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THE ALGEBRAIC FACE OF MINIMALITY

Abstract. Operators which map subsets of a given set to the set of their minimal elements with respect to some relation R form the basis of a semantic approach in non-monotonic logic, belief revision, conditional logic and updating. In this paper we investigate operators of this type from an algebraic viewpoint. A representation theorem is proved and various properties of the resulting algebras are investigated. It is shown that they behave quite differently from known algebras related to logics, e.g. modal algebras and Heyting algebras.

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In "Five Faces of Minimality" D. Makinson [12] has surveyed the use of operators $\mathfrak{m}_R$ which — when applied to a set X — form the set of minimal elements $\mathfrak{m}_R X$ in X with respect to some relation R in non-monotonic logic, belief revision, conditional logic, updating, and conditional deontic logic. By now it is generally accepted that operators of this type form the basis of a semantic approach in all those areas.

In this paper we shall abstract from specific applications and investigate the behaviour of minimality operators from an algebraic perspective. Switching from the intended semantics for a language (e.g., models for first order logics, Kripke frames for modal logics) to algebras is a well understood and rather useful move, see e.g. Henkin et al. [11], Blok [1], and Goldblatt [7]. It enables us to use techniques from universal algebra to solve problems formulated in terms of the underlying logic. There is, however, at least one more motivation to investigate algebras induced by minimality operators: non-monotonic logic as well as logics involving conditionals are known to behave quite differently from standard logics like classical propositional logic, intuitionistic logic, or modal logics. This difference should be reflected (and at least partially explained) by means of the algebraic properties of the algebras induced by the minimality operator. For example, members of varieties (equationally definable classes) of algebras related to logics mostly have rather well behaved congruences (e.g. equationally definable principal congruences or at least first order definable congruences, see e.g. Blok and Pigozzi [2] and [3]). Properties of the congruences often reflect interesting properties of the associated logic. If it is true that there is an essential difference between non-monotonic and monotonic logics, then we should expect congruences to show some unusual features which do not appear in, say, Boolean algebras, Heyting algebras, or Modal algebras.

In order to explain the objective of our investigation more precisely, some notation is required. Form for any relational structure $\mathcal{F} = \langle W, R\rangle$ the boolean algebra with an operator $\mathcal{F}^+ = \langle 2^W, \cap, -, \mathfrak{m}_R, \emptyset, W\rangle$, where

$$\mathfrak{m}_R X = \{y \in X : (\forall x \in W)(yRx \Rightarrow x \notin X)\}.$$

Interpreting the elements of W as worlds and the relation R as a preference relation (or normality relation) between worlds, then — following the basic idea of Kraus, Lehmann and Magidor [10] — a defeasible inference relation $X \vdash Y$ between propositions X and Y holds iff Y is true in every world that is minimally abnormal among those satisfying X. That is to say, $X \vdash Y$

iff $\mathfrak{m}_R X \subseteq Y$. We have a reduction of $\vdash$ to the minimality operator $\mathfrak{m}_R$. The situation is a bit more complex in the other areas mentioned above (cf. [12]), but still the minimality operator is the basic operation to which the inference relation is reduced.

Above we moved from the relational structure $\mathcal{F}$ to the algebra $\mathcal{F}^+$. By omitting also the reference to the set W we obtain the variety $\mathscr{M}$ of algebras $\mathfrak{A} = \langle A, \wedge, \neg, f, 0, 1\rangle$ generated by the algebras of the form $\mathcal{F}^+$. $\mathscr{M}$ will turn out to coincide with the class of representable algebras; that is to say, $\mathfrak{A} \in \mathscr{M}$ iff $\mathfrak{A}$ is isomorphic to a subalgebra of an algebra of the form $\langle W, R\rangle^+$.[1] The members of $\mathscr{M}$ are called *min-algebras*.

In this paper we are going to address the following problems:

- Axiomatize the variety of min-algebras. That is to say, characterize in algebraic terms the algebras for which f can be interpreted as a minimality operator.
- Axiomatize the varieties generated by interesting classes of relational structures, e.g., transitive structures, linear structures, and noetherian structures.
- Which properties of the relational structure $\langle W, R\rangle$ can be described by means of algebraic properties of $\langle W, R\rangle^+$?
- Develop duality theory for min-algebras and relational structures.
- Investigate the min-algebras from an algebraic point of view. Here we shall consider only the congruences of min-algebras. They turn out to be not first order definable and behave differently from known varieties related to logics.
- Finally we briefly study splittings of lattices of subvarieties of the variety of min-algebras. This concept enables us to give rather intuitive axiomatizations of various varieties of min-algebras.

We close the introduction with a remark about the relation between modal algebras and min-algebras. Min-algebras are ordinary Boolean algebras with an operator. However, from this class of algebras only those with an operator f validating the equation $fx \wedge fy = f(x \wedge y)$ have been investigated intensively, see e.g. [7] and [4]. This equality does not hold for min-algebras. Moreover, the operator f in min-algebras is not monotonic (i.e., we do not

[1] In this paper we consider *boolean algebras* with the minimality operator. Of course, it would be of interest to consider algebras with less structure, e.g., distributive lattice or Heyting algebras with the minimality operator. We decided to take Boolean algebras because we should like to concentrate on the minimality operator and therefore want the simplest underlying algebra for the remaining operations.

have $x \leq y \Rightarrow fx \leq fy$) and it is this property which enables us to model non-monotonic reasoning: the inference relation $\vdash$ defined by putting $x \vdash y$ iff $fx \leq y$ is non-monotonic iff f is not monotonic.

To keep the paper reasonably short we assume basic knowledge of algebraic notions and duality between boolean algebras and Stone spaces (or modal algebras and descriptive frames) see e.g. [8] and [7].

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1. Axiomatization

We are first going to axiomatize the variety of min-algebras and show that it coincides with the class of representable algebras. To this end we require the following set of equations Ax:

(sub) $fx \leq x$,

(dis) $f(x \vee y) \leq f(x) \vee f(y)$ and $fx \wedge fy \leq f(x \vee y)$,

(ex) $y \wedge f(x \vee y) \leq fy$.

We shall prove that Ax (together with a set of equations axiomatizing the variety of Boolean algebras) axiomatizes the variety $\mathscr{M}$. Observe first the easily proved

Proposition 1.1. (Soundness) *For all relational structures $\langle W, R\rangle$ the algebra $\langle W, R\rangle^+$ validates all equations in* Ax.

Lemma 1.2. *If $\mathfrak{A} = \langle A, \wedge, \neg, f, 0, 1\rangle$ is a boolean algebra with an operator which validates all equations in* Ax*, then the following holds for all a, b_1, $\dots$, b_n in A:*

$$fb_1 \wedge \dots \wedge fb_n \wedge a \wedge \neg fa > 0 \text{ implies } \neg b_1 \wedge \dots \wedge \neg b_n \wedge a > 0.$$

Proof. Suppose that $\neg b_1 \wedge \dots \wedge \neg b_n \wedge a = 0$. Then $a \vee b_1 \vee \dots \vee b_n = b_1 \vee \dots \vee b_n$. Now we derive

$$\begin{aligned} fb_1 \wedge \dots \wedge fb_n \wedge a \wedge \neg fa &\leq f(b_1 \vee \dots \vee b_n) \wedge a \wedge \neg fa \\ &= f(b_1 \vee \dots \vee b_n \vee a) \wedge a \wedge \neg fa \\ &\leq fa \wedge \neg fa \\ &= 0 \end{aligned}$$ □

To prove completeness we extend the notion of a relational structure to the notion of generalized min-structures. This will also be useful in the section on duality.

A generalized min-structure is a tuple $\mathcal{G} = \langle W, R, \mathcal{P}\rangle$ such that $\langle W, R\rangle$ is a structure and $\mathcal{P}$ is a set of subsets of W containing W and closed under intersection, complement, and the operation $\mathfrak{m}_R$. It follows from Proposition 1.1 that the algebra $\mathcal{G}^+ = \langle \mathcal{P}, \cap, -, \mathfrak{m}_R, \emptyset, W\rangle$ is a min-algebra whenever $\mathcal{G}$ is a generalized min-structure. A structure $\langle W, R\rangle$ is identified with the generalized min-structure $\langle W, R, 2^W\rangle$.

Conversely, define for an algebra $\mathfrak{A}$ validating all equations in Ax the structure $\mathfrak{A}_{\max} = \langle W, R, \mathcal{P}\rangle$ as follows:

- W is the set of all ultrafilters in the boolean reduct of $\mathfrak{A}$.
- uRv iff $(\forall a \in A)(fa \in u \Rightarrow a \notin v)$.
- $\mathcal{P} = \{\beta(a) : a \in A\}$, where $\beta(a) = \{u \in W : a \in u\}$.

Consider now an ultrafilter $u \in W$ such that $\neg fa \in u$, for all $a \in A$. Then uRv for all $v \in W$. This follows from the definition of R. We have $fx \leq x$, for all $x \in A$, and so the following conditions are equivalent (in $\mathfrak{A}_{\max}$) for any $u \in W$:

- uRu,
- $\neg fa \in u$, for all $a \in A$
- uRv, for all $v \in W$.

Sometimes it will turn out to be useful to omit some of the arrows starting at a reflexive u. Define $\mathfrak{A}_{\min} = \langle W, S, \mathcal{P}\rangle$ in such way that W and $\mathcal{P}$ are defined as before but uSv iff uRv & not uRu or $u = v$ and uRv. Certainly $S \subseteq R$ and $\mathfrak{A}_{\min} = \mathfrak{A}_{\max}$ whenever there is no reflexive point in $\mathfrak{A}_{\max}$.

A *dual min-structur* is any $\langle W, R', \mathcal{P}\rangle$ such that $S \subseteq R' \subseteq R$. Observe

$$\mathfrak{m}_R X = \mathfrak{m}_S X = \mathfrak{m}_{R'} X,$$

for any R' with $S \subseteq R' \subseteq R$ and any $X \subseteq W$. In conclusion there exists a dual min-structure of $\mathfrak{A}$ which is a generalized min-structure iff all dual min-structures of $\mathfrak{A}$ are generalized min-structures. This turns out to be the case:

Theorem 1.3. *For any algebra $\mathfrak{A}$ validating all equations in Ax any dual min-structure $\mathfrak{A}_+ = \langle W, R', \mathcal{P}\rangle$ of $\mathfrak{A}$ is a generalized min-structure and the mapping $\beta\colon \mathfrak{A} \to (\mathfrak{A}_+)^+$ is an isomorphism.*

Proof. Most parts of the proof are standard, see e.g. Goldblatt [7]. So we only show $\beta(fa) = \mathfrak{m}_{R'}\beta(a)$, for all $a \in A$ and leave the rest to the reader. Since $\mathfrak{m}_R X = \mathfrak{m}_{R'} X$ whenever R is the relation in $\mathfrak{A}_{\max}$ and $X \subseteq W$, it suffices to show

$$\beta(fa) = \mathfrak{m}_R\beta(a), \text{ for all } a \in A.$$

Let $a \in A$. We have

$$\begin{aligned} u \in \beta(fa) &\Rightarrow fa \in u \\ &\Rightarrow a \in u \text{ and } (\forall v)(uRv \Rightarrow a \notin v) \\ &\Rightarrow u \in \beta(a) \text{ and } (\forall v)(uRv \Rightarrow v \notin \beta(a)) \\ &\Rightarrow u \in \mathfrak{m}_R\beta(a). \end{aligned}$$

For the converse direction assume $u \notin \beta(fa)$. Then $fa \notin u$. If $a \notin u$, then $u \notin \mathfrak{m}_R\beta(a)$. Assume $a \in u$. We show that there exists $v \in W$ with $a \in v$ and uRv. To this end we prove the finite meet property of

$$F = \{\neg b : fb \in u\} \cup \{a\}.$$

But suppose there are b_i with $fb_i \in u$ $(i = 1, \dots, n)$ such that $\neg b_1 \wedge \dots \wedge \neg b_n \wedge a = 0$. Then, by Lemma 1.2, $fb_1 \wedge \dots \wedge fb_n \wedge a \wedge \neg fa = 0$ which is impossible. Any ultrafilter v containing F is as required. □

Theorem 1.4. (Completeness) *Let* $\mathfrak{A} = \langle A, \wedge, \neg, f, 0, 1\rangle$ *be a boolean algebra with an operator* f*. The following conditions are equivalent:*
(1) $\mathfrak{A}$ *validates all equations in* Ax*;*
(2) *there exists* $\langle W, R\rangle$ *such that* $\mathfrak{A}$ *is a subalgebra of* $\langle W, R\rangle^+$*;*
(3) $\mathfrak{A}$ *is a min-algebra.*

Proof. (1) implies (2). Suppose that $\mathfrak{A} \models Ax$. Let $\mathfrak{A}_{\min} = \langle W, R, \mathcal{P}\rangle$. Then, by the previous theorem, $\langle W, R, \mathcal{P}\rangle^+$ is isomorphic to $\mathfrak{A}$ and so $\mathfrak{A}$ is isomorphic to a subalgebra of $\langle W, R\rangle^+$.

(2) implies (3) is trivial.

(3) implies (1) follows from Proposition 1.1. □

For a class of algebras $\mathscr{A}$ we denote by $\mathrm{V}(\mathscr{A})$ the variety generated by $\mathscr{A}$. Put $\mathrm{V}(\mathfrak{A}) := \mathrm{V}(\{\mathfrak{A}\})$, for any algebra $\mathfrak{A}$. For a class of relational structures $\mathscr{R}$ we denote by $\mathrm{V}(\mathscr{R})$ the subvariety of $\mathscr{M}$ generated by $\{\mathcal{F}^+ : \mathcal{F} \in \mathscr{R}\}$. One of the most interesting subvarieties of $\mathscr{M}$ is of course the variety generated by the class of transitive relational structures. Denote this variety by $\mathscr{TR}$. The class of transitive and linear structures is denoted by $\mathscr{L}$ and the class of transitive and noetherian structures (i.e., structures without infinite strictly ascending chains) is denoted by $\mathscr{N}$.

In what follows we require some notation for *valuations* in an algebra. A valuation γ in an algebra $\mathfrak{A}$ is a homomorphism from the algebra of all terms (over the signature $\wedge$, $\neg$, f, 0, 1) into the algebra $\mathfrak{A}$. Let $\mathcal{G} = \langle W, R, \mathcal{P}\rangle$ be a generalized min-structure. A mapping from the algebra of terms into $\mathcal{P}$ is called a valuation in $\mathcal{G}$ iff it is a valuation in $\mathcal{G}^+$. We axiomatize the variety $\mathscr{TR}$.

Theorem 1.5. *For any min-algebra* $\mathfrak{A}$ *the following conditions are equivalent:*
(1) $\mathfrak{A} \models \varphi$, *where* $\varphi = fx \leq \neg((\chi \wedge \neg f\chi) \wedge \neg f(\chi \wedge \neg f\chi))$ *and* $\chi = fy \vee x$;
(2) $\mathfrak{A} \in \mathscr{TR}$.

Proof. We leave it to the reader to check that φ is valid in all duals of transitive structures. Conversely, we show that $\mathfrak{A}_{\mathrm{min}}$ is transitive whenever $\mathfrak{A} \models \varphi$. Assume that $\mathfrak{A}_{\mathrm{min}}$ is not transitive. We find ultrafilters u_1, u_2, and u_3 such that $u_1 R u_2 R u_3$ but $\neg(u_1 R u_3)$. Notice that we took $\mathfrak{A}_{\mathrm{min}}$ and therefore u_1 and u_2 are irreflexive. This means that we find $a, b \in A$ with $fa \in u_1$ but $a \in u_3$ and $fb \in u_2$. Define a valuation γ in $\mathfrak{A}$ by putting $\gamma(x) = a$ and $\gamma(y) = b$. We show

$$\gamma(fx \wedge ((\chi \wedge \neg f\chi) \wedge \neg f(\chi \wedge \neg f\chi))) \in u_1.$$

Clearly $fa \in u_1$, $fb \vee a \in u_1$, and $\neg f(fb \vee a) \in u_1$ since $fb \vee a \in u_2$. It remains to show

$$\neg f((fb \vee a) \wedge \neg f(fb \vee a)) \in u_1.$$

To this end it suffices to show

$$(fb \vee a) \wedge \neg f(fb \vee a) \in u_2.$$

Clearly $fb \vee a \in u_2$. Moreover, $fb \vee a \in u_3$ and so $\neg f(fb \vee a) \in u_2$. □

In the proof above it is essential to take $\mathfrak{A}_{\mathrm{min}}$ since, for example, $\mathfrak{A}_{\mathrm{max}}$ is mostly not transitive. We discuss this in more detail and thereby give a partial answer to the question which properties of $\langle W, R\rangle$ can be characterized by means of algebraic properties of $\langle W, R\rangle^+$. For a generalized min-structure $\mathcal{G} = \langle W, R, \mathcal{P}\rangle$ we can always form

$$\mathcal{G}_{\mathrm{max}} = \langle W, R_{\mathrm{max}}, \mathcal{P}\rangle,$$

where $R_{\mathrm{max}} = R \cup \{\langle u, v\rangle \in W \times W : uRu\}$, and

$$\mathcal{G}_{\mathrm{min}} = \langle W, R_{\mathrm{min}}, \mathcal{P}\rangle,$$

where $R_{\mathrm{min}} = \{\langle u, u\rangle \in W \times W : uRu\} \cup \{\langle u, v\rangle \in W \times W : uRv \;\&\; \text{not } uRu\}$. Both $\mathcal{G}_{\mathrm{min}}$ as well as $\mathcal{G}_{\mathrm{max}}$ are generalized min-structures and

$$\mathcal{G}^+ = \mathcal{G}^+_{\mathrm{max}} = \mathcal{G}^+_{\mathrm{min}}.$$

In other words, the algebraic language is not expressive enough to feel whether arrows start from a reflexive point or not. It follows, for example, that various natural classes of relational structures $\langle W, R\rangle$ — like the class of transitive structures and the class of linear structures — cannot be characterized by means of algebraic conditions for $\langle W, R\rangle^+$.

2. Duality

In this section we shall develop some pieces of duality theory for generalized min-structures and min-algebras. We are mainly interested in the relational duals of algebraic homomorphisms. Given their characterization standard duality theory between, say, modal algebras and Kripke-frames (see Goldblatt [7]) is easily translated into duality results between min-algebras and generalized min-structures. We leave this to the interested reader. First we characterize the relational dual of subalgebras.

Definition 2.1. *Suppose that $\mathcal{G} = \langle W, R, \mathcal{P}\rangle$ and $\mathcal{F} = \langle V, S, \mathcal{Q}\rangle$ are generalized min-structures. A mapping g from W onto V is a p-morphism iff*

$$\begin{aligned} uR_{\min}v &\Rightarrow g(u)S_{\max}g(v), \\ g(u)S_{\min}v &\Rightarrow \exists w\ uR_{\max}w\ \&\ g(w) = v, \\ X \in \mathcal{Q} &\Rightarrow g^{-1}(X) \in \mathcal{P}. \end{aligned}$$

Modulo the difference between $R_{\min}$ and $R_{\max}$ this is the usual definition of p-morphisms for Kripke-frames. It follows that again the main difference between the modal language and the minimality operator is based upon reflexive points.

Theorem 2.2. (1) *If a min-algebra $\mathfrak{B}$ is a subalgebra of a min-algebra $\mathfrak{A}$ then $g: \mathfrak{A}_+ \to \mathfrak{B}_+$ defined by*

$$g(u) = u \cap B, \text{ for all ultrafilters } u \text{ in } \mathfrak{A},$$

is a p-morphism from $\mathfrak{A}_+$ onto $\mathfrak{B}_+$.

(2) *If g is a p-morphism from $\mathcal{G}$ onto $\mathcal{F}$, then the mapping $g^+: \mathcal{F}^+ \to \mathcal{G}^+$ defined by*

$$g^+(X) = g^{-1}(X), \text{ for all } X \in \mathcal{F}^+,$$

is an embedding of $\mathcal{F}^+$ into $\mathcal{G}^+$.

Proof. (1) Assume that $\mathfrak{A}_+ = \langle W, R, \mathcal{P}\rangle$ and $\mathfrak{B}_+ = \langle V, S, \mathcal{Q}\rangle$. Suppose $uR_{\min}v$. We show $g(u)S_{\max}g(v)$. To this end assume $g(u) \in \mathfrak{m}_S\beta a$. Then $g(u) \in \beta f a$ and so $u \in \beta f a$. Hence $fa \in u$ and so $u \in \mathfrak{m}_R\beta a$ and $v \notin \beta a$. This holds for all $a \in B$ and so $g(u)S_{\max}g(v)$.

Now let $g(u)S_{\min}v$. To construct w s.t. $uR_{\max}w$ & $g(w) = v$, it suffices to show that the set

$$\{\neg a : fa \in u\} \cup (v \cap B)$$

has the finite meet property. Suppose otherwise. Then

$$\neg a_1 \wedge \ldots \wedge \neg a_n \wedge b = 0$$

for some $a_i \in u$ and $b \in v$. Since $g(u)Sv$, we have $\neg fb \in u$.

Case 1: $b \in u$. Then $fa_1 \wedge \ldots \wedge fa_n \wedge b \wedge \neg fb \neq 0$ and we arrive at a contradiction with Lemma 1.2.

Case 2: $b \notin u$. Then $g(u) \neq v$. Since $g(u)S_{\min}v$, $g(u)$ is irreflexive. So there is c such that $fc \in u \cap B$. Then

$$fc \wedge fa_1 \wedge \ldots \wedge fa_n \wedge (c \vee b) \wedge \neg f(c \vee b) \in u.$$

So by Lemma 1.2 above,

$$\neg c \wedge \neg a_1 \wedge \ldots \wedge \neg a_n \wedge (c \vee b) > 0,$$

in contrast to our assumption.

(2) is easy and left to the reader. □

Consider now the duals of homomorphisms.

Definition 2.3. *Let $\mathcal{F} = \langle W, R, \mathcal{P}\rangle$ be a generalized min-structure. Let $V \subseteq W$ be $R_{\min}$-closed, i.e. $v \in V$ whenever $u \in V$, $v \in W$ and $uR_{\min}v$. Then*

$$\mathcal{G} = \langle V, R \cap (V \times V), \{X \cap V : X \in \mathcal{P}\}\rangle$$

is a generalized min-structure as well and we call it a generated subframe of $\mathcal{F}$.

Theorem 2.4. (1) *If $\mathcal{G} = \langle V, S, \mathcal{Q}\rangle$ is a generated subframe of $\mathcal{F} = \langle W, R, \mathcal{P}\rangle$, then the mapping g defined by*

$$g(X) = X \cap V, \text{ for } X \in \mathcal{P},$$

is a homomorphism from $\mathcal{F}^+$ onto $\mathcal{G}^+$.

(2) *If g is a homomorphism from a min-algebra $\mathfrak{A}$ onto a min-algebra $\mathfrak{B}$, then g_+ defined by*

$$g_+(u) = g^{-1}(u), \ u \text{ an ultrafilter in } \mathfrak{B},$$

is an isomorphism from $\mathfrak{B}_{\min}$ onto a generated subframe of $\mathfrak{A}_{\min}$.

Proof. The proof of (1) is easy and left to the reader. (2) Assume that $\mathfrak{A}_{\min} = \langle W, R_{\min}, \mathcal{P}\rangle$ and $\mathfrak{B}_{\min} = \langle V, S_{\min}, \mathcal{Q}\rangle$. Now let V' denote the set of ultrafilters u in W such that $g^{-1}(1) = \{b \in A : g(b) = 1\} \subseteq u$. The claim is shown if (a) V' is $R_{\min}$-closed, i.e., if $g^{-1}(1) \subseteq u$ and $uR_{\min}v$, then $g^{-1}(1) \subseteq v$. (b) g^{+} is an isomorphism from $\mathfrak{B}_{\min}$ onto the generated subframe of $\mathfrak{A}_{\min}$ induced by V'. (b) is proved in a straightforward manner. (a) Suppose $g^{-1}(1) \subseteq u$ and $uR_{\min}v$. The claim is trivial for $u = v$. To prove the claim for $u \neq v$ observe that we find — because $uR_{\min}v$ — an $a \in A$ with $fa \in u$. Let $g(b) = 1$. We are going to show $b \in v$. We have $g(f(a \vee \neg b)) = g(fa)$. So $g(fa \wedge \neg f(a \vee \neg b)) = 0$. Since $fa \in u$, we have $\neg f(a \vee \neg b) \notin u$. Thus $f(a \vee \neg b) \in u$ and so $b \in v$. □

3. Congruences

In this section we characterize the congruences in min-algebras and prove some basic properties. In Boolean algebras we have a one-one correspondence between ideals and congruences, see e.g. [8]. It does not come as a surprise that the congruences of a min-algebra correspond to ideals of the boolean reducts which satisfy one more condition.

Definition 3.1. *Let $\mathfrak{A}$ be a min-algebra. A subset I of A is called a min-ideal iff the following holds:*

- $0 \in I$,
- *if* $a, b \in I$, *then* $a \vee b \in I$,
- *if* $a \in I$ *and* $b \leq a$, *then* $b \in I$,
- *if* $a \in I$ *and* $b \in A$, *then* $fb \wedge \neg f(b \vee a) \in I$.

In what follows we sometime write $a - b$ for $a \wedge \neg b$.

Theorem 3.2. *Let $\mathfrak{A}$ be a min-algebra. Let σ be a mapping from the lattice of congruences of $\mathfrak{A}$ to the lattice of min-ideals defined by*

$$\sigma\Theta = \{a : (a, 0) \in \Theta\}.$$

Then σ is a surjective isomorphism. The inverse mapping is given by

$$\sigma^{-1}(I) = \{(a, b) : (a - b) \vee (b - a) \in I\}.$$

Proof. Clearly $\sigma\Theta$ is a min-ideal whenever Θ is a congruence. Conversely, suppose that I is a min-ideal. We show $\sigma^{-1}(I)$ is a congruence. Modulo some boolean considerations the essential step is to prove for all $a, b \in A$ that $(a, b) \in \sigma^{-1}(I)$ implies $(fa, fb) \in \sigma^{-1}(I)$. To this end assume $(a, b) \in \sigma^{-1}(I)$. It suffices to show $fa - fb \in I$. But

$$fa - fb \leq (fa \wedge \neg f(a \vee (b - a)) \vee f(a - b)$$

can be checked easily using duality. We have $f(a - b) \in I$ since $a - b \in I$. Also $fa \wedge \neg f(a \vee (b - a)) \in I$ since $b - a \in I$ and I is a min-ideal. □

This characterization of congruences by means of min-ideals is rather convenient. However, congruences in min-algebras do not behave as well as congruences in standard varieties corresponding to logics. The main reason is that principal ones are not first order definable even for the variety $\mathscr{TR}$. For a min-algebra $\mathfrak{A}$ and an element b of $\mathfrak{A}$ we denote by $\langle b \rangle$ the min-ideal generated by b. (This corresponds to the congruence generated by $(b, 0)$). We write $\mathfrak{A} \models a \in \langle b \rangle$ if $a \in \langle b \rangle$.

Theorem 3.3. *There does not exist a first order formula $\chi(x, y)$ such that for all $\mathfrak{A} \in \mathscr{TR}$ and all $a, b \in A$*

$$\mathfrak{A} \models \chi(a, b) \Leftrightarrow \mathfrak{A} \models a \in \langle b \rangle.$$

Proof. Assume that there exists a formula $\chi(x, y)$ which defines principal congruences. Let

$$\mathfrak{A}_n = \langle \{0, 1, \ldots, n\}, < \rangle^+,$$

for $n \in \omega$. Then $\mathfrak{A}_n \models 1 \in \langle f1 \rangle$, as is easily checked. So $\mathfrak{A}_n \models \chi(1, f1)$, for $n \in \omega$. Take a non-principal ultrafilter U in 2^ω and form the ultraproduct

$$\mathfrak{A} = \prod_U \langle \mathfrak{A}_n : n \in \omega \rangle.$$

We have $\mathfrak{A} \models \chi(1, f1)$. But notice that the set

$$\{c \leq \bigvee \langle fa : a \in X \rangle : X \subseteq A \text{ finite } \}$$

is a min-ideal in $\mathfrak{A}$ containing $f1$ which does not contain 1. This can be checked by using well-known properties of ultraproducts, see e.g. [5]. Thus $\mathfrak{A} \not\models 1 \in \langle f1 \rangle$ and we have a contradiction. □

We actually proved a stronger result. The algebras $\mathfrak{A}_n$ are in $\mathrm{V}(\mathscr{N} \cap \mathscr{L})$. Thus even for this variety the principal congruences are not first order definable.

The sequence $\mathfrak{A}_n$ is of interest also for another reason: notice firstly that it is easy to show that

$$\mathrm{V}(\{\mathfrak{A}_n : n \in \omega\}) = \mathrm{V}(\mathscr{L} \cap \mathscr{N}).$$

Define $\langle W, R, \mathcal{P}\rangle$ by putting:

- $W = \omega \cup \{\infty\}$.
- uRv if $u, v \in \omega$ and $u > v$ or $u = \infty$.
- $X \in \mathcal{P}$ iff X is finite and $\infty \notin X$ or X is cofinite and $\infty \in X$.

We have $\langle W, R, \mathcal{P}\rangle^+ \in \mathrm{V}(\mathscr{L} \cap \mathscr{N})$ and so $\langle W, R_{\min}, \mathcal{P}\rangle^+ \in \mathrm{V}(\mathscr{L} \cap \mathscr{N})$. $\{\infty\}$ is $R_{\min}$-closed. By duality, Theorem 2.4, $\langle\{\infty\}, \langle\infty,\infty\rangle\rangle^+ \in \mathrm{V}(\mathscr{L}\cap\mathscr{N})$. That is to say, the dual of the reflexive point is in the variety generated by the class of duals of structures in $\mathscr{N} \cap \mathscr{L}$. It follows that the classes $\mathscr{N}$ and $\mathscr{L} \cap \mathscr{N}$ cannot be characterized by means of algebraic properties of min-algebras. (Recall that both classes can be characterized by means of algebraic properties of modal algebras, cf. [4]).

We close this section with a remark about subdirectly irreducible (s.i.) min-algebras. Recall that an algebra $\mathfrak{A}$ is s.i. iff there exists a smallest non-trivial congruence Θ in $\mathfrak{A}$. For modal algebras there is a convenient characterization of finite s.i. algebras by means of their duals. For a finite structure $\langle W, R\rangle$ we call $r \in W$ a root of $\langle W, R\rangle$ if rR^*u, for any $u \in W$. Here R^* is the transitive and reflexive closure of R. $\langle W, R\rangle$ is rooted iff it has a root. A modal algebra $\mathfrak{A}$ is s.i. iff its dual has a root, see e.g. [4]. In the case of min-algebras we have to take care of reflexive points again:

Theorem 3.4. *Let $\mathfrak{A}$ be a finite min-algebra. The following conditions are equivalent:*
(1) *$\mathfrak{A}$ is subdirectly irreducible;*
(2) *$\mathfrak{A}_{\min}$ has a root.*

The simple proof is left to the reader.

4. Splittings

The equation which axiomatizes $\mathscr{TR}$ is rather lengthy and certainly not intuitive. This turns out to be the case for many interesting subvarieties of $\mathscr{M}$. In this section we (briefly) present an alternative geometrical way to characterize varieties, namely by means of splittings or subframe splittings. We shall not go into the details but sketch the main ideas. Since the varieties of interest are contained in $\mathscr{TR}$ we restrict the investigation to splittings in the lattice of subvarieties of $\mathscr{TR}$.

Definition 4.1. *Let $\mathfrak{A} \in \mathscr{TR}$ be a finite and subdirectly irreducible algebra. We say that $\mathfrak{A}$ splits $\mathscr{TR}$ if there exists a largest variety $\mathscr{V} \subseteq \mathscr{TR}$ such that $\mathfrak{A} \notin \mathscr{V}$. The variety $\mathscr{V}$ is then denoted by $\mathscr{TR}/\mathfrak{A}$.*

For information about splittings and their use for studying lattices of logics we refer the reader to [13], [1], [9], and [14].

In contrast to the situation in modal logic not every finite s.i. algebra splits $\mathscr{TR}$. Let $\mathfrak{B} = \langle\{0\}, \{\langle 0, 0\rangle\}\rangle^+$. $\mathfrak{B}$ is s.i. but does not split. For assume that $\mathfrak{B}$ splits and let $\mathrm{V} = \mathscr{TR}/\mathfrak{B}$. Then $\mathfrak{A}_n \in \mathrm{V}$ for all $n \in \omega$, where the algebras $\mathfrak{A}_n$ are from the proof of Theorem 3.3. But we have shown already that $\mathfrak{B} \in \mathrm{V}(\{\mathfrak{A}_n : n \in \omega\})$ and so $\mathfrak{B} \in \mathrm{V}$ which is a contradiction.

$\mathfrak{B}$ turns out to be the only finite s.i. algebra which does not split $\mathscr{TR}$. To sketch the proof we shall work with generalized min-structures instead of algebras. We know, by Theorem 3.4, that a finite min-algebra $\mathfrak{A}$ is s.i. iff $\mathfrak{A}_{\min}$ has a root. Notice also that $\mathfrak{A}_{\min}$ is actually a relational structure. That is to say, for $\mathfrak{A}_{\min} = \langle W, R, \mathcal{P}\rangle$ we have $\mathcal{P} = 2^W$. So, in order to study splittings by finite s.i. algebras it suffices to study splitting by algebras $\langle W, R\rangle^+$ such that $\langle W, R_{\min}\rangle$ has a root.

Consider a finite structure $\mathcal{G} = \langle W, R\rangle$ with root 0, take for any $u \in W$ a variable x_u, and define the following terms:

- $t_1 = \bigwedge\langle \neg f x_u : uRu\rangle$
- $t_2 = \bigwedge\langle x_u \to \neg f(x_u \vee x_v) : uRv\rangle$
- $t_3 = \bigwedge\langle x_u \to f(x_u \vee x_v) : \neg(uRv), \neg(uRu)\rangle$
- $t_4 = \bigwedge\langle x_u \to \neg x_v : u \neq v\rangle$
- $t_5 = \bigvee\langle x_u : u \in W\rangle$

Put $S(\mathcal{G}) = \bigwedge\langle t_i : 1 \leq i \leq 5\rangle$ and $SP(\mathcal{G}) = S(\mathcal{G}) \wedge f x_0 \wedge f(x_0 \vee \neg S(\mathcal{G}))$.

Theorem 4.2. *Let $\mathcal{G} = \langle W, R_{\min}\rangle$ be a finite and rooted transitive structure. Then $\mathcal{G}^+$ splits $\mathscr{TR}$ iff the root of $\mathcal{G}$ is irreflexive (iff W is the irreflexive point or has at least two points.)*

Moreover, if $\mathcal{G}^+$ splits $\mathscr{TR}$, then $\mathscr{TR}/\mathcal{G}^+$ is axiomatized by adding the equation $SP(\mathcal{G}) = 0$ to the axiomatization of $\mathscr{TR}$.

Proof. Let $\mathcal{G}$ have an irreflexive root. The Theorem follows immediately from the following

Claim. For all $\mathfrak{A} \in \mathscr{TR}$: $\mathfrak{A} \not\models SP = 0$ iff $\mathcal{G}^+ \in \mathrm{V}(\mathfrak{A})$.

The proof of the direction from right to left is easy: define a valuation γ in $\mathcal{G}$ by putting $\gamma(x_u) = \{u\}$, for all $u \in W$. Then $\gamma(S(\mathcal{G})) = 1$ and therefore $\gamma(SP(\mathcal{G})) = \{0\}$. We have proved $\mathcal{G}^+ \not\models SP(\mathcal{G}) = 0$. Now suppose $\mathcal{G}^+ \in \mathrm{V}(\mathfrak{A})$. Then any equation which is valid in $\mathfrak{A}$ is also valid in $\mathcal{G}^+$ and so $\mathfrak{A} \not\models SP(\mathcal{G}) = 0$.

Conversely, suppose that $\mathfrak{A} \not\models SP(\mathcal{G}) = 0$. Let $\mathfrak{A}_+ = \langle V, S, \mathcal{Q}\rangle$. We find a valuation γ in $\mathfrak{A}_+$ such that $\gamma(SP(\mathcal{G})) \neq \emptyset$. Take $w \in \gamma(SP(\mathcal{G}))$ and consider

$$\mathcal{F} = \langle V', S \cap (V' \times V'), \{V' \cap X : X \in \mathcal{P}\}\rangle$$

where $V' = \{w' \in V : wS^*w'\}$. $\mathcal{F}$ is a generated substructure of $\mathfrak{A}_+$. Let

$$\gamma^*(x_u) = \gamma(x_u) \cap V', \text{ for } u \in W.$$

Clearly γ^* is a valuation in $\mathcal{F}$. Moreover, $\gamma^*(S(\mathcal{G})) = V'$. Define a mapping g from V' onto W by putting

$$g(w) = u \text{ iff } w \in \gamma^*(x_u).$$

Using the conjuncts t_4 and t_5 of $S(\mathcal{G})$ it is readily checked that g is well defined and onto. Using the conjuncts t_2, t_3, and t_4 one can show that g is a p-morphism from $\mathcal{F}$ onto $\langle W, R\rangle$. It follows that $\langle W, R\rangle$ is a p-morphic image of a generated substructure of $\mathfrak{A}_+$. By duality, Theorems 2.2 and 2.4, $\mathcal{G}^+$ is a subalgebra of a homomorphic image of $\mathfrak{A}$, and so $\mathcal{G}^+ \in \mathrm{V}(\mathfrak{A})$. □

We easily obtain axiomatizations of various subvarieties of $\mathscr{TR}$. For example, the variety $\mathrm{V}(\{\langle W, R\rangle : \forall w \exists v (wRv)\})$ coincides with $\mathscr{TR}/\mathfrak{A}$, where $\mathfrak{A} = \langle \{0\}, \emptyset\rangle^+$.

However, to axiomatize $\mathrm{V}(\mathscr{N})$ and $\mathrm{V}(\mathscr{L})$ another form of splittings is more useful.[2] We are alluding to the notion of subframe splittings introduced for varieties of modal algebras in [6] and [14]: for a finite structure $\mathcal{G} = \langle W, R\rangle$ let $S'(\mathcal{G}) = \bigwedge\langle t_i : 1 \le i \le 4\rangle$ and

$$SP'(\mathcal{G}) = S'(\mathcal{G}) \wedge f x_0 \wedge f(x_0 \vee \neg S'(\mathcal{G})).$$

The only difference between $S(\mathcal{G})$ and $S'(\mathcal{G})$ consists in the omission of the conjunct t_5. In other words, from $u \in \gamma(SP'(\mathcal{G}))$ it does not follow that $w \in \bigcup\{\gamma(x_u) : u \in W\}$ for all w with uS^*w. We explain the meaning of

[2] We note that it is possible to axiomatize the variety $\mathrm{V}(\mathscr{L})$ by means of (iterated) splittings, but that $\mathrm{V}(\mathscr{N})$ is not axiomatizable in this manner.

$SP'(\mathcal{G})$ by means of the notion of a *substructure*: Consider a generalized min-structure $\mathcal{F} = \langle V, S, \mathcal{Q} \rangle$ and $V' \in \mathcal{Q}$. Then the structure

$$\langle V', S \cap (V' \times V'), \{X \cap V' : X \in \mathcal{Q}\} \rangle$$

is a generalized min-structure as well and we call it a substructure of $\mathcal{F}$. Now one can easily show that for any finite transitive $\mathcal{G} = \langle W, R_{\min} \rangle$ with an irreflexive root and all transitive $\mathcal{F} = \langle V, S, \mathcal{Q} \rangle$ the following conditions are equivalent:

- $\mathcal{F}^+ \not\models SP'(\mathcal{G}) = 0$,
- there exists a substructure $\mathcal{F}'$ of $\mathcal{F}$ such that $\mathcal{G}$ is a p-morphic image of a generated subframe of $\mathcal{F}$.

The following axiomatizations are easily proved with the help of this observation:

Theorem 4.3. (1) *Let* $\mathcal{G} = \langle \{0, 1\}, \{\langle 0, 1 \rangle, \langle 1, 1 \rangle\} \rangle$. *Then* $\mathrm{V}(\mathscr{N})$ *is axiomatized by adding* $SP'(\mathcal{G}) = 0$ *to the axiomatization of* $\mathscr{TR}$.

(2) *Let* $\mathcal{F} = \langle \{0, 1, 2\}, \{\langle 0, 1 \rangle, \langle 0, 2 \rangle\} \rangle$. *Then* $\mathrm{V}(\mathscr{L})$ *is axiomatized by adding* $SP'(\mathcal{F}) = 0$ *to the axiomatization of* $\mathscr{TR}$.

5. Conclusion

In this paper we have investigated basic properties of algebras induced by minimality operators. It turned out that — when compared with standard algebras related to logics — the resulting min-algebras show some unusual and interesting features. However, from the algebraic perspective we certainly scratched the surface only and various questions remain. We mention here the following problems:

- Investigate the lattice of subvarieties of $\mathscr{M}$ in more detail. Compare it with the lattice of modal varieties.
- Characterize the definable relational structures. That is to say, classes of structures of the form $\{\langle W, R \rangle : \langle W, R \rangle^+ \in \mathscr{V}\}$, for some variety $\mathscr{V} \subseteq \mathscr{M}$.

In this paper we did not apply min-algebras to obtain directly new insight into non-monotonic logics or conditional logics. However, we believe that the algebraic perspective should form an interesting tool to understand those logics and that the results presented here form a good basis to start such an enterprise. But this claim remains to be justified.

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