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THEOREM PROVING WITH BUILT-IN HYBRID THEORIES

Abstract. A growing number of applications of automated reasoning exhibits the necessity of flexible deduction systems. A deduction system should be able to execute inference rules which are appropriate to the given problem. One way to achieve this behavior is the integration of different calculi. This led to so called hybrid reasoning [22, 1, 10, 20] which means the integration of a general purpose foreground reasoner with a specialized background reasoner. A typical task of a background reasoner is to perform special purpose inference rules according to a built-in theory. The aim of this paper is to go a step further, i.e. to treat the background reasoner as a hybrid system itself. The paper formulates sufficient criteria for the construction of complete calculi which enable reasoning under hybrid theories combined from sub-theories. For this purpose we use a generic approach described in [20]. This more detailed view on built-in theories is not covered by the known general approaches [1, 3, 6, 20] for building in theories into theorem provers. The approach is demonstrated by its application to the target calculi of the algebraic translation [9] of multi-modal and extended multi-modal [7] logic to first-order logic.

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1. Introduction

Hybrid reasoning is usually understood as the cooperation of a foreground reasoner with a background reasoner. The foreground reasoner takes care of the general logical structure of a formula to be proved or refuted. The background reasoner is consulted whenever the meaning of special built-ins must be considered. There is a number of results for building in theories [1, 20, 6, 3]. Those results form a general framework for building in theories. However, those approaches consider the built-in theory as homogeneous.

For our application, the translations of (extended) multi-modal logic into fragments of first-order logic following [7, 9], we have to take care of the internal structure of the built-in theory, which is the combination of two sub-theories. For the target logic of the translation of multi-modal logic one sub-theory is a definite theory without equality, and the other sub-theory is an equational theory. For the target logic of the algebraic translation of the extended multi-modal logic one sub-theory is a definite theory without equality, and the other part of the theory is an equational theory. Reasoning within this equational theory may be reduced to associative unification with unit. Because of the restricted form of the queries to the Horn theory we just have an instance of constraint reasoning. In the case of the algebraic translation of extended multi-modal logic the sub-theories are an equational and definite theory with equality. Both sub-theories share an equational theory as sub-theory.

Nevertheless, in both cases the constituent parts the unification problems of the hybrid theories do not interfere. Moreover sufficient criteria for constructing a complete set of theory connections for the hybrid theory may be applied. Now, having at hand a complete set of theory connections with solvable unification problem we may apply the general technique surveyed in [20] in order to obtain a complete calculus for reasoning within the hybrid theory. The calculus is based on the connection method. This approach has been realized in an implementation of the mentioned hybrid theory. The implementation provides an automatic translation of a given multi-modal problem into first-order clause logic and the construction of the corresponding constraint theory.

This paper is organized as follows. In Section 2 will be introduced necessary general notions. The algebraic translation of (extended) multi-modal logic to first-order logic and the resulting target calculi will be discussed in

Sections 3.1 and 3.2. Section 4 is devoted to the presentation of the generic approach to building in theories into theorem provers. The application of the general approach to reasoning under hybrid theories will be presented in Section 5. The implementation is briefly described in Section 6.

Related work. Theory connections are a generalization of Wolfgang Bibel's eq-connections (cf. [4]). The extension of resolution to theory resolution is due to Mark Stickel. Many improvements of resolution have been shown as special kinds of theory resolution in [22]. For a treatment of the lifting to the full first-order calculus see [1] or [19]. Another approach considering theories given by classes of models has been presented by Hans-Jürgen Bürckert [6]. Our approach carries over to that case if one considers a complete set for theory connections of each model of the considered class. The translation of Hans-Jürgen Bürckert's approach to theory model elimination may be found in a more recent paper of Peter Baumgartner and Frieder Stolzenburg [3]. The case of constraint reasoning may be seen as a special case of reasoning in a hybrid theory with one theory being the empty theory. An alternative translation of modal logic to first-order logic, the relational one, has been described by Alan Frisch and Richard Scherl as an instance of constraint reasoning [11]. Similar to the algebraic translation as described in [9] is the functional translation due to Hans Jürgen Ohlbach [18].

2. Preliminaries

For keeping the paper self-contained we recall basic notions concerning logic in general and theory reasoning in particular. We assume that the reader is familiar with the basic notions of first-order logic in clause form (cf. [14]). We consider only formulas that are conjunctions of universally closed disjunctions of literals and we will ask for the unsatisfiability of those formulas in a theory. That is we formulate our results in the refutational or, in other words, negative setting. Though our presentation is formulated for clause logic it may be carried over to full first-order logic. A clause with at most (exactly) one positive literal will be called a *Horn (definite) clause*. A definite clause consisting only of equational literals will be called a *conditional equation*. A *clause* is represented as a multi-set of literals. A *matrix* is a multi-set of clauses. Multi-sets will be denoted as sequences of their elements. A set of copies of clauses of a matrix M will be called an *amplification* of M (see [15] for a more general definition of this notion). Clauses will be abbreviated also by Γ, C, D etc. Γ_1, Γ_2 denotes the union $\Gamma_1 \cup \Gamma_2$, whereas

Γ, L denotes $\Gamma \cup \{L\}$ etc. A clause $L_1, \ldots, L_n$ means the universal closure $\bar{\forall}(L_1 \vee \cdots \vee L_n)$ of the disjunction of its elements. The meaning of a matrix $C_1, \ldots, C_n$ is the conjunction $\bar{\forall} C_1 \wedge \cdots \wedge \bar{\forall} C_n$.

This paper will focus on a family of proof procedures that generate goal driven a set of instances of clauses such that its unsatisfiability in a given theory may be proved by checking a simple sufficient criterion. In order to formulate this criterion we first of all need the notions of a path and of a spanning theory mating. A *(partial) path (in) through* a matrix M is a multi-set containing (at most) exactly one literal from each clause of M. Paths will be abbreviated also by p or q. A set of partial paths in a matrix M is called a *mating* in M. A partial path u in a matrix M is *spanning* a path p through M if $u \subseteq p$. A mating U in a matrix M is *spanning* if for every path p through M exists an element $u \in U$ which is spanning p. If L is a positive literal then $\bar{L}$ denotes the literal $\neg L$. If L has the form $\neg K$ then $\bar{L}$ denotes the literal K. If p is the path $L_1, \ldots, L_n$ then $\bar{p}$ denotes the clause $\bar{L_1}, \ldots, \bar{L_n}$. And, vice versa, if Γ is the clause $L_1, \ldots, L_n$ then $\bar{p}$ denotes the path $\bar{L_1}, \ldots, \bar{L_n}$. The set of variables occurring in a term t, literal L, clause Γ or path p will be denoted by $\mathrm{Var}(t)$, $\mathrm{Var}(L)$, $\mathrm{Var}(\Gamma)$ or $\mathrm{Var}(p)$ respectively.

A *substitution* is a mapping from the set of variables into the set of terms which is almost everywhere equal to the identity. The *domain* of a substitution σ is the set $\mathrm{D}(\sigma) = \{X \mid \sigma(X) \neq X\}$. The set of *variables introduced by* σ is the set $\mathrm{I}(\sigma) = \bigcup_{x \in \mathrm{D}(\sigma)} \mathrm{Var}(\sigma(X))$. If the variables $X_1 \ldots, X_n$ are the elements of the domain of a substitution σ and the terms $t_1, \ldots, t_n$ are the corresponding values then σ will be denoted by $\{X_1 \mapsto t_1, \ldots, X_n \mapsto t_n\}$. A substitution σ may be extended canonically to a mapping from the set of terms into the set of terms. This extension will be denoted by σ too. For a set of variables V and substitutions σ and ρ we write $\sigma =_V \rho$ if for every element $X \in V$ holds $\sigma(X) = \rho(X)$. In the previous equation the lower index V may be omitted if V is the set of all variables. The *composition* $\sigma\theta$ of substitutions σ and θ is the substitution which assigns to every variable X the term $\theta(\sigma(X))$. A substitution σ is called *idempotent* if $\sigma = \sigma\sigma$. A substitution σ is idempotent iff $\mathrm{D}(\sigma) \cap \mathrm{I}(\sigma) = \emptyset$. If M is the multi-set of clauses $C_1, \ldots, C_n$ then $M' = C'_1, \ldots, C'_k$ is a *sub-matrix* of M iff there is a sequence of pairwise disjoint indices $i_1, \ldots, i_k$ s.t. C'_l is a sub-multi-set of C_{i_l} for each l with $1 \leq l \leq k$. A set of matrices which is closed w.r.t. the application of substitutions, forming amplifications and sub-matrices will be called a *query language*. For a path $p = L_1, \ldots, L_n$ and a query language $\mathcal{Q}$ we will write $p \in \mathcal{Q}$ in order to abbreviate $\{\{L_1\}, \ldots, \{L_n\}\} \in \mathcal{Q}$.

Let $\mathscr{T}$ be an open, i.e. quantifier-free, theory. A $\mathscr{T}$-model is an interpretation satisfying $\mathscr{T}$. A query (a clause, a path, a literal) S is *$\mathscr{T}$-satisfiable* if there is a $\mathscr{T}$-model satisfying S. It is *$\mathscr{T}$-unsatisfiable* else. Let $\mathscr{E}$ be the theory of equality, i.e. the clause set consisting of clauses expressing reflexivity, symmetry, transitivity and functional and predicative substitutivity. Let $\mathscr{T}$ be an arbitrary theory. Then the set of predicate (function) symbols occurring in the formulas of a theory $\mathscr{T}$ be denoted by $\mathrm{P}(\mathscr{T})$ ($\mathrm{F}(\mathscr{T})$ respectively).

3. Sample hybrid theories

In this section we discuss two sample classes of hybrid theories. They are related to the target logics of the algebraic translation of certain multi-modal logics into first-order logic. Those translations are of great practical interest because they allow us to use provers, which have been designed for classical first-order logic, for proving theorems in non-classical logics as well.

Following the Kripke semantics [13] the algebraic translation of modal logics introduces new semantical items, so called possible worlds. Moreover, so called transitions — semantical items of a further kind — are introduced. Transitions allow to pass from one world to another. The features of specific modal logics are expressed by first-order theories in terms of the target logic. Those theories consist of certain sub-theories. Thus, they may be considered as hybrid theories.

3.1. The algebraic translation of multi-modal logic

For a detailed presentation of the algebraic translation the reader is referred to [9]. Here we can illustrate only basic features of the target logics of this translation. As an example we consider a slightly more complicated version of the simplest instance of the wise men puzzle [12].

Puzzle 3.1. *There are two wise men, a and b. Both are wearing a hat that can be either black or white. A wise man cannot see the color of his own hat. He can only see the color of the other's hat. Wise man b says that he does not know the color of his hat.* Question: *Is this information sufficient for wise man a in order to determine the color of his own hat?*

Figure 1 shows a formalization of the wise men puzzle. Some facts, which are either given explicitly or common sense consequences of explicitly given facts, are formulated in natural language and has been written on the left

a knows that a knows that if a isn't wearing a white hat then b knows this.	$\Box_a\Box_a(\neg w(a) \to \Box_b\neg w(a))$

$$\forall_{\alpha_1:k(a,\alpha_1)}\forall_{\alpha_2:k(a,\alpha_2)}(\neg w(\varepsilon!\alpha_1!\alpha_2, a) \to \forall_{\alpha_3:k(b,\alpha_3)}\neg w(\varepsilon!\alpha_1!\alpha_2!\alpha_3, a))$$

a knows that b knows that a or b are wearing a white hat.	$\Box_a\Box_b(w(a) \vee w(b))$

$$\forall_{\alpha_4:k(a,\alpha_4)}\forall_{\alpha_5:k(b,\alpha_5)}(w(\varepsilon!\alpha_4!\alpha_5, a) \vee w(\varepsilon!\alpha_4!\alpha_5, b))$$

a knows that b doesn't know that the color of his own hat is white.	$\Box_a\neg\Box_b w(b)$

$$\forall_{\alpha_6:k(a,\alpha_6)}\neg\forall_{\beta_1:k(b,\beta_1)}w(\varepsilon!\alpha_6!\beta_1, b)$$

Hypothesis: a knows that his hat is white.	$\Box_a w(a)$

$$\neg\forall_{\beta_2:k(a,\beta_2)}w(\varepsilon!\beta_2, a)$$

Figure 1. Multi-modal formalization of the wise men puzzle

hand side of this figure. On the right hand side a modal logic formalization of these sentences has been given. A statement of the form "Wise man a knows ..." can be coded by writing $\Box_a$ .... The question of the puzzle will be formulated as the hypothesis $\Box_a w(a)$, saying that wise man a knows that he is wearing a white hat. The negation[1] of the hypothesis should be refuted assuming the facts formulated in the puzzle and logical properties of the modal operators. The modalities $\Box_a$ and $\Box_b$ are characterized by the axiom schemes

(1)	$\Box_a\Phi \to \Box_a\Box_a\Phi$	(3)	$\Box_b\Phi \to \Box_b\Box_b\Phi$
(2)	$\Box_a\Phi \to \Phi$	(4)	$\Box_b\Phi \to \Phi$

where Φ denotes an arbitrary formula. Readers which are familiar with modal logic will observe that both modalities $\Box_a$ and $\Box_b$ are characterized by the modal system S4. Let us consider the translation of the multi-modal

[1] For rather accidental reasons often the task of proving that a formula, say $\Box_a w(a)$, is a consequence of some assumptions, say $A_1, \dots, A_n$, is not attacked directly. Rather it is demonstrated that the conjunction of these assumptions and the negation of the hypothesis, $\neg\Box_a w(a)$, is contradictory to the underlying theory. This, so called, refutational setting will be used below.

formulas into those of first-order logic with restricted quantifiers. Compare the multi-modal formalization in the first line in Figure 1 and the first-order sentence below of it. Modal operators, like $\Box_a$, have been translated by a restricted quantifier, like $\forall_{\alpha_1 : k(a,\alpha_1)}$, and each predicate symbol has obtained an additional argument, $\varepsilon!\alpha_1!\alpha_2$ for example. First of all let us discuss the role of this additional argument. It represents a possible world, which has been coded by a term. The term $\varepsilon!\alpha_1$ represents a world, which is accessible from the initial world ε via the transition α_1. Formally this has been expressed by the operator !, which takes two arguments, a world (here ε) and a possible transition (here α_1), and returns a world accessible from the given world via that transition. The operator ! associates to the left, therefore brackets will be omitted wherever possible. Transitions can be combined by the binary associative operator $*$. Moreover, there is a distinguished transition, which is denoted by 1. The operations $*$, ! and 1 form a monoid operating on the set of worlds, i.e. we have the equational theory $\mathscr{T}$ consisting of the axioms (5), …, (9) introduced below.

$$(5)\ w!1 = w \qquad \begin{array}{lrcl} (6) & w!(\alpha_1 * \alpha_2) & = & (w!\alpha_1)!\alpha_2 \\ (7) & (\alpha_1 * \alpha_2) * \alpha_3 & = & \alpha_1 * (\alpha_2 * \alpha_3) \end{array} \qquad \begin{array}{lrcl} (8) & 1 * \alpha & = & \alpha \\ (9) & \alpha * 1 & = & \alpha \end{array}$$

Now let us consider the restricted quantifiers, which have been introduced by the translation, in more detail. The restricted quantifier $\forall_{\alpha_1 : k(a,\alpha_1)}$ is the translation of the modal operator $\Box_a$. The sort information $\alpha_1 : k(a, \alpha_1)$ given by the restricted quantification of variable α_1 just says that this variable is related to the interpretation of the modality $\Box_a$. The term $\varepsilon!\alpha_1!\alpha_2!\alpha_3$ represents a world which may be accessed by transitions of different sorts which are related to different modalities. The first two transitions, α_1 and α_2, are related to modality $\Box_a$, whereas the third transition, i.e. α_3, is related to the modality $\Box_b$. The properties of those modalities will be expressed by the four definite clauses (13), (12), (14), and (15) given below. Finally, let us mention that relations between different modalities may be expressed as well. For example, the following, so called interaction axioms schema, $\Box_a\Phi \rightarrow \Box_b\Phi$ corresponds in the target logic to the first-order axiom $k(b, \alpha) \rightarrow k(a, \alpha)$. Interaction axioms appear in the example in Section 3.2. We just draw the reader's attention to the fact that in the target logic also those relations between different modalities can be expressed by definite clauses.

Now we consider the translation of the formulas from Figure 1 into clause normal form. In Figure 2 we write the clause normal form as a matrix with the literals of each clause forming a row. The 4 clauses in Figure 2 correspond to the 4 formulas in Figure 1.

$$\left[\;\begin{array}{|cc|}\hline w(\varepsilon!\alpha_1!\alpha_2,a), & \neg w(\varepsilon!\alpha_1!\alpha_2!\alpha_3,a)\\ w(\varepsilon!\alpha_4!\alpha_5,a), & w(\varepsilon!\alpha_4!\alpha_5,b)\\ \multicolumn{2}{|c|}{\neg w(\varepsilon!\alpha_6!f_1(\varepsilon!\alpha_6),b)}\\ \multicolumn{2}{|c|}{\neg w(\varepsilon!f_2(\varepsilon),a)}\\ \hline\end{array}\;\begin{array}{c} \neg k(a,\alpha_1),\neg k(a,\alpha_2),\neg k(b,\alpha_3)\\ \neg k(a,\alpha_4),\neg k(b,\alpha_5)\\ \neg k(a,\alpha_6)\\ \\ \end{array}\;\right]$$

Figure 2. Matrix form of the translated wise men puzzle

The matrix in Figure 2 has to be proved under the union of the theories $\mathscr{T}$, consisting of Formulas (5), …, (9), and $\Re$, consisting of Formulas (10), …, (15) as axioms. Let us recall, that the properties (1), …, (4) of the modal system S4, which characterizes the modalities $\Box_a$ and $\Box_b$, are expressed by Clauses (12), …, (15) in terms of the target logic. Clauses (10) and (11) characterize the properties of the Skolem functions f_1 and f_2 which had to be introduced for the negated universal quantifiers $\neg\forall_{\beta_1:k(b,\beta_1)}$ and $\neg\forall_{\beta_2:k(a,\beta_2)}$. For details see [9].

(10) $\neg k(a,\alpha),\ k(b,f_1(\varepsilon!\alpha))$	(13) $\neg k(a,\alpha_1),\ \neg k(a,\alpha_2),\ k(a,\alpha_1 * \alpha_2)$
(11) $k(a,f_2(\varepsilon))$	(14) $\neg k(b,\alpha_1),\ \neg k(b,\alpha_2),\ k(b,\alpha_1 * \alpha_2)$
(12) $k(a,1)$	(15) $k(b,1)$

The proof task in the target logic is to show that the matrix in Figure 2 is unsatisfiable in the union of the theories $\mathscr{T}$ and $\Re$. The last mentioned matrix has the following syntactic properties. (1) Equality does not occur as a predicate symbol in the matrix. (2) Sort literals occurring in clauses with non-sort literals are negative. From the first observation we deduce that all theory connections within the boxed part of the matrix in Figure 2 are binary connections of the form $p(\underline{t}), \neg\, p(\underline{s})$ where the tuples of terms $\underline{t}$ and $\underline{s}$ are component-wise $\mathscr{T}$-unifiable. The second syntactic property and the form of theory $\Re$ make sure that sort literals occurring in the matrix in Figure 2 may be elements only of unary $\Re$-connections. Now we can discuss the remaining

$\Re$-connection	Used axioms
$\neg k(a, \alpha_1)$	(13)
$\neg k(a, \alpha_2)$, $\neg k(a, \alpha_4)$, $\neg k(a, \alpha_6)$	(11)
$\neg k(a, \alpha_3)$, $\neg k(a, \alpha_5)$	(10), (11)

Table 1. Solving constraints of the matrix in Figure 2

details of Figure 2. Three $\mathscr{T}$-connections are indicated by arcs in the boxed part. They may be simultaneously $\mathscr{T}$-unified by the substitution

$$\{\alpha_1 \mapsto 1,\ \alpha_2, \alpha_4, \alpha_6 \mapsto f_2(\varepsilon),\ \alpha_3, \alpha_5 \mapsto f_1(\varepsilon ! f_2(\varepsilon))\} \tag{16}$$

It is easy to verify that every of the sort literals in the dashed boxed sub-matrix in Figure 2 is a $\Re$-connection. Substitution 16 is also a simultaneous $\Re$-unifier for these $\Re$-connections. Table 1 gives for each of those $\Re$-connections the axioms which have to be used for proving this statement. The mentioned theory connections may be found subsequently by theory inference steps of the form given in Example 4.6. An appropriate calculus will be introduced in Section 4.3. The reader may have observed that none of the equational axioms (5), ..., (9) has been mentioned in Table 1. Indeed, the following proposition holds.

Proposition 3.1. *Suppose that the equation $t = s$ is valid in the equational theory $\mathscr{T}$. Then for the literal $\neg k(a, s)$ (and analogously for $\neg k(b, s)$) holds that $\neg k(a, s)$ is $\Re$-unsatisfiable iff $\neg k(a, t)$ is $\Re$-unsatisfiable.*

From this observation follows that for proving the $\Re$-unsatisfiability of a literal $\neg k(a, t)$ we don't need to apply equational axioms. Speaking more operationally, when inferencing within the constraint theory $\Re$ one does not need to apply $\mathscr{T}$-unification but only syntactical unification. This is a useful feature of the target logic of multi-modal logic. Assumption (1) of Proposition 5.1 is related to this feature.

3.2. The algebraic translation of extended multi-modal logic

Our next example illustrates the extended multi-modal logic. While modalities in the multi-modal logic have been indexed only by constants, now they may be indexed by arbitrary terms of the discourse domain. Let us consider a simplified version of the "safe puzzle" from [7].

It is always true that if something is written on a paper P, then the helper $h(X)$ of an agent X knows the combination N of some safe S after he has been instructed by X about the paper P.

$$[u]\forall P, X(w(P) \to [a]_{X,i(P,h(X))}\exists S, N[k]_{h(X)}c(N,S))$$

$$\begin{aligned} &\forall_{\alpha:u(\alpha)}\forall P, X(\, w(\varepsilon!\alpha, P) \\ &\to \forall_{\beta:a(X,i(P,h(X)),\beta)}\,\exists\, S, N\,\forall_{\gamma:k(h(X),\gamma)}\, c(\varepsilon!\alpha!\beta!\gamma, N, S)) \end{aligned} \tag{17}$$

It is always true that if N is the combination of a safe S then the safe will be open after some agent X has dialed N on S.

$$[u]\forall S, N, X(c(N,S) \to [a]_{X,d(N,S)}o(S))$$

$$\forall_{\alpha:u(\alpha)}\,\forall S, N, X(c(\varepsilon!\alpha, N, S) \to \forall_{\beta:a(X,d(N,S),\beta)}\, o(\varepsilon!\alpha!\beta, S)) \tag{18}$$

Agent Joe knows that there is a paper P with something written on it.

$$[k]_{joe}\exists P w(P)$$

$$\forall_{\alpha:k(joe,\alpha)}\exists P w(\varepsilon!\alpha, P) \tag{19}$$

Hypothesis: Agent Joe knows that some safe can be opened by him and his helper.

$$[k]_{joe}\exists S\langle as\rangle_{joe,h(joe)}o(S)$$

$$\neg\forall_{\alpha:k(joe,\alpha)}\exists S\exists_{\beta:as(joe,h(joe),\beta)}o(\varepsilon!\alpha!\beta, S) \tag{20}$$

Figure 3. A safe puzzle formulated in extended multi-modal logic

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Puzzle 3.2. *There is a room with several safes in it. There are some sheets of paper. Sentences like "N is the combination of the safe S" have been written on those sheets. Two agents are co-operating. One of them is the master, the other his helper. The master knows everything his helper knows. Any master can read sentences (written on a paper) and instruct his helper. Any helper can try to open a safe dialing the number he has been told. A safe opens if its number has been dialed. Those informations are known to all agents and remain true whatever actions are performed.* Question: *Does agent Joe know that there is a sequence of actions allowing him and his helper to open some safe?*

Figure 3 presents a more detailed formulation of that problem and a formalization of the four statements using the extended multi-modal logic [7].

The *signature* of the extended multi-modal logic provides the following modalities: $[u]$ modeling universal truth, $[a]_{X,A}$ modeling the result of an action A performed by an agent X, $[as]_{X,Y}$ modeling the result of a sequence of actions performed by agents X and Y, and $[k]_X$ modeling the knowledge of an agent X. The *sorts of the discourse domain* are: X for agents, A for actions performed by agents, P for sheets of papers with information on them, S for safes, and N for numbers. The operation symbols of the discourse domain are $i{:}\,\mathsf{P}\times\mathsf{X}\to\mathsf{A}$ for the action of instructing an agent about (the information on) a paper, $d{:}\,\mathsf{N}\times\mathsf{S}\to\mathsf{A}$ for the action of dialing a number on a safe, and $h{:}\,\mathsf{X}\to\mathsf{X}$ for an helper associated with an agent. *Predicate symbols of the discourse domain* and their intended meaning are $c{:}\,\mathsf{N}\times\mathsf{S}$ where the first argument is the combination of the safe being the second argument, $o{:}\,\mathsf{S}$ expressing that a safe is open, and $w{:}\,\mathsf{P}$ expressing that something has been written on a paper.

The formalization given in Figure 3 will be translated into a many-sorted first-order logic (see Figure 3). The different modalities are mirrored by the following predicate symbols: u for universal truth, a for actions, as for action sequences, and k for the knowledge of agents. The translation of the safe puzzle into clause logic is given in Figure 4. Skolemization adds the function symbols s, n, p, and ϕ for the variables S and N (from Formula (17)), P (from Formula (19)), and α (from Formula (20)) respectively. The notation $\underline{r}$ occurring in the first clause of Figure 4 is a shorthand for the arguments $\varepsilon!\alpha_1!\beta_1!\gamma_1, n(\underline{t}), s(\underline{t})$ of the literal $c(\underline{r})$ where $\underline{t}$ abbreviates the arguments $\varepsilon!\alpha_1!\beta_1, P_1, X_1$ of the Skolem functions s and n. Moreover, we had to abbreviate $i(P_1, h(X_1))$ by $i(\underline{r'})$. Again the matrix has been partitioned into two sub-matrices. The left one, surrounded by a box, is related to a sub-theory $\mathscr{T}$ which is an equational theory. The right one, surrounded by a dashed line, corresponds to the constraint theory.

The equational theory $\mathscr{T}$ consists again of the equations (5) to (9) from Subsection 3.1. The constraint theory $\Re$ which has been obtained by the translation is more complicated now. For the matrix in Figure 4 we obtain the definite theory $\Re$ consisting of the axioms (21), …, (32).

$$k(joe, \phi(\varepsilon)) \tag{21}$$

$$\neg u(\alpha_1),\ \neg u(\alpha_2),\ u(\alpha_1 * \alpha_2) \tag{22}$$

$$u(1) \tag{23}$$

$$\neg k(X,\alpha_1),\ \neg k(X,\alpha_2),\ k(X,\alpha_1 * \alpha_2) \tag{24}$$

$$k(X,1) \tag{25}$$

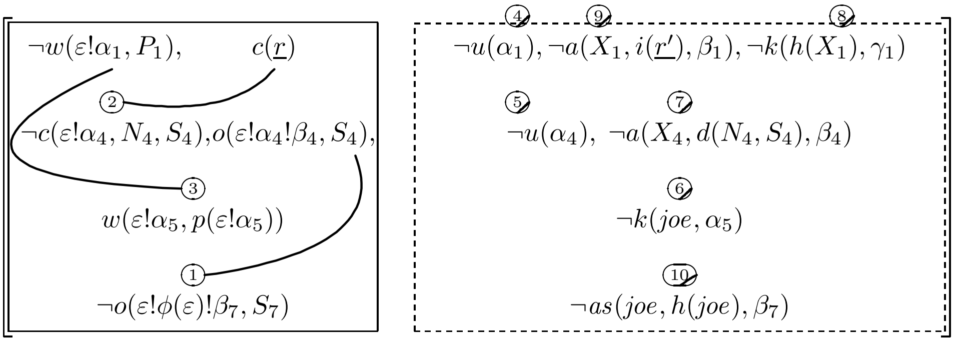

Figure 4. A spanning mating for the translated safe puzzle in matrix form

(26) $\neg as(X, Y, \alpha_1),\ \neg as(X, Y, \alpha_2),\ as(X, Y, \alpha_1 * \alpha_2)$

(27) $as(X, Y, 1)$

(28) $a(X, A, \psi(X, A))$

(29) $\neg a(X, A, \alpha),\ as(X, Y, \alpha)$

(30) $\neg a(Y, A, \alpha),\ as(X, Y, \alpha)$

(31) $\neg as(X, Y, \alpha),\ u(\alpha)$

(32) $\neg k(X, \alpha),\ u(\alpha)$

Let us discuss the subsequent axioms. Clause (21) had to be included because of the Skolemization of the goal clause (20). Clauses (22), ..., (27) express that the modalities $[u]$, $[k]_X$ and $[as]_{X,Y}$ are characterized by the modal system S4. The appropriate axiom schemes are analogous to Formulas (1) and (2). Clause (28) expresses the seriality of modality $[a]_{X,A}$. Clauses (29), ..., (32) express interaction axioms between the modalities $[as]_{X,Y}$, $[a]_{X,A}$, $[u]$, and $[k]_X$ given by the following axiom schemes.

(33) $[as]_{X,Y}\Phi \to [a]_{X,A}\Phi$ (35) $[u]\Phi \to [as]_{X,Y}\Phi$

(34) $[as]_{X,Y}\Phi \to [a]_{Y,A}\Phi$ (36) $[u]\Phi \to [k]_X\Phi$

In Figure 4 we display a mating which is spanning the matrix. The boxed sub-matrix is spanned already be the connections labeled by 1, 2, 3. Below we give the partial unifiers computed in the subsequent inference steps finding the indicated connections. The resulting simultaneous $\mathscr{T}$-unifier is $\theta = \theta_1\theta_2\theta_3$.

$\Re$-connections		Axioms needed
4, 5	$\neg u(\phi(\varepsilon))$	(21), (32)
6	$\neg k(joe, \phi(\varepsilon))$	(21)
7	$\neg a(X_4, d(n(\varepsilon!\phi(\varepsilon)!\beta_1, p(\varepsilon!\phi(\varepsilon))), X_1,$ $s(\varepsilon!\phi(\varepsilon)!\beta_1, p(\varepsilon!\phi(\varepsilon)))), X_1, \beta_4)$	(28)
8	$\neg k(h(X_1), \gamma_1)$	(25)
9	$\neg a(X_1, i(p(\varepsilon!\phi(\varepsilon)), h(X_1)), \beta_1)$	(28)
10	$\neg as(joe, h(joe), (\beta_1 * \gamma_1) * \beta_4)$	(26), ..., (29)

Table 2. Solving constraints of the matrix in Figure 4

$$(37)\quad \theta_1 \;=\; \{\alpha_4 \mapsto \phi(\varepsilon) * \delta,\ \beta_7 \mapsto \delta * \beta_4,\ S_4 \mapsto S_7\}$$

$$(38)\quad \theta_2 \;=\; \left\{\begin{array}{l} \alpha_1 \mapsto \phi(\varepsilon),\ \delta \mapsto \beta_1 * \gamma_1,\ N_4 \mapsto n(\varepsilon!\phi(\varepsilon)!\beta_1, P_1, X_1), \\ S_7 \mapsto s(\varepsilon!\phi(\varepsilon)!\beta_1, P_1, X_1) \end{array}\right\}$$

$$(39)\quad \theta_3 \;=\; \{\alpha_5 \mapsto \phi(\varepsilon),\ P_1 \mapsto p(\varepsilon!\phi(\varepsilon))\}$$

With this substitution θ the literals in the dash boxed part of the matrix in Figure 4 become simultaneously $\Re$-unifiable. However, different to the case discussed in the previous subsection, variables have to be instantiated further, even after the application of the substitution θ. The composition $\theta_1\theta_2'$ of the substitutions given by Equations (40) and (41) will solve the problem.

$$(40)\qquad \theta_1' \;=\; \left\{\beta_4 \mapsto \begin{array}{r} \psi(X_4, d(n(\varepsilon!\phi(\varepsilon)!\beta_1, X_1, p(\varepsilon!\phi(\varepsilon))), \\ h(\varepsilon!\phi(\varepsilon)!\beta_1, X_1, p(\varepsilon!\phi(\varepsilon))))) \end{array}\right\}$$

$$(41)\qquad \theta_2' \;=\; \{X_1 \mapsto joe,\ \gamma_1 \mapsto 1,\ \beta_1 \mapsto \psi(joe, i(p(\varepsilon!\phi(\varepsilon)), s(joe)))\}$$

Table 2 shows each of the $\Re$-connections 4, ..., 10 from Figure 4 after the substitution θ has been applied. For each of those $\Re$-connections Table 2 gives the axioms which have to be used for proving their $\Re$-complementarity. For the $\Re$-connections 4, 5 and 6 no further instantiation is necessary. $\Re$-connection 7 needs the application of substitution (40), whereas the remaining $\Re$-connections 8, 9 and 10 need the application of the substitution $\theta_1'\theta_2'$. Let us conclude this section with the remark that the simultaneous $\Re$-unifiability problem for a wide class of extended multi-modal logics is decidable (see [7]).

4. A generic approach to theory reasoning

In the present section we introduce a formal framework for constructing complete total theory reasoning calculi for open, i.e. quantifier free, theories. A complete theory reasoning calculus for an open theory needs the following key capabilities: (1) finding theory connections, (2) computing unifiers for theory connections, and (3) managing amplifications and representations of sets of paths which are not spanned by a currently found theory mating. The ingredients for constructing a complete theory reasoning calculus — a complete set of theory connections (Definition 4.3) with a solvable unification problem (Definition 4.4) and a calculus managing amplifications of matrices and keeping track of unsolved goals — will be introduced in the subsections 4.1, 4.2 and 4.3 respectively. Implementation issues are discussed in Section 6. In the present section we formalize what it means to have for a given theory "enough" theory connections in order to refute all theory unsatisfiable matrices which belong to a given query language. We formulate a Herbrand theorem by use of this notion (cf. Subsection 4.1). The notion of a complete set of unifiers for a theory connection generalizes the notion of complete set of theory unifiers of a pair of terms.

4.1. Complete sets of theory connections

In order to formulate sufficient conditions for the completeness of a theory reasoning calculus we introduce the notion of a set of theory connections which is complete with respect to a given query language. For an open theory first-order $\mathscr{T}$, given as a set of clauses we formalize (see Definition 4.3), what it means, to have "enough" theory connections in order to refute all theory unsatisfiable matrices, which belong to a given query language.

Definition 4.1. ($\mathscr{T}$-complementary, $\mathscr{T}$-unifier) A path u is called *$\mathscr{T}$-complementary* if and only if the existential closure of the conjunction of the elements of u, $\bar{\exists}\,(\bigwedge_{L\in u} L)$, is $\mathscr{T}$-unsatisfiable. A substitution σ is a *$\mathscr{T}$-unifier* of u if and only if $\sigma(u)$ is $\mathscr{T}$-complementary.

Remark 4.1. The $\mathscr{T}$-complementarity of a path u has been defined via the $\mathscr{T}$-unsatisfiability of the existential closure of the conjunction of the elements of u according to the negative representation which has been chosen in the present paper. In the positive representation $\mathscr{T}$-complementarity of a path u we would have been defined via the $\mathscr{T}$-validity of the disjunction of the elements of u. The remaining notions and results may be defined independently on the chosen (positive or negative) representation.

Definition 4.2. (Connection, $\mathscr{T}$-Connection) Let $\mathscr{T}$ be a theory, M a matrix, $\mathcal{U}$ a set of multi-sets of literals and $\mathcal{Q}$ a query language. Any partial path u in M will be called a *$\mathscr{T}$-connection* in M if there exists a $\mathscr{T}$-unifier for u. If $\mathscr{T}$ is the empty theory then the prefix $\mathscr{T}$ may be omitted.

Definition 4.3. (Complete set of theory connections) Let $\mathscr{T}$ be a theory, M a matrix, $\mathcal{U}$ a set of $\mathscr{T}$-connections and $\mathcal{Q}$ a query language.

(1) Any set of $\mathscr{T}$-connections in a matrix M which are elements of $\mathcal{U}$ is called a *$\mathcal{U}$-mating* in M.

(2) A decidable set $\mathcal{U}$ of $\mathscr{T}$-connections which is closed w.r.t. application of substitutions will be called *$\mathscr{T}$-complete w.r.t. $\mathcal{Q}$* if

  (2.1) for each $\mathscr{T}$-complementary ground path $p \in \mathcal{Q}$ exists $u \in \mathcal{U}$ such that $u \subseteq p$ and

  (2.2) for each $\mathscr{T}$-complementary ground path of the form $\sigma(u) \in \mathcal{U}$ such that $u \in \mathcal{Q}$ holds $u \in \mathcal{U}$.

Example 4.1. In the simplified version of equational reasoning, discussed in subsections 3.1 and 3.2, the equality symbol does not occur in the query language. In terms of Definition 4.3 the set $\mathcal{U}_{\mathscr{T}}$ of connections of the form $p(t_1, \ldots, t_n), \neg p(s_1, \ldots, s_n)$ for simultaneously pairwise $\mathscr{T}$-unifiable terms t_i and s_i is complete w.r.t. to the query language $\mathcal{Q}_{\mathscr{T}}$.

Example 4.2. Let us now consider the query language $\mathcal{Q}_{\Re}$ discussed in Subsection 3.1. It contains only negative clauses with a single predicate symbol k and the function symbols as in Example 4.1. The theory $\Re$ is formed from the definite clauses (10) and (11) given in Subsection 3.1. As an example consider the negative clause $\neg k(b, \alpha_5), \neg k(a, \alpha_4)$ which occurs as a fragment of the second clause in Figure 2. Each literal of this clause becomes $\Re$-unsatisfiable after applying the substitution $\{\alpha_5 \mapsto f_1(\varepsilon ! f_2(\varepsilon)), \alpha_4 \mapsto f_2(\varepsilon)\}$. Since $\Re$ is definite all $\Re$-connections in queries from $\mathcal{Q}_{\Re}$ are units. Thus, the set $\mathcal{U}_{\Re}$ of negative literals with predicate symbol k having a $\Re$-unifier form a set of $\Re$-connections complete with respect to the query language $\mathcal{Q}_{\Re}$.

The less literals a connection consists of the more paths it may span. Therefore, we are interested to find theory connections which are minimal with respect to set-theoretical inclusion. Every extra literal may cause that additional sub-goals have to be solved. The following proposition makes sure that a complete set of theory connections contains also all minimal connections.

Proposition 4.1. (Properties of complete sets of theory connections) *Let the set of $\mathscr{T}$-connections $\mathcal{U}$ be $\mathscr{T}$-complete with respect to the query language $\mathcal{Q}$. Let u be a path such that $u \in \mathcal{Q}$. If u is minimal $\mathscr{T}$-complementary then $u \in \mathcal{U}$.*

Having a complete set of theory connections a Herbrand theorem may be proved. The following version of Herbrand's theorem applies to the discussed examples 4.1 and 4.2.

Theorem 4.1. (Herbrand's theorem) *Let $\mathscr{T}$ be an open theory, $\mathcal{Q}$ a query language, $\mathcal{U}$ a set of $\mathscr{T}$-connections which is complete w.r.t. to $\mathcal{Q}$. Then for every $\mathscr{T}$-unsatisfiable matrix $M \in \mathcal{Q}$ there exists an amplification M' of M, a $\mathcal{U}$-mating U which is spanning in M' and a substitution σ such that $\sigma(u)$ is $\mathscr{T}$-complementary for each $u \in U$.*

4.2. The unification problem for sets of theory connections

The Herbrand theorem gives neither a hint how to find the substitution σ nor how to decide the existence of σ. In order to obtain a proof calculus for a given complete set of $\mathscr{T}$-connections $\mathcal{U}$ we also need to be able to compute or to represent for every $u \in \mathcal{U}$ all substitutions σ such that $\sigma(u)$ is $\mathscr{T}$-unsatisfiable. This will be formulated in the following definition.

Definition 4.4. (more general $\mathscr{T}$-unifier, $\mathscr{T}$-unification problem in $\mathcal{U}$) Let $\mathcal{U}$ be a set of multi-sets of literals.

(1) Let ϱ and σ be $\mathscr{T}$-unifiers of a path $u \in \mathcal{U}$ such that $\mathrm{D}(\varrho), \mathrm{D}(\sigma) \subseteq \mathrm{Var}(u)$. Then ϱ is called *more general than* σ if there exists η such that $\varrho\eta =_{\mathrm{Var}(u)} \sigma$. This will be denoted by $\varrho \leq \sigma$.

(2) A set S of $\mathscr{T}$-unifiers of a multi-set $u \in \mathcal{U}$ will be called *complete* if for each $\mathscr{T}$-unifier σ of u exists a substitution $\varrho \in S$ such that $\varrho \leq \sigma$.

(3) We say that the *$\mathscr{T}$-unification problem in $\mathcal{U}$ is solvable* if
   (3a) for every $u \in \mathcal{U}$ there exists an enumerable complete set S_u of $\mathscr{T}$-unifiers for u and
   (3b) for a given $u \in \mathcal{U}$ it is decidable whether $S_u \neq \emptyset$.

(4) A substitution σ will be called a *simultaneous $\mathscr{T}$-unifier* of a set U of multi-sets of literals if and only if $\sigma(u)$ is $\mathscr{T}$-complementary for every $u \in U$.

Example 4.3. Let $\mathcal{U}^{upp}_{\mathscr{T}}$ be the subset of $\mathscr{T}$-connections defined in Example 4.1 obeying the following so called unique prefix property (cf. [8]). A formula or a term has the unique prefix property if the binary symbol $*$ does not occur and for each variable α introduced for a modal operator holds that it occurs always in the same left context. Formulas obtained by the algebraic translation have this property. This corresponds to the fact that each variable introduced for a modal operator occurs always in the same modal context. For those restricted $\mathscr{T}$-unification problems exists an efficient unification algorithm [8].

In the case of syntactical unification there exist unification algorithms which have the following nice property. Whenever for a particular unification problem is solvable then the algorithm computes such a most general unifier which introduces only variables occurring already in the unification problem. Unfortunately, this is not the case for theory unification. Here it is possible that most general unifiers introduce "new" variables, i.e. those not occurring already in the unification problem. The following example illustrates the problem of new variables introduced by a unifier of a theory connection.

Example 4.4. Let $\mathscr{T}$ be the theory given in Example 4.1. Then the $\mathscr{T}$-equational connection (42) has the most general $\mathscr{T}$-unifier (43).

$$\left\{ \begin{array}{c} p(\varepsilon!\alpha_4!\beta_4) \\ \neg p(\varepsilon!\phi(\varepsilon)!\beta_7) \end{array} \right\} \tag{42}$$

$$\{\alpha_4 \mapsto \phi(\varepsilon) * \delta,\ \beta_7 \mapsto \delta * \beta_4\} \tag{43}$$

This unifier introduces a new variable, i.e. δ. From the semantical point of view it is important to realize that "new" variables are as other variables universally quantified. Therefore, introducing "new" variables does not change the meaning of a matrix. In other words introducing new variables is semantically safe.

In Prolog based implementations (for example see [21]) the problem is solved rather immediately in that way that "new" variables are new with respect to the current problem state. "new" variables must not occur already in the matrix under consideration. Otherwise they may prevent the solution of further unification problems. It remains to show that the solvability of the $\mathscr{T}$-unification problem within a set of theory connections $\mathcal{U}$ implies that for each given finite set X of variables and $\mathscr{T}$-connection $u \in \mathcal{U}$ it is always possible to enumerate a complete set of unifiers such that each of its members introduces only those variables not being element of X.

Proposition 4.2. *Suppose the set of multi-sets of literals $\mathcal{U}$ has a solvable $\mathscr{T}$-unification problem. Then for every $u \in \mathcal{U}$ and every finite set of variables X such that $X \cap \mathrm{Var}(u) = \emptyset$ may be enumerated a complete set S_u of $\mathscr{T}$-unifiers such that for each $\sigma \in S_u$ holds $\mathrm{I}(\sigma) \cap X = \emptyset$.*

Remark 4.2. Proposition 4.2 can be proved relying only on the notion of solvable unification problem (cf. Definition 4.4). Thus, Proposition 4.2 holds on the basis of the notions in our previous papers (e.g. [20]), though, it has not been formulated there. It closes a gap in the completeness proof of our theory connection calculus which has been observed in [2].

In a connection calculus we have to find a *simultaneous $\mathscr{T}$-unifier* of a spanning mating of $\mathscr{T}$-connections incrementally. The solvability of the unification problem in a set of theory connections $\mathcal{U}$ implies the solvability of the simultaneous unification problem in $\mathcal{U}$.

Proposition 4.3. *Suppose that the $\mathscr{T}$-unification problem is solvable for the set of $\mathscr{T}$-connections $\mathcal{U}$ and that S_u denotes the complete set of $\mathscr{T}$-unifiers for each $u \in \mathcal{U}$. Then every simultaneous $\mathscr{T}$-unifier θ of a set of $\mathscr{T}$-connections $U \subseteq \mathcal{U}$ may be found incrementally. Indeed, for each enumeration $u_1, \ldots, u_n$ of the elements of U may be constructed sequences $\{\sigma_i\}_{i=1}^n$, $\{\eta_i\}_{i=0}^n$, $\{\varrho_i\}_{i=0}^n$ such that*

(1) *$\eta_0 = \theta$ and $\varrho_0 = \{\ \}$ and for every $i, 1 \leq i \leq n$*

(2) *$\sigma_i \in S_{\varrho_{i-1}(u_i)}$,*

(3) *$\sigma_i \eta_i = \eta_{i-1}$ and*

(4) *$\varrho_i = \varrho_{i-1}\sigma_i$.*

Example 4.5. Let θ be the simultaneous $\mathscr{T}$-unifier (16) of the 3 connections indicated by arcs in Fig. 2. That three connections may be found in 3 deduction steps. Those inferences determine the unifiers σ_1, σ_2 and σ_3 which are subsequent approximations of the simultaneous $\mathscr{T}$-unifier of the three connections. We have $\theta = \sigma_1\sigma_2\sigma_3$ for $\sigma_1 = \{\alpha_1 \mapsto 1, \alpha_2 \mapsto f_2(\varepsilon)\}$, $\sigma_2 = \{\alpha_3 \mapsto \alpha_5, \alpha_4 \mapsto f_2(\varepsilon)\}$ and $\sigma_3 = \{\alpha_6 \mapsto f_2(\varepsilon), \alpha_5 \mapsto f_1(\varepsilon! f_2(\varepsilon))\}$.

4.3. The pool calculus with built-in theory

In this section we introduce a generalization of the pool calculus [17] towards theory reasoning. For an amplification M' of the matrix M to be proved a *pool* of so-called *hooks* represents the set of paths through M' which are

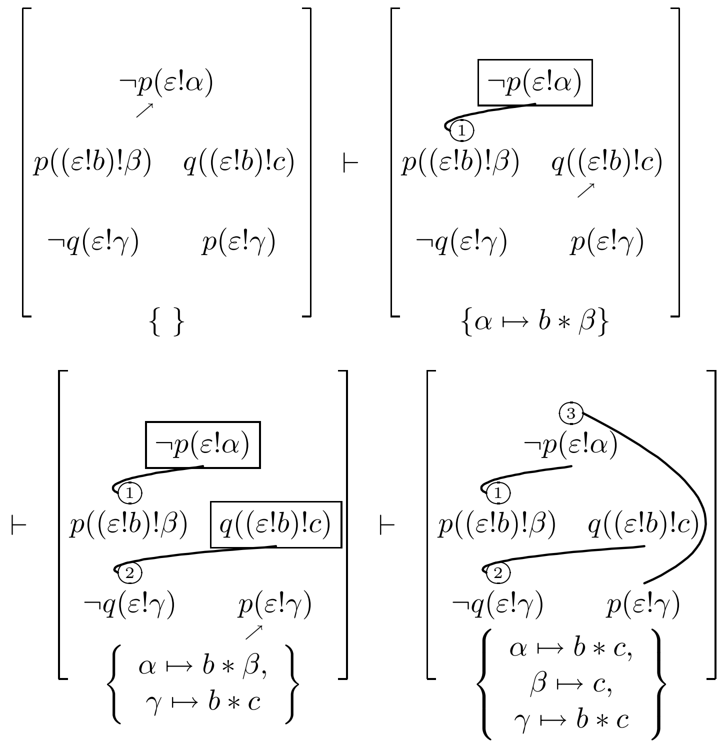

Figure 5. A Sample Deduction

not spanned by the set of theory connections which have been found in the current proof state. Each hook, denoted by $(p \perp \Gamma)$, and consisting of a partial path p in M' and a partial clause Γ in M', represents all paths through M' continuing p via some literal of Γ. Figure 5 shows a three-step derivation under the built-in theory $\mathscr{T}$ given by the equations (5), …, (9). In each deduction step a $\mathscr{T}$-connection is detected and a most general $\mathscr{T}$-unifier has to be computed. The three $\mathscr{T}$-connections are drawn as arcs in the final proof state (the rightmost in Figure 5). The mating U formed by those $\mathscr{T}$-connections is spanning the matrix in Figure 5. A diagonal arrow pointing to the *current goal* appears in every but the rightmost matrix. The *current path* p is given by the set of the boxed literals. In each of the inference steps a $\mathscr{T}$-connection is found which contains the current goal L. In the last inference the found connection is subset of $p \cup \{L\}$. This so-called *reduction step* does not generate additional goals. This is not the case in the first two inference steps, so-called *extension steps*. The first extension step solves the initial goal $\neg p(\varepsilon!\alpha)$ with the substitution $\{\alpha \mapsto b * \beta\}$ but opens a new goal $q((\varepsilon!b)!c)$. This goal is solved by extension step 2, which in turn opens the goal $p(\varepsilon!\gamma)$. The latter is solved by the last inference. For more details see [17] and [20].

Definition 4.5. (Pools, hooks) A *hook* for a matrix M is a pair (p, Γ) where p is a partial path in an amplification M' of M and Γ is a sub-clause of a clause $\Gamma' \in M'$ such that $p \cap \Gamma = \emptyset$. The hook (p, Γ) will be denoted by $(p \perp \Gamma)$. The partial path p is called the *current path*. The elements of Γ are called *goals*. The *set of paths represented by the hook* $(p \perp \Gamma)$ is the set $\{p' \mid \exists L(p \cup \{L\} \subset p', p' \cap \Gamma = \{L\}), p' \text{ is a path through } M'\}$. It will be denoted by $Paths_{(p\perp\Gamma)}$. A hook $(p \perp \emptyset)$ will be called a *solved hook*, and a hook of the form $(\emptyset \perp \Gamma)$ is called an *initial hook*.

An inference step chooses a hook, removes it from the pool, and eventually produces some new hooks. The rules of a calculus describe how to construct new hooks from a chosen hook.

Definition 4.6. ($\mathscr{T}$-connection inference) Let $\mathcal{U}$ be a complete set of $\mathscr{T}$-connections and M a matrix. A *$\mathscr{T}$-connection inference* is an inference rule of the form

$$\frac{(p \perp \Gamma_0, L_0) \qquad \Gamma_1 \cup \{L_1\}, \ldots, \Gamma_n \cup \{L_n\}}{(p \perp \Gamma_0), (p, L_0 \perp \Gamma_1), \ldots, (p, L_0, \ldots, L_{n-1} \perp \Gamma_n)} \quad \sigma$$

where (1) $(p \perp \Gamma_0, L_0)$ is a hook, called the *chosen hook*, (2) if $0 < n$ then the clauses $\Gamma_1 \cup \{L_1\}, \ldots, \Gamma_n \cup \{L_n\}$ are copies of clauses from M, called the *extension clauses*, (3) σ is a substitution, (4) the hooks $(p \perp \Gamma_0)$, $(p, L_0 \perp \Gamma_1)$, …, $(p, L_0, \ldots, L_{n-1} \perp \Gamma_n)$ are called *new hooks* and (5) there exists a sub-path q of p such that $u \in \mathcal{U}$ and $\sigma(u)$ is $\mathscr{T}$-complementary for the partial path $u = q \cup \{L_0, \ldots, L_n\}$. A $\mathscr{T}$-connection inference is called an *extension step* if $n \neq 0$ and a *reduction step* else.

Example 4.6. Let us return to the sample derivation in Figure 5. In that example an equational theory $\mathscr{T}$ has been assumed which contains the equation $(\varepsilon!\alpha)!\beta = \varepsilon!(\alpha * \beta)$. Let $\mathcal{U}$ be the set of all unordered pairs of literals $\{p(t_1, \ldots, t_n), \neg p(t'_1, \ldots, t'_n)\}$ such that for each i with $1 \leq i \leq n$ the terms t_i and t'_i are $\mathscr{T}$-unifiable. A theory extension is an inference rule of the form

$$\frac{(p \perp L_0, \Gamma_0) \qquad L_1, \Gamma_1}{(p \perp \Gamma_0),\ (p, L_0 \perp \Gamma_1)} \ \sigma$$

where (1) L_1, Γ_1 is a copy of a clause from M, called the *extension clause* and (2) for $u = \{L_0, L_1\}$ holds $u \in \mathcal{U}$ and $\sigma(u)$ is $\mathscr{T}$-complementary. A theory reduction rule has the form

$$\frac{(p \perp L_0, \Gamma_0)}{(p \perp \Gamma_0)} \ \sigma$$

where for some literal $L_1 \in p$ and $u = \{L_0, L_1\}$ holds $u \in \mathcal{U}$ and $\sigma(u)$ is theory complementary.

Definition 4.7. (Rule application) A rule

$$\frac{h \qquad \Gamma_1, \ldots, \Gamma_n}{H} \ \sigma$$

may be applied to a pool P if $h \in P$. The new pool is obtained from P by removing h, then adjoining those hooks from H which are not solved and finally applying the substitution σ to the resulting pool. The clause copies used in an inference within a derivation must have always a set of new variables, i.e. those not occurring already in the pool. Moreover if $u \in \mathcal{U}$ is the $\mathscr{T}$-connection chosen in the considered rule application then the variables from $\mathrm{Var}(\sigma) \setminus \mathrm{Var}(u)$ must not occur in P.

An *initial pool* in a derivation consists of a single initial hook. Now a *derivation* may be defined as a sequence of rule applications which starts from an initial pool. A derivation is called *ground* if the unifier in every $\mathscr{T}$-connection step is empty. A derivation is *successful* if its last element is the empty pool. The calculus is sound, because in every state of a derivation the pool represents all paths, such that there still have to be found theory connections spanning them.

Proposition 4.4. (Soundness) *The theory connection calculus is sound.*

The completeness proof consists of the steps Herbrand theorem, ground completeness and lifting lemma. The Herbrand theorem (4.1) and the lifting lemma rely on the completeness of a given set of theory connections $\mathcal{U}$ and the solvability of the theory unification problem in $\mathcal{U}$. The proof of the ground completeness relies on the properties of minimal spanning matings. The following result may be found already in [20].

Theorem 4.2. (General Completeness theorem) *Suppose that for a theory $\mathscr{T}$ and a query language $\mathcal{Q}$ there is given a decidable set $\mathcal{U}$ of $\mathscr{T}$-connections which is $\mathscr{T}$-complete w.r.t. $\mathcal{Q}$ and the $\mathscr{T}$-unification problem in $\mathcal{U}$ is solvable.*

Then for every $\mathscr{T}$-unsatisfiable query from $\mathcal{Q}$ exists a clause $\Gamma \in \mathcal{Q}$ and a successful derivation starting from the initial pool $\{(\ \perp \Gamma)\}$ such that in each inference according to Definition 4.6 for the chosen connection u holds $u \in \mathcal{U}$ and the chosen T-unifier σ is an element of the complete set of $\mathscr{T}$-unifiers S_u for u.

For an outline of the completeness proof we conclude from the Herbrand theorem (4.1) that for each $\mathscr{T}$-unsatisfiable matrix $M \in \mathcal{Q}$ exist an amplification M', a ground substitution σ and a $\mathcal{U}$-mating U spanning M' such that σ is a minimal $\mathscr{T}$-unifier of u for each $u \in U$. We assume U to be minimal with respect to inclusion and construct a derivation satisfying the following invariant: There exist a minimal subset $U' \subseteq U$ and a substitution σ' such that for each unsolved goal L in a hook $(p \perp L, \Gamma)$ exist $u \in U'$, σ'' and σ''' such that $L \in \sigma'(u)$, σ'' is a minimal $\mathscr{T}$-unifier, $\sigma'' \in S_{\sigma'(u)}$, and $\sigma =_{\mathsf{var}(M')} \sigma'\sigma''\sigma'''$. This general theorem will be specialized to the case of hybrid theories. In Section 5 we introduce sufficient criteria for obtaining a complete set of theory connections for a hybrid theory if those are given for its constituents. The criteria can be applied to the target logic of the algebraic translation of multi-modal logic and of extended multi-modal logic of [9] (cf. Section 3.1).

5. Combining theories

In Section 3 we have discussed examples justifying the treatment of the background reasoner as a hybrid system itself. Let us now forge precise notions from the observations made for the target logic of the algebraic translation of multi-modal reasoning. Our goal is to construct a $\mathscr{T} \cup \Re$-reasoner from a $\mathscr{T}$-reasoner and a $\Re$-reasoner. A formula will be considered as consisting of a $\mathscr{T}$-layer and an $\Re$-layer. The intended $\mathscr{T} \cup \Re$-reasoner should try to find a $\mathcal{U}_{\mathscr{T}}$-connection if the current goal is in the $\mathscr{T}$-layer and a $\mathcal{U}_{\Re}$-connection if the current goal is in the $\Re$-layer. We formulate sufficient conditions such $\mathcal{U}_{\mathscr{T}} \cup \mathcal{U}_{\Re}$ is a complete set of T$\cup\Re$-connections for $\mathcal{Q}$ if so are $\mathcal{U}_{\mathscr{T}}$ for $\mathcal{Q}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ for $\mathcal{Q}_{\Re}$. Moreover the theory unification problems in both $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ should not interfere. The last condition will us allow to use just the unification algorithms for the connections belonging to one of both layers without change.

Definition 5.1. Let a theory be given by its sub-theories $\mathscr{T}$ and $\Re$ which are formulated within the signatures Σ and Δ respectively. Then we say that $\mathscr{T}$ and $\Re$ *form a hybrid theory* in the union $\Sigma \cup \Delta$ of both signatures.

Definition 5.2. Let the theories $\mathscr{T}$ and $\Re$ form a hybrid theory in the union $\Sigma \cup \Delta$ of their signatures and let $\mathcal{Q}$ be a query language formulated in a signature which contains $\Sigma \cup \Delta$.

Every clause C in a matrix $M \in \mathcal{Q}$ contains then two sub-clauses $C_{\mathscr{T}}$ and $C_{\Re}$ consisting of literals L expressed in signature Σ (respectively L' expressed in signature Δ). The set of nonempty sub-clauses $C_{\mathscr{T}}$ of M will be called the $\mathscr{T}$-layer of M. Analogously will be defined the $\Re$-layer of M. By $\mathcal{Q}_{\mathscr{T}}$ (analogously $\mathcal{Q}_{\Re}$) will be denoted the set of all matrices being the $\mathscr{T}$-layer (respectively the $\Re$-layer) of a query from $\mathcal{Q}$. $\mathcal{Q}_{\mathscr{T}}$ (analogously $\mathcal{Q}_{\Re}$) will be called the *$\mathscr{T}$-layer* (respectively the $\Re$-layer) of $\mathcal{Q}$. If for a matrix $M \in \mathcal{Q}$ every of its clauses is the union of its $\mathscr{T}$- and $\Re$-layers then M will be called *covered by its $\mathscr{T}$- and $\Re$-layers*. If every matrix $M \in \mathcal{Q}$ is covered by its $\mathscr{T}$- and $\Re$-layers then query language $\mathcal{Q}$ is said to be *covered by its $\mathscr{T}$- and $\Re$-layers*.

Example 5.1. In the wise men puzzle in Figure 2 signatures Σ of the $\mathscr{T}$-layer and Δ of the $\Re$-layer share the function symbols $!$, ε, f_1, f_2, a and b. Σ contains w as the single predicate symbol, Δ contains k and the equality symbol $=$. The target language of the algebraic translation of multi-modal logic is covered by its $\mathscr{T}$- and $\Re$-layers. Since the sets of predicate symbols of the $\mathscr{T}$-layer and the $\Re$-layer are disjoint, for each literal L the sets of $\mathscr{T}$- and of $\Re$-connections L might belong to are disjoint.

Example 5.2. In the safe puzzle from Figure 4 the signatures Σ of the $\mathscr{T}$-layer and Δ of the $\Re$-layer share the equality symbol $=$ and the function symbols ϕ, ψ, h, i, n, s, d, p, joe, 1 and ε. Thus, the sets of predicate symbols for the $\mathscr{T}$-layer and the $\Re$-layer are not disjoint. Σ contains moreover w, c and o and Δ contains k, u, a and as as predicate symbols. The target language of the algebraic translation of extended multi-modal logic is covered by its $\mathscr{T}$- and $\Re$-layers. Again we may show that for each literal L the sets of $\mathscr{T}$-connections and of $\Re$-connections L can belong to are disjoint. Definition 5.3 introduces a notion for this property.

Definition 5.3. Let $\mathscr{T}$ and $\Re$ form a hybrid theory in the union $\Sigma \cup \Delta$ of signatures. Let $\mathcal{Q}$ be a query language formulated in a signature containing both signatures Σ and Δ. Moreover, let $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ be sets of $\mathscr{T}$-connections and of $\Re$-connections. We say that $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ are *separated w.r.t. Q* if and only if there does not exist connections $u \in \mathcal{U}_{\mathscr{T}}$ and $u' \in \mathcal{U}_{\Re}$ with $\emptyset \neq u \cap u'$.

The following propositions 5.1 and 5.2 give sufficient criteria for the theory completeness of the union of sets of theory connections that are theory complete with respect to the constituent sub-theories of a hybrid theory. The case of the target logic of the multi-modal logic will be covered by Proposition 5.1. The criterion Proposition 5.2 covers the case of the target logic of the algebraic translation of extended multi-modal logic.

Definition 5.4. Let M be a set of instances of clauses and U a mating in M. For every literal L in M we define the *set* R_L *of clauses reachable from* L *via* U as the least set being closed with respect to the following condition: If there exists a connection $u \in U$ such that one of the literals of u is L or a literal in a clause being element of R_L then also any clause containing a literal of u different from L belongs to R_L.

Proposition 5.1. *Let theories $\mathscr{T}$ and $\Re$ be expressed in the signatures Σ and Δ respectively form a hybrid theory such that $\mathscr{T} \cup \Re$ is consistent. The query language $\mathcal{Q}$ is formulated in the union $\Sigma \cup \Delta$ of signatures. Moreover suppose that:*

(1) *The sets of $\mathscr{T}$-connections $\mathcal{U}_{\mathscr{T}}$ and of $\Re$-connections $\mathcal{U}_{\Re}$ are complete w.r.t. $\mathcal{Q}_{\mathscr{T}}$ and $\mathcal{Q}_{\Re}$ respectively.*
(2) *In $\mathcal{Q}$ equality literals occur only negative.*
(3) *In both theories positive equality literals may occur only within conditional equations.*
(4) *The sets of predicate symbols occurring in $\mathscr{T} \cup \mathcal{Q}_{\mathscr{T}}$ and $\Re \cup \mathcal{Q}_{\Re}$ are disjoint.*
(5) *If equality occurs in $\mathscr{T} \cup \Re$ then let $\mathscr{T}_1$ be that of the sub-theories $\mathscr{T}$ and $\Re$ that does not contain equality and $\mathcal{U}_1$ be the set of theory connections for that sub-theory. Moreover let $\mathscr{E}$ be the set of equational axioms in $\mathscr{T} \cup \Re$. For every $u \in \mathcal{U}_1$ and substitution σ holds $\mathscr{E} \cup \mathscr{T}_1 \vDash \sigma(\bigvee \bar{u})$ if and only if $\mathscr{T}_1 \vDash \sigma(\bigvee \bar{u})$.*

Then the sets of $\mathscr{T}$-connections $\mathcal{U}_{\mathscr{T}}$ and $\Re$-connections $\mathcal{U}_{\Re}$ are separated with respect to $\mathcal{Q}$ and $\mathcal{U}_{\mathscr{T}} \cup \mathcal{U}_{\Re}$ is $\mathscr{T}, \Re$-complete with respect to $\mathcal{Q}$.

Proof. Let us suppose that theories $\mathscr{T}$ and $\Re$, signatures Σ and Δ, query language $\mathcal{Q}$ and the sets of $\mathscr{T}$-connections $\mathcal{U}_{\mathscr{T}}$ and of $\Re$-connections $\mathcal{U}_{\Re}$ satisfy the assumptions of the proposition. In order to show that $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ are separated with respect to $\mathcal{Q}$ it is sufficient to observe that the sets of predicate symbols occurring in $\mathscr{T} \cup \mathcal{Q}_{\mathscr{T}}$ and $\Re \cup \mathcal{Q}_{\Re}$ are disjoint. In order to show that $\mathcal{U}_{\mathscr{T}} \cup \mathcal{U}_{\Re}$ is $\mathscr{T}, \Re$-complete with respect to $\mathcal{Q}$ we show first of all that $\mathcal{U}_{\mathscr{T}} \cup \mathcal{U}_{\Re}$ has property 2.1 formulated in Definition 4.3. Let

p be a $\mathscr{T}, \Re$-complementary ground path. We have to show there exists a sub-path such that $u \in \mathcal{U}_{\mathscr{T}} \cup \mathcal{U}_{\Re}$. We consider p as a set of unit clauses. By the compactness theorem for first-order logic there exists a finite set M of instances of clauses of $\mathscr{T}$ and of $\Re$ and a minimal mating U spanning $M \cup p$. Let u be the multi-set of all literals of p which are element of a connection in U. Then u is not empty because of the consistency of $\mathscr{T} \cup \Re$. Because the sets of predicate symbols occurring in $\mathscr{T} \cup \mathcal{Q}_{\mathscr{T}}$ and $\Re \cup \mathcal{Q}_{\Re}$ are disjoint either for every connection $u' \in U$ holds $u \in \mathcal{Q}_{\Sigma}$ or for every connection $u' \in U$ holds $u \in \mathcal{Q}_{\Delta}$. Therefore, u is either element of $\mathcal{Q}_{\mathscr{T}}$ or of $\mathcal{Q}_{\Re}$. If $u \in \mathcal{Q}_{\mathscr{T}}$ (the case $u \in \mathcal{Q}_{\Re}$ may be treated analogously) then there exists $u'' \in \mathcal{U}_{\mathscr{T}}$ such that $u'' \subseteq u$, and therefore $u'' \subseteq p$, because $\mathcal{U}_{\mathscr{T}}$ is $\mathscr{T}$-complete with respect to $\mathcal{Q}_{\mathscr{T}}$. Both $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ satisfy condition 2.2 of Definition 4.3. Therefore also $\mathcal{U}_{\mathscr{T}} \cup \mathcal{U}_{\Re}$ has this property. □

Example 5.3. Let us observe that in a matrix belonging to the target language of the algebraic translation of multi-modal logic theory connections either are in the non-sort part, i.e. those discussed in Example 4.1, or in the sort part, i.e. those discussed in Example 4.2. This is obvious because both parts of the hybrid theory are expressed by use of disjoint sub-sets of predicate symbols and equality does not occur in the query language. Therefore, in order to obtain a complete set of theory connections for the hybrid theory consisting of $\mathscr{T}$ and $\Re$ it is sufficient to take just the union of the complete sets of theory connections $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$.

Proposition 5.2. *Let theories $\mathscr{T}$ and $\Re$ be expressed in the signatures Σ and Δ respectively form a hybrid theory such that $\mathscr{T} \cup \Re$ is consistent. The query language $\mathcal{Q}$ is formulated in the union $\Sigma \cup \Delta$ of signatures. Moreover suppose that:*

(1) *The sets of $\mathscr{T}$-connections $\mathcal{U}_{\mathscr{T}}$ and of $\Re$-connections $\mathcal{U}_{\Re}$ are complete w.r.t. $\mathcal{Q}_{\mathscr{T}}$ and $\mathcal{Q}_{\Re}$ respectively.*

(2) *In $\mathcal{Q}$ equality literals may occur only negative.*

(3) *In both theories $\mathscr{T}$ and $\Re$ positive equality literals may occur only within conditional equations.*

(4) *The sets of non-equational predicate symbols occurring in $\mathscr{T} \cup \mathcal{Q}_{\mathscr{T}}$ and $\Re \cup \mathcal{Q}_{\Re}$ are disjoint.*

(5) *If $\mathscr{T}_{=+}$ (and $\Re_{=+}$) are the sets of non-negative equational clauses in $\mathscr{T}$ (and $\Re$ respectively) then hold $\mathscr{T} \vDash \Re_{=+}$ and $\Re \vDash \mathscr{T}_{=+}$.*

Then the set $\mathcal{U}_{\mathscr{T}} \cup \mathcal{U}_{\Re}$ is $\mathscr{T}, \Re$-complete with respect to $\mathcal{Q}$.

Proof. Analogously to the proof of Proposition 5.1 it is sufficient to show that for every $\mathscr{T}, \Re$-complementary ground path $p \in \mathcal{Q}$ exists $u \in \mathcal{U}_{\mathscr{T}} \cup \mathcal{U}_{\Re}$ such that $u \subseteq p$. Let p be such path. We consider p as a set of unit clauses. By the compactness theorem for first-order logic there exists a finite set M of ground instances of clauses of $\mathscr{T}$ and $\Re$ and a minimal mating U spanning $M \cup p$. Let u' be the multi-set of all literals of p which are element of a connection in U. Then u' is not empty because of the consistency of $\mathscr{T} \cup \Re$. In order to complete the proof it will be sufficient to show that u' is $\mathscr{T}$-unsatisfiable or $\Re$-unsatisfiable. We prepare this proof by the following three claims.

Claim 1: Let L be a negative equational literal in $M \cup p$. Then every clause reachable from L via U is a conditional equation and $R_L \vDash \neg L$.

Proof : Immediately from assumption (3) follows that R_L consists of conditional equations only. The minimality of U ensures that U also is spanning $R_L \cup \bar{L}$. Therefore $R_L \vDash \neg L$.

Claim 2: Under the assumptions of claim 1 holds: If $L \in \mathcal{Q}_{\mathscr{T}}$ — the opposite case may be treated by symmetry — then every conditional equation e reachable from L and being element of $\Re$ may be substituted by a set of clauses $R'_e \subseteq \mathscr{T}$ such that $R'_e \vDash e$.

Proof : Follows immediately from assumption (5).

Claim 3: Let $L \in p$ be an non-equational literal. If $L \in \mathcal{Q}_{\mathscr{T}}$ — again the opposite case may be treated by symmetry — then every clause $\Gamma \in (\mathscr{T} \cup \Re) \cap R_L$ containing non-equational literals satisfies $\Gamma \in \mathscr{T}$.

Proof: From the assumptions (2), (3) and (4) and follows that the predicate symbol of every non-equational literal in any clause reachable from L belongs to Σ. The assumptions (3) and (4) are important for this conclusion because they ensure that any clause reachable from an equational literal in a non-equational clause reachable from L is a conditional equation. We complete the proof by the following case analysis.

Case 1: The sub-path u' of p contains an equality literal L. By assumption (2) L is negative and by claim 1 all clauses in R_L are conditional equations and $\mathcal{R}_L \vDash \neg L$. Therefore $u' = \{L\}$ because U is a minimal mating spanning $M \cup p$. Suppose that R_L contains clauses being instances of clauses from $\mathscr{T}$ (the case symmetric case may be proved analogously). Then according to assumption (4) every element e of R_L being not an instance of a clause in $\mathscr{T}$ may be substituted by a set of conditional equations implying e. Therefore

$\mathscr{T} \models \bar{u}'$. Because $\mathcal{U}_{\mathscr{T}}$ is $\mathscr{T}$-complete with respect to $\mathcal{Q}_{\mathscr{T}}$ there exists a $\mathscr{T}$-connection $u'' \subseteq u'$ and therefore $u'' \subseteq p$.

Case 2: The subpath u' of p does not contain any equality literal L. We suppose that u' contains a $\mathscr{T}$-layer literal — the opposite case may be treated by symmetry reasoning. Then by claim 3 every non-equational clause in $R_L \cap (\mathscr{T} \cup \Re)$ is element of $\mathscr{T}$. By claim 2 every equational clause e being element of $R_L \setminus \mathscr{T}$ may be substituted by a finite subset $R'_e \subseteq \mathscr{T}$ such that $R'_e \models e$. Therefore $\mathscr{T} \models \bigvee \bar{u}'$. □

Now we discuss briefly the unification problem in sets of hybrid theory connections. We restrict our attention to the case that for given theories $\mathscr{T}$ and $\Re$ a complete set of theory connections is given by the union of sets of theory connections that are complete with respect to the respective theories. What we have in mind is that unification of a theory connection u is either $\mathscr{T}$-unification if u is a $\mathscr{T}$-connection or $\Re$-unification otherwise. This leads to the notion of non-interfering unification problems.

Definition 5.5. Let $\mathcal{U}_{\Re}$ and $\mathcal{U}_{\mathscr{T}}$ be sets of theory connections for the components of a hybrid theory $\mathscr{T}, \Re$. We say that *the unification problems in $\mathcal{U}_{\Re}$ and $\mathcal{U}_{\mathscr{T}}$ do not interfere* if and only if

(1) for every $u \in \mathcal{U}_{\mathscr{T}}$ and for every substitution σ holds: σ is a $\mathscr{T}$-unifier of u if and only if σ is $\mathscr{T}, \Re$-unifier of u and

(2) for every $u \in \mathcal{U}_{\Re}$ and for every substitution σ holds: σ is a $\Re$-unifier of u if and only if σ is $\mathscr{T}, \Re$-unifier of u.

Let $\mathcal{U}_{\mathscr{T}} \cup \mathcal{U}_{\Re}$ be the set of theory connections discussed in Section 3.1 for the target logic of the algebraic translation of multi-modal logic. Then the unification problems in $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ do not interfere. Let $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ be the sets of theory connections discussed in Section 3.2 for the sub-theories $\mathscr{T}$ and $\Re$ of the target logic of the algebraic translation of extended multi-modal logic. Then the unification problems in $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ do not interfere.

Proposition 5.3. *Let theories $\mathscr{T}$ and $\Re$, which are expressed in the signatures Σ and Δ respectively, form a hybrid theory, such that $\mathscr{T} \cup \Re$ is consistent. The query language $\mathcal{Q}$ is formulated in the union $\Sigma \cup \Delta$ of signatures. Moreover suppose that the assumptions (1)–(5) of Proposition 5.1 (resp. 5.2) are satisfied. Then the unification problems in $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ do not interfere.*

Proof. In the non-trivial direction of the equivalence to be proved we have to show that every $\mathscr{T} \cup \Re$-unifier of a $\mathscr{T}$-connection $u \in \mathcal{U}_{\mathscr{T}}$ is a $\mathscr{T}$-unifier of u and that every $\mathscr{T} \cup \Re$-unifier of a $\Re$-connection $u \in \mathcal{U}_{\Re}$ is a $\Re$-unifier of u. The latter claim is satisfied because $\mathscr{T}$ and $\Re$ have no common predicate symbols and $\Re$ does not contain the equality sign. The former claim follows from assumption (5). □

Now a completeness theorem for hybrid theories may be proved.

Theorem 5.1. (Completeness theorem for hybrid theories) *Let $\mathcal{Q}$ be a query language expressed in a signature containing Σ and Δ. Moreover, let $\mathcal{Q}_{\Re}$ and $\mathcal{Q}_{\mathscr{T}}$ be the $\Re$-layer and $\mathscr{T}$-layer of $\mathcal{Q}$ respectively. Let $\mathcal{U}_{\Re}$ and $\mathcal{U}_{\mathscr{T}}$ be complete sets of $\Re$-connections and $\mathscr{T}$-connections which satisfy the assumptions either of Proposition 5.1 or 5.2. Then for every $\mathscr{T}, \Re$-unsatisfiable query $M \in \mathcal{Q}$ exists a clause $\Gamma \in M$ and a successful derivation starting from the initial pool $\{(\perp \Gamma)\}$ such that in each inference according to Definition 4.6 for the chosen connection u holds either $u \in U_{\Re}$ or $u \in \mathcal{U}_{\mathscr{T}}$ and for the chosen theory unifier $\sigma \in S_u$, with S_u being the set of $\mathscr{T}$-unifiers or, respectively, $\Re$-unifiers.*

Proof. Due to Proposition 5.1 and 5.2 the set of $\mathscr{T}, \Re$-connections $\mathcal{U}_{\Re} \cup \mathcal{U}_{\mathscr{T}}$ is $\mathscr{T}, \Re$-complete w.r.t. query language $\mathcal{Q}$. Due to Proposition 5.3 the unification problem in $\mathcal{U}_{\Re} \cup \mathcal{U}_{\mathscr{T}}$ is solvable and applying the $\mathscr{T}$-unification procedure to $\mathcal{U}_{\mathscr{T}}$-connections and the $\Re$-unification procedure to $\mathcal{U}_{\Re}$-connections provides a solution to the $\mathcal{U}_{\Re} \cup \mathcal{U}_{\mathscr{T}}$-unification problem. Thus the assumptions of Theorem 4.2 are satisfied and the calculus for the hybrid theory is complete. □

Let $\mathcal{U}_{\mathscr{T}}$ and $\mathcal{U}_{\Re}$ be either the set of theory connections discussed in Section 3.1 for the target logic of the algebraic translation of multi-modal logic or those discussed in Section 3.2 for the sub-theories $\mathscr{T}$ and $\Re$ of the target logic of the algebraic translation of extended multi-modal logic. Then for both cases we obtain a complete calculi instantiating the theory pool calculus (cf. Section 4.3) as a corollary of Theorem 5.1.

6. Concluding remarks

A prover for multi-modal logic has been implemented by a joint effort of research groups in Leipzig and Caen. We used the calculi description interface

CaPrI of the PTTP-prover ProCom [16]. The algebraic translation of Francoise Debart and Patrice Enjalbert from multi-modal logic to a language of constrained clauses has been implemented by Zoltán Rigó [21]. The translation generates a constraint theory that provides information about the interaction between modalities, the properties of the occurring modalities and the dependencies introduced by Skolemization. For reasoning in the non-constraint part of a matrix being element of the target language an A1-unification algorithm due to Francoise Debart and Patrice Enjalbert [8] is used. The algorithm has been tuned for this application. The used implementation is due to Gilbert Boyreau [5]. ProCom and his interface has been implemented by Gerd Neugebauer. He also integrated constraint reasoning into ProCom.

We examined the algebraic translation of multi-modal logic into a fragment of first-order logic. To the target of this translation we applied a general framework which allows to build-in theories into provers which are based on the connection method. For this purpose we introduced the notion of a hybrid theory. We obtained a completeness result for a connection method based calculus dealing with hybrid theories. A brief overview about an implementation has been given. Ongoing research considers the combination of theories given syntactically with those given semantically.

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