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IS STOIC LOGIC CLASSICAL?

Abstract. In this paper I would like to argue that Stoic logic is a kind of relevant logic rather than the classical logic. To realize this purpose I will try to keep as close as possible to Stoic calculus as expressed with the help of their arguments.

CONTENTS

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1. Introduction

What in fact is Stoic logic? Is it classical? This is a long-standing question. Before the contribution of Peirce and Łukasiewicz Stoic logic was considered to be merely a continuation of the ideas of Aristotle and the Peripatetic school. However, a series of works by Łukasiewicz initiated the consideration of Stoic logic as a propositional calculus [2].

Now it is rather common to consider Stoic logic as equal to the classical calculus. Is it true however?

2. Data about Stoic logic

Let us shortly resume what we know about Stoic logic:

1. The Stoics paid attention to the form of an expression, seeing that in some cases this determines meaning.
2. The Stoics had deep insights about what is important in logic. They discussed, for instance, the nature of implication and distinguished at least four ways of understanding conditional statements.
3. The Stoics had created the calculus for the purpose of reasoning and focused their attention on what conclusion follows from premises. So they used arguments. This is a prototype of today's sequent, meaning an ordered pair of sequences of formulas.

Thus when we consider Stoic calculus we should talk about a calculus of sequents rather than about sets of true expressions. Some of these arguments the Stoics considered as undemonstrated. They believed that these arguments did not demand proof. Today we would say that they treated them as primitive rules of inference in a system of natural deduction. They accepted at least five of them and used some kind of variables to express these rules.

We have examples of substitution in their arguments[1], we can therefore either consider schemas of sequents instead of sequents or add the rule of substitution.

[1] For example [5], VIII, 233 and 236.

Let me use X, Y for sequences of formulas and A, B for a single formula. The five undemonstrated stoic arguments are as follows:

(R1) $A \to B,\ A \vdash B$ *modus ponens*

(R2) $A \to B,\ \neg B \vdash \neg A$ *rule of contraposition*

(R3) $\neg(A \wedge B),\ A \vdash \neg B$
(R3′) $\neg(A \wedge B),\ B \vdash \neg A$ *disjunctive syllogism*

(R4) $A \veebar B,\ A \vdash \neg B$
(R4′) $A \veebar B,\ B \vdash \neg A$
(R5) $A \veebar B,\ \neg A \vdash B$
(R5′) $A \veebar B,\ \neg B \vdash A$ *rules for the (excluding) disjunction*

Stoic proofs probably looked as follows: They assumed premises of the argument under consideration and used their undemonstrated arguments to obtain the required conclusion. Thus their proofs were similar to the proofs in modern natural deduction systems.

Besides the above-mentioned undemonstrated (primitive) arguments, the Stoics also used four rules called *themata*. They tried to reduce the other arguments to the five axioms using these four rules of inference of arguments. We know only two of these four rules[2]:

$$\text{(MT1)}\quad \frac{X, A, B \vdash C}{X, A, \neg C \vdash \neg B} \qquad\qquad \text{(MT1}'\text{)}\quad \frac{X, A, B \vdash C}{X, B, \neg C \vdash \neg A}$$

$$\text{(MT3)}\quad \frac{X, A \vdash C \qquad Y \vdash A}{X, Y \vdash C}$$

The rule (MT3) is known as the *cut rule*.[3]

According to Benson Mates ([3]) a certain version of the third *thema* or maybe one of the missing *themata* was the so-called *theorema*

$$\text{(TH)}\quad \frac{X \vdash A \qquad X, A \vdash C}{X \vdash C}$$

[2] Apuleius, *In de Interp. Comm.*, ed. Oud., 277–278, Alexander of Aphrodisias, *In An. Pr. Comm.*, ed. Wallies, p. 278 and Simplicius, *In De Caelo*, ed. Heiberg, p. 236.

[3] Notice, that in the original version of the rules (MT1) and (MT3) instead of an arbitrary sequence X, A, B and X, A respectively, were just two premises. This restriction is not essential, however.

To resume, we could understand the Stoics' undemonstrated arguments as axioms and their *themata* as rules of inference ofthe sequents calculus. From the natural deduction point of view, they could also be understood as metarules.

To be more strict, let us accept the following formal definition of the notion of *proof* which formalise the intuitive understanding of proofs in the sequents calculus:

Definition 1. An argument $X \vdash A$ *has the proof* (in the sequents calculus) iff there is a sequence of sequents $S_1, S_2, \ldots, S_n$, such that:

1° S_n is identical with $X \vdash A$ and for each $1 \leqslant i \leqslant n$ either:
2° S_i is a sequent which belongs to the schema (Rk) or (Rk'), for some $1 \leqslant k \leqslant 5$, or
3° S_i is a seqent obtained from sequents previosly accepted by use of (MT3).

We also know[4] that Stoics used also a so-called *principle of conditionalisation*, which was a sort of the deduction theorem. Roughly speaking, it states that a conclusion is derivable from the premises iff the conditional, in which the antecedent is a conjunction of premises and the consequent is the conclusion, is logically valid. We have to say that unfortunately we don't have any example of the use of this rule as a rule of derivation.

(DT1) $$\frac{A_1,\ A_2,\ \ldots,\ A_n\ \vdash\ B}{\vdash (A_1 \wedge A_2 \wedge \cdots \wedge A_n) \to C}$$ *Deduction Theorem*

(DT2) $$\frac{\vdash (A_1 \wedge A_2 \wedge \cdots \wedge A_n) \to C}{A_1,\ A_2,\ \ldots,\ A_n\ \vdash\ B}$$ *Detachment*

3. Stoic logic = the relevant logic R?

Let me restrict myself to a consideration of the implicational part of Stoic logic. Thanks to Sextus Emiricus we know, that the Stoics were not ready to accept redundant arguments, i.e. arguments which among their premises have at least one which is not necessary to derive conclusion. Stoics didn't like to accept sequents of the form: $A, B \vdash A$.[5] Notice, that Benson Mates perhaps defending thesis that Stoic logic was classical, supposes that Sextus

[4] See [4], II, 135 and [5], VIII, 415.
[5] See [4], II, 147 and [5], VIII, 431.

is wrong here. On the other hand we haven't got possibility of weakening sequents. These two observations bring to mind the calculae of entailment and relevant logics.

Let us remind ourselves of the implicational part of relevant logic **R** expressed in the sequent language:

(Id)	$A \vdash A$	*Identity*
(Contr)	$\dfrac{X, A, A, Y \vdash C}{X, A, Y \vdash C}$	*Contraction*
(Perm)	$\dfrac{X, A, B, Y \vdash C}{X, B, A, Y \vdash C}$	*Permutation*
$(\to\vdash)$	$\dfrac{X \vdash A \qquad B, Y \vdash C}{X, Y, A \to B \vdash C}$	*Implication on the left*
$(\vdash\to)$	$\dfrac{X, A \vdash B}{X \vdash A \to B}$	*Implication on the right*

We know that at least some Stoics acknowledged rules (Id) since they considered identity low: $A \to A$, so by detachment rule (DT2) they should accept (Id) as well.

It is rather clear that without refusing the spirit of Stoic logic we can add to the Stoics rules the rule of contraction (Contr) and permutation (Perm).

The rule *implication on the right* needs more attention. Besides the rule of conditionalisation we don't have any possibility of proving any sequent with the implication in the conclusion. But since (DT1) is a version of the deduction theorem, it is acceptable to add $(\vdash\to)$ to the reconstructed Stoic system.

Formally we have to change our definition of the notion of *the proof*:

Definition 2. An argument $X \vdash A$ *has the proof* (in the implicational part of sequents calculus) iff there is a sequence of sequents $S_1, S_2, \dots, S_n$, such:

1° S_n is identical with $X \vdash A$ and for each $1 \leqslant i \leqslant n$ either:
2° S_i is a sequent which belongs to the schema (R1) or (Id), or
3° S_i is a sequent obtained from sequents previously accepted by use of (MT3), (Contr), (Perm) or $(\vdash\to)$.

We have the following theorem:

Theorem 1. *An argument has the proof in* **R** *iff it has the proof in the sense of the Definition 2.*

Proof (draft). Firstly, we have to show that each sequent provable in our system has also proof in the logic **R**. Since (Contr), (Perm) and ($\vdash\rightarrow$) are just the same in both calculi, it is enough to show that (R1) and (MT3) can be proved in the logic **R**. Using **($\rightarrow\vdash$)** we can prove the sequent (R1):

$$(\rightarrow\vdash)\quad \frac{A \vdash A \qquad B \vdash B}{A,\ A \rightarrow B\ \vdash B}$$

By our last observation it is easy to see, that each proof in the sense of the Definition 2, can be rewritten as a proof in the implicational part of (R1) with aditional (MT3), but by the "Elimination Theorem" *the cut rule* can be eliminated[6], therefore the system **R** contains the implicational part of our system.

In the opposite direction, we have to show that the third rule ($\rightarrow\vdash$) can be proved in the Stoic system. By using their third metarule and the first undemonstrated argument we indeed obtain:

$$\text{(MT3) \& (Perm)}\quad \frac{A \rightarrow B,\ A\ \vdash B \qquad X \vdash A}{X,\ A \rightarrow B\ \vdash B}$$

and once more

$$\text{(MT3) \& (Perm)}\quad \frac{X,\ A \rightarrow B\ \vdash B \qquad B,\ Y\ \vdash C}{X,\ Y,\ A \rightarrow B\ \vdash C}$$

□

4. Conclusion

The Stoics started investigation of sententional calculae in general. They also provided the foundations for modal propositional logic and, of course, the classical logic. Their main calculus was however more close to the relevant logic rather then to the classical one.

References

[1] Anderson, A. R., N. D. Belnap, Jr., *Entailment. The Logic of Relevance and Necessity*, vol. I, Princeton University Press, Princeton 1975.

[6] See [1], p. 62–67 and [6], p. 88–103. You can find there approriate proof for implicational part of **E**, the proof for **R** is going in the similar way.

[2] Łukasiewicz, J., "Z historii logiki zdań", Przegląd Filozoficzny XXXVII, 1934; also in *Z zagadnień logiki i filozofii*, PWN, Warszawa 1961; in German: "Zur Geschichte der Aussagenlogik", Erkenntnis V, 1935, 2nd issue.

[3] Mates, B., *Stoic Logic*, University of California Press, Berkeley & Los Angeles 1971.

[4] Sexstus Empirikus, *Zarysów pirrońskich księga pierwsza, druga i trzecia*, nakładem Polskiej Akademii Umiejętności, Kraków 1931; Sexstus Empiricus: *Hyp. Pyrrh., Opera*, Vol. 1, Teubner, Leipzig 1912, H. Mutschmann (ed.).

[5] Sekstus Empiryk, "Przeciw logikom", (*Przeciw uczonym*, ks VII i VIII), PWN, Warszawa 1970; Sextus Empiricus: *Adv. Math., Opera*, Vol. 2, Teubner, Leipzig 1914, E. Mutschmann (ed.).

[6] Szabo, M. E. (ed.), *The Collected Papers of Gerhard Gentzen*, North-Holland Publishing Company, Amsterdam 1969.

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