $1 \leqslant n \leqslant \omega$. Then $\Gamma \vdash \varphi$ in $\mathcal{C}_n^*$ if and only if $\Gamma \vdash \varphi$ in $\mathcal{C}_n^=$, $1 \leqslant n < \omega$.*

Corollary 19.1. *The schemes of the first corollary to Theorem 18 are not valid in $\mathcal{C}_n^=$, $1 \leqslant n \leqslant \omega$.*

THEOREM 20. *If $\Gamma \vdash \varphi$ in $\mathcal{C}_1^*$, then all of the P-k-transforms of φ are deducible from the P-k-transforms of the formulas of Γ in $\mathcal{C}_1$.*

COROLLARY 20.1. *If $\vdash \varphi$ in $\mathcal{C}_1^*$, then the P-k-transforms of φ are theorems of $\mathcal{C}_1$.*

COROLLARY 20.2. *If $\Gamma \vdash \varphi$ in $\mathcal{C}_n^*$, $0 \leqslant n < \omega$, then all of the P-k-transforms of φ are deducible in $\mathcal{C}_n^*$ from the P-k-transforms of the formulas of Γ.*

We now consider the systems $\mathcal{T}_n$, $1 \leqslant n \leqslant \omega$.

THEOREM 21. *The schemes of the first corollary to Theorem 18 are not provable in $\mathcal{T}_n$ (or in $\mathcal{T}_{n,u}$), $1 \leqslant n \leqslant \omega$ and $u \leqslant n$.*

THEOREM 22. *$\mathcal{T}_n$ is a conservative extension of $\mathcal{C}_n^*$, $1 \leqslant n \leqslant \omega$, and $\mathcal{C}_n^=$, $0 \leqslant n \leqslant \omega$.*

THEOREM 23. *$\mathcal{T}_{n,u}$, $1 \leqslant n \leqslant \omega$ and $u \leqslant n$, is a conservative extension of $\mathcal{T}_{n,j}$, where $j < u$, and $n = 1, 2, \ldots, \omega$.*

THEOREM 24. *If the set $\Gamma \cup \{\varphi\}$ contains only formulas without equality, then $\Gamma \vdash \varphi$ in $\mathcal{T}_n$ with equality if and only if $\Gamma \vdash \varphi$ in $\mathcal{T}_n$ without equality, $1 \leqslant n \leqslant \omega$.*

It is possible to adapt the above notions and results in order to be applied to sequent versions of and algebraic approaches to most logical systems.

It is worthwhile to note that all systems investigated in this paper have appropriate semantics of valuations relative to which they are sound and complete. In general, our results can also be proved by semantical devices, though these devices are not finitary (on the theory of valuation see, for example, [13]).

4.3. *k*-transforms in other logics

Using the preceding methods we are able to prove numerous results connected with some non classical logics, other than paraconsistent logics. For instance, we have:

THEOREM 25. *Classical and intuitionistic implicative logics are not finitely trivializable.*

PROOF. Classical implicative propositional logic, based on postulates $\to_{1,2,3}$ plus Peirce's law $((\varphi \to \beta) \to \varphi) \to \varphi$, is characterized by the standard truth table of implication. No pure implicative formula can assume only the value false; therefore, if φ is a formula that trivializes implicative logic, $\varphi \to \beta$, where β is a propositional variable that does not occur in φ, is not a tautology.

Since intuitionistic implicative propositional logic (postulates $\to_{1,2,3}$) is part of classical implicative logic, it is not finitely trivializable. □

By the means of k-transforms, we extend the result to the quantificational level (adding the postulates for quantification) .

THEOREM 26. *Classical and intuitionistic positive logics are not finitely trivializable.*

PROOF. Adjoining to classical and intuitionistic implicative logics the postulates of conjunction and of disjunction, we obtain the classical and the intuitionistic positive logics respectively. Clearly the proof of the preceding theorem can be adapted to the case of positive logics. □

THEOREM 27. *If we add Peirce's law to $\mathcal{C}_\omega$, then the resulting calculus is not finitely trivializable.*

COROLLARY 27.1. *$\mathcal{C}_\omega^*$ and $\mathcal{C}_\omega^=$ are not finitely trivializable.*

THEOREM 28. *The scheme $\beta \leftrightarrow (\beta \to \alpha)$, where β is a propositional variable and α is any formula whatever, joined to positive intuitionistic logic, makes this logic trivial.*

PROOF. In effect, we have from $\beta \leftrightarrow (\beta \to \alpha)$:

1) $\beta \to (\beta \to \alpha)$;

2) And $(\beta \to \alpha) \to \beta$;

3) From 1, it follows that $\beta \to \alpha$;

4) On the other hand, 2 and 3 imply that β;

5) Finally, from 3 and 4, we get α.

So, trivialization, since α is any formula. □

THEOREM 29. *The scheme of separation of usual set theory, without any restriction, trivializes, in an obvious sense, positive intuitionistic logic (with quantification).*

PROOF. The language of set theory contains the logical symbols of positive intuitionistic logic (with quantification). The scheme of separation is the following:

$$\exists y \forall x (x \in y \leftrightarrow \varphi(x)).$$

Therefore, replacing $\varphi(x)$ by $x \in x \to \alpha$, where α is any formula, we have that

$$\exists y \forall x (x \in y \leftrightarrow (x \in x \to \alpha))$$

and, in consequence,

$$\forall x (x \in y \leftrightarrow (x \in x \to \alpha));$$

and

$$y \in y \leftrightarrow (y \in y \to \alpha)).$$

However, this scheme, by the previous theorem, makes the system trivial. □

Remark. The preceding theorem is related to Curry's paradox (see [4] and [12]).

THEOREM 30. *Positive intuitionistic and positive classical logics with a finite number of predicate symbols are finitely trivializable. The same is true of the corresponding propositional calculi with only a finite number of propositional variables.*

PROOF. In the propositional case, if $\beta_1, \beta_2, \ldots, \beta_n$ are the propositional variables, then $\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_n$ trivializables positive logic (In addition, it is easy to define in these logics a negation, which, in classical positive logic, has all the properties of classical negation). □

So, if we are interested in a paraconsistent system of logic, compatible with the scheme of separation without the usual restrictions, one way is to weaken positive (quantificational) logic, as we shall show in a forthcoming paper.

Acknowledgments. The authors would like to thank professor Marcel Guillaume and João Marcos for their remarks and criticisms on an earlier draft of this paper.

References

[1] Arruda, A. I. and da Costa, N. C. A. "Sur un théorème de Hilbert et Bernays", *Comptes Rendus de l'Académie de Sciences de Paris*, série A, 258, 1964, pp. 6311–6312.

[2] Arruda, A. I. and da Costa, N. C. A. "Sur une hiérarchie de systèmes formels", *Comptes Rendus de l'Académie de Sciences de Paris*, série A, 259, 1964, pp. 2943–2945.

[3] Arruda, A. I. and da Costa, N. C. A. "Transformadas no cálculo restrito de predicados", *Anais da Academia Brasileira de Ciências*, 38, 1966, pp. 385–390.

[4] Arruda, A. I. and da Costa, N. C. A. "O paradoxo de Curry-Moh Shaw Kwei", *Boletim da Sociedade Matemática de São Paulo*, 18, 1966, pp. 83–89.

[5] Church, A. *Introduction to Mathematical Logic*. Princeton University, 1956.

[6] da Costa, N. C. A. "Calculus de prédicats pour les systèmes formels inconsistants", *Comptes Rendus de l'Académie de Sciences de Paris*, série A, 258, 1964, pp. 27–29.

[7] da Costa, N. C. A. "Calculus de prédicats avec égalité pour les systèmes formels inconsistants", *Comptes Rendus de l'Académie de Sciences de Paris*, série A, 258, 1964, pp. 1111–1113.

[8] da Costa, N. C. A. "Sobre o conceito de transformada no cálculo de predicados", *Notas do IME*, Universidade de São Paulo, 1973, pp. 53–59.

[9] da Costa, N. C. A. "On the theory of inconsistent formal systems", *Notre Dame Journal of Formal Logic*, 15, 1974, pp. 497–510.

[10] da Costa, N. C. A. *Sistemas Formais Inconsistentes*. Curitiba, Universidade Federal do Paraná, second edition, 1993 (first edition 1963).

[11] da Costa, N. C. A. *Logiques Classiques et non Classiques*. Masson, 1997.

[12] Curry, H. B. "The inconsistency of certain formal logics", *Journal of Symbolic Logic*, 7, 1942, pp. 115–117.

[13] Grana, N. *Sulla Teoria delle Valutazioni di N. C. A. da Costa*. Ligouri, 1990.

[14] Hilbert, D. and Bernays, P. *Grundlagen der Mathematik*. Springer Verlag, vol. 1, 1934 and vol. 2, 1939.

[15] Kleene, S. C. *Introduction to Metamathematics*. Van Nostrand, 1952.

[16] Leisering, A. C. *Mathematical Logic and Hilbert ε-symbol*. MacDonald, 1969.

[17] Shoenfield, J. *Mathematical Logic*. Addison Wesley, 1967.

Newton C. A. da Costa
Department of Philosophy
University of São Paulo
São Paulo – SP 05508-900, Brazil
ncacosta@usp.br

Roque da C. Caiero
Rua José Albinatti, 54
São Paulo – SP 02751-090, Brazil
rccaiero@usp.br