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# SOME REMARKS CONCERNING MODAL PROPOSITIONAL LOGIC OF QUESTIONS

**Abstract.** Recently, it has become a custom to treat questions (or, better, questioning) as a game between two subjects. Unfortunately, one rarely goes beyond the scheme of Questioner-Scientist and Answerer-Nature, although the Interlocutor so conceived displays some undesirable features. This paper argues for the idea that logic of questions can be build as a logic of the game between "knowledge resources" persons or theories, rather than errant Scientist and omniscient Nature. To this end the concept of epistemically-possible worlds is discussed, which is conceived as analogous to that of possible worlds in modal logic. And, furthermore, the concepts of relation of epistemic alternativeness and of epistemically-alternative worlds are introduced. On this basis a version of semantics for propositional, three-valued logic of questions is offered and semantic proofs of some theses are given.

The aim of this paper is to construct a certain logic of yes–no questions. This logic will be based on three-valued propositional logic. We will also make use of some concepts which are similar to those which are often used in modal logic.

In the first section of this paper, we shall introduce some auxiliary concepts, at the first place the concept of epistemically-possible worlds. To this end, let us consider three kinds of utterance expression: propositions, assertions, and questions. Unlike propositions, assertions and questions involve a subject. Any assertion presupposes a person who asserts (an "asserter"), any question — a questioner. Yet, there is no such an animal as a "propositioner". Assertions differ from questions in that they involve only one subject, whereas questions have both questioners and addressees. So in the case of questions we can talk about "bi-subjectivity".

In short, a question is always asked by someone to someone, by someone, who wants to get an answer to someone, who is supposed to know the answer. So questioning (or, better, questioning-answering) is a game between two subjects, and there is nothing new in that. Unfortunately, one rarely goes beyond the scheme of a Questioner-Scientist (explorer, discoverer) and an Answerer-Nature (or whatever name we want to call the Second Side). The Interlocutor so conceived displays some undesirable features: it is secretive and taciturn, and not always answers our questions, although it is supposed to be omniscient and frank.

In fact, we hardly deal with such interlocutors. Usually our questioning-answering games are played between "knowledge resources": between persons (e.g. when John asks Peter whether it is raining), or persons and theories (e.g. when John asks about the cause of snowfall on the basis of a certain meteorological theory); perhaps even between theories themselves.[1] We can express this intuitions by introducing a semantic construction which employs an epistemological counterpart of the ontological idea of possible worlds. Possible worlds so conceived will be called *epistemically-possible worlds*.

As a matter of fact, in the case of modal logic, the notion of possible worlds ("ontically-possible" from now on) is used to introduce the al-

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[1] We mean "knowledge" in the objective sense, e.g. as a set of propositions. Therefore, in the case of "knowledge resources" of persons one is in need of some logic of assertions or acknowledgements, which avoids the omniscience paradox. Theories are unprotected against that (we use the term "theory" in the wide sense).

ternativeness relation and, furthermore, the notion of alternative worlds ("ontically-alternative", resp.). Similarly, we are mainly interested here in epistemically- alternative worlds. And we shall try to characterize this notion on the analogy of certain modal logic constructions.

By "ontically-possible worlds" one usually means certain (actual or possible, concrete or abstract) temporal and spatial systems — arrangements of objects, courses of events (local or global) — related in some way to the structure which is usually called "the real world". Similarly, epistemically-possible worlds will also be related to some kind of reality. Yet, in this particular context by "reality" we mean the true-and-complete knowledge about states of affairs rather than the states of affairs themselves. One can say that an epistemically-possible world is a world of knowledge about what obtains and what does not. In this sense, epistemic constructions are founded on ontical ones: the former are the look down on the latter. Or perhaps a multitude of looks, for the true-and-complete one — God's one — is beyond our reach. Of course, we will not try to define the notion of "epistemically-possible worlds" in detail. Such a definition will obviously depend on the concept of knowledge. For example, "epistemologically-possible worlds" can be conceived as sets of propositions accepted by some subject(s) — provided that the concept of knowledge is understood in terms of a mere acceptance.

Whether the world of the true-and-complete knowledge exists at all, and what the relation of epistemic alternativeness is, are metaphysical problems. The relation under consideration holds between those epistemically-possible worlds which can enter the game of questioning-answering. One problem is that saying that a world $\alpha$ is epistemically alternative to a world $\beta$ if $\alpha$ is cognitively-possible with respect to $\beta$ (or perhaps commutable with it in some sense) is as instructive as saying that "ontically-possible" means "that one, which can happen instead". But it is not our ambition to solve this problem at this point. And following the tradition here, we shall confine ourselves to what is without doubt namely, that every epistemically-possible world is alternative to itself, and this is justified by the intuition that we can ask questions to ourselves.

The next question is who and what can ask about. Two problems are worthy of attention here. First, the problem of reason: in what epistemic situation it is reasonable (or who is epistemically entitled) to ask a question; second, the problem of goal: what is the epistemical aim of posing questions. Let us start from the latter.

The goal of asking questions is to get answers — true or false, but always direct (both possible and just-sufficient). We can distinguish two basic kinds

of questions: *correct questions* and *incorrect questions*. At a first approximation we may say that a correct question is a question which has at least one direct answer with a determinate truth value (i.e. true or false). An incorrect question, in turn, is simply a question which is not correct. (Note that a question can be incorrect for two reasons: it has no direct, i.e. possible and just-sufficient, answer at all, or it has direct answer(s), but there is no direct answer to the question which has a determinate truth value.) Yet, the above definitions are not complete: the phenomenon of "bi-subjectivity" of questions must be taken into account. The goal of asking questions is to get answers, and it is impossible to achieve this goal if in the epistemic world of an addressee no direct answer has a determinate truth value. So it is the epistemic world of the addressee which counts here. Moreover, the epistemic world of the addressee must be compatible with that of the questioner. And finally, there must be at least one epistemic world which fulfils the above conditions. So the correctness of a question $Q$ posed in an epistemic world $\alpha$ (of the questioner) is dependent upon the existence of an epistemically alternative world $\beta$ (of a possible addressee) in which at least one direct answer to $Q$ is true or false.

In turn, there are two satisfactory solutions of the problem of reason. One can argue that it is pointless to ask about what is known to the questioner.[2] It is reasonable to ask somebody if an addressee can fill a truth-value gap in the questioner's knowledge: the informativeness of an answer with respect to the epistemically-possible world of the questioner is what is important here. Of course this approach would of impose some additional conditions on our (informal, as yet) definition of the correctness of a question.

The second possible solution of the problem of reason is to admit rhetorical and "examination" questions as a solid kind of questions. In this case it does not matter whether the questioner himself knows the answer in advance or not. In what follows we will adopt this latter solution.

Let us now analyze the expression "It is a question whether". It is easily seen that this expression belongs to the same syntactic category as "It is possible that", "It is necessary that", "It is not the case that", etc. On the other hand, as far as (simple) yes-no questions are concerned, the expression "It is a question whether $A$" seems to be true just in case the question "Is it the case that $A$?" is correct. Taking this intuition for granted, let us now construct a certain logic of the operator "It is a question whether".

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[2] Aristotle, for example, in *Analytica Posteriora*, 90a, says that we do not ask about what is self-evident.

Let $L_0$ be the language of Classical Propositional Calculus (CPC). Let us now supplement the vocabulary of $L_0$ with the operator '?', which can be interpreted as an erotetic one-place operator "It is a question whether". The vocabulary of $L_0$ enriched with the operator '?' is the vocabulary of a new language L. There are two categories of well-formed formulas of L: *declarative well-formed formulas* (d-wffs) and *erotetic well-formed formulas* (e-wffs). The d-wffs of L are simply the well-formed formulas of $L_0$. The set $\Phi_E$ of e-wffs is the smallest set which fulfils the following conditions:

 (i) if $A$ is a d-wff, then $?\,A$ is in $\Phi_E$,

(ii) if $A$, $B$ are in $\Phi_E$, then $\neg A$, $?\,A$, $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, $(A \leftrightarrow B)$ are in $\Phi_E$.

Note that among the meaningful expressions of L there are no "mixed" formulas, i.e. there are no expressions of the form $?\,A * B$ and $A * ?\,B$, where $*$ stands for a propositional connective and $A$, $B$ are d-wffs. Yet, we allow iterations of the operator '?'. And among e-wffs we can distinguish (i) *simple* e-wffs (that is e-wffs of the form $?\,A$, where $A$ is a d-wff) and (ii) *compound* e-wffs, which are the remaining e-wffs.

Let us now supplement the language L with some semantics.

**Definition 1.** An *interpretation* of the language L is an ordered quadruple $\mathfrak{I} = \langle \mathsf{U}, \mathsf{A}, \eta, \eta^* \rangle$, where $\mathsf{U}$ is a non-empty set, $\mathsf{A}$ is a dyadic, reflexive relation defined over the members of $\mathsf{U}$ ($\mathsf{A} \subseteq \mathsf{U} \times \mathsf{U}$), and $\eta$, $\eta^*$ are mappings of the set Var of propositional variables into the set of all subsets of $\mathsf{U}$ which fulfil the following conditions: for any propositional variable $p_i$, $\eta(p_i) \cap \eta^*(p_i) = \emptyset$ and $\eta(p_i) \cup \eta^*(p_i)$ is a proper subset of $\mathsf{U}$.

Intuitively, $\mathsf{U}$ is the set of epistemically-possible worlds (and thus can be infinite) and $\mathsf{A}$ is the relation of epistemic alternativeness.

We can now define a valuation function $\mathsf{V}$ for the declarative part of the language L (i.e. for the d-wffs of L). The initial conditions are as follows (we assume that $p_i \in$ Var and $\alpha \in \mathsf{U}$):

**Definition 2.** $\mathsf{V}(p_i, \alpha) = 1$ iff $\alpha \in \eta(p_i)$;
$\mathsf{V}(p_i, \alpha) = 0$ iff $\alpha \in \eta^*(p_i)$;
$\mathsf{V}(p_i, \alpha) = \mathrm{n}$ iff $\alpha \notin \eta(p_i)$ and $\alpha \notin \eta^*(p_i)$.

The clauses for formulas with propositional connectives (i.e. negation, conjunction, disjunction, implication and equivalence) can be built as semantic counterparts of the following truth tables of the Kleene's "strong" three valued logic:

| $A$ | $\neg A$ |
|---|---|
| 1 | 0 |
| 0 | 1 |
| n | n |

| $A$ | $B$ | $A \wedge B$ | $A \vee B$ | $A \to B$ | $A \leftrightarrow B$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | n | n | 1 | n | n |
| 0 | 1 | 0 | 1 | 1 | 0 |
| 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | n | 0 | n | 1 | n |
| n | 1 | n | 1 | 1 | n |
| n | 0 | 0 | n | n | n |
| n | n | n | n | n | n |

But this was a preliminary step. The next one is as follows. Using the above definition we will define a second valuation function $\mathsf{V}'$ for simple e-wffs. This function will only be a two valued one, according to the intuitions that any question is correct or incorrect. Moreover, its definition will involve two elements of the set $\mathsf{U}$, due to the phenomenon of the "bi-subjectivity" of questions mentioned above (an expression '$? A, \alpha/\beta$' can be read as: "it is a question whether $A$ in a world $\alpha$ with respect to a world $\beta$").

**Definition 3.** If $p_i \in \mathrm{Var}$, $\alpha, \beta \in \mathsf{U}$, $\alpha \mathrel{\mathsf{A}} \beta$ and $A$, $B$ are d-wffs of L, then:

1) $\mathsf{V}'(? p_i, \alpha/\beta) = 1$ iff $\mathsf{V}(p_i, \beta) = 1$ or $\mathsf{V}(p_i, \beta) = 0$ (i.e. iff $\mathsf{V}(p_i, \beta)$ is determinate),
   $\mathsf{V}'(? p_i, \alpha/\beta) = 0$ iff $\mathsf{V}(p_i, \beta) = \mathrm{n}$ (i.e. iff $\mathsf{V}(p_i, \beta)$ is undeterminate);

2) $\mathsf{V}'(? A, \alpha/\beta) = 1$ iff $\mathsf{V}(A, \beta) = 1$ or $\mathsf{V}(A, \beta) = 0$,
   $\mathsf{V}'(? A, \alpha/\beta) = 0$ iff $\mathsf{V}(A, \beta) = \mathrm{n}$;

3) $\mathsf{V}'(? \neg A, \alpha/\beta) = 1$ iff $\mathsf{V}(\neg A, \beta) = 1$ or $\mathsf{V}(\neg A, \beta) = 0$
   $\mathsf{V}'(? \neg A, \alpha/\beta) = 0$ iff $\mathsf{V}(\neg A, \beta) = \mathrm{n}$;

4) $\mathsf{V}'(?(A \wedge B), \alpha/\beta) = 1$ iff $\mathsf{V}((A \wedge B), \beta) = 1$ or $\mathsf{V}((A \wedge B), \beta) = 0$
   $\mathsf{V}'(?(A \wedge B), \alpha/\beta) = 0$ iff $\mathsf{V}((A \wedge B), \beta) = \mathrm{n}$;

and so on for other connectives.

Let us now introduce the definition of the concept of *truth* of a formula of L in a world $\alpha$ of an interpretation $\mathfrak{I}$:

**Definition 4.** 1. if $A$ is a d-wff, then $A \in \mathrm{Ver}_{\mathrm{L}}(\mathfrak{I}, \alpha)$ iff $\mathsf{V}(A, \alpha) = 1$;

2. if $A$ is a simple e-wff, then $A \in \mathrm{Ver}_{\mathrm{L}}(\mathfrak{I}, \alpha)$ iff there is a $\beta \in \mathsf{U}$ such that $\alpha \mathrel{\mathsf{A}} \beta$ and $\mathsf{V}'(A, \alpha/\beta) = 1$;

3. if $A$ is a compound e-wff of the form $?B$, then $A \in \mathrm{Ver_L}(\mathfrak{I}, \alpha)$ iff $B \in \mathrm{Ver_L}(\mathfrak{I}, \alpha)$ or $B \notin \mathrm{Ver_L}(\mathfrak{I}, \alpha)$;

4. if $A$ is a compound e-wff of the form $\neg B$, then $A \in \mathrm{Ver_L}(\mathfrak{I}, \alpha)$ iff $B \notin \mathrm{Ver_L}(\mathfrak{I}, \alpha)$;

5. if $A$ is a compound e-wff of the form $(B \wedge C)$, then $A \in \mathrm{Ver_L}(\mathfrak{I}, \alpha)$ iff $B \in \mathrm{Ver_L}(\mathfrak{I}, \alpha)$ and $C \in \mathrm{Ver_L}(\mathfrak{I}, \alpha)$;

6. if $A$ is a compound e-wff of the form $(B \vee C)$, then $A \in \mathrm{Ver_L}(\mathfrak{I}, \alpha)$ iff $B \in \mathrm{Ver_L}(\mathfrak{I}, \alpha)$ or $C \in \mathrm{Ver_L}(\mathfrak{I}, \alpha)$;

7. if $A$ is a compound e-wff of the form $(B \rightarrow C)$, then $A \in \mathrm{Ver_L}(\mathfrak{I}, \alpha)$ iff $B \notin \mathrm{Ver_L}(\mathfrak{I}, \alpha)$ or $C \in \mathrm{Ver_L}(\mathfrak{I}, \alpha)$;

8. if $A$ is a compound e-wff of the form $(B \leftrightarrow C)$, then $A \in \mathrm{Ver_L}(\mathfrak{I}, \alpha)$ iff $B \in \mathrm{Ver_L}(\mathfrak{I}, \alpha)$ just in case that $C \in \mathrm{Ver_L}(\mathfrak{I}, \alpha)$.

Let us add some explanatory remarks here. A formula which says that it is a question whether $A$ is true in a world $\alpha$ of an interpretation $\mathfrak{I}$ iff there exists a world $\beta$ (of the interpretation $\mathfrak{I}$) which is epistemically alternative to $\alpha$ and such that it is a question whether $A$ in the world $\alpha$ with respect to the world $\beta$. It follows that a formula of the form "it is a question whether $A$", where $A$ is a d-wff, is true in a world $\alpha$ just in case the d-wff $A$ has a determinate truth value (i.e. is true or false) in some world which is epistemically alternative to the world $\alpha$. If, however, $A$ is an e-wff, then a formula of the form "it is a question whether $A$" represents in fact a question about justification of a question and, according to the above definition, is true in a world $\alpha$ if an only if the "inner" erotetic wff (-s) is (are) true or false in $\alpha$. Since in every world any erotetic formula is true or false it follows that a formula of the form $?A$ where $A$ is an e-wff is always true (note, that our relation of epistemic alternativeness is reflexive).

According to the intuitions presented above, we can say that a question of the form "Is it the case that $A$?" is correct in a world $\alpha$ of an interpretation $\mathfrak{I}$ iff $?A \in \mathrm{Ver_L}(\mathfrak{I}, \alpha)$.

Let us now introduce the following definitions:

**Definition 5.** $A \in \mathrm{Ver_L}(\mathfrak{I})$ iff for any $\alpha \in \mathsf{U}$: $A \in \mathrm{Ver_L}(\mathfrak{I}, \alpha)$.

**Definition 6.** $A \in \mathrm{Ver_L}$ iff for any $\mathfrak{I}$: $A \in \mathrm{Ver_L}(\mathfrak{I})$.

The expression $A \in \mathrm{Ver_L}(\mathfrak{I})$ should be read "$A$ is true in an interpretation $\mathfrak{I}$". The expression $A \in \mathrm{Ver_L}$, in turn, is to be read "$A$ is a L-tautology".

If $A$ is an e-wff and $A \in \text{Ver}_\text{L}$, we will be saying that $A$ is an *erotetic tautology*. We can easily prove what has been noted above:

**Lemma 1.** *Any formula of the form '$? \, A$', where $A$ is an e-wff, is an erotetic tautology.*

Sketch of proof: If $A$ is an e-wff, then in any world $\alpha$ of any interpretation $\mathfrak{I}$: $A \in \text{Ver}_\text{L}(\mathfrak{I}, \alpha)$ or $A \notin \text{Ver}_\text{L}(\mathfrak{I}, \alpha)$ (according to the Definition 4). Then, according to clause 3 of Definition 4, $? \, A \in \text{Ver}_\text{L}(\mathfrak{I}, \alpha)$.

Informally, one can say that a question about correctness of a question is always correct.

It can also be proved, that among L-tautologies there are, e.g. formulas of the following forms:

$$? \, A \vee \neg \, ? \, A$$
$$?(A \rightarrow B) \leftrightarrow ?(\neg A \rightarrow \neg B)$$
$$?(A \vee B) \rightarrow (? \, A \vee ? \, B)$$
$$? \, A \leftrightarrow ? \, \neg A$$
$$\neg(? \, A \wedge \neg \, ? \, A)$$
$$?(A \wedge B) \rightarrow (? \, A \vee ? \, B)$$

Some examples of interesting formulas which are *not* L-tautologies are all the formulas of the following forms:

$$?(A \wedge B) \rightarrow (? \, A \wedge ? \, B)$$
$$(? \, A \vee ? \, B) \rightarrow ?(A \vee B)$$
$$(? \, A \rightarrow ? \, B) \leftrightarrow ?(A \rightarrow B)$$
$$? \, A \rightarrow ?(A \vee B)$$
$$?(A \wedge B) \rightarrow ? \, A$$
$$(?(A \rightarrow B) \wedge ? \, A) \rightarrow ? \, B$$
$$(?(A \rightarrow B) \wedge \neg \, ? \, B) \rightarrow \neg \, ? \, A$$
$$(?(A \rightarrow B) \wedge ? \, \neg B) \rightarrow ? \, \neg A$$
$$? \, A \vee ? \, \neg A$$

Finally, let us turn to the problem of erotetic modalities. We define an erotetic modality (on the analogy of the definition of modality in modal logic, which can be found e.g. in Hughes and Cresswell's *Introduction to Modal Logic*) as any unbroken sequence of zero or more one-place operators '$\neg$' and '$?$'.

An intuitive meaning of iteration of the erotetic operator '?' can be briefly described as follows. Questions of the form: "It is a question whether it is a question whether ..." (and so on) can be conceived as questions about the justification of a question: it is a question whether a certain question is correct or incorrect.

Now, the problem arises, how many different and non-reducible erotetic modalities can be distinguished in language L. It can be proved that every erotetic modality is equivalent to one or other of the following or their negations:

(i)          − (a sequence of zero operators)

(ii)         ?

(iii)        ??

In particular, it can be proved that every wff of the following form is a L-tautology :

$$\underbrace{?\cdots?}_{k \text{ times}} A \leftrightarrow \underbrace{?\cdots?}_{m \text{ times}} A$$

where $k, m > 1$. If $k = 1$, then the implication holds:

$$? A \rightarrow \underbrace{?\cdots?}_{m \text{ times}} A \,;$$

but its converse does not hold. And this means, that we can always multiply questions (from a simple: "It is a question, whether" to "It is a question whether it is a question whether" and so on) and we can always ask about the correctness of posing questions. But the question about the justification of a question cannot be reduced to the initial question.

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