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ON RUSSELL'S DEFINITION OF MOMENTS OF TIME

Abstract. In the paper two definitions of moments of time as the sets of events are considered. The first one is Russell's definition based on a relation simultaneity of events. The second one is my construction of moments of time grounded on a relation of being immediately preceding.

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In the philosophy of time, "reductionism" refers to a standpoint concerning the nature of moments of time that, although ontologically primitive, are time-extended processes called events, whereas moments are derivative entities. Adherents of such mean reductionism come from different, often opposed philosophical traditions. To them, reduction has varied meanings. In this paper the problem of reduction of moments of time to events is meant according to standards of logical philosophy, i.e. as logical definability of moments of time in terms of sets of events.

One pioneer of such an approach to the problem was Bertrand Russell. In the book *Our Knowledge of the External World as a Field for Scientific Method in Philosophy* (published in 1912) he presented a definition of moment which has become a specific "paradigm" for other research in the domain, for example Whitehead's and Newton-Smith's. To be sure, the main idea of Russell's theory of events and his definition of moment are well known, but they are usually presented superficially, with omissions of many important assumptions on which his reduction of moments to events is grounded. A more detailed analysis of his theory reveals some weak points and suggests the need to search for alternative solutions. The first part of the paper is devoted to a critical analysis of Russell's theory. In the second part, I present my own theory of events which, is in a sense, an answer to the objections to Russell's theory raised in the first part.

1. Russell's proposal

Let 'E' denote the set of all events[1]. We shall assume that the set is nonempty. Individual variables ranking over the set E will be the letters 'u', 'w', 'x', 'y' and 'z', whereas the capital letters 'X' and 'Y' will be variables for the subsets of the set E.

The primitive notion of this theory is a binary relation P defined in the set E, i.e. $P \subseteq E \times E$. The expression '$x\,P\,y$' we read: "the event x WHOLLY PRECEDES the event y" or "x is EARLIER then y". By means of the relation P we shall define in the set E a relation of simultaneity (denoted by 'S')

(def S) $$x\,S\,y \overset{\text{df}}{\iff} \neg\, x\,P\,y \wedge \neg\, y\,P\,x.$$

The expression '$x\,S\,y$' we read: "x is SIMULTANEOUS with y".

[1] By an *event* we shall understand a one-dimensionally extended process conceiving as a MEREOLOGICAL SET.

The following theorems are immediate consequences of the above definition.

(T1) $$\forall_{x,y}(x\,S\,y \Leftrightarrow y\,S\,x),$$

i.e. S is a symmetric relation to the set E.

(T2) $$\forall_{x,y}(x\,P\,y \vee y\,P\,x \vee x\,S\,y),$$

i.e. the sum of relation P and S (i.e. $P \cup S$) is a connected relation to E.

The following two axioms express the fundamental properties of the relations P and S.

(A1) $$\forall_x \neg\, x\,P\,x,$$

i.e. P is irreflexive in E.

(A2) $$\forall_{u,x,y,z}(u\,P\,x \wedge x\,S\,y \wedge y\,P\,z \Rightarrow u\,P\,z).$$

Remark 1. Instead of the axiom (A2) Russell has assumed only the asymmetricity and transitivity of the relation P. However the axioms are too weak to prove the theorems (T12) and (T13). Hence my proposal to strengthen them to the form expressed by (A2). □

An immediate consequence of (A1) is the theorem:

(T3) $$\forall_x\, x\,S\,x,$$

i.e. the relation S is reflexive in E. From the axioms (A1) and (A2) it is easy to obtain the following theorems:

(T4) $$\forall_{x,y}(x\,P\,y \Rightarrow \neg\, y\,P\,x),$$

i.e. the relation P is asymmetric in E. Indeed, let $x\,P\,y$ and suppose that $y\,P\,x$. Hence and from (T3) we have $x\,P\,y \wedge y\,S\,y \wedge y\,P\,x$. So, by virtue of (A2), $x\,P\,x$ holds, which contradicts the axiom (A1).

(T5) $$\forall_{x,y,z}(x\,P\,y \wedge y\,P\,z \Rightarrow x\,P\,z),$$

i.e. P is transitive in E. Suppose that $x\,P\,y$ and $y\,P\,z$. Then $x\,P\,y \wedge y\,S\,y \wedge y\,P\,z$. Hence, by virtue of (A2), we have $x\,P\,z$.

(T6) $$\forall_{x,y,z}(x\,P\,y \wedge y\,S\,z \Rightarrow \neg\, z\,P\,x).$$

Let $x\,P\,y$ and $y\,S\,y$ and suppose, in spite of our thesis, that $z\,P\,x$. Thus, in accordance with (A2), we have $x\,P\,x$ what contradicts the axiom (A1).

Combining[2] the relation S with the relation P and with its converse[3] $\breve{P}$ we obtain the following four relations: (a) $S \circ P$, (b) $S \circ \breve{P}$, (c) $P \circ S$ and (d) $\breve{P} \circ S$ such that their holding between any events x and y mean respectively: (a) x BEGINS BEFORE y, (b) x ENDS AFTER y, (c) x ENDS BEFORE y, (d) x BEGINS AFTER y.

(T7) $$\forall_x \neg\, x\, S \circ P\, x,$$

(T8) $$\forall_{x,y}(x\, S \circ P\, y \;\Rightarrow\; \neg\, y\, S \circ P\, x),$$

(T9) $$\forall_{x,y}(x\, S \circ P\, y \;\Leftrightarrow\; y\, \breve{P} \circ S\, x),$$

(T10) $$\forall_{x,y}(x\, S \circ \breve{P}\, y \;\Leftrightarrow\; y\, P \circ S\, x).$$

Let $\mathsf{S} := \{X \in \mathcal{P}(E) \setminus \{\emptyset\} \,:\, \forall_{x,y \in X}\, x\, S\, y\}$. Thus the set S is a family of nonempty sets consisting of simultaneous events.

A moment we shall define indirectly as any elements of the set M which are a family of maximal (in the sense of inclusion) sets of simultaneous events.

$$\mathsf{M} := \{X \in \mathsf{S} \,:\, \forall_{Y \in \mathsf{S}}(X \subseteq Y \Rightarrow X = Y)\}.$$

A MOMENT is to be said any element of the set M. Thus a moment is any nonempty and maximal, in the sense of the inclusion, set of events such that any two of its elements are simultaneous events. The maximality condition of the set X ensures that any moment is uniquely determined by the set of simultaneous events.

Remark 2. A moment can be also defined directly (i.e. without mediation of the family M) as any set X fulfilling the following condition:

$$X \in \mathcal{P}(E) \setminus \{\emptyset\} \;\wedge\; X = \bigcap\{\, Y \,:\, \exists_{z \in X}\, Y = S[\{z\}]\},$$

where $S[X]$ is an image of the set X under the relation S, i.e. $S[X] := \{\, y \in E \;:\, \exists_{x \in X}\, x\, S\, y\}$ (in particular $S[\{x\}] = \{\, y \in E \;:\; x\, S\, y\}$). Thus moments are fixed points of the mapping $f\colon \mathcal{P}(E) \setminus \{\emptyset\} \to \mathcal{P}(E)$ having form $f(X) := \bigcap\{\, Y \,:\, \exists_{z \in X}\, Y = S[\{z\}]\}$. □

The letter '$\mathfrak{m}$' (possibly with indices) will be the variable ranking over the set of moments M.

In the set M a counterpart of the relation P is a relation $\prec$ defined in the following way:

(def $\prec$) $$\mathfrak{m}_1 \prec \mathfrak{m}_2 \;\overset{\text{df}}{\Longleftrightarrow}\; \exists_{x \in \mathfrak{m}_1} \exists_{y \in \mathfrak{m}_2}\, x\, P\, y.$$

[2] $x\, S \circ P\, y \overset{\text{df}}{\Longleftrightarrow} \exists_z (x\, S\, z \wedge z\, P\, y)$.

[3] $x\, \breve{P} y \overset{\text{df}}{\Longleftrightarrow} y\, \breve{P}\, x$.

A moment $\mathfrak{m}_1$ precedes a moment $\mathfrak{m}_2$ iff there is some event belonging to $\mathfrak{m}_1$ and preceding some event from $\mathfrak{m}_2$.

Immediate consequences of definition (def $\prec$) and axiom (A1) is theorem (T11) expressing irreflexivity of the relation $\prec$:

(T11) $$\forall_{\mathfrak{m}} \neg\, \mathfrak{m} \prec \mathfrak{m}.$$

From (def $\prec$) and (A2) we have transitivity of the relation $\prec$:

(T12) $$\forall_{\mathfrak{m},\mathfrak{m}_1,\mathfrak{m}_2}(\mathfrak{m}_1 \prec \mathfrak{m} \wedge \mathfrak{m} \prec \mathfrak{m}_2 \;\Rightarrow\; \mathfrak{m}_1 \prec \mathfrak{m}_2).$$

From (T11) and (T12) we have asymmetricity of the relation $\prec$:

(T13) $$\forall_{\mathfrak{m}_1,\mathfrak{m}_2}(\mathfrak{m}_1 \prec \mathfrak{m}_2 \;\Rightarrow\; \neg\, \mathfrak{m}_2 \prec \mathfrak{m}_1).$$

Moreover, we have connectivity of the relation $\prec$:

(T14) $$\forall_{\mathfrak{m}_1,\mathfrak{m}_2}(\mathfrak{m}_1 \neq \mathfrak{m}_2 \;\Leftrightarrow\; \mathfrak{m}_1 \prec \mathfrak{m}_2 \vee \mathfrak{m}_2 \prec \mathfrak{m}_1).$$

Proof. Since the sets $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are maximal in the sense of the relation of inclusion, so $\mathfrak{m}_1 \neq \mathfrak{m}_2$ iff $\exists_x(x \in \mathfrak{m}_1 \wedge x \notin \mathfrak{m}_2$ iff $\exists_{x,y}(x \in \mathfrak{m}_1 \wedge y \in \mathfrak{m}_2 \wedge \neg x\, S\, y)$ iff $\exists_{x,y}(x \in \mathfrak{m}_1 \wedge y \in \mathfrak{m}_2 \wedge (x\, P\, y \vee y\, P\, x))$ iff $\exists_{x,y}((x \in \mathfrak{m}_1 \wedge y \in \mathfrak{m}_2 \wedge x\, P\, y) \vee (x \in \mathfrak{m}_1 \wedge y \in \mathfrak{m}_2 \wedge y\, P\, x))$ iff $(\mathfrak{m}_1 \prec \mathfrak{m}_2 \vee (\mathfrak{m}_1 \prec \mathfrak{m}_2)$. □

Remark 3. The simple model below shows that the assumptions of irreflexivity, asymmetricity and transitivity of the relation P do not suffice to prove the theorems (T12) and (T13). For this purpose we need the axiom (A2).

Indeed, let E be the following set of numbers $\{1, 2, 3, 4\}$ and put $P = \{\langle 1, 2\rangle, \langle 4, 3\rangle\}$, i.e. only $1\,P\,2$ and $4\,P\,3$ hold. It is obvious that a relation defined as P is irreflexive, asymmetric and transitive. But it does not fulfill the axiom (A2). Indeed, we have $1\,P\,2$, $2\,S\,4$ and $4\,P\,3$, but it is not true that $1P3$ holds. Notice that in this interpretation the theorem (T13) is also untrue. Indeed, it suffices to take into account the moments $\{1, 3\}$ and $\{2, 4\}$. For these we have: $\{1, 3\} \prec \{2, 4\}$ (because $1\,P\,2$) and $\{2, 4\} \prec \{1, 3\}$ (because $4\,P\,3$), what contradicts the theorem (T13). Moreover, this fact contradicts the theorem (T12) (because $\{1, 3\} \not\prec \{1, 3\}$, by (T11)). □

We shall say that an event x HAPPENS AT the moment $\mathfrak{m}$ iff $x \in \mathfrak{m}$. This relationship of an occurrence of an event x at a moment $\mathfrak{m}$ we shall denote by '$\rhd$'. We shall prove that every event happens at some moment.

(T15) $$\forall_x \exists_{\mathfrak{m}}\, x \rhd \mathfrak{m}.$$

Proof. Let x be an event and $\boldsymbol{S}_x$ be a family of sets of events defined in the following way:

$$\boldsymbol{S}_x := \{X \in \mathsf{S} \ : \ \forall_{y \in X}\, y\, S\, x\}.$$

The set $\boldsymbol{S}_x$ be nonempty because it contains the singleton $\{x\}$. Moreover it is partially ordered by the relation of inclusion. Then, by virtue of Hausdorff's lemma, the set $\boldsymbol{S}_x$ contains some maximal chain $\boldsymbol{S}_0$. Making $\mathfrak{m} = \bigcup \boldsymbol{S}_0$, we have the thesis (T15). □

Remark 4. Notice that the theory of events does not exclude the existence of an ATOMIC event i.e. such event u that $\forall_{x,y}(x\, S\, u \wedge y\, S\, u \ \Rightarrow \ x\, S\, y)$. For any atomic event u a moment $\mathfrak{m}$ such that $u \rhd \mathfrak{m}$ is identical with the set $\bigcup \boldsymbol{S}_u$ defined in the proof of the theorem (T15). In this case the existence of the moment is independent from the axiom of choice. Moreover, for any atomic event there exists exactly one moment such that the event happens at that moment. □

The theorems (T11)–(T14) show that the relation $\prec$ LINEARLY orders the set M. In order to ensure DENSE ordering of the set M by relation $\prec$ Russell has assumed the axiom:

(A3) $$\forall_{x,y}(x\, P\, y \ \Rightarrow \ \exists_z(x\, P\, z \wedge z\, P\, y)).$$

The below theorem is an immediate consequence of the axiom, the definition of relation $\prec$ and the theorem (T15).

(T16) $$\forall_{\mathfrak{m}_1,\mathfrak{m}_2}(\mathfrak{m}_1 \prec \mathfrak{m}_2 \ \Rightarrow \ \exists_{\mathfrak{m}}(\mathfrak{m}_1 \prec \mathfrak{m} \wedge \mathfrak{m} \prec \mathfrak{m}_2)).$$

2. Critical comments

Russell's pioneering attempt to reduce moments of time to the set of events, although formally and logically quite correct, is less satisfactory from a philosophical point of view. First of all some reservations arise from the fact that, in general, in order to prove the existence of moments of time we have to use the axiom of choice. To be sure, at present the axiom does not raise any doubts amongst mathematicians. Nevertheless its strong non-constructive character brings Russell's attempt into a question

> … to show the kind of way in which, given a world with the kind of properties that psychologists find in the world of sense, it may be possible, by means of purely logical constructions, to make it amenable to mathematical treatment by defining series or classes of amenable

> to sense-data which can be called respectively particles, points, and instants. If such construction are possible, then mathematical physics is applicable to the real world, in spite of the fact that its particles, points, and instants are not to be found among actually existing entities.[4]

I think that the building of a bridge between the world of sense and the entities of theoretical physics by means of constructions grounded on this axiom is rather not too sound. In such constructions it is important not only that individual events should be sensually perceivable but that these constructions should be an idealization of some sensually observable procedures. It seems to me that any construction based on the axiom of choice fails to fulfill the last condition.

For any nonatomical event the theorem (T15) leaves unanswered the question of existence of its initial and its terminal moment. In order to ensure their existence Russell had to assume some additional axioms which limit the set of admissible — in his theory — events, and in this way they lessen its generality. In order for any event X to have its initial and its terminal moment, the following two conditions have to hold: $\breve{P}[S[\{x\}]] = \breve{P}[S[\{x\}] \setminus \breve{P}[S[\{x\}]]]$ and $P[S[\{x\}]] = P[S[\{x\}] \setminus P[S[\{x\}]]]$. The first one excludes the possibility that for any event y different from x and simultaneous with x there is an event earlier than y and simultaneous with x. Similarly for the second condition.

The axiom (A3) also has a restrictive character, which excludes the existence of tangential events. It seems to me that the axiom is an *ad hoc* hypothesis assumed by Russell only to ensure a dense ordering of the set of moments. But it is not clear if we should retain such kind of ordering of moments in the case of atomical events.

In the latter part of the paper I shall present a theory of events which on the one hand is grounded on some rather natural axioms and on the other hand makes possible such a definition of moment to which none of the above reservations applies.

3. An alternative proposal

The starting point for an alternative — in comparison with the Russellian — definition of moment as a set of events is the rejection of the axiom (A3). This give us the possibility to define in terms of relation P a next relation

[4] Bertrand Russell: *Our Knowledge of the External World*, revised ed. George Allen & Unwin, London 1929, p. 122.

between events which in Russell's theory — by virtue of axiom (A3) — is an empty relation. This new relation we shall denote by 'IP'. The expression '$x\, IP\, y$' we read "x IMMEDIATELY PRECEDES y".

$$\text{(def } IP\text{)} \qquad x\, IP\, y \overset{\text{df}}{\iff} x\, P\, y \wedge \neg\exists_z (x\, P\, z \wedge z\, P\, y).$$

Analogicaly we define the next relation holding between the events x and y iff y happens IMMEDIATELY AFTER x. This relation we denote by 'IA'.[5]

$$\text{(def } IA\text{)} \qquad x\, IA\, y \overset{\text{df}}{\iff} x\breve{P}y \wedge \neg\exists_z (x\breve{P}z \wedge z\breve{P}y).$$

From these definitions we have:

$$\text{(t1)} \qquad \forall_{x,y}(x\, IP\, y \Rightarrow x\, P\, y),$$

$$\text{(t2)} \qquad \forall_{x,y}(x\, IA\, y \Rightarrow y\, P\, x),$$

$$\text{(t3)} \qquad \forall_{x,y}(x\, IP\, y \Leftrightarrow y\, IA\, x),$$

$$\text{(t4)} \qquad \forall_{x,y,z}(x\, IP\, y \wedge y\, IP\, z \Rightarrow \neg\, x\, IP\, z).$$

The axiomatic foundations of this theory of events are axioms (A1) and (A2) from the first section and an additional axiom expressing a fundamental property of the relation IP.

$$\text{(A3}^*\text{)} \qquad \forall_{u,x,y,z}(u\, IP\, x \wedge u\, IP\, y \wedge z\, IP\, x \Rightarrow z\, IP\, y).$$

An analogical axiom for the relation IA in not needed because the following theorem holds:

$$\text{(t5)} \qquad \forall_{u,x,y,z}(u\, IA\, x \wedge u\, IA\, y \wedge z\, IA\, x \Rightarrow z\, IA\, y).$$

Indeed, let $u\, IA\, x \wedge u\, IA\, y \wedge z\, IA\, x$. This conjunction — by (t3) — is equivalent to $x\, IP\, u \wedge y\, IP\, u \wedge x\, IP\, z$. Hence — by virtue of (A3*) — we have $y\, IP\, z$, i.e. $z\, IA\, y$. We shall assume another axiom ensuring nonemptiness of the relations IP and IA.

$$\text{(A4)} \qquad \forall_x \exists_{y,z}\; y\, IP\, x \wedge z\, IA\, x.$$

Remark 5. (a) The theory grounded on the axioms (A1), (A2), (A3*) and (A4) is consistent, what follows from the following simple model: $E = \mathbb{Z}$ (the set of rational numbers) and P is the relation $<$ in the set $\mathbb{Z}$. So S is the identity relation in the set $\mathbb{Z}$, $x\, IP\, y \Leftrightarrow y = x + 1$ and $x\, IA\, y \Leftrightarrow x = y + 1$. Under this interpretation: the truth of (A1) it follows from the condition

[5] A prototype of our relations IP and IA is the Whiteheadian relation of EXTENSIVE CONNECTION). Cf. A. N. Whitehead, *Process and Reality*, Corrected edition, The Free Press, New York–London 1978, p. 294 and n.

$\forall_{x\in\mathbb{Z}} \neg\, x < x$; the implication (A2) has a false antecedent; (A3*) it follows from transitivity of identity; (A4) is expressing a self-evident property of integers $\forall_{x\in\mathbb{Z}} \exists_{y,z\in\mathbb{Z}} (x = y+1 \wedge z = x+1)$. In the model all events are atomical.

In this case the moments (in the sense from the first section) are all singleton subsets of the set $\mathbb{Z}$. It is obvious that $\{x\} \prec \{y\} \Leftrightarrow x < y$.

(b) It is also possible to construct a more «intuitive» model of the presenting theory. For the rationals $p, q \in \mathbb{Q}$ such, that $p < q$ we define an «open rational segment» making $(p, q) := \{r \in \mathbb{Q} : p < r < q\}$. Let E be the set of all such segments. In this set the relation P is defined by condition: $(p_1, q_1)\, P\, (p_2, q_2) \Leftrightarrow q_1 \leqslant p_2$. With such an interpretation the truth of (A1) follows from $\forall_{p,q\in\mathbb{Q}}(p < q \Rightarrow \neg\, q \leqslant p)$. In this model the relation of simultaneity is expressed by condition: $(p_1, q_1)\, S\, (p_2, q_2) \Leftrightarrow q_1 > p_2 \wedge q_2 > p_1$. Then the truth of the axiom (A2) follows from the property: $\forall_{p,q,r,s\in\mathbb{Q}}(p \leqslant q \wedge q < r \wedge r \leqslant s \Rightarrow p < s)$. Under this interpretation the relations IP and IA are expressed respectively by the following conditions: $(p_1, q_1)\, IP\, (p_2, q_2) \Leftrightarrow q_1 = p_2$ and $(p_1, q_1)\, IA\, (p_2, q_2) \Leftrightarrow p_1 = q_2$. The truth of axioms (A3*) is (A4) self-evident. In the model there is no atomical events.

In the model the set of moments (in the sense of Russell's proposal) is equinumerous with the set $\mathbb{Q}$. More exactly: the pair $\langle \mathsf{M}, \prec\rangle$ is izomorfic with the pair $\langle \mathbb{Q}, <\rangle$. For each moment $\mathfrak{m}$ corresponds (in a one-to-one manner) to such a number $r \in \mathbb{Q}$, that $\mathfrak{m} = \{\,(p, q) \in E : p < r < q\}$, i.e. $\mathsf{M} = \{\{\,(p, q) : p < r < q\} : r \in \mathbb{Q}\}$. Moreover, for $r_1, r_2 \in \mathbb{Q}$: $\{\,(p, q) : p < r_1 < q\} \prec \{\,(p, q) : p < r_2 < q\} \Leftrightarrow r_1 < r_2$.

(c) In order to obtain a model with atomical events it is enough to extend the universe from (b) on all singletons from the set $\mathbb{Q}$ and to add (to the definition of the relation P) the following conditions: $(p, q)\, P\, \{r\} \Leftrightarrow q \leqslant r$, $\{r\}\, P\, (p, q) \Leftrightarrow r \leqslant p$ and $\{r_1\}\, P\, \{r_2\} \Leftrightarrow r_1 < r_2$. Then, besides the condition from (b), we also have: $\{r_1\}\, S\, \{r_2\} \Leftrightarrow r_1 = r_2$ and $(p, q)\, S\, \{r\} \Leftrightarrow \{r\} S (p, q) \Leftrightarrow p < r < q$. In this model the «singletons» are just atomical events. It is easy to see that $\{r\}\, IP\, (p, q) \Leftrightarrow r = p$, $(p, q)\, IP\, \{r\} \Leftrightarrow q = r$ and «singletons» do not precede themselves immediately. Analogicaly we may extend the relation IA. The truth of axioms in such an interpretation we verify in the same manner as in point (b).

As in (b), for every moment $\mathfrak{m}$ corresponds (in a one-to-one manner) to such a number $r \in \mathbb{Q}$, that $\mathfrak{m} = \{\,(p, q) \in E : p < r < q\} \cup \{r\}$. □

We shall define in the set IP (included in $E \times E$) some auxiliary, binary relation $\backsimeq$ (included in $IP \times IP$):

(def $\backsimeq$) $$\langle u, w\rangle \backsimeq \langle x, y\rangle \overset{\text{df}}{\iff} u\, IP\, y.$$

We shall prove that the relation $\backsimeq$ is reflexive, symmetric and transitive in the set *IP*, i.e. we shall prove that the relation is an equivalency relation in the set *IP*. Reflexivity is an immediate consequence of definition (def $\backsimeq$):

$$\text{(t6)} \qquad \langle u, w\rangle \backsimeq \langle u, w\rangle.$$

Symmetricity follows from axiom (A3*):

$$\text{(t7)} \qquad \langle u, w\rangle \backsimeq \langle x, y\rangle \Rightarrow \langle x, y\rangle \backsimeq \langle u, w\rangle.$$

Indeed, if $\langle u, w\rangle \backsimeq \langle x, y\rangle$, then $u\, IP\, y$, $u\, IP\, w$ and $x\, IP\, y$. Hence — by virtue of (A3*) — we have $x\, IP\, w$, i.e. $\langle x, y\rangle \backsimeq \langle u, w\rangle$. Transitivity is easy to obtain from (A3*):

$$\text{(t8)} \qquad \langle u, w\rangle \backsimeq \langle x, y\rangle \wedge \langle x, y\rangle \backsimeq \langle z, z_1\rangle \Rightarrow \langle u, w\rangle \backsimeq \langle z, z_1\rangle.$$

Indeed, by hypothesis $u\, IP\, y$ and $x\, IP\, z_1$. Moreover $x\, IP\, y$. Hence, by virtue of (A3*), we have $u\, IP\, z_1$.

By MOMENT we shall mean any class of abstraction of relation $\backsimeq$, i.e. a moment is any element of set $IP/_{\backsimeq}$. In the set of all moments the relation $\prec$ is defined in the following way:

$$\mathfrak{m}_1 \prec \mathfrak{m}_2 \overset{\text{df}}{\iff} \exists_{x,y,u,w}(\langle x, y\rangle \in \mathfrak{m}_1 \wedge \langle u, w\rangle \in \mathfrak{m}_2 \wedge x\, P\, w \wedge \neg\, x\, IP\, w).$$

Remark 6. (a) In the model presented in Remark 5a the relation $\backsimeq$ is an identity relation to the set *IP*. This follows from transitivity of identity in $\mathbb{Z}$. So, the moments, in the model, are all singletons from the set *IP*, i.e. $\mathsf{M} = \{\{\langle x, x+1\rangle\} : x \in \mathbb{Z}\}$. Moreover, for any $x, y \in \mathbb{Z}$: $\{\langle x, x+1\rangle\} \prec \{\langle y, y+1\rangle\} \Leftrightarrow x < y$.

(b) In the model presented in remark 5b the relation $\backsimeq$ is expressed in the terms of *IP* by the condition: $\langle (p_1, q_1), (q_1, r_1)\rangle \backsimeq \langle (p_2, q_2), (q_2, r_2)\rangle \Leftrightarrow q_1 = q_2$. For every moment $\mathfrak{m}$ corresponds (in a one-to-one manner) to such a number $r \in \mathbb{Q}$, that $\mathfrak{m} = \{\langle (p, r), (r, q)\rangle : p, q \in \mathbb{Q} \wedge p < r < q\}$, i.e. $\mathsf{M} = \{\{\langle (p, r), (r, q)\rangle : p, q \in \mathbb{Q} \wedge p < r < q\} : r \in \mathbb{Q}\}$. Moreover $[\langle (p_1, r_1), (r_1, q_1)\rangle]_{\backsimeq} \prec [\langle (p_2, r_2), (r_2, q_2)\rangle]_{\backsimeq} \Leftrightarrow r_1 < r_2$. Then in this model any pair $\langle \mathsf{M}, \prec\rangle$ is izomorfic with $\langle \mathbb{Q}, <\rangle$.

(c) In the model from remark 4c the following situations $\langle (p_1, q_1), (q_1, r_1)\rangle \backsimeq \langle (p_2, q_2), \{q_2\}\rangle$, $\langle (p_1, q_1), (q_1, r_1)\rangle \backsimeq \langle \{q_2\}, (q_2, r_2)\rangle$, $\langle \{q_1\}, (q_1, r_1)\rangle \backsimeq \langle \{q_2\}, (q_2, r_2)\rangle$, $\langle (p_1, q_1), \{q_1\}\rangle \backsimeq \langle (p_2, q_2), \{q_2\}\rangle$, $\langle (p_1, q_1), \{q_1\}\rangle \backsimeq \langle \{q_2\}, (q_2, r_2)\rangle$ hold iff $q_1 = q_2$. Moreover, the pair $\langle \{q_1\}, (q_1, r_1)\rangle$ is never in relation $\backsimeq$ with the pair $\langle (p_2, q_2), \{q_2\}\rangle$. Such as in (b), for every $\mathfrak{m}$ corresponds in a one-to-one manner such $r \in \mathbb{Q}$, that $\mathfrak{m} = \{\langle (p, r), (r, q)\rangle : p < r < q\} \cup \{\langle (p, r), \{r\}\rangle : p < r\} \cup \{\langle \{r\}, (r, q)\rangle : r < q\}$. □

An immediate consequence of the definition of $\prec$ is the theorem:

(t9) $$\forall_{\mathfrak{m}} \neg\, \mathfrak{m} \prec \mathfrak{m}.$$

To prove the further theorems about moments we shall need the following auxiliary theorem:

(t10) $$\forall_{x,y,z,u}(x\,P\,y \wedge \neg\, x\,IP\,y \wedge x\,IP\,z \wedge u\,IP\,y \;\Rightarrow\; \neg\, u\,P\,z).$$

Proof. Let $x\,P\,y$, (1) $\neg\, x\,IP\,y$ and (2) $x\,IP\,z$. From $x\,P\,y$, (1) and the definition of IP we get $\exists_{z_1}(x\,P\,z_1 \wedge z_1\,P\,y)$. From this and from (2) we have: $\neg z_1\,P\,z$, i.e. $z_1\,S\,z \vee z\,P\,z_1$. Let u be any event such that (3) $u\,IP\,y$. Suppose — in spite of the thesis — that $u\,P\,z$. If $u\,IP\,z$, then from (2), (3) and the axiom (A3*) it follows $x\,IP\,y$, contrary to (1). If $\neg\, u\,IP\,z$, then $\exists_{z_2}(u\,P\,z_2 \wedge z_2\,P\,z)$. For $z_1\,S\,z$ we have $z_2\,P\,z \wedge z_1\,S\,z \wedge z_1\,P\,y$. Hence — by (A2) — $z_2\,P\,y$. If $z\,P\,z_1$ then $z_2\,P\,z \wedge z\,P\,z_1 \wedge z_1\,P\,y$. So — by transitivity of P — $z_2\,P\,y$. In this way we have proved that $\exists_{z_2}(u\,P\,z_2 \wedge z_2\,P\,y)$, contrary to (3). □

(t11) $$\forall_{x,y,z,u}(x\,IP\,z \wedge y\,IP\,z \wedge x\,P\,u \;\Rightarrow\; y\,P\,u).$$

Proof. Let $x\,IP\,z$, (1) $y\,IP\,z$ and $x\,P\,u$. If $x\,IP\,u$ then — by (A3*) — $y\,IP\,u$. Hence $y\,P\,u$. Suppose that (2) $\neg\, x\,IP\,u$. By (A4) there exists an event z_1 such that (3) $z_1\,IP\,u$. From this and from $x\,P\,u$, $x\,IP\,z$ and (2) — by virtue of (t10) — we have $\neg\, z_1\,P\,z$. Therefore either $z\,P\,z_1$ or $z\,S\,z_1$. If $z\,P\,z_1$ then (by (1), (3)) we get $y\,P\,z \wedge z\,P\,z_1 \wedge z_1\,P\,u$. Hence $y\,P\,u$. If $z\,S\,z_1$ then — also by (1), (3) — we have $y\,P\,z \wedge z\,S\,z_1 \wedge z_1\,P\,u$. So, by (A2), we get $y\,P\,u$. □

Analogicaly we may prove the theorem:

(t12) $$\forall_{x,y,z,u}(z\,IP\,x \wedge z\,IP\,y \wedge u\,P\,x \;\Rightarrow\; y\,P\,x).$$

By means of the above theorems we shall prove the asymmetricity of relation $\prec$:

(t13) $$\forall_{\mathfrak{m}_1,\mathfrak{m}_2}(\mathfrak{m}_1 \prec \mathfrak{m}_2 \;\Rightarrow\; \neg\, \mathfrak{m}_2 \prec \mathfrak{m}_1).$$

Proof. Let $\mathfrak{m}_1 \prec \mathfrak{m}_2$ and suppose — contrary to the thesis — $\mathfrak{m}_2 \prec \mathfrak{m}_1$. So there are pairs $\langle x, y\rangle, \langle x_1, y_1\rangle \in \mathfrak{m}_1$ and pairs $\langle u, w\rangle, \langle u_1, w_1\rangle \in \mathfrak{m}_1$ such that (1) $x\,P\,w \wedge \neg\, x\,IP\,w$ and $u_1\,P\,y_1$. Moreover, in accordance with the definition of moment, we have $u\,IP\,w$, $u_1\,IP\,w_1$, $u\,IP\,w_1$, and $x\,IP\,y$, $x_1\,IP\,y_1$ and $x\,IP\,y_1$. From this — by (A3*) — it follows that $u_1\,IP\,w$ and $x_1\,IP\,y$. By (t11), from $u_1\,IP\,w$, $u\,IP\,w$ and $u_1\,P\,y_1$ we get $u\,P\,y_1$. By (t12) from $x_1\,IP\,y$, $x_1\,IP\,y_1$ and $u\,P\,y_1$ we have $u\,P\,y$. As a consequence, by the fact that (1), $x\,IP\,y$ and $u\,IP\,w$ hold, we have a contradiction with (t10). □

Finally we shall prove the transitivity and connectivity of relation $\prec$:

(t14) $$\forall_{\mathfrak{m},\mathfrak{m}_1,\mathfrak{m}_2}(\mathfrak{m}_1 \prec \mathfrak{m} \wedge \mathfrak{m} \prec \mathfrak{m}_2 \Rightarrow \mathfrak{m}_1 \prec \mathfrak{m}_2).$$

Proof. Suppose that $\mathfrak{m}_1 \prec \mathfrak{m}$ and $\mathfrak{m} \prec \mathfrak{m}_2$. So, there are pairs $\langle x, y\rangle \in \mathfrak{m}_1$, $\langle u, w\rangle, \langle u_1, w_1\rangle \in \mathfrak{m}$ and $\langle z, z_1\rangle \in \mathfrak{m}_2$ such that (1) $x\,P\,w \wedge \neg\, x\,IP\,w$ and $u_1\,P\,z_1 \wedge \neg\, u_1\,IP\,z_1$. From the first argument of the second conjunction and from $u\,IP\,w_1$ and $u_1\,IP\,w_1$ — by virtue of (t11) — we get (2) $u\,P\,z_1$. Analogicaly we prove that (3) $\neg\, u\,IP\,z_1$. Since (2), (3), $u\,IP\,w$ and (4) $z\,IP\,z_1$ hold, so — by (t10) — we have $\neg\, z\,P\,w$, i.e. either $w\,P\,z$ or $w\,S\,z$. In the first case from (1), (4) and the transitivity of P we have $x\,P\,z_1$. In the second case from (1), (4) and the axiom (A2) we also get $x\,P\,z_1$. Moreover, it is easy to check that from $\neg\, x\,IP\,w$ follows $\neg\, x\,IP\,z_1$. So, $\mathfrak{m}_1 \prec \mathfrak{m}_2$. □

(t15) $$\forall_{\mathfrak{m}_1,\mathfrak{m}_2}(\mathfrak{m}_1 \neq \mathfrak{m}_2 \Leftrightarrow \mathfrak{m}_1 \prec \mathfrak{m}_2 \vee \mathfrak{m}_2 \prec \mathfrak{m}_1).$$

Proof. The implication "$\Leftarrow$" is an immediate consequence of the definition of moment. Suppose that $\mathfrak{m}_1 \neq \mathfrak{m}_2$, i.e. there are pairs $\langle x, y\rangle \in \mathfrak{m}_1$ and $\langle u, w\rangle \in \mathfrak{m}_2$ such that $\neg\, x\,IP\,w$. We shall consider only the case when $x\,S\,w$. Then — by (A2) — $u\,P\,y$, because $u\,P\,w \wedge x\,S\,w \wedge x\,P\,y$. So $\mathfrak{m}_2 \prec \mathfrak{m}_1$. □

We have shown that $\prec$ is an irreflexive, anti-symmetric, transitive and connected relation on the set of moments — as we might expect in the case of time-preceding relation.

A moment $\mathfrak{m}$ is said to be an INITIAL (*resp.* a TERMINAL) moment of the event u iff there exists in $\mathfrak{m}$ such ordered pair that the event u is its second (*resp.* first) element. From the axiom (A4) and from the definition of moment it immediately follows that for every event there exists its initial and its terminal moment and these moments are different.

At present the occurrence of an event u at a moment $\mathfrak{m}$ will be defined in another way than previously. Namely

$$u \rhd \mathfrak{m} \overset{\text{df}}{\Longleftrightarrow} \exists_{x,y}(\langle x, y\rangle \in \mathfrak{m} \wedge x\,S\,u \wedge y\,S\,u).$$

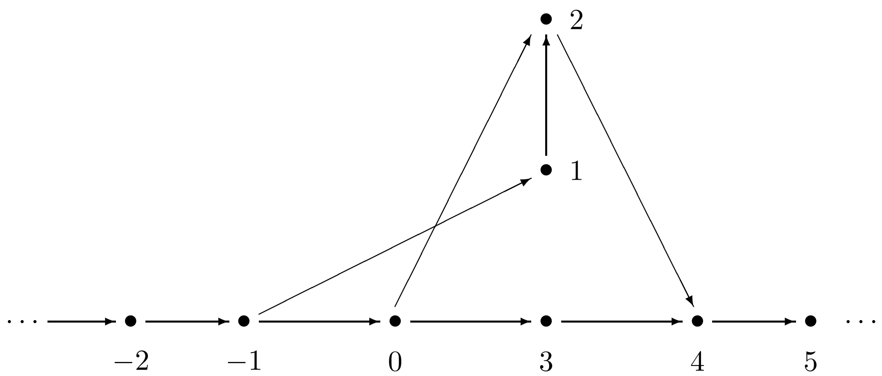
If the relation $\rhd$ is meant in the above way, the existence of a moment at which an event u happens is equivalent to nonatomicity of the event u, i.e. $\exists_{x,y}(x\,S\,u \wedge y\,S\,u \wedge \neg\, x\,S\,y)$. Notice that — according to this definition — no event happens at its initial and its terminal moment.

(t16) $$\forall_u\,(u \textit{ is nonatomic} \Leftrightarrow \exists_{\mathfrak{m}}\, u \rhd \mathfrak{m}).$$

Proof. Let u be nonatomic event. So there exist some events x and y such that $x\,S\,u$, $y\,S\,u$ and $\neg\,x\,S\,y$. If $x\,IP\,y$ or $y\,IP\,x$ then it is enough to put $\mathfrak{m} = [\langle x, z\rangle]_{\backsimeq}$. In the opposite case there exists — by virtue of (A4) — an event z such that $x\,IP\,z$. Let $\mathfrak{m} = [\langle x, z\rangle]_{\backsimeq}$. In order to prove that $u \rhd \mathfrak{m}$ we have to show that $z\,S\,u$. Let us assume the opposite, i.e. $\neg\,z\,S\,u$. If $z\,P\,u$, then — by $x\,S\,u$, $x\,P\,z$ and (A2) — we have $z\,P\,z$. So $u\,P\,z$. Without the loosing of the generality of the proof we may assume that $x\,P\,y$. From the axiom (A4) follows the existence of w such that $w\,IP\,y$. Since $w\,P\,y$, $y\,S\,u$ and $u\,P\,z$ hold, so $w\,P\,z$. Finally we have got contradiction with the theorem (t10), because $x\,P\,y \wedge \neg\,x\,IP\,y \wedge x\,IP\,z \wedge w\,IP\,y$ holds. The implication in the opposite direction is obvious. □

It is easy to check that the set of moments is linearly and densely ordered by the relation $\prec$ iff there is no atom in the set E. In the opposite case there exists such pairs of different moments that between them there is no moment. These moments are the initial and terminal moments of an event. In the frame of the theory — in opposition to Russell's — existence of moments of time is insured independently from the axiom of choice. Moreover every event has its initial and its terminal moment. The existence of these moments is insured by the axiom (A4).

Remark 7. The axioms (A3*) and (A4) are independent from the remaining ones. In the case of axiom (A4) it is obvious. The independence of the axiom (A3*) shows the following — constructed in the set of integral numbers $\mathbb{Z}$ — graph, where the arrow '$\rightarrow$' denotes the transitive relation P.



It is easy to see that the axioms (A1) and (A4) are satisfied in this graph.

In order to verify the truth of the axiom (A2) in this model it is enough to notice that $S = \{\langle 1,3\rangle, \langle 3,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$, i.e. only $1\,S\,3$, $3\,S\,1$, $2\,S\,3$ and $3\,S\,2$ hold. It is easy to see (from the diagram) that for any $u, z \in \mathbb{Z}$ if $u\,P\,2$ and $3\,P\,z$ then $x\,P\,y$. Similarly for remaining pairs of simultaneous events $\langle 3,2\rangle$, $\langle 1,3\rangle$ and $\langle 3,1\rangle$. So it is true that for any $u, x, y, z \in \mathbb{Z}$ if $u\,P\,x$ and $x\,S\,y$ and $y\,P\,z$, then $u\,P\,z$.

Notice finally that in the model we have: $0\,IP\,2$, $0\,IP\,3$ and $1\,IP\,2$ although it is not the case that $1\,IP\,3$ holds. So, the axiom (A3*) is untrue in this interpretation i.e. it is independent from the remaining axioms. □

4. Concluding remarks

In this paper we have presented two ways of defining of moments of time as distributive sets of events.

In the interpretation intended here, in which the events are meant as one-dimensional mereological sets, Russell's definitions reflect the idea that a moment is determined by events having shorter and shorter (converging to zero) «duration». So the main idea of Russell's conception is the concept of approximation carried to the basis of mereological sets and based on a timely extension of events.

In my theory the existence of moments is a consequence of the fact that every event has its beginning and its end, and every event is tangentially connected with another event. In such treatment of the existence of moments, their construction is effective. It is not clear whether and how effective the theories based on the idea of approximation are, i.e. how far they are depending on the axiom of choice.

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