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FINITELY MANY-VALUED PARACONSISTENT SYSTEMS

Abstract. In the paper n -valued paraconsistent matrices are defined by an adaptation of the well-known Łukasiewicz's matrices. An appropriate set of axioms is presented and the 3-valued case is examined.

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In this paper we shall make an attempt to find a solution to a problem concerned with finitely many-valued paraconsistent logics. The idea of developing them was one of the items on the list of open problems posed by D'Ottaviano in [3] (for related aspects of many-valued paraconsistent logics, see [2]).

We shall define n -valued matrices, for any natural number $n > 2$ under which a theory may be both syntactically inconsistent and non-trivial. The way the class of finitely many-valued logics (understood as sets of tautologies) is obtained appears to be a simple and elegant modification of Łukasiewicz's well-known idea, which seems advantageous.

1. Definition

We define a paraconsistent matrix by simple adaptation of the well-known n -valued Łukasiewicz matrix. Moreover, the n -valued Łukasiewicz matrix is a submatrix of the matrix we define, which entails that all the tautologies we obtain are n -valued tautologies.

The n -valued Łukasiewicz matrix is the following structure

$$(1) \quad \mathfrak{M}_n = \langle \langle \mathcal{A}_n, f^\rightarrow, f^{\&}, f^\vee, f^\neg \rangle, \{1\} \rangle$$

for any natural number $n > 1$, where

$$\mathcal{A}_n = \{k/(n-1) : k = 0, 1, \dots, n-1\}.$$

1 is the designated element, and for $x, y \in \mathcal{A}_n$, we define

$$\begin{aligned} f^\rightarrow(x, y) &= \min\{1 - x + y, 1\}, \\ f^{\&}(x, y) &= \min\{x, y\}, \\ f^\vee(x, y) &= \max\{x, y\}, \\ f^\neg(x) &= 1 - x. \end{aligned}$$

The present plan is to modify the n -valued matrix by supplementing \mathcal{A}_n , with an additional value $p > 1$; in this way we obtain an $(n+1)$ -valued matrix \mathfrak{M}_{np} , still with one designated element 1. We define the matrix's domain

$$\mathcal{A}_{np} = \mathcal{A}_n \cup \{p\}$$

and extend the truth tables by setting for all $x, y \in \mathcal{A}_{np}$:

$$\begin{aligned}
 f^{\rightarrow}(x, y) &= \begin{cases} y & \text{if } x = p \text{ and } y \in \mathcal{A}_n \\ \min\{1 - x + y, 1\} & \text{otherwise} \end{cases} \\
 f^{\&}(x, y) &= \min\{\min\{x, y\}, 1\} \\
 f^{\vee}(x, y) &= \min\{\max\{x, y\}, 1\} \\
 f^{\neg}(x) &= \begin{cases} 1 & \text{if } x = p \\ 1 - x & \text{otherwise} \end{cases}
 \end{aligned}$$

The set of tautologies of \mathfrak{M}_{np} turns out to be quite rich. Of the axioms of n -valued Łukasiewicz logic, contraposition $(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$ is the only one that fails (whenever A takes the value p and B takes a value less than 1). The intuitionistic law of double negation $A \rightarrow \neg\neg A$ and the Law of Overcompleteness $A \rightarrow (\neg A \rightarrow B)$ are not tautologous either (although their variants can be tautologous, e.g. whenever $A = \alpha$, for some compound formula α). Formulas such as $\neg(A \& \neg A)$ and $A \vee \neg A$ fail, since they fail already in the matrix \mathfrak{M}_n .

As we know, the formula $A \& \neg A$ is anti-tautologous in the matrix \mathfrak{M}_n ; that is, $V(A \& \neg A) = 1$ for no valuation V . But it is contingent in \mathfrak{M}_{np} , for it takes the designated value 1 if A takes the value p .

At the same time the formula $A \rightarrow (\neg A \rightarrow B)$ (which is tautologous in \mathfrak{M}_n) fails whenever A takes the value p and B takes a value less than 1. That is to say, the matrix \mathfrak{M}_{np} behaves paraconsistently at the level of p only. That is, if the set of values we assign to the variables does not contain p , then we obtain the n -valued Łukasiewicz calculus. For this reason it may be preferable to treat \mathfrak{M}_{np} as an n -valued matrix with a parameter p , rather than as an $(n + 1)$ -valued matrix.

It has to be stressed that only atomic formulas can take the value p , and this fact seems to be an important feature of the matrix \mathfrak{M}_{np} . It denies the possibility of having the formula $\alpha \& \neg\alpha$ satisfied for a more complex α . For example, if we take $\alpha = A \& \neg A$, where A is an atomic formula, and a valuation V that assigns the value p to A , then of course $V(\alpha) = 1$, but $V(\alpha \& \neg\alpha) = 0$. This fact prevents the system determined by the matrix \mathfrak{M}_{np} from being trivial in a different sense: from the possibility of containing infinitely many true contradictions ($\alpha, \alpha \& \neg\alpha, (\alpha \& \neg\alpha) \& \neg(\alpha \& \neg\alpha)$, etc.) generated by a single one (i.e. α).

So far, so good. Problems appear when we try to define the so-called matrix consequence operation in a standard way (a formula is a consequence of

the set of formulas X iff for any valuation V , if $V(X)$ is included in a set of designated values, then $V(\alpha)$ belongs to this set). Since there is only one designated value in our matrix, and negation is not an identity operator, we shall have an arbitrary formula B as a (matrix) consequence of the set $\{A, \neg A\}$ (A and $\neg A$ cannot simultaneously take the designated value), which is not desirable from the paraconsistency viewpoint. The other problem is that the Modus Ponens rule is not “normal” in the matrix \mathfrak{M}_{np} ; that is, for some atomic formula β we can have both $V(\beta) = p$ and $V(\alpha) = V(a \rightarrow \beta) = 1$. Both these drawbacks, however, can be eliminated by extending the set of designated values to $\{1, p\}$. The matrix obtained in this way still preserves the most important properties: the n -valued matrix is its submatrix, and the set of tautologies is exactly the same as in the case of one designated value. Because the value p cannot be taken by compound formulas, no extra tautologies can be produced in a matrix with two designated values.

Although the Modus Ponens rule is not normal in the matrix \mathfrak{M}_{np} , it is a “sound” (theorem preserving) rule of inference, i.e. if the premises are tautologies, so is the conclusion. Thus we can prove the following.

Theorem 1. *The Modus Ponens rule is sound in the matrix \mathfrak{M}_{np} .*

Proof. We have to show that if α and $\alpha \rightarrow \beta$ are tautologous in \mathfrak{M}_{np} , then β is a tautology. Let us assume that β is not a tautology in this matrix. Since we know that the Modus Ponens rule is sound in the matrix \mathfrak{M}_n , we need to consider the case when β takes the value p only. In that case β has to be an atomic formula, which entails that α and $\alpha \rightarrow \beta$ cannot be tautologous at the same time. \square

2. Axioms

The paraconsistent calculus obtained is the calculus generated by the matrix \mathfrak{M}_{np} , that is the set of all the tautologies of the matrix. Now we shall make an attempt to axiomatize this matrix, i.e. to present a system of formulas, from which all the tautologies of \mathfrak{M}_{np} , can be derived by means of Modus Ponens.

We shall use the standard notation: $E(\mathfrak{M})$ for the set of tautologies of the matrix \mathfrak{M} , and $Cn(X)$ for the set of consequences of X by means of the Modus Ponens rule. We shall also use the abbreviations $(A \rightarrow^n B) = A \rightarrow (A \rightarrow^{n-1} B)$, where $(A \rightarrow^0 B) = B$, and $A \leftrightarrow B$ instead of $(A \rightarrow B) \& (B \rightarrow A)$. Let us consider the following set \mathcal{E}_n of formulas:

- (A1) $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
 (A2) $A \rightarrow (B \rightarrow A)$
 (A3) $((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)$
 (A4) $(A \rightarrow^n B) \rightarrow (A \rightarrow^{n-1} B)$
 (A5) $A \& B \rightarrow A$
 (A6) $A \& B \rightarrow B$
 (A7) $(A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow B \& C))$
 (A8) $A \rightarrow A \vee B$
 (A9) $B \rightarrow A \vee B$
 (A10) $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$
 (A11) $(A \leftrightarrow (A \rightarrow^{s-2} \neg A)) \rightarrow^{n-1} A$
 for any $2 \leq s \leq n-1$, such that s is not a divisor of $n-1$
 (A12) $(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$
 (A12a) $(\neg A \rightarrow \neg \beta) \rightarrow (\beta \rightarrow A)$
 (A12b) $(\neg A \rightarrow \beta) \rightarrow (\neg \beta \rightarrow A)$
 where β is any compound formula.

3. Theorems

In [4] we have proved that the set $\mathcal{L}_n = \{(A1), \dots, (A12)\}$ is an axiom system for the n -valued Łukasiewicz logic, that is

Theorem 2. $E(\mathfrak{M}_n) = \text{Cn}(\mathcal{L}_n)$, for any $n \geq 2$.

Now we shall formulate the soundness theorem for a paraconsistent system defined above. \mathcal{L}_{np} stands for the set $\mathcal{E}_n \setminus \{(A12)\}$.

Theorem 3. (soundness) *All the consequences of the set of axioms \mathcal{L}_{np} are tautologous in the matrix \mathfrak{M}_{np} , i.e. for any $n \geq 2$, $\text{Cn}(\mathcal{L}_{np})$ is included in $E(\mathfrak{M}_{np})$.*

Proof. We have to show that all the formulas of \mathcal{L}_{np} are tautologous, and that the Modus Ponens rule is sound in \mathfrak{M}_{np} . We have already proved the latter in Theorem 1; let us display, for example, a truth table test on one of the axioms. Since the axioms are n -valued tautologies, it suffices to consider

the cases when only the value p is taken. If we cannot compute the value of some connectives, then we shall write the formulas of the same value under them. We abbreviate $V(A) = V(B) = p$ to $V(A, B) = p$.

Axiom 1.	$(A \rightarrow B)$	\rightarrow	$((B \rightarrow C)$	\rightarrow	$(A \rightarrow C))$
1. $V(A, B, C) = p$	1	1	1	1	1
2. $V(A, B) = p$	1	1	C	1	1
3. $V(A, C) = p$	B	1	1	1	1
4. $V(B, C) = p$	1	1	1	1	1
5. $V(A) = p$	B	1	$B \rightarrow C$		C
6. $V(B) = p$	1	1	C	1	$A \rightarrow C$
7. $V(C) = p$	$A \rightarrow B$	1	1	1	1

The value of the formula for the assignment of line 5 is obtained by means of the n -valued tautologies $A \rightarrow A$ (i.e. its substitution instance $(B \rightarrow C) \rightarrow (B \rightarrow C)$) and $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$. In line 6 we use axiom 2.

The validity of the other axioms can be shown in a similar way. □

As we know, the set \mathcal{L}_2 is the axiom system for Classical Logic, that is $E(\mathfrak{M}_2) = \text{Cn}(\mathcal{L}_2)$. In order to obtain a paraconsistent system it suffices to replace classical contraposition (A12) with some of its variants. If we replace (A12) with (A12a,b) and take $n = 2$, we get a 3-valued paraconsistent logic. There is an open question whether the completeness theorem (the converse inclusion of Theorem 3) is true for all $n \geq 2$. We have not managed to prove it in general, but there is a result, due to A. Arruda (cf. [1]), that gives a partial solution to the problem in its 3-valued case.

Arruda in [1] compiles the set of axiom schemas of the system she calls V1, defines a 3-valued matrix M1, and proves the following completeness theorem.

Theorem 4. $\text{Cn}(V1) = E(M1)$.

But the truth functions of the matrix M1 are exactly the same as those of the matrix \mathfrak{M}_{2p} . Hence, both matrices have the same sets of tautologies. A natural question arises, whether the axiom systems V1 and \mathcal{L}_{2p} lead to the same sets of theorems. Since Modus Ponens is the sole rule of inference in both systems and their axioms are interderivable, the answer to this question is affirmative. It can be expressed as follows:

Theorem 5. $\text{Cn}(V1) = \text{Cn}(\mathcal{L}_{2p})$.

Now we can go back to the completeness problem for the system \mathcal{L}_{np} , and conclude that it is true for $n = 2$. That is,

Theorem 6. $\text{Cn}(\mathcal{L}_{2p}) = \text{E}(\mathfrak{M}_{2p})$.

On the other hand, by Theorem 5 together with Theorem 6, we obtain a new axiomatization of the matrix \mathfrak{M}_{2p} .

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