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ANTINOMICITY AND THE AXIOM OF CHOICE

A Chapter in Antinomic Mathematics

To the memory of Stanisław Jaśkowski

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I. Introduction and Motivation

§ 1. A positive view of antinomies

Russell's discovery in 1902 of the antinomy of the set of all sets which are not members of themselves prompted a profound and widespread examination of the foundations of mathematics for many years to come.[1] No other discovery has shaken mathematics and logic more deeply than Russell's antinomy, which came just when set theory was beginning to be widely accepted after years of rejection. From that time on antinomies have been treated seriously — to be avoided, to be sure, but nevertheless constituting a stimulating logical phenomenon at the very heart of mathematical reasoning.

The next natural step was the acceptance of antinomies in their own right. For this to occur, the basic logical assumptions had to be changed; this was accomplished in various ingenious ways, setting aside in the process the many acrobatic pirouettes that logic had been required to perform in order to jump over antinomies without tripping.

Underlying this acceptance is the belief that there is something intrinsically valuable in antinomies. Evolving from being merely a strong motivating force for deep analysis of the foundations of mathematics, antinomies now became a significant center of attention in themselves, a positive part of reason with their own legitimacy. This legitimacy arises from the fact that, although not always so, our thought processes are often antinomic, which in turn reflects the parallel fact that reality itself is often antinomic — hence why not logic and mathematics?

What began as a few timid investigations today has proliferated into a vast variety of logical approaches, different in point of view and method but all sharing in common the objective of using antinomies positively as valuable, intelligible, and rational parts of the logical discourse (cf. [17]).

The many antinomic logics now in existence prove beyond question the feasibility of the formal incorporation of antinomicity as an extension of rationality. What is still missing, though, are the strictly mathematical applications of this logical approach. In order to obtain acceptance of antinomic

[1] "Many of the most profound results in modern logic have arisen from the analysis of the paradoxes." [8, p. 481]

logic as more than a curiosity, new and effective mathematical structures must be developed — as happened with nonstandard models, in the limbo of curiosa before A. Robinson put them to good use. The present work is an attempt to break ground in mathematics proper, armed with the accepting view just described. Specifically, we shall examine various versions of antinomic set theory, in particular the axiom of choice, keeping the presentation as intuitive as possible, more in the manner of a nineteenth century paper than as a thoroughly formalized system. The reason for such a presentation is the conviction that at this point it should be the mathematics that eventually determines the logic, rather than the other way around.

§ 2. Some antecedents of this view

Kant was the first modern thinker to make the point that antinomies are not to be "solved" but accepted as constructive rational elements. In his *Critique of Pure Reason* he presents them not only as a reflection of the nature of the mind but also as a force to awake reason from its consuetudinal state of slumber.

Cantor was the first mathematician to acknowledge the presence of inconsistencies in set theory but he left them alone, only mentioning them casually in a letter published for the first time in 1932. He said [11, p. 114]: "For a multiplicity can be such that the assumption that *all* of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call *absolutely infinite* or *inconsistent multiplicities*." Also, "Two equivalent multiplicities either are both 'sets' or are both inconsistent."[11, p. 114] Further, Cantor was not particularly upset by Russell's discovery (as Frege was), having himself discovered in 1895 the "paradox of the largest cardinal number."

Although a Platonist and therefore a believer in the reality of correct mathematical propositions, Gödel admitted "the amazing fact that our logical intuitions (i.e., intuitions concerning such notions as truth, concept, being, class, etc.) are self-contradictory" [10, p. 131]. He added that it is "not self-contradictory that a proper part should be identical (not merely equal) to the whole, ... and it is easily seen that there exist also structures containing infinitely many different parts, each containing the whole structure as a part."[10, p. 139] "Furthermore, there exist sentences referring to a totality of sentences to which they themselves belong."[10, p. 140.]

II. Logic for Antinomies

§ 3. Prelogical antinomies

Before the antinomies of simultaneous truth and falsity, we have the antinomies of sense and nonsense — Zeno's paradox being one such prelogical antinomy. And, before the latter, we still have concrete apophantic antinomies, i.e., factual antinomies in which two opposite qualities are displayed simultaneously in the same given entity or event; for example, when something is both plural and unitary, or when a movement both helps and hinders reaching an objective, etc. However, even though an antinomic logic can be based on an antinomic semantics of sense and nonsense, so too can such semantics be based on the phenomenological apophantic description of a contradictory reality. We shall put aside these last two important prelogical areas, for they must be treated at length elsewhere. Here, we shall deal only with the true-and-false kind of antinomy, treating it as the absolute beginning. We must always keep in mind, however, that any talk of difference in identity — say, of something being simultaneously the same and different, which occurs naturally and correctly in ordinary language — already involves antinomic thinking and points implicitly to the coexistence of truth and falsity.

§ 4. Truth-and-falsity not a third logical value

Sometimes truth is simple and so is falsity, but at other times we hit upon the true by way of the false in a way that makes the false a necessary component of the true. To see the true in the true-and-false as different from the true in truth alone is not an accurate conception of antinomicity. To fit the facts, the logic of antinomicity should not be conceived as a three-valued logic but as a complex two-valued one in which truth valuations are not functions but rather one-to-one or one-to-two correspondences between sentences and the unordered pair $\{\mathbf{T}, \mathbf{F}\}$. That is, some valuations assign to a sentence A the value $\mathbf{T}$, to a sentence B the value $\mathbf{F}$, and to a sentence C both $\mathbf{T}$ and $\mathbf{F}$.

§ 5. Assertion and negation independent of truth and falsity

As mentioned, there are already many antinomic logics in the literature, with more to come. The logic outlined here is clearly not the only possible one. In accordance with the comment at the end of § 1 to the effect that we do not want to reflect prejudice towards any of the new directions that will arise from mathematical applications, we shall set here only a minimum list of conditions that are indispensable for our own objectives, leaving the rest indefinite. It should be obvious, for example, that reduction to the absurd as a method of proof cannot be allowed if one is to avoid the derivability of every well-formed formula from a single contradiction, i.e., a fall into absolute inconsistency.

It is indeed extraordinary that even the most sophisticated definitions of truth and falsity, as well as those of assertion and negation, rely on one another in a blatant vicious circle. Thus, for example, the truth of an atomic predicate formula is defined in accordance with the interpreted terms of the formula belonging or *not* belonging to the set-theoretic relation that interprets the predicate, a metalinguistic definition that leans on negation and set theory as much as it does on the law of excluded middle — the n-tuple of terms $\langle \boldsymbol{t}_1, \dots, \boldsymbol{t}_n \rangle$ in $P(t_1, \dots, t_n)$ is a member of the relation R that interprets the n-ary predicate P, or it is not, one or the other, with no third alternative possible. In turn, the semantic definition of negation is given in terms of truth values as follows: If a sentence A is true, its negation $\neg A$ is false, and if A is false, then $\neg A$ is true. This definition is taken to be the last word on the matter, and is uncritically used whenever negation is used (except that within many-valued logics — which lie beyond our scope — a different approach to negation is considered). In the classical propositional calculus, then, given a sentence A and a structure $\mathscr{I}$ that interprets the language in which the sentence is formed, the sentence is either simply true or simply false; in symbols, $\models A$ or not-$\models A$, but not both. Further, syntactically only A or $\neg A$ are provable; again, no third alternative is allowed.

Here, however, negation will be looked at differently, accepting the principle that negation is not a logical operation definable in terms of truth and falsity, but that its meaning, in effect, stands prior and beyond whatever any truth table rule can provide. Russell has already observed that there is something primitive and peculiarly irreducible in the notion of negation that escapes the truth-table approach; he believed in the necessity of some "negative basic propositions" side by side with the positive ones (see [20] pp. 99, 174) — fundamental propositions, atomic in their own way. This belief, followed systematically, sets negation apart from the other connectives — which is precisely the objective of this section. Our reasoning, however, arises from the antinomic approach, that is, by the acceptance of the fact that there are cases in which one can see contradictions in a negation — that $\neg A$ is true-and-false — and hence, that truth is not necessarily simply the negation of a falsity. Or even more strikingly, we come to the same reasoning by recognizing the existence of cases in which a negation is neither true nor false. Antinomies, which themselves do not necessarily depend on negation, force upon us the inevitable conclusion that truth and falsity must be divorced from assertion and negation, that $\neg A$ may be simply true, simply false, true and false, or neither true nor false.

To make all this intuitive let us say informally that the negation of a sentence A refers to all assertions A_i that are in opposition or disagreement with A. Negation, therefore, is a form of indirect assertion; as such, it can be characterized as a mapping on the class $\boldsymbol{S}$ of all well-formed sentences of a given language $\mathscr{L}$ into the power set of $\boldsymbol{S}$ as follows: $\neg A = \{A_i : i \in I\}$ where the indexed A_i's (a finite or infinite family) are all the assertions in $\mathscr{L}$ which stand in opposition to A. If, for example, we consider A to be the sentence 'four is even', there is of course only one sentence in opposition to A: 'four is odd', hence, the semantic meaning of the expression $\neg A$ is in this case the singleton {'four is odd'}. In fuzzy set theory on the other hand, if A stands for $a \in_0 b$ (a is a member of b with probability zero), then $\neg A$ is the uncountable set of sentences $\{a \in_r b$: where r is a real number such that $0 < r \leqslant 1\}$. It has been suggested that $\neg A$ means to assert the disjunction $A_1 \vee A_2 \vee \ldots \vee A_n$ of a finite sequence of sentences in opposition to A. Apart from the A_i's possibly being infinite in number as in the last example, this interpretation of negation would subject it to the truth-table definition of disjunction. Since we want to have negation fully independent of all other connectives, we shall adhere to the "neutral" set-theoretic characterization given above — not a definition proper but an informal intuitive one similar to Cantor's characterization of a set as "a multiplicity taken as a unit", (which, incidentally, was Kant's characterization of a totality).

As for the truth of a sentence A, we can simply say — also informally — that A is the case in a given context. Now, A can be fully the case — simply true — or only partly the case — true-and-false. Further, A is simply false if the context fully opposes A, and neither true nor false if the context is fully irrelevant to A. This is all we shall say here to make understandable our having four possible truth valuations (**T**, **F**, **T**&**F**, neither **T** nor **F**) for a given sentence, plus four truth valuations for its negation. Note, again, that negation does not determine truth and falsity but is given either a single value, two values, or none, regardless of the truth values for the corresponding assertion. In other words, each of the four cases for A branches out into four additional cases for $\neg A$: negation extends assertion, does not exclude it. Needless to say, we shall make room for the preservation of the standard two-value situation of ordinary sentences, i.e., "two is even" is to remain true and "two is not even" false, etc. Whether the expression above, "to be the case", is an assumption or is to be determined by an effective rule whenever possible depends on mathematical considerations that cannot be established in advance.

Now, a theory can be considered complete in two metalinguistic senses: (i) every sentence A is tautologically true or logically valid if and only if it is provable from the theory's axioms; in symbols, $\models A$ iff $\vdash A$ (completeness theorem) and (ii) given a sentence A in a theory $\mathscr{T}$, either A is a theorem or $\neg A$ is: $\vdash A$ or $\vdash \neg A$ (definition of complete theory). In antinomic logic (i) and (ii) are to be considered hypotheses independent of one another. All this means that while $\vdash A$ and $\vdash \neg A$ are both possible simultaneously, if $\models A$ is the case, not-$\models A$ cannot also be the case; although A can be true and false, not-$\models A$ (A is not true) is not synonymous with false. The metalinguistic contradiction $\models A$ and not-$\models A$ is not allowed: contradictions belong to the object language. Furthermore, (i) applies to true formulas but says nothing about false ones: a false formula may be provable or not, even if (i) is assumed. Also, the law of excluded middle does not extend to the metalinguistic statement '$\models A$ or $\models \neg A$', for neither $\models A$ nor $\models \neg A$ may be the case if both A and $\neg A$ are simply false, say; on the other hand, $\models A$ and $\models \neg A$ may also be compatible. Finally, not-$\models A$ and not-$\models \neg A$ may both be the case simultaneously (again, keep in mind that not-$\models A$ is not the same as A is false).

Metalinguistically, then, negation preserves some of the characteristics of its classical use; for example, as already indicated, no metalinguistic assertion or negation is both true and not true, and it must be either one or the other. The metalinguistic 'not', then, abides by no-contradiction and excluded middle, in contrast to the object-language negation '$\neg$' which will not satisfy either. However, the laws of contraposition and double negation (in both directions) and the proofs by contradiction will not be valid in either the metalanguage or the object-language of our antinomic logic. These proof-theoretic limitations have been adopted to broaden negation's meaning and are not more restrictive than those of intuitionism, which rejects the laws and proof method just mentioned, except for $A \Rightarrow \neg\neg A$ and $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$. In addition, antinomic logics are for the most part nonconstructive and are therefore in a stronger proof-theoretic position than intuitionism to find deductive replacements for the laws and proof method in question. For example, we shall assume completeness in the sense of (i), i.e., $\models A$ iff $\vdash A$; thus, the proof of the semantic truth of A will automatically entail A's syntactic provability. In addition, the systems to be proposed can be extended to complete extensions in the sense that for every sentence A

either $\vdash A$, or $\vdash \neg A$, or both, extensions in which because of (i), either $\models A$, or $\models \neg A$, or both must be the case respectively. (Once more, note that '$\vdash A$ and $\vdash \neg A$' does not imply '$\models A$ and not-$\models A$' but merely '$\models A$ and $\models \neg A$' as stated.)

For A false let us use the symbol $\mathrel{|\!\equiv} A$, given that falsity, just as negation, will have a positive meaning here, not a negative one: A false may mean A holds in a different context, or in the same context relative to a different rule than the one that makes A hold, if the latter is indeed the case. This is patently clear in model theory, where the truth of a sentence is a function of the domain of interpretation (the universe of discourse or context of the moment) and the specific rules attached to that interpretation. This relativity of truth and falsity will be expanded: not only will there be no absolute truth and no absolute falsity but also truth and falsity will not be truth values rigidly connected to one another.

Metamathematically, then, although simultaneously we can have $\models A$ and $\mathrel{|\!\equiv} A$, we cannot have $\mathrel{|\!\equiv} A$ and not-$\mathrel{|\!\equiv} A$, and at most we can have one or the other; nor can we have $\mathrel{|\!\equiv} \neg A$ and not-$\mathrel{|\!\equiv} \neg A$, but at most only one or the other. The metalinguistic rule of excluded middle which applies to 'not' does not, however, extend to the following: $\models A$ or $\mathrel{|\!\equiv} A$, $\mathrel{|\!\equiv} A$ or $\mathrel{|\!\equiv} \neg A$ (A and $\neg A$ may both be not false), not-$\models A$ or not-$\mathrel{|\!\equiv} A$ (A may be true and false). The following are possible though: (i) $\mathrel{|\!\equiv} A$ and $\mathrel{|\!\equiv} \neg A$, (ii) not-$\models A$ and $\mathrel{|\!\equiv} \neg A$, and (iii) $\models A$ and not-$\mathrel{|\!\equiv} \neg A$. Similarly, we cannot have $\vdash A$ and not-$\vdash A$, or $\vdash \neg A$ and not-$\vdash \neg A$. Nor is it the case that if $\vdash A$ then not-$\vdash \neg A$, or that if $\vdash \neg A$, then not-$\vdash A$. (Incidentally, were we to have several truth values $t_1, t_2, \dots, t_m$, and several false values $f_1, f_2, \dots, f_n$, using them to distinguish $\models_{\mathrm{t1}} A, \dots, \models_{\mathrm{tm}} A$, $\mathrel{|\!\equiv}_{\mathrm{f1}} A, \dots, \mathrel{|\!\equiv}_{\mathrm{fm}} A$, the metalinguistic law of excluded middle would extend in the sense that $\models_{\mathrm{ti}} A$ or not-$\models_{\mathrm{ti}} A$ but not both, and either $\mathrel{|\!\equiv}_{\mathrm{fi}} A$ or not-$\mathrel{|\!\equiv}_{\mathrm{fi}} A$ but not both).

We shall emphasize that, although a sentence in a given language is designated as true or false, or both, or neither, in accordance with context and interpretation, these designations need not be understood in set-theoretic terms. An explicit assumption, or a constructive or nonconstructive rule is indispensable of course, but assumptions and rules can be presented in many forms that are decidedly independent of set theory. Also, whereas in the classical propositional calculus the class of all the negations of tautologies is a disjoint mirror image of the class of tautologies, here — with the broader meaning of negation — the two classes intersect and in the class of tautologies we shall find both propositions and their negations.

Finally, antinomic logic makes room for an included middle, which intuitionism will abhor. In antinomic logic if A and $\neg A$ are both simply false, and $A \vee \neg A$ is also simply false, the latter may not exclude a *tertium datur*, say $\neg\neg A$, which is a consequence of the fact that true and not-true, false and not-false, are metalinguistic assessments — and compatible ones at that. In other words, $A \vee \neg A$ is neither a tautology nor a contradictory statement (always false). It is a contingent statement, true or false as the case may be. Metamathematically, even both not-$\models A \vee \neg A$ and not-$\models \neg(A \vee \neg A)$ are contingent.

§ 6. Truth tables for the positive fragment of logic, and other assumptions

Having made the point that antinomicity is better off with a revised negation that makes it a nonexclusive operation independent of any truth table, we must now turn to the remaining connectives of the propositional calculus — the positive fragment — as well as to the definitions of truth, of an antinomic model for the predicate calculus, and of first-order theories. We shall adopt the following tables for the four positive connectives.

A	B	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
T	**T**	**T**	**T**	**T**	**T**
T	**F**	**F**	**T**	**F**	**F**
T	**T&F**	**T&F**	**T**	**T&F**	**T&F**
F	**T**	**F**	**T**	**T**	**F**
F	**F**	**F**	**F**	**T**	**T**
F	**T&F**	**F**	**T&F**	**T**	**T&F**
T&F	**T**	**T&F**	**T**	**T**	**T&F**
T&F	**F**	**F**	**T&F**	**T&F**	**T&F**
T&F	**T&F**	**T&F**	**T&F**	**T&F**	**T&F**

The following example shows how these tables were generated for the antinomic cases. If classically A is either true or false and B is true, then the compound statement $A \Rightarrow B$ is true for both cases; hence, if A is antinomic (**T&F**) and B true, $A \Rightarrow B$ is true. If A is antinomic and B is false, $A \Rightarrow B$ is antinomic since it is false if A is true, and true if A is false. If A is false, the truth value of B is irrelevant, $A \Rightarrow B$ is therefore true; hence, if A is false and B antinomic, $A \Rightarrow B$ is true. These tables are the same as those proposed in a previous paper (cf. [1] and [2, p 103]).

As for the syntax, we shall keep the two positive propositional axiom schemes given in Mendelson [14, p. 29]: (i) $A \Rightarrow (B \Rightarrow A)$ and (ii) $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$, dropping the axiom scheme (iii)

$(\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)$, the last being the only propositional axiom scheme involving negation. We shall also keep $A, A \Rightarrow B \vdash B$ (modus ponens), applicable exclusively to positive statements, that is, statements in which '$\neg$' does not occur at all.

Let us call positive tautology any positive propositional statement that is either true or antinomic by construction as determined by the above truth tables. In turn, let us call negative tautology any propositional statement that involves at least one occurrence of negation and that is either true or antinomic by specific designation, whatever the truth values of all the positive statements involved. Schemes (i) and (ii) and modus ponens generate positive tautologies only, i.e., $\vdash A$ implies $\models A$, understanding $\models A$ to mean A is true or true-and-false.

We shall assume the completeness theorem as a meta-axiom for the propositional calculus, i.e., we now add $\vdash A$ if and only if $\models A$ for all well-formed statements, positive or negative. As a consequence, a negative statement automatically becomes a syntactic axiom whenever it is declared true or antinomic by specific designation, i.e., by an ad hoc assumption or rule, since no truth table or general axiom scheme regulates negation. In this manner, we are able to move freely not only from syntax to semantics but also from semantics to syntax.

For the predicate calculus, the usual notion of interpretation is to be expanded as follows. Given a domain of interpretation or universe D (a set--theoretic particularization of the broader concept of context), each predicate P of a formal language $\mathscr{L}$ is associated not only with one but with four relations R_1, R_2, R_3, R_4 such that if P is an n-ary predicate, the relations R_i $(i = 1, 2, 3, 4)$ are all n-ary, and each is a subset of the Cartesian product D^n. In addition, formal terms $t_1, t_2, \dots$ are interpreted in the domain D by specific individuals of D in the usual way. The interpreted terms will be denoted by $\boldsymbol{t}_1, \boldsymbol{t}_2, \dots$, etc. Now to the definitions of truth, falsity, and negation in a given interpretation $\mathscr{I}$ with domain D:

Definition 1. $P(t_1, \dots, t_n)$ is true in $\mathscr{I}$ iff $\langle \boldsymbol{t}_1, \dots, \boldsymbol{t}_n \rangle \in R_1$; in symbols: $\models_{\mathscr{I}} P(t_1, \dots, t_n)$.

Definition 2. $P(t_1, \dots, t_n)$ is false in $\mathscr{I}$ iff $\langle \boldsymbol{t}_1, \dots, \boldsymbol{t}_n \rangle \in R_2$; in symbols $\models\!\!\!\mid_{\mathscr{I}} P(t_1, \dots, t_n)$.

Definition 3. $\neg P(t_1, \dots, t_n)$ is true in $\mathscr{I}$ iff $\langle \boldsymbol{t}_1, \dots, \boldsymbol{t}_n \rangle \in R_3$; in symbols $\models_{\mathscr{I}} \neg P(t_1, \dots, t_n)$.

Definition 4. $\neg P(t_1, \dots, t_n)$ is false in $\mathscr{I}$ iff $\langle \boldsymbol{t}_1, \dots, \boldsymbol{t}_n \rangle \in R_4$; in symbols $\models\!\!\!\mid_{\mathscr{I}} \neg P(t_1, \dots, t_n)$.

The set-theoretic relations R_i are not fixed beforehand; they can be pairwise disjoint, intersect, be included one in another, etc. Thus we can have the following cases for a given predicate P.

1. $R_1 \cup R_2 = D^n$; $P(t_1, \dots, t_n)$ is either true or false.
2. $R_1 \cup R_2 \subset D^n$; for some n-tuples $\langle \boldsymbol{t}_1, \dots, \boldsymbol{t}_n \rangle$, $P(t_1, \dots, t_n)$ is neither true nor false.
3. $R_1 \cup R_3 = D^n$; $P(t_1, \dots, t_n)$ is true or $\neg P(t_1, \dots, t_n)$ is true.
4. $R_1 \cup R_2 = D^n \wedge R_1 \cap R_2 \neq \emptyset$; $P(t_1, \dots, t_n)$ is true, false, or antinomic.
5. $R_2 \subseteq R_3$; if $P(t_1, \dots, t_n)$ is false, $\neg P(t_1, \dots, t_n)$ is true, the converse is not necessarily the case.
6. $R_3 \subseteq R_2$; if $\neg P(t_1, \dots, t_n)$ is true, $P(t_1, \dots, t_n)$ is false, the converse is not necessarily the case.
7. $R_1 \cap R_3 \neq \emptyset$; for some n-tuples $\langle \boldsymbol{t}_1, \dots, \boldsymbol{t}_n \rangle$, $P(t_1, \dots, t_n)$ is true and so is $\neg P(t_1, \dots, t_n)$.
8. $R_2 \cap R_4 \neq \emptyset$; for some n-tuples $\langle \boldsymbol{t}_1, \dots, \boldsymbol{t}_n \rangle$, $P(t_1, \dots, t_n)$ is false and so is $\neg P(t_1, \dots, t_n)$.
9. $R_2 = R_3$; $P(t_1, \dots, t_n)$ is false iff $\neg P(t_1, \dots, t_n)$ is true.
10. $R_3 \cup R_4 \subset D^n$; for some n-tuples $\langle \boldsymbol{t}_1, \dots, \boldsymbol{t}_n \rangle$, $\neg P(t_1, \dots, t_n)$ is neither true nor false.

As these examples show, there is no truth and falsity in the abstract but only in reference to a specific interpretation: R_1 to R_4 and their set-theoretic relationships can be assigned to P very differently in different domains.

Having considered the atomic predicate formulas, we can now use Definitions 1 and 2 to extend the notion of satisfiability to all positive well-formed formulas.

Definition 5. $A(x_1, \dots, x_n)$ is a well-formed positive predicate formula iff it is formed in accordance with the usual rules of formation and neither '$\neg$' nor the existential quantifier '$\exists_{x_i}$' occur in the formula.

Definition 6. A well-formed positive predicate formula $A(x_1, \dots, x_n)$ is satisfiable in a given interpretation $\mathscr{I}$ iff for some n-tuple $\langle \boldsymbol{t}_1, \dots, \boldsymbol{t}_n \rangle$, $A(x_1, \dots, x_n)$ meets the usual definition of satisfiability. Then, $A(x_1, \dots, x_n)$ is true in $\mathscr{I}$ iff it is satisfied by all n-tuples in $\mathscr{I}$, and logically valid iff it is true in all interpretations.

As with the truth of negation in the propositional calculus, expressions involving negation and the existential quantifier are to be considered satisfiable, true, or logically valid in an ad hoc manner. Although much of the meaning of classical existential quantification is meant to be retained, the usual definition '$\exists_x A(x)$ stands for $\neg\forall_x \neg A(x)$' does not hold in this work, that is, $\exists_x$ is to be taken as a primitive operator. The usual relation between universal and existential quantification '$\forall_x A(x) \Rightarrow \exists_x A(x)$', which is intuitionistically acceptable, will be the case here occasionally but not always. It is possible to have $\models \forall_x A(x)$ and $\models \neg\exists_x A(x)$. Since intuitionistically $\forall_x A(x)$ must be constructively determined, it stands to reason that $\exists_x A(x)$ follows, i.e., that what is true for all must be true for some. But if nonconstructive methods are accepted (excluded middle, axiom of choice, and the like), $\forall_x A(x)$ may be deducible without our having any method to find an x such that $A(x)$, thus opening the possibility that no such x exists.

Here, then, asserted formulas involving existential quantification will each have the status of a proper axiom. Only the positive fragment of the antinomic predicate calculus will retain its classical deductive generality, it being understood that 'true' in Definition 6 above includes the case in which a positive well-formed formula is both true and false.

Further, since we do not have the classical satisfiability rules for existential quantification, it is possible to have $\models \exists_x A(x)$ and $\models \neg\exists_x A(x)$: an x that satisfies $A(x)$ may exist and not exist. For example, to say that a function f mapping the set A onto the set B in a one-to-one manner exists means: for every $a \in A$ there is a unique $b \in B$ such that $\langle a, b\rangle \in f$. Yet, if there is an $a \in A$ such that not only $\langle a, b\rangle \in f$ but also $\neg\langle a, b\rangle \in f$, then we must conclude that f simultaneously exists and does not exist, and that the image $f(a)$ of a exists and does not exist at the same time. The existence or not of f means, precisely, the membership or not of the appropriate ordered pairs $\langle a, b\rangle$ to f. In general, $\forall_x A(x)$ may be true in an abstract sense without having any concrete individual x satisfying $A(x)$; in these cases, $\neg\exists_x A(x)$ is asserted — a possibility that is not counterintuitive but rather is the natural result of the acceptance of nonconstructive methods.

Definition 7. The model for a well-formed predicate formula A is any interpretation $\mathscr{I}$ in which A is true or antinomic according to the following.

(i) If A is positive, A is true in an interpretation $\mathscr{I}$ iff A fulfills in $\mathscr{I}$ the usual definition of truth restricted to such positive formulas; note that A may also be false, i.e., antinomic.

(ii) If A is negative then A is asserted as true, antinomic, or logically valid by specific designation.[2]

Note that for both the positive and negative formulas we can have A (i) antinomic for some valuations in the given interpretation $\mathscr{I}$; (ii) antinomic for

[2] This expands the idea of antinomic model introduced in [3].

all valuations in $\mathscr{I}$; in other words, true and false in $\mathscr{I}$ or fully antinomic in $\mathscr{I}$; (iii) false for some valuations in $\mathscr{I}$; (iv) false for all valuations in $\mathscr{I}$; (v) logically false, i.e., false in all interpretations, a notion that is independent of negation since, again, false is not necessarily "not true"; and (vi) logically antinomic, i.e., fully antinomic (true and false) in all interpretations.

As for syntax, the axioms for the positive fragment of the predicate calculus are the same classical positive axiom schemes:

(i) $\forall_x A(x) \Rightarrow A(t)$, with t a term free for x in $A(x)$.

(ii) $\forall_x (A \Rightarrow B) \Rightarrow (A \Rightarrow \forall_x B)$, with A having no free occurrence of x.

No axiom scheme for negative formulas will be added: negative formulas will be asserted as needed, not inferred, much as one chooses proper axioms for a given first-order theory.

In addition, let us postulate the rule of inference of generalization, from A, $\forall_x A$ follows; in symbols, $A \vdash \forall_x A$, where A is a positive well-formed formula (only positive formulas can be inferred).

An alternative way to define a positive predicate logic would be to retain $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, but substitute the universal quantifier with the existential one. Whereas the positive logic with $\forall_x$ exclusively is a logic of generalities, the positive logic with $\exists_x$ exclusively is a logic of particular cases. In the latter, the axiom schemes would be different, including, for example, $A(x) \Rightarrow \exists_x A(x)$; also, the rule of generalization would be replaced by the introduction of the existential quantifier as follows: if B does not contain x free, then $A(x) \Rightarrow B \vdash \exists_x A(x) \Rightarrow B$. The positive predicate logic thus obtained would be different from the previous one, of course, and the negative formulas would be those in which '$\neg$' or '$\forall_x$' occur. Once more, the negative fragment of this predicate calculus would share some of the characteristics of a first--order theory, with any asserted negative formula having the status of an ad-hoc axiom. These axioms would be all the negative theorems since, again, we would have no rule of inference for negative formulas, a situation that is similar to the way in which one defines a complete theory by postulating as axioms all the well-formed formulas true in a given model.

Whether one selects $\forall_x$ or $\exists_x$ as the positive quantifier, neither one can be defined in terms of the other and negation in the usual way. We have already made the point that, with $\forall_x$ as the positive quantifier, we cannot automatically transfer the validity of a property for a whole class of individuals to the validity of that property for a single specific individual. In the second case, with $\exists_x$ as the positive quantifier, it is possible to have

$\models \exists_x A(x)$ and $\models \forall_x \neg A(x)$, that is, local validity does not necessarily have any of the usual consequences for global validity. Here, we shall stay with the first case, i.e., with $\forall_x$ as positive, and add the completeness theorem of the predicate calculus as a meta-axiom for all well-formed formulas, positive and negative. Thus A is logically valid if and only if it is a theorem, $\models A$ iff $\vdash A$. As a result, a negative formula that is true or antinomic in all interpretations is automatically an axiom of the predicate calculus. For the negative fragment of the predicate calculus, then, semantics fully determines the syntax; the positive fragment remains close to the classical two-way form of completeness (allowing, of course, for the possibility of true formulas that are also false).

Let us pause now to elaborate on the meaning of the existential quantifier in the context of this antinomic logic. Informally, we shall characterize the existential quantification $\exists_x P(x)$ not as the disjunction $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_k)$ (extendable to an infinite number of disjuncts), nor as the class $\{x_i : P(x_i)\}$ of all individuals x_i for which $P(x_i)$ holds, but as one single individual choice from the collection of all individuals satisfying $P(x)$: in symbols, $\imath_x P(x)$, extending the meaning of the iota symbol (introduced by Russell for the description of individuals) from referring only to the unique x such that $P(x)$ (cf. [21], Vol. 1, p. 30]) to referring to a nonspecified individual chosen from $\{x_i : P(x_i)\}$. Having put aside the usual definition '$\exists_x P(x)$ stands for $\neg\forall_x \neg P(x)$' allows us to map the well-formed formula $\exists_x P(x)$ into one single individual as the formula's meaning (if no x satisfies $P(x)$ in a given interpretation, $\imath_x P(x)$ is the empty set). All this is similar to the above mentioned informal characterization of set given by Cantor; i.e., it is intended to provide an intuitive justification for the cleavage we have drawn between $\forall_x$ and $\exists_x$. Note that since $\models \forall_x P(x)$ and not-$\models \exists_x P(x)$ are simultaneously possible, a property can be generally true without being true specifically: $\models \forall_x P(x)$ is compatible with not being able to find an individual value a for x such that $\models P(a)$. Let us look at an example of this situation, still informally.

Let $C(x)$ be a function that determines the cardinality of a set x, that is, a set $|x|$ that can be defined with or without the axiom of choice. Let $x \cong y$ indicate that x can be mapped in a one-to-one manner onto y. We shall assume that there may be several such cardinality functions, but that if C and C' are any two of such functions, then $C(x) \cong C'(x)$. Assume that universal and existential quantification is restricted to these cardinality functions. Then $\models \forall_C((C(x) = C(y) \Leftrightarrow C(x) \cong C(y))$ obtains, but not-$\models \exists_C((C(x) = C(y) \Leftrightarrow C(x) \cong C(y))$ can be the case at the same time, since there are models of Zermelo-Fraenkel's set theory with a proper class of atoms in which no function C can be defined for all x with the property $C(x) = C(y) \Leftrightarrow C(x) \cong C(y)$ (cf. [12] p. 153).

It is advisable now to point out explicitly some classical theorems and metatheorems which will hold in some cases but definitely not in all. For example, classically, if in any theory **T** it is the case that $\vdash A \vee B$ implies $\vdash A$ or $\vdash B$, then and only then **T** is syntactically complete, i.e., $\vdash A$ or $\vdash \neg A$ for any well-formed formula A. The proof of this equivalence requires excluded middle, contraposition, and the tautology $\neg A \Rightarrow ((A \vee B) \Rightarrow B)$, all of which are not valid here, both in the object language and in the metalanguage. In addition, the following negative formulas can be true, or false, or both, or neither: $\neg(A \wedge \neg A \Rightarrow B), \neg((A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A))$, $\neg(A \vee \neg A)$, $\neg(\neg\neg A \Rightarrow A)$, $\neg(A \Rightarrow \neg\neg A)$, $\neg((\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A))$. There will be cases in which $A \vee \neg A$ is a good choice for some A's and $\neg(B \vee \neg B)$ is a good choice for some B's. The same applies to the other formulas just listed. In particular, the law of excluded middle, a negative metatheorem not itself responsible for contradictions and not assumed here in general, as we mentioned, could be assumed in particular to make room for the conclusion that every real number has a decimal expansion, even though Brouwer actually exhibited a definite number for which it is not known if there is a first digit in its decimal expansion (cf. [13], pp. 431, 436). The prime ideal theorem, used in the proof of Gödel's completeness theorem, is also a negative metatheorem which will not be assumed here, although the completeness theorem will be assumed in general as a meta-axiom for every first-order theory.

Finally, as already stated, both $\exists_x A(x)$ and $\neg\exists_x A(x)$ may be true, and hence axioms. But it is also possible that not-$\models \exists_x A(x)$ and not-$\models \neg\exists_x A(x)$ are the case, together with not-$\models\!\!\!\models \exists_x A(x)$ and not-$\models\!\!\!\models \neg\exists_x A(x)$; that is, $\exists_x A(x)$ and its negation are neither true nor false. Thus, instead of saying that the sentence 'there is a white unicorn' is false because unicorns do not exist in reality, here, precisely because unicorns cannot be found in reality and therefore a white one cannot be selected, the sentence is neither true nor false. If A. Robinson's definition of the complete diagram of a given model is extended to include not only those sentences which are true in that model but also those which are antinomic in the model, then we must also exclude from the diagram not only the simply false sentences but also those which are neither true nor false in the model. Note that whereas the positive fragment of this complete diagram can be considered deductively predetermined, the negative fragment is always open to enlargement when negative formulas are axiomatically added as needed (again, the interpretation of negative atomic formulas does not predetermine the truth or falsity of the compound ones).

§ 7. Equality as an antinomic predicate

The motivation behind antinomic logic lies in the conviction that, irreducibly, there is identity in difference in many realms, including nature. As a consequence, $\models x = y$ and $\not\models x = y$ together must be considered possible for some values of x and y. Since here equality is to be defined in terms of membership, we shall not add equality as a primitive antinomic predicate because it will turn out to be antinomic as a derived one (cf. [5], p. 407).

§ 8. Other kinds of antinomicity

It is a mistake to think that antinomicity is exclusively caused by negation: negationless systems can harbor their own forms of antinomicity. Nor must antinomic statements be defined in terms of truth and falsity. Any kind of opposition can produce its own form of antinomicity — whole and part, one and many, and a host of other contrasting concepts which do not necessarily involve negation and which can be considered independently of truth and falsity. Here we shall restrict ourselves solely to antinomic sentences and formulas in the sense in which they have been introduced above.

III. Antinomic Set Theories

§ 9. Antinomic membership

Some sets will be antinomic in the sense that they belong and do not belong to another set, that is, $\models x \in y \wedge x \notin y$ and $\not\models x \in y$, regardless of whether $\not\models x \notin y$ or not-$\not\models x \notin y$, abbreviated $x \in\!\!\!\notin y$, which will be read "x is an antinomic member of y", or "y contains x as an antinomic member". The set y need not be an antinomic member of another set. In what follows, some sets will be antinomic members of other sets and nonantinomic members of still others; some sets will not be antinomic members of any other set; and other sets will be antinomic members of any set to which they belong. Symmetrically, some sets may have some antinomic members and some nonantinomic ones; others may not have a single antinomic member; and still others may have only antinomic members.

The language of set theory will include variables $x, y, z, u, v, w, x_1, x_2, x_3, \ldots$, to range in given domains, and also constants $a, b, c, a_1, a_2, a_3, \ldots$, to represent single fixed sets. We shall postulate set-theoretic completeness below; as a consequence, if $a \epsilon b$ is true ($\models a \epsilon b$), then $a \epsilon b$ is an axiom or a theorem ($\vdash a \epsilon b$), and vice versa. Also, if $\models a \not\epsilon b$, then $\vdash a \not\epsilon b$, and vice versa.

In addition to the notation $a \mathrel{\epsilon\!\not\epsilon} b$ already introduced, we shall represent by $a \in b$ the case in which $\models a \epsilon b$ but not-$\models a \not\epsilon b$ and not-$\mathrel{|\!\equiv} a \epsilon b$, regardless of whether $\mathrel{|\!\equiv} a \not\epsilon b$ or not-$\mathrel{|\!\equiv} a \not\epsilon b$. The metamathematical negation 'not-$\models A$' stands for 'A is not true', and is equivalent metamathematically to not-$\vdash A$, 'A is not provable', given completeness; not-$\mathrel{|\!\equiv} A$ means A is not false. Therefore, $a \in b$ implies that the sentence $a \not\epsilon b$ is neither true nor a theorem. Finally, let us use $a \notin b$ to represent the case in which $\models a \not\epsilon b$ but not-$\models a \epsilon b$ and not-$\mathrel{|\!\equiv} a \not\epsilon b$, regardless of whether $\mathrel{|\!\equiv} a \epsilon b$ or not-$\mathrel{|\!\equiv} a \epsilon b$ ('$a \epsilon b$' is neither true nor is it therefore a theorem). As determined in Part II, the metamathematical negation 'not' must be distinguished from the formal negation '$\neg$' in that the metamathematics of antinomic set theory is not antinomic in the following sense: although A may be true and false, it is not the case that A is and is not true ($\models A$ and not-$\models A$); or, correspondingly, that A is both provable and unprovable ($\vdash A$ and not-$\vdash A$).

Given an arbitrary set b and a member a both in a given universe w such that $a \epsilon w$ and $b \epsilon w$ or b is included in w (see Definition 10 below), we shall assume that it is always determined which of these three mutually exclusive cases is in order: (i) $a \in b$, (ii) $a \notin b$, or (iii) $a \mathrel{\epsilon\!\not\epsilon} b$. These cases are relative to the given universe w; that is, $a \in b$ in w_1 is compatible with $a \notin b$ in w_2, and with $a \mathrel{\epsilon\!\not\epsilon} b$ in w_3: although the antinomicity of membership is a matter between a set a and the set b to which a belongs, it is dependent on the universe w in which both are being considered. Further, within the same universe w, a may be an antinomic member of b and a nonantinomic member of a proper subset or a proper superset of b. In a relative universe w, antinomicity is strictly an internal relation between a and b, a complex kind of membership and not a property that is intrinsic to the member a or the set b. Thus, we can say that a is a "circumstantially" antinomic member of b which can be "de-antinomized" by changing the universe w, or simply by considering a as a nonantinomic member of another set c in the same universe. Antinomicity is a variable, not an absolute condition.

§ 10. Axioms for an antinomic set theory $\mathbf{AS_1}$ based on membership

The presence of antinomic sets in a given universe w forces us to review the usual axioms to make room for the new cases. Let us begin by considering equality.

Definition 8. $y = z$ stands for $\forall_x (x \epsilon y \Leftrightarrow x \epsilon z)$. If y and z are included in w (see Definition 10 below), this definition thoroughly defines equality in w. If $y \epsilon w \wedge z \epsilon w$, the following becomes necessary.

Axiom 1. $y = z \Rightarrow \forall_u (y \epsilon u \Leftrightarrow z \epsilon u)$. This extensionality in terms of ϵ obtains in the relative universe w within which x, y, z, and u are considered. But extensionality determines uniqueness of sets only insofar as the all-inclusive membership ϵ is concerned — that is, uniqueness must be understood as "modulo" antinomicity, disregarding the branching of $x \epsilon y$ into either $x \in y$ or $x \epsilon\!\notin y$.

Each specific unique set in this sense will be represented by a constant a, b, c, etc., as mentioned. But although $y = z$ is an equivalent relation that implies that y and z have the same ϵ-members in a given universe w and may be represented therefore by the same constant a, because the type of membership of x to z may vary from universe w_1 to universe w_2, then if $y = z$ in both w_1 and w_2, the following two cases are compatible with $x \epsilon a$ (the meaning of $z \subseteq w$ is given in the usual way in Definition 10 below).

$\models x \epsilon w_1 \wedge (a \epsilon w_1 \vee a \subseteq w_1) \wedge x \in a$, and $\models x \epsilon w_2 \wedge (a \epsilon w_2 \vee a \subseteq w_2) \wedge x \epsilon\!\notin a$.

Definition 9. $y \neq z$ stands for $\exists_x ((x \epsilon y \wedge x \not\epsilon z) \vee (x \not\epsilon y \wedge x \epsilon z)) \vee \exists_u ((y \epsilon u \wedge z \not\epsilon u) \vee (y \not\epsilon u \wedge z \epsilon u))$.

Let us then distinguish the following particular cases:

(i) $y \doteq z$ stands for $y = z \wedge \forall_x (x \in y \Leftrightarrow x \in z) \wedge \forall_u (y \in u \Leftrightarrow z \in u) \wedge \forall_x (x \epsilon\!\notin y \Leftrightarrow x \epsilon\!\notin z) \wedge \forall_u (y \epsilon\!\notin u \Leftrightarrow z \epsilon\!\notin u)$.

(ii) $y\ {}_a{=}\ z$ stands for $y = z \wedge \exists_x (x \epsilon\!\notin y) \wedge \forall_v (v \epsilon z \Rightarrow v \in z)$. Symmetrically, the meaning of $y =_a z$ is obvious.

(iii) $y\ {}_a{=}_a\ z$ stands for $y = z \wedge \exists_x (x \epsilon\!\notin y) \wedge \exists_v (v \epsilon\!\notin z) \wedge x \neq v$.

(iv) $y\ {}^a{=}\ z$ stands for $y = z \wedge \exists_u (y \epsilon\!\notin u) \wedge \forall_v (z \epsilon v \Rightarrow z \in v)$. The meaning of $y =^a z$ is obvious.

(v) $y\ {}^a{=}^a\ z$ stands for $y = z \wedge \exists_u (y \epsilon\!\notin u) \wedge \exists_v (z \epsilon\!\notin v) \wedge u \neq v$.

(vi) $y\ {}^a_a{=}^a_a\ z$ stands for $y\ {}_a{=}_a\ z \wedge y\ {}^a{=}^a\ z$.

These different cases show that equality is a type of equivalence relation that can be interpreted as strict regular identity in terms of $\in$ and $\epsilon\!\notin$ if and only if $y \doteq z$. Thus, even if $y = z$ obtains in all universes w, the kind of extensionality of y and z may vary from one universe to another, say, $y\ {}_a{=}\ z$ in w_1 and $y =^a z$ in w_2, even if y and z are not only equal but have exactly the same ϵ-members in w_1 and w_2. Also, since $y \neq z$ obtains if $\exists_x(x \in y \wedge x \notin z)$, then $y =_a z$ entails $y = z \wedge y \neq z$, i.e., equality is antinomic in such cases. In particular, two relative universes w_1 and w_2 may be equal and different at the same time if, say $w_1 = w_2$ but $w_1 {}_a{=} w_2$ specifically. All this necessarily affects the application of any of the forthcoming axioms in which the existence of a set is relativized to a given universe.

§ 11. Inclusion

Let us now define inclusion in the usual way.

Definition 10. $y \subseteq z$ stands for $\forall_x(x \in y \Rightarrow x \in z)$, with proper inclusion, $y \subset z$, meaning $y \subseteq z \wedge \exists_x(x \in z \wedge x \notin y)$.

With this definition, $y = z$ is compatible with $y \subset z$ and $z \subset y$; obviously, $y {}_a{=} z$ implies $y \subset z$. We shall distinguish the following cases:

(i) $y {}_a{\subset} z$ for $y \subset z \wedge \exists_x(x \in\!\notin y \wedge x \in z)$.

(ii) $y \subset_a z$ for $y \subset z \wedge \exists_x(x \in y \wedge x \in\!\notin y)$.

(iii) $y {}_a{\subset}_a z$ for $y {}_a{\subset} z \wedge y \subset_a z$.

(iv) $y \subseteqq z$ for $y \subseteq z \wedge \forall_x(x \in y \Rightarrow x \in y \wedge x \in z)$.

To repeat, note that membership of a set x to a set y, being strictly a matter between x and y relative to the universe in which they are considered, has nothing to do with the kind of membership of x to the proper subsets of y or to the proper supersets of y. That is, if $x \in u \subset y \subset z$, it is possible to have $x \in\!\notin u \wedge x \in y \wedge x \in\!\notin z$, etc. In addition, if we change the relative universe w in which x and y are considered, $x \in y$ may become $x \in\!\notin y$. As a particular case, if in any universe w a set x is an antinomic member of every set to which it belongs, $\forall_w\forall_y(x \in w \wedge (y \in w \vee y \subseteq w \Rightarrow (x \in y \Rightarrow x \in\!\notin y)))$, we can represent this situation with the one-place predicate $\mathsf{Ant}(x)$, defined by the last formula which reads "x is universally antinomic".

Finally, antinomicity makes possible mutual proper inclusion. In other words, if proper inclusion is taken as the set-theoretic meaning of the phrase 'being a part of', then it is possible for two sets to each be a part of the other. Further, we can even say that the whole can be part of the part, i.e., $y \subset z \subset y$ if $y = z$ and $\exists_x(x \notin y \wedge x \in z) \wedge \exists_v(v \in y \wedge v \notin z)$. This is also the case if we use the expression 'being a part of' in the set-theoretic sense of being a member of, that is, $x \in y \in x$. We shall not assume the axiom of foundations that rules out $x \epsilon x$, $x \epsilon y \epsilon x$, etc., hence, $x \not\epsilon x$, $x \in x$, $x \epsilon\!\!\notin x$, $x \epsilon y \epsilon x$, etc., all are distinctly possible.

§ 12. Axiom of comprehension

One good mathematical reason for building antinomic set theories is to retrieve Cantor's comprehension axiom in its original unrestricted form; this return to "Cantor's paradise" would have significant consequences for the mathematical usefulness of such theories. Here, however, since we want to relativize membership as much as possible, we shall use an antinomic version of Zermelo's axiom of separation, the standard form of which is expressible as follows: Given a set y and an arbitrary set-theoretic formula $A(x)$ in which y does not occur and x is a free variable, there exists a set z such that $x \epsilon z \Leftrightarrow (x \epsilon y \wedge A(x))$. In this form, several possibilities are in order in accordance with the two mutually exclusive meanings of membership, that is, whether $x \epsilon y$ is interpreted as $x \in y$ or $x \epsilon\!\!\notin y$, and whether $x \epsilon z$ is interpreted as $x \in z$ or $x \epsilon\!\!\notin z$. To leave the ambiguity unresolved would mean that z would not be strictly unique; in effect, we could have as many z's as there are ways in which these four possibilities can be combined. In order to make z uniquely determined in each relative universe w, we postulate specifically:

Axiom 2. $\forall_w \forall_y (y \epsilon w \vee y \subseteq w \Rightarrow \exists_z (z \subseteq y \wedge \forall_x (x \epsilon w \Rightarrow ((x \in z \Leftrightarrow A(x) \wedge x \in y) \wedge (x \epsilon\!\!\notin z \Leftrightarrow A(x) \wedge x \epsilon\!\!\notin y)))))$, in which $A(x)$ does not involve any of the quantified variables w, y, and z, and in which x is a free variable. In other words, the kind of membership of x to z is determined by the kind of membership of x to y. The notation $z = \{x : (x \epsilon w \wedge x \epsilon y) \wedge (y \epsilon w \vee y \subseteq w) \wedge A(x)\}$ is now in order, and its meaning is unambiguously determined by Axiom Scheme 2. If w is fixed exclusively, then the expression $z = \{x : x \epsilon y \wedge A(x)\}$ suffices; and if in addition y is w, then $z = \{x : A(x)\}$ suffices, and z will gather those sets x which are members of w and satisfy $A(x)$, with w fixed.

§ 13. Russell's paradox

If $A(x)$ is $x \not\epsilon x$, then $z = \{x : x \epsilon y \wedge x \not\epsilon x\}$. If $z \epsilon y$ and $z \not\epsilon z$, then $z \epsilon z$, that is, $z \epsilon\!\!\notin z$ — z is an antinomic member of itself. If y is also an antinomic member of itself, then $y \epsilon z$, although $y \not\epsilon z$ remains undetermined. If $\forall_x (x \epsilon y \Rightarrow x \not\epsilon x)$, then $z = y$, even if $x \epsilon\!\!\notin x$ for some x. If, on the other hand, there is an x such that $x \in x$, then $z \subset y$. In any event, Russell's paradox is harmless even if it leads to contradictions.

§ 14. Other axioms and the Boolean operations

The following axioms are not all independent and each is relativized to a circumstantial universe w in which the sets involved are (i) members of w, (ii) members of members of w, or (iii) subsets of w. We shall not make this relativization to w explicit in all the axioms nor for all the sets, and will assume w fixed when it does not occur in the expressions that follow. Note once more that the kind of membership of x to w does not determine the kind of membership of x to any member of w.

Axiom 3. $\forall_y \forall_z (y \,\epsilon\, w \wedge z \,\epsilon\, w \Rightarrow \exists_u (u \,\epsilon\, w \wedge \forall_x (x \,\epsilon\, u \Leftrightarrow x = y \vee x = z)))$. Pairing.

Axiom 4. $\forall_y (y \,\epsilon\, w \Rightarrow \exists_u (u \,\epsilon\, w \wedge \forall_x (x \,\epsilon\, u \Leftrightarrow x \subseteq y)))$. Power set.

Axiom 5. $\forall_y \exists_u \forall_x (x \,\epsilon\, u \Leftrightarrow \exists_z (x \,\epsilon\, z \wedge z \,\epsilon\, y))$. Union.

Axiom 6. $\exists_y (\forall_x (x \notin y)) \wedge \forall_y \forall_z (\forall_u (u \notin y) \wedge \forall_v (v \notin z) \Rightarrow y \doteq z)$. Null set.

Definition 11. $\{y, z\}$ represents the unique set modulo antinomicity determined by Axiom 3; $\{y\}$ stands for $\{y, y\}$. $\mathcal{P}y$ represents the unique power set modulo antinomicity determined by Axiom 4. The expression 'modulo antinomicity' already used in connection with equality here means, precisely, that in applying Axioms 3, 4, and 5 as well, two sets u and u' may exist in each of these three cases that satisfy the axiom but such that $x \in u \wedge x \,\epsilon\!\!\!\notin u'$, say, and yet, $u = u'$ in each case. Finally, $\emptyset$ represents the unique null set; $\emptyset$ does not have antinomic members, although it may be the antinomic member of other sets; further, $\emptyset \in a$ may be true in the universe w_1 but $\emptyset \,\epsilon\!\!\!\notin a$ may also be true in w_2.

Note that the various kinds of inclusion, together with Axiom 2, allow us to distinguish special power sets as follows.

(i) $\mathscr{P}y = \{x \,:\, x \,\epsilon\, \mathcal{P}y \wedge x \subseteqq y\}$,

(ii) $\mathcal{P}_a y = \{x \,:\, x \,\epsilon\, \mathcal{P}y \wedge x \subseteq_a y\}$,

(iii) ${}_a\mathcal{P}y = \{x \,:\, x \,\epsilon\, \mathcal{P}y \wedge x \,{}_a\!\subseteq y\}$,

(iv) ${}_a\mathcal{P}_a y = \{x \,:\, x \,\epsilon\, \mathcal{P}y \wedge x \,{}_a\!\subseteq_a y\}$.

The kind of membership of x to $\mathcal{P}_a y$, etc., is determined by the kind of membership of x to $\mathcal{P}y$ in accordance with Axiom 2.

Axiom 2 also guarantees the existence of the usual set-theoretic operations, but some restrictions should apply on the possible kinds of membership. For the case of intersection, for example, the usual Boolean definition $x \in y \cap z \Leftrightarrow x \in y \wedge x \in z$ will hold in general, but whether $x \in y \cap z$ or $x \in\!\!\!\notin y \cap z$ will depend on the kind of membership of x to the universe w in which the intersection is considered. To make certain that the operations are single-valued in each universe, we then define the following:

Definition 12. (i) $y \cap_w z = \{x : x \in y \wedge x \in z\}$, which implicitly means $((x \in w \Rightarrow x \in y \cap z) \wedge (x \in\!\!\!\notin w \Rightarrow x \in\!\!\!\notin y \cap z))$. The kind of membership of x to y and to z is irrelevant; note also that y and z are each either a member or a subset of w, given that Axiom 2 relativizes comprehension to a fixed universe w. The subindex w in $y \cap_w z$ can be dropped when w is taken for granted. In fact, given the final remark in § 12, $y \cap z = \{x : x \in y \wedge x \in z\}$ is sufficient as a definition of intersection if we take $A(x)$ to mean $x \in y \wedge x \in z$ with y and z as fixed parameters.

(ii) $y \cup_w z = \{x : x \in y \vee x \in z\}$, which implicitly means $(x \in w \Rightarrow x \in y \cup z) \wedge (x \in\!\!\!\notin w \Rightarrow x \in\!\!\!\notin y \cup z)$.

(iii) $y'_w = \{x : x \notin y\}$, which implicitly means $(x \in w \Rightarrow x \in y') \wedge (x \in\!\!\!\notin w \Rightarrow x \in\!\!\!\notin y')$. Again, note that the kind of membership of x to the complement of y is determined not by the kind of nonmembership of x to y but by the kind of membership of x to w. Thus, the two mutually exclusive cases follow: first, if $x \in w \wedge x \in y'$, then $x \in y'$, whether $x \notin y$ or $x \in\!\!\!\notin y$; second, if $x \in\!\!\!\notin w \wedge x \in y'$, then $x \in\!\!\!\notin y'$, whether $x \notin y$ or $x \in\!\!\!\notin y$. The expression $y'_w = \{x : x \notin y\}$ implicitly assumes this distinction.

(iv) $\mathsf{s}_w y = \{x : x \in y \vee x \doteq y\}$, where $x \in \mathsf{s}_w y$ if $x \in w$, and $x \in\!\!\!\notin \mathsf{s}_w y$ if $x \in\!\!\!\notin w$. In addition, $\mathsf{S}_w y = \{x : x \in y \vee x \doteq y\}$ where $y \in w$ and hence $y \in \mathsf{S}y$ also. If w is fixed, we simply write $\mathsf{s}y$ and $\mathsf{S}y$. For $\mathsf{s}y$, and $\mathsf{S}y$ in particular, we shall assume $\models (y \doteq z \Rightarrow \mathsf{S}y \doteq \mathsf{S}z) \wedge (\mathsf{S}y \doteq \mathsf{S}z \Rightarrow y \doteq z)$.

(v) $\mathsf{Nat}(x)$ iff $x \doteq \emptyset \vee (x \doteq \mathsf{S}y \wedge \mathsf{Nat}(y))$, x is a natural number.

Axiom 7. $\exists_y(\emptyset \in y \wedge \forall_x(x \in y \Rightarrow \mathsf{S}x \in y))$. Existence of an infinite set with an infinity of nonantinomic members. The axiom also guarantees the existence of an infinity of natural numbers.

Axiom 8 (meta-axiom). The antinomic set theory AS_1 satisfies completeness in the sense that A is an axiom or a theorem of AS_1 if and only if it is true in all models of AS_1: $\vdash A$ iff $\models A$. It should be re-emphasized that $\models A$

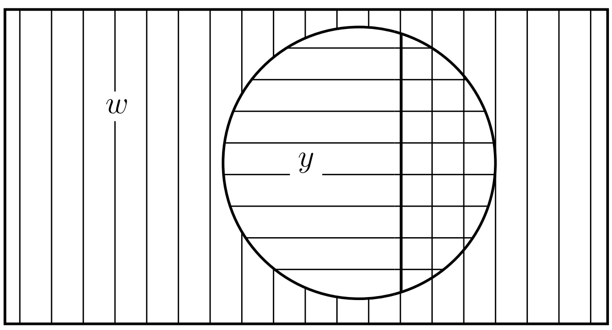
includes these two mutually exclusive cases: (i) A is simply true, $\models A$ but not-$|\!\equiv A$, and (ii) A is true-and-false, $\models A$ and $|\!\equiv A$, noting that A could be simply true in one model and antinomic in another despite being true in all models of AS_1. For positive formulas in AS_1 the only change with respect to the classical situation is the addition of semantic antinomicity in some cases. For negative formulas in AS_1 the application of Axiom 8 is ad hoc and goes from semantics to syntax. Again, the positive diagram of a given model of AS_1 is predetermined by the axioms. The negative diagram, i.e., the collection of all negative formulas true or antinomic in such a model, remains incomplete and open to successive additions.

Axiom 8 does not imply that AS_1 is syntactically complete, although the existence of a complete extension of AS_1 can certainly be assumed. Since AS_1 is far from having a recursive set of axioms, Gödel's first incompleteness theorem does not apply; but even if AS_1 could be presented as an axiomatizable extension of formal number theory, once one gives up the premise of consistency Gödel's second incompleteness theorem does not apply either.

§ 15. Relative complementation and Venn diagrams

Definition 13. $z - y = \{x : x \epsilon z \wedge x \not\epsilon y\}$, complement of y relative to z for all sets y and z that are either members or subsets of the implicit universe w.

Because of antinomicity, some members x of a relative universe w, which in turn contains y as a subset, may belong to y and to its complement. The Venn diagram for the complement of y (y represented by the horizontally shaded area inside the circle) would look like the following vertically shaded area.



That is, y and y' intersect, and the members of this nonempty intersection are those within the doubly shaded area inside the circle. The area inside the circle not in $y \cap y'$ corresponds to $y - y' = \{x : x \epsilon y \wedge x \notin y'\} = \{x : x \in y\}$.

Note that $x \epsilon\!\notin y$ is compatible with $x \notin y'$ (if $x \in w$), and precisely because the antinomic member of a set is not necessarily the antinomic member of its complement, $y \neq y''$ is possible. In effect, y'' may be a proper subset of y if y' has no antinomic members, but if it does, then again y' and y'' would intersect and y'' would not be contained in y; if $x \in y'$, then $x \notin y''$. Also, $z \subset y$ does not imply $y' \subset z'$, since for the same x we may have $x \epsilon\!\notin y$ and $x \in z$, i.e., $x \epsilon y' \wedge x \notin z'$. Further, since $x \epsilon\!\notin y \wedge x \epsilon\!\notin z \wedge x \in y \cap z$ is possible (if $x \in w$), then $x \notin (y \cap z)'$ even though $x \epsilon y' \wedge x \epsilon z'$. If we define in the usual way the generalized intersection $\cap_i y_i$ (relative to a universe w) of a family of sets (each included in w) indexed by an index set I, then if there is a set x such that $\forall_i (i \epsilon I \Rightarrow x \epsilon\!\notin y_i), x \epsilon \cap_i y_i$ but also $x \epsilon \cap_i y'_i$. If the relative universe w contains a single antinomic member $x \epsilon\!\notin w$, then the complement of w is not empty. If $y \subseteq w$ and all the members of y are antinomic, $y \subseteq y'$; also, many subsets y_i of w could have the same complement, and if for each y_i all its members were antinomic, then $y'_i = w$ for all i. In extreme cases, if $y \subseteq w$ and $\forall_x (x \epsilon y' \Rightarrow x \epsilon y)$, then $y' \subseteq y = w$, and if $\forall_x (x \epsilon y' \Leftrightarrow x \epsilon y)$, then $y = y' = w$.

Because the laws of double negation are not valid, the logical De Morgan laws do not obtain, and neither do the set-theoretic De Morgan laws. For example, (i) $(y \cap z)' \subset y' \cup z'$ and (ii) $(y \cap z)' \supset y' \cup z'$ are both possible cases. To see this, consider that the kind of membership of a set x to y', z', $(y \cap z)'$, and $(y \cup z)'$ is determined according to Axiom 2 and Definition 12 by the kind of membership of x to the relative universe w of which y and z are members or subsets. The proper inclusion (i) is possible because members x of $y' \cup z'$ may not be members of $(y \cap z)'$ if the following obtains: if $x \in w$, $x \epsilon\!\notin y$, $x \in z$, then $x \epsilon y'$ and $x \in y' \cup z'$, but $x \in y \cap z$, hence $x \notin (y \cap z)'$. As for (ii), it may obtain if $x \epsilon\!\notin w$, $x \in y$, and $x \in z$, for then $x \epsilon\!\notin y \cap z$ but $x \notin y'$ and $x \notin z'$, hence $x \notin y' \cup z'$. Also, neither $(y \cap z)'$ nor $y' \cup z'$ may be included in the other.

As hinted, if we iterate the operation of complementation various cases are possible. (i) If $x \notin y$ and $x \epsilon\!\notin y'$, we cannot in general assert $y \subseteq y''$ or $y'' \subseteq y$. Similar situations arise if (ii) $x \epsilon\!\notin y \wedge x \in y'$, and (iii) $x \epsilon\!\notin y \wedge x \epsilon\!\notin y'$. Cases (iv) $x \notin y \wedge x \in y'$, and (v) $x \in y \wedge x \notin y'$ are the usual nonantinomic ones. In turn, $x \in y''$, $x \notin y''$, and $x \epsilon\!\notin y''$ yield corresponding cases for y''. All these cases are of course determined by the kind of membership of x to the relative universe w; that is, if $x \in w$, only cases (ii) and (iv) can obtain; and if $x \epsilon\!\notin w$, only cases (i) and (iii) are possible. A change in universe may change not only the members x of y', y'', etc., but also the kind of membership of each x to y', y'', etc.

§16. Ordered pairs, relations, functions, cardinalities, Sierpiński's theorem

Definition 14. (i) $\langle y, z\rangle$ stands for $\{\{y\}\}, \{\{y, z\}\}$. The existence of the ordered pair follows Axiom 3; $\langle x, y\rangle$ could have antinomic members if $\{y\} \epsilon\!\notin \langle y, z\rangle$ or $\{y, z\} \epsilon\!\notin \langle y, z\rangle$.

(ii) $y \times z$ stands for $\{\langle u, v\rangle : u \in y \wedge v \in z\}$. The existence of the Cartesian product follows Axiom 2; again, $y \times z$ could have antinomic members.

(iii) $\mathsf{Rel}(R)$ iff $R \subseteq y \times z$ for some y and z members or subsets of w. If R is a binary relation and $y = z$, then R is a relation on y. R is antinomic iff there exists a pair $\langle u, v\rangle$ such that $\langle u, v\rangle \epsilon\!\notin R$, nonantinomic otherwise. We shall only consider binary relations here.

(iv) If R is a relation, $\mathsf{Dom}\, R = \{u : \langle u, v\rangle \in R\}$, $\mathsf{Rng}\, R = \{v : \langle u, v\rangle \in R\}$. Domain and range of a binary relation.

(v) Given the sets y and z in w, $\mathsf{Func}(F)$ iff $F \subseteq y \times z \wedge \mathsf{Dom}\, F \doteq y \wedge \mathsf{Rng}\, F \subseteq z \Rightarrow (\langle u, v\rangle \in F \wedge \langle u, t\rangle \in F \Rightarrow v \doteq t)$. F is a function on y into z. F is antinomic iff it is an antinomic relation, nonantinomic otherwise.

(vi) $\mathsf{Inj}(F)$ iff $F \subseteq y \times z \wedge \mathsf{Func}(F) \wedge (\langle u, v\rangle \in F \Rightarrow \exists!_u \langle u, v\rangle \in F)$. F is a one-to-one or injective function on y into z. $\exists!_u A(u)$ is defined by $\exists_u A(u) \wedge \forall_r \forall_s (A(r) \wedge A(s) \Rightarrow r \doteq s)$.

(vii) $\mathsf{Sur}(F)$ iff $F \subseteq y \times z \wedge \mathsf{Func}(F) \wedge \mathsf{Rng}\, F \doteq z$. F is a function on y onto z, or surjective.

(viii) $\mathsf{Bij}(F)$ iff $F \subseteq y \times z \wedge \mathsf{Inj}(F) \wedge \mathsf{Sur}(F)$. F is a one-to-one function on y onto z, or bijective.

(ix) Two sets y and z have the same cardinality (or are equinumerous) iff $\exists_F(\mathsf{Bij}(F) \wedge \mathsf{Dom}\, F \doteq y \wedge \mathsf{Rng}\, F \doteq z)$, denoted by $\mathsf{Card}\, y = \mathsf{Card}\, z$. A set x is inductive, $\mathsf{Ind}(x)$, iff $\exists_y \exists_F(\mathsf{Nat}(y) \wedge \mathsf{Bij}(F) \wedge \mathsf{Dom}\, F \doteq x \wedge \mathsf{Rng}\, F \doteq y)$.

(x) A set y is of cardinality less than or equal to that of z iff $\exists_F(F \subseteq y \times z \wedge \mathsf{Inj}(F) \wedge \mathsf{Dom}\, F \doteq y \wedge \mathsf{Rng}\, F \subseteq z)$, denoted $\mathsf{Card}\, y \leqslant \mathsf{Card}\, z$. $\mathsf{Card}\, y < \mathsf{Card}\, z$ stands for $\mathsf{Card}\, y \leqslant \mathsf{Card}\, z \wedge \mathsf{Card}\, y \neq \mathsf{Card}\, z$. A set y is reflexive, $\mathsf{Ref}(y)$, iff $\exists_x \exists_F(x \subset y \wedge \mathsf{Bij}(F) \wedge \mathsf{Dom}\, F \doteq y \wedge \mathsf{Rng}\, F \doteq x)$.

The equivalence relation $\mathsf{Card}\, y = \mathsf{Card}\, z$ is antinomic iff $\exists_{F_1}(\mathsf{Bij}(F_1) \wedge \mathsf{Dom}\, F_1 \doteq y \wedge \mathsf{Rng}\, F_1 \doteq z \wedge \exists_{\langle u,v\rangle} \langle u, v\rangle \epsilon\!\notin F_1) \vee \exists_{F_2}(\mathsf{Bij}(F_2) \wedge \mathsf{Dom}\, F_2 \doteq z \wedge \mathsf{Rng}\, F_2 \doteq y \wedge \exists_{\langle u,v\rangle} \langle u, v\rangle \epsilon\!\notin F_2)$, which allows for three possibilities:

(i) $\mathsf{Card}\, y\,{}_a\!\!= \mathsf{Card}\, z$ iff the first disjunct is true, but the second one is not in the disjunction just given; similarly, (ii) $\mathsf{Card}\, y =_a \mathsf{Card}\, z$ iff there is no F_1 but there is a F_2 for the same formula, and (iii) $\mathsf{Card}\, y\,{}_a\!\!=_a \mathsf{Card}\, z$ iff there is both a F_1 and a $F2$.

Correspondingly, $\mathsf{Card}\, y\,{}_a\!\!\leqslant \mathsf{Card}\, z$ iff $\exists_F(\mathsf{Inj}(F) \wedge \mathsf{Dom}\, F \doteq y \wedge \mathsf{Rng}\, F \subseteq z \wedge \exists_{\langle u,v\rangle} \langle u, v\rangle \in\!\!\!\notin F)$. The meaning of $\mathsf{Card}\, y\,{}_a\!\!< \mathsf{Card}\, z$ is obvious.

Definition 15. The cardinal number of a set y, denoted $\mathsf{Card}\, y$, is the equivalence class of all sets u equinumerous to y in a universe w ($\mathsf{Card}\, u = \mathsf{Card}\, y$). $\mathsf{Card}\, y$ is a subset of w and hence relative to the given universe: from universe w_1 to universe w_2, $\mathsf{Card}\, y$ may change its members, and y its relative cardinality vis-a-vis other sets. $\mathsf{Card}\, y$ may contain antinomic members as well as members which are and are not equinumerous to y.

In 1947 Sierpiński showed that given a function F on y into z, it is not possible to prove without the axiom of choice that the cardinality of the range of F is not greater than the cardinality of the domain of F. That is, without the axiom of choice, $\mathsf{Card}\, y < \mathsf{Card}\, \mathsf{Rng}\, F$ is not inconsistent with set theory. Clearly, with the antinomic comparability of cardinalities if $\forall_F((F \subseteq y \times z \wedge \mathsf{Func}(F) \wedge \mathsf{Dom}\, F \doteq y \wedge \mathsf{Rng}\, F \doteq z \wedge \mathsf{Bij}(F)) \Rightarrow \exists_{\langle u,v\rangle}(\langle u, v\rangle \in\!\!\!\notin F))$, then $\mathsf{Card}\, z > \mathsf{Card}\, y$ as well as $\mathsf{Card}\, y \geqslant \mathsf{Card}\, z$. Thus, even with the axiom of choice $\mathsf{Card}\, z > \mathsf{Card}\, y$ is not excluded.

§ 17. Mediate sets

It was Bolzano in his *Paradoxes of the Infinite* who first distinguished between a set being finite if (i) it is inductive, i.e., counted by a terminal sequence of positive integers, or (ii) it is not reflexive, i.e., equinumerous to a proper subset of itself (today a nonreflexive set is also called Dedekind finite). A mediate set is defined in [21] (vol. II, p. 280) as one which is noninductive and nonreflexive. The existence of such sets is ruled out by the axiom of choice; without the axiom of choice, their existence is possible. The cardinality of a mediate set μ is comparable to that of an inductive set x in the sense that $\mathsf{Card}\, x < \mathsf{Card}\, \mu$ (mediate sets contain finite subsets), but it is not comparable to the cardinality of a reflexive (Dedekind infinite) set; that is, $\neg(\mathsf{Card}\, \mu < \aleph_0) \wedge \neg(\mathsf{Card}\, \mu \geqslant \aleph_0)$, a mediate cardinal being the cardinal number of a mediate set. There is neither a minimum nor a maximum mediate cardinal; also, $\mathsf{Card}\, \mu \neq \mathsf{Card}\, \mu + 1$ and $\mathsf{Card}\, \mu \neq \mathsf{Card}\, \mu - 1$. The mediate cardinals are closed under addition and under multiplication by a mediate cardinal or by an inductive cardinal different from zero. Further, if $\mathsf{Card}\,\mu^{\mathsf{Card}\,\nu}$ is mediate, then μ or ν is mediate. However, if μ is mediate, then $2^{2^{\mathsf{Card}\mu}}$ is not mediate but reflexive (the power set of the power set of a nonempty, noninductive, and nonreflexive set is reflexive). As for $2^{\mathsf{Card}\,\mu}$ with μ mediate, sometimes it is mediate, sometimes it is reflexive (see [21], Vol II, p. 280).

A paper by Dorothy Wrinch [22] generalizes the notion of mediate cardinals to those which are comparable to all the usual cardinals up to an aleph greater than or equal to $\aleph_0$. The negation of the existence of such generalized mediate cardinals implies the axiom of choice (cf. [22]) and is therefore equivalent to such axiom, since the latter implies the nonexistence of all mediate cardinals. Axiom 7 above asserts the existence of nonmediate infinite sets but leaves open the possible existence of mediate sets. One of the various axioms of choice to be proposed here will be relativized to nonmediate sets; yet, choice and mediate sets will be compatible.

Since classically mediate cardinals do not satisfy the axiom of choice, they need not necessarily be comparable. The existence of incomparable mediate cardinals is still an open question. Mediate cardinals have been described as “small” in that they share with inductive sets the property of being nonreflexive, and as “large” because they cannot be obtained by adding 1 to 0 a finite number of times (“finite” used in the intuitive sense that such addition has an effective end). Here, because functions can be antinomic, a set y can be mediate and nonmediate if every function F that maps y onto a proper subset of y contains a pair $\langle u, v\rangle$ which belongs and does not belong to F. As a consequence, in such a case, $\forall_z(z \subset y \wedge \mathsf{Card}\,y = \mathsf{Card}\,z \Rightarrow \mathsf{Card}\,z < \mathsf{Card}\,y)$. A mediate set y which is not nonmediate shall be called *strictly mediate*; if y is both mediate and nonmediate it shall be called *antinomically mediate*. A set can be simultaneously antinomically mediate and the antinomic member of another set. Note that a reflexive set y may have an injective image in an antinomically mediate set z: since z is reflexive and nonreflexive, then y may be comparable and noncomparable to z. In fact, if every function F that compares y to z is not only antinomic and injective but bijective as well, and such that $\forall_u\forall_v(\langle u, v\rangle \mathrel{\epsilon} y \times z \wedge \langle u, v\rangle \mathrel{\epsilon} F) \Rightarrow \langle u, v\rangle \not\mathrel{\epsilon} F)$, then z is both mediate and equinumerous to a reflexive set. One should keep in mind, though, that being finite, infinite, or mediate in any sense are properties relative to the universe w. Changing the universe may make a set reflexive, if it was not, by adding the appropriate function, or make it antinomically reflexive if it was simply reflexive, etc. It is a prejudice to think that mediate sets are useless; like the generic sets produced by forcing methods, they throw light on the understanding of sets in general and on the axiom of choice in particular. More about this later.

§ 18. Amorphous sets

The standard definition of infinite set is that of a set not equinumerous with a natural number, and finite if it is. A set is called amorphous if it is infinite in the standard sense but it is not the union of two disjoint infinite sets. There are models of set theory in which the axiom of choice fails and which have amorphous sets; one such is the basic Fraenkel model.

A set y is called Tarski finite (T-finite) iff every nonempty $\subseteq$-monotone chain $X \subseteq \mathcal{P}y$ has a $\subseteq$-maximal element. Every amorphous set is T-finite; hence, not every T-finite set is finite in the standard sense. Further, every T-finite set is nonreflexive (Dedekind finite) but the converse is not true.

Since amorphous sets are infinite in the standard sense they are noninductive, and because they are also nonreflexive, they are mediate. One must remember that if we do not assume the axiom of choice, there are several nonequivalent ways of defining infinite sets, as well as finite sets. Thus, a set may be nonfinite in one sense and finite in another. According to Von Neumann, this situation raises serious objections to constructive philosophies of mathematics intuitionism and the like (cf. [16]). The fact is that without the axiom of choice we do not know exactly what finite means, the one concept that constructivism deems fundamental and unmistakable. We must face this issue: without the axiom of choice the idea of finite becomes ambiguous and hazy, and a set can be finite in one sense and infinite in another, as well as being both finite and infinite in the same sense, as is the case with an antinomically mediate set, both Dedekind finite and Dedekind infinite.

§ 19. An antinomic set theory $\mathbf{AS_2}$ based on inclusion

In a previous paper [4] we took inclusion instead of membership as the one basic primitive set-theoretic predicate; the other primitive ideas were those of set (x, y, z,…, variable sets; a, b, a_1, a_2, a_3,…, constant sets), and binary relations (R, R_1, R_2,…). The definitions and axioms offered there are as follows.

Definition 1. $y = z$ iff $\forall_x(x \subseteq y \Leftrightarrow x \subseteq z)$. Equality.

Axiom 1. $\forall_y \forall_z(y = z \Rightarrow \forall_u(y \subseteq u \Leftrightarrow z \subseteq u))$. Extensionality.

Definition 2. $y \subset z$ iff $y \subseteq z \wedge y \neq z$. Proper inclusion.

Axiom 2. $\exists_y \forall_z(z \nsubseteq y \wedge \forall_u(u \neq y \Rightarrow y \subseteq u))$. Null set, denoted by $\emptyset$ and not included in itself.

Axiom 3. Reflexivity (for all sets other than $\emptyset$), antisymmetry, and transitivity of inclusion.

Axiom 4. $\forall_y \exists_z ((z \subseteq y \wedge \forall_x (x \subseteq y \wedge \varphi(x) \Rightarrow x \subseteq z) \wedge \forall_u \forall_v ((v \subseteq y \wedge \varphi(v) \Rightarrow v \subseteq u) \Rightarrow z \subseteq u))$. Separation, where $\varphi(x)$ is any well-formed formula in the language of AS_2 in which y, z, u, and v do not occur and x is a free variable. If $\varphi(y)$ is also the case, then $y \subseteq z$, i.e., $y = z$ by antisymmetry. Since some subsets of y may not satisfy φ, "separation" does not have the clear-cut meaning that it has in Zermelo's set theory, i.e., it is possible for z to have as subsets sets without the property φ.

Definition 3. The notation $z = \{x : x \subseteq y \wedge \varphi(x)\}$ represents the least set u that contains all the sets included in y having the property φ.

Axiom 5. $\forall_x \exists_y \exists_z (x \subseteq y \wedge z \subseteq y \wedge z \nsubseteq x \wedge x \nsubseteq z)$. Expansion. There is no class of all sets.

Now let a_1 be an arbitrary but fixed set, and a_2 a nonspecified but fixed superset of a_1 satisfying the condition that y satisfies in Axiom 5, that is, $a_1 \subseteq a_2 \wedge \exists_z (z \subseteq a_2 \wedge z \nsubseteq a_1 \wedge a_1 \nsubseteq z)$; the existence of this a_2 is guaranteed by the axiom. Let a_3 be a nonspecified but fixed superset of a_2 satisfying the same condition. In general, let a_{n+1} be a similar superset of a_n. The finite sequence $a_1, a_2, \ldots, a_n, a_{n+1}$ can be made as long as one wishes by successive application of Expansion. However, in order to assert the existence of an infinite set that contains as subsets all the possible terms of this sequence, we need the following additional axiom scheme.

Axiom 6. For any sequence $a_1, a_2, \ldots, a_n, \ldots$ satisfying the description just given $\exists_y (a_1 \subseteq y \wedge (a_n \subseteq y \Rightarrow a_{n+1} \subseteq y))$. Infinity. The infinite set y contains all the sets a_i of the sequence, plus all the subsets of each of these terms.

Axiom 7. $\forall_x \forall_y \exists_z \forall_u (u \subseteq x \vee u \subseteq y \Leftrightarrow u \subseteq z)$. Union.

Axiom 8. $\forall_x \forall_y \exists_z \forall_u (u \subseteq x \wedge u \subseteq y \Leftrightarrow u \subseteq z)$. Intersection. Union and intersection as determined by these axioms differ from their usual definitions as operations given in terms of membership; for example, no new subsets can be obtained in z by the union of x and y other than those already in x and y.

Definition 4. $\mathsf{E}(x)$ stands for $x \neq \emptyset \wedge \forall_y (y \neq \emptyset \Rightarrow y \nsubseteq x \vee y = x)$. Elementhood. Elements are nonempty sets without nonempty proper subsets. The null set is not an element, although it can be the term of a predicate formula and is certainly the subset of every set except itself.

Schröder asserted that "nothing" is a subject of every predicate, to which Frege objected, drawing the contradictory conclusion that if so, then $\varphi(\emptyset) \wedge \neg\varphi(\emptyset)$ would obtain, suggesting that if one must have the null set at all, it is better to have it as a subset of every set (cf. [9]). From an antinomic point of view both positions can be made simultaneously acceptable.

Axiom 9. $\forall_x(x \neq \emptyset \Rightarrow \exists_y(y \subseteq x \wedge \mathsf{E}(y))$. Regularity. Every set contains at least one element.

Axiom 10. $\forall_x \exists_y \exists_z(x \subseteq y \wedge z \subseteq y \wedge z \nsubseteq x \wedge \mathsf{E}(z))$. Element expansion. There is no set of all elements, and there is an infinity of them.

Definition 5. $x \in y$ stands for $x \subseteq y \wedge \mathsf{E}(x)$. Membership. Only elements are members. Also, every element is a member of itself, and given two distinct elements, neither one is a member of the other.

Axiom 11. $\exists_y \forall_x(x \in y \Leftrightarrow \varphi(x))$. Comprehension for elements. $\varphi(x)$ is a well-formed formula in which y does not occur and x is a free variable. This axiom asserts the existence of a set containing as members all the elements that have the property φ. The set of all elements which are not members of themselves is empty, i.e., Russell's paradox cannot be transferred to AS_2.

The objective of the approach just described is to have a set-theoretic base on which to build a topology of multiple location (cf. [4]). Here, we shall outline briefly how to use inclusion as an antinomic predicate. Assume that some sets are antinomic in the sense that they are included and not included in another set, that is, $\models x \subseteq y \wedge x \nsubseteq y$ and $\mathrel{|}\!\equiv x \subseteq y$, regardless of whether $\mathrel{|}\!\equiv x \nsubseteq y$ or not-$\mathrel{|}\!\equiv x \nsubseteq y$, abbreviated $x \subseteq\!\nsubseteq y$, which reads "$x$ is an antinomic subset of y" or "y is an antinomic superset of x". The set y need not be an antinomic subset of another set. In fact, (i) some sets can be antinomic subsets of other sets, $x \subseteq\!\nsubseteq x$ included as a possibility, and (ii) some sets may not be antinomic subsets of any set. Symmetrically, (iii) some sets may have only antinomic subsets, (iv) others may have some antinomic subsets and some nonantinomic ones, and (v) some sets, finally, may not have a single antinomic subset.

Similar to the notation proposed for membership, given the constants a and b representing fixed sets, $a \subseteqq b$ stands for $\models a \subseteq b$ but not-$\models a \subseteq b$ and not-$\mathrel{|}\!\equiv a \subseteq b$, regardless of whether $\mathrel{|}\!\equiv a \nsubseteq b$ or not-$\mathrel{|}\!\equiv a \nsubseteq b$. Also, $a \nsubseteqq b$ stands for $\models a \nsubseteq b$ but not-$\models a \subseteq b$ and not-$\mathrel{|}\!\equiv a \nsubseteq b$, regardless of whether $\mathrel{|}\!\equiv a \subseteq b$ or not-$\mathrel{|}\!\equiv a \subseteq b$. Assuming completeness as we did with AS_1, $\models a \subseteq b$ is metamathematically equivalent to $\vdash a \subseteq b$, which also holds for every positive or negative formula of AS_2, whose axioms can now be extended to include antinomic cases. Thus, for example, $y = z \wedge y \neq z$ may obtain, and the existence of $\mathcal{P}_a y = \{x : x \subseteq\not\subseteq y\}$ be justified by Separation, even though not every set in $\mathcal{P}_a y$ must be an antinomic subset of y. The notions of ordered pair, Cartesian product, relation, function, equinumerosity, and comparability of cardinals given in the earlier paper (cf. [4]) can also be antinomically extended and an antinomic topology based on inclusion developed. Further, as with membership, the kind of inclusion in $x \subseteq y$ may be relativized to a universe w, and modified from universe to universe. Incidentally, the null set can also be the antinomic subset of other sets.

It should be remarked that Frege, following Schröder, considered inclusion as "the most important relation between sets" [9], fully identifying the part-whole relation with set-theoretic inclusion. On the other hand, Hao Wang observed that an unavoidable conclusion of the independence of the continuum hypothesis is that, from the point of view of classical set theory, we still do not know what being a subset really means.

§ 20. An antinomic set theory $\mathbf{AS_3}$ based on union taken as a primitive predicate

In a previous paper [6] union was used as a primitive binary predicate rather than as an operator. Here we shall expand the predicate of union and make it antinomic. Let us assume a universe of sets $x, y, z, u, v, s, t, x_1, x_2, x_3, \ldots$ in which for some sets x, y, $\mathsf{u}(x, y)$ holds ("x is united to y"), for other sets z, v, $\neg\mathsf{u}(z, v)$ holds ("z is disunited from v") and for still other sets s, t, $\mathsf{u}(s,t) \wedge \neg\mathsf{u}(s,t)$ holds ("s is united to and disunited from t").

As with ϵ, u is an ambiguous notation to cover both the nonantinomic and the antinomic cases. Accordingly, we shall identify the following possibilities: (i) $\mathsf{u}\neg\mathsf{u}(x, y)$ represents the case $\models \mathsf{u}(x, y) \wedge \neg\mathsf{u}(x, y)$ and $\models\!\!\!\equiv \mathsf{u}(x, y)$, regardless of whether $\models\!\!\!\equiv \neg\mathsf{u}(x, y)$ or not-$\models\!\!\!\equiv \neg\mathsf{u}(x, y)$; (ii) $\mathsf{U}(x, y)$ for the case $\models \mathsf{u}(x, y)$ but not-$\models \neg\mathsf{u}(x, y)$ and not-$\models\!\!\!\equiv \mathsf{u}(x, y)$, whether $\models\!\!\!\equiv \neg\mathsf{u}(x, y)$ or not-$\models\!\!\!\equiv \neg\mathsf{u}(x, y)$; (iii) $\neg\mathsf{U}(x, y)$ for the case $\models \neg\mathsf{u}(x, y)$ but not-$\models \mathsf{u}(x, y)$ and not-$\models\!\!\!\equiv \neg\mathsf{u}(x, y)$, whether $\models\!\!\!\equiv \mathsf{u}(x, y)$ or not-$\models\!\!\!\equiv \mathsf{u}(x, y)$.

Axiom 1. $\exists_x \forall_y \neg\mathsf{U}(x, y)$. There is at least one isolated set strictly disunited from all other sets including itself. If x is one such set, we write $\mathsf{Iso}(x)$.

Axiom 2. $\forall_x(\neg\mathsf{Iso}(x) \Rightarrow \mathsf{u}(x, x))$; $\forall_x \forall_y(\mathsf{u}(x, y) \Rightarrow \mathsf{u}(y, x))$. (Union is not necessarily transitive.)

Axiom 3. $\forall_y \exists_x(\neg\mathsf{Iso}(x) \wedge \neg\mathsf{u}(x, y))$. Unity of sets is not universal for non-isolated sets.

Definition 1. $y = z$ iff $\forall_x(\mathsf{u}(x,y) \Leftrightarrow \mathsf{u}(x,z))$. In particular: $y \doteq z$ iff $\forall_x(\mathsf{U}(x,y) \Leftrightarrow \mathsf{U}(x,z)) \wedge \forall_u(\mathsf{u}\neg\mathsf{u}(u,y) \Leftrightarrow \mathsf{u}\neg\mathsf{u}(u,z))$.

Definition 2. $y \subseteq z$ iff $\forall_x(\mathsf{u}(x,y) \Rightarrow \mathsf{u}(x,z))$. Note that $y \subseteq z$ is compatible with $\exists_u(\mathsf{u}\neg\mathsf{u}(u,y) \wedge \mathsf{U}(u,z))$.

Axiom 4. $\forall_y\exists_z\forall_x((\mathsf{U}(x,z) \Leftrightarrow \mathsf{U}(x,y) \wedge A(x)) \wedge (\mathsf{u}\neg\mathsf{u}(x,z) \Leftrightarrow \mathsf{u}\neg\mathsf{u}(x,y) \wedge A(x)))$, where $A(x)$ is a well-formed formula in the language of AS_3 in which y and z do not occur and x is a free variable. Separation scheme. Since z is uniquely determined, the notation $z = \{x : \mathsf{u}(x,y) \wedge A(x)\}$ is justified. Note that if $\mathsf{Iso}(u) \wedge A(u)$ is the case, still $\neg\mathsf{U}(u,z)$ obtains: z does not gather isolated sets.

Axiom 5. For any positive integer k, $\exists_{x_1}\exists_{x_2}\ldots\exists_{x_k}(x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \ldots \wedge x_{k-1} \neq x_k)$. This scheme guarantees the existence of an infinity of sets.

Axiom 6. $\forall_x\forall_y\exists!_z(\mathsf{u}(x,z) \wedge \mathsf{u}(y,z) \wedge \forall_u(\mathsf{u}(u,z) \Leftrightarrow u = x \vee u = y))$. Paring. The notations $\{x,y\}$ and $\{x\}$ are now justified if we define $\exists!_x A(x)$, "there exists one and only one x such that $A(x)$", by $\exists_x A(x) \wedge \forall_u\forall_v(A(u) \wedge A(v) \Rightarrow u \doteq v)$.

Definition 3. $\langle x,y\rangle$ stands for $\{\{x\},\{x,y\}\}$. Ordered pair.

Axiom 7. $\forall_u\forall_v\exists!_z\forall_x\forall_y(\mathsf{u}(x,u) \wedge \mathsf{u}(y,v) \Leftrightarrow \mathsf{u}(\langle x,y\rangle,z)$. Binary Cartesian product denoted by $u \times v$.

Definition 4. (i) R is a binary relation in $u \times v$ means $R \subseteq u \times v$.

(ii) Given $R \subseteq u \times v$, $\mathsf{Dom}\,R = \{x : \mathsf{u}(x,u) \wedge \mathsf{u}(\langle x,y\rangle,R)\}$ and $\mathsf{Rng}\,R = \{y : \mathsf{u}(y,u) \wedge \mathsf{u}(\langle x,y\rangle,R)\}$. Domain and range of a relation.

(iii) R^{-1} is the inverse relation of R iff $R \subseteq u \times v \wedge R^{-1} \subseteq v \times u \wedge \forall_x\forall_y(\mathsf{u}(\langle x,y\rangle,R) \Leftrightarrow \mathsf{u}(\langle y,x\rangle,R^{-1}))$.

(iv) F is a function in $u \times v$ iff F is a binary relation in $u \times v$ and $\forall_x\forall_y\forall_z(\mathsf{u}(\langle x,y\rangle,F) \wedge \mathsf{u}(\langle x,z\rangle,F) \Rightarrow y \doteq z)$.

(v) F is a bijection on $\mathsf{Dom}\,F$ onto $\mathsf{Rng}\,F$ iff F and F^{-1} are both functions.

(vi) $\mathsf{Card}\,u = \mathsf{Card}\,v$ iff there exists a function F which is a bijection on u onto v with $u \doteq \mathsf{Dom}\,F \wedge v \doteq \mathsf{Rng}\,F$.

(vii) If $z \doteq \mathsf{Dom}\,R \doteq \mathsf{Rng}\,R$ with R a relation in $u \times v$, then R is a linear ordering on z iff $(\mathsf{u}(x,z) \Rightarrow \mathsf{u}(\langle x,x\rangle,R)) \wedge (\mathsf{u}(\langle x,y\rangle,R) \wedge \mathsf{u}(\langle y,x\rangle,R) \Rightarrow x \doteq y) \wedge (\mathsf{u}(\langle x,y\rangle,R) \wedge \mathsf{u}(\langle y,z\rangle,R) \Rightarrow \mathsf{u}(\langle x,z\rangle,R)) \wedge \forall_x\forall_y(\mathsf{u}(x,z) \wedge \mathsf{u}(y,z) \Rightarrow \mathsf{u}(\langle x,y\rangle,R) \vee \mathsf{u}(\langle y,x\rangle,R))$. In particular, R is a well-ordering on z, denoted

$\mathsf{wo}_R(z)$ or $\mathsf{wo}(z)$ if R is tacitly assumed to exist, iff R is a linear ordering on z and, in addition, $\forall_w(w \subseteq z \wedge \neg\mathsf{Iso}(w) \Rightarrow \exists!_s(\mathsf{u}(s,w) \wedge \forall_t(\mathsf{u}(t,w) \Rightarrow \mathsf{u}(\langle s,t\rangle, R))))$.

It should be mentioned that instead of union, intersection can be taken as an antinomic predicate, antinomic sets being those that satisfy $\mathsf{i}(x,y) \wedge \neg\mathsf{i}(x,y)$. This will not be pursued here.

IV. Antinomic Axioms of Choice

E. Hobson proved in 1905 that the standard axiom of choice does not rule out the existence of antinomic sets (cf. [15], pp. 128–129). Obviously, an antinomic axiom of choice should be based on such sets. More recently, it has been shown that the standard axiom of choice implies the law of excluded middle (cf. [7] p. 163). The proof, however, breaks down if one assumes the logic outlined in this paper; thus, the antinomic versions of the axiom of choice to be proposed will not imply excluded middle, although they will be compatible with specific instances of this law. (The proof that standard choice implies excluded middle uses contradiction, which is why it fails here.) More important, antinomic versions of the axiom of choice are compatible with sequences of more than two alternatives: $\varphi_1 \vee \neg\varphi_1 \vee \varphi_2 \vee \varphi_3 \vee \ldots \vee \varphi_n$. To bring antinomicity to choice, then, is in keeping with the fact that although $\varphi \vee \neg\varphi$ understood as an exclusive alternative simplifies logic, the situation in mathematics and the natural sciences is replete with instances in which φ and $\neg\varphi$ are far from being the only options available.

§ 21. Antinomic axioms of choice for AS_1

Because the standard proof of the equivalence of the axiom of choice with, say, the well-ordering principle relies on contradiction, we cannot assume here that the axiom implies the principle or vice versa — as is the case with most equivalent forms of the axiom of choice. This nonequivalence has, in effect, its positive consequences in that it returns to each of these forms some of the independence, strength, and breadth of scope with which they were originally conceived.

Well ordering can be defined as follows within AS_1.

Definition 1. $\mathsf{wo}(z)$ (or $\mathsf{wo}_R(z)$) iff $\exists_R(\mathsf{Rel}(R) \wedge \mathsf{Dom}\, R \doteq \mathsf{Rng} R \doteq z \wedge \forall_x(x \in z \Rightarrow \langle x,x\rangle \in R) \wedge \forall_x\forall_y(\langle x,y\rangle \in R \wedge \langle y,x\rangle \in R \Rightarrow x \doteq y) \wedge \forall_x\forall_y\forall_u(\langle x,y\rangle \in R \wedge \langle y,u\rangle \in R \Rightarrow \langle x,u\rangle \in R) \wedge \forall_x\forall_y(x \in z \wedge y \in z \Rightarrow \langle x,y\rangle \in R \vee \langle y,x\rangle \in R) \wedge \forall_v(v \subseteq z \wedge v \neq \emptyset \Rightarrow \exists_s(s \in z \wedge \forall_t(t \in z \Rightarrow \langle s,t\rangle \in R)))$. The expression $\mathsf{WO}(z)$

means $\models \mathsf{wo}(z)$ but not-$\models \neg\mathsf{wo}(z)$ and not-$\models\!\!\!\equiv \mathsf{wo}(z)$, whether $\models\!\!\!\equiv \neg\mathsf{wo}(z)$ or not-$\models\!\!\!\equiv \neg\mathsf{wo}(z)$; $\neg\mathsf{WO}(z)$ means $\models \neg\mathsf{wo}(z)$ but not-$\models \mathsf{wo}(z)$ and not-$\models\!\!\!\equiv \neg\mathsf{wo}(z)$, whether $\models\!\!\!\equiv \mathsf{wo}(z)$ or not-$\models\!\!\!\equiv \mathsf{wo}(z)$; $\mathsf{wo}\neg\mathsf{wo}(z)$ means $\models \mathsf{wo}(z) \wedge \neg\mathsf{wo}(z)$ and $\models\!\!\!\equiv \mathsf{wo}(z)$, whether $\models\!\!\!\equiv \neg\mathsf{wo}(z)$ or not-$\models\!\!\!\equiv \neg\mathsf{wo}(z)$. $\mathsf{WO}(z)$ will read "z is strictly well-ordered", $\mathsf{wo}\neg\mathsf{wo}(z)$ will read "z is antinomically well-ordered".

Similarly, the predicate $\mathsf{m}(z)$, "z is mediate", can be defined in AS_1 as $\neg\mathsf{Ind}(z) \wedge \neg\mathsf{Ref}(z)$, using the definitions of $\mathsf{Ind}(z)$, "z is inductive", and $\mathsf{Ref}(z)$, "z is reflexive", given in § 16. $\mathsf{M}(z)$, $\neg\mathsf{M}(z)$, and $\mathsf{m}\neg\mathsf{m}(z)$ stand for "z is strictly mediate", "z is strictly nonmediate", and "z is antinomically mediate", defined respectively as above for $\mathsf{WO}(z)$, and $\neg\mathsf{WO}(z)$, and $\mathsf{wo}\neg\mathsf{wo}(z)$.

An antinomic axiom of choice AAC for AS_1 may be introduced in a number of ways; we shall select two of them. The idea is that AAC should not apply to all sets but only to those which are, for example, well-ordered or nonmediate. That is, we shall break the universe w into two classes not necessarily disjoint; in one case, the class of well-ordered sets and the class of non-well-ordered sets; in the second case, the class of nonmediate sets and the class of mediate sets. Accordingly, we have the following two axioms in which $\mathcal{F}$ is a given family of sets, S is a member of the family $\mathcal{F}$ and $\mathcal{C}$ is the choice set.

Axiom 1. $\mathsf{wo}(\mathcal{F}) \Rightarrow \exists_{\mathcal{C}_1}\forall_x(x \,\epsilon\, \mathcal{C}_1 \Leftrightarrow \exists_S(S \,\epsilon\, \mathcal{F} \wedge x \,\epsilon\, S \wedge \forall_y(y \,\epsilon\, \mathcal{C}_1 \wedge y \,\epsilon\, S \Rightarrow x = y)))$. Choice for well-ordered sets.

Axiom 2. $\neg\mathsf{m}(\mathcal{F}) \Rightarrow \exists_{\mathcal{C}_2}\forall_x(x \,\epsilon\, \mathcal{C}_2 \Leftrightarrow \exists_S(S \,\epsilon\, \mathcal{F} \wedge x \,\epsilon\, S \wedge \forall_y(y \,\epsilon\, \mathcal{C}_2 \wedge y \,\epsilon\, S \Rightarrow x = y)))$. Choice for nonmediate sets.

Both axioms leave room for subuniverses in which AAC does not apply; both are also ambiguous in the sense that all the following are possible: (i) $\mathsf{WO}(\mathcal{F})$ or $\mathsf{wo}\neg\mathsf{wo}(\mathcal{F})$, (ii) $x \in \mathcal{C}_1$ or $x \,\epsilon\!\!\not\in \mathcal{C}_1$, (iii) $S \in \mathcal{F}$ or $S \,\epsilon\!\!\not\in \mathcal{F}$, (iv) $x \in S$ or $x \,\epsilon\!\!\not\in S$, (v) $\neg\mathsf{M}(\mathcal{F})$ or $\mathsf{m}\neg\mathsf{m}(\mathcal{F})$, and (vi) $x \in \mathcal{C}_2$ or $x \,\epsilon\!\!\not\in \mathcal{C}_2$. Classically, (i) $\mathsf{wo}(\mathcal{F}) \Rightarrow \mathsf{wo}(\mathcal{C}_1)$ and (ii) $\neg\mathsf{m}(\mathcal{F}) \Rightarrow \neg\mathsf{m}(\mathcal{C}_2)$, since the injective correspondence given by $F : S \to x$ mapping $\mathcal{F}$ onto $\mathcal{C}_1$ (or $\mathcal{C}_2$), with x the unique representative of S in $\mathcal{C}_1$ (or $\mathcal{C}_2$), makes $\mathcal{C}_1$ well-ordered (and $\mathcal{C}_2$ nonmediate). We shall assume implications (i) and (ii); hence, $\mathsf{WO}(\mathcal{F})$ and $\neg\mathsf{WO}(\mathcal{C}_1)$ will be incompatible (and so will $\neg\mathsf{M}(\mathcal{F})$ and $\mathsf{M}(\mathcal{C}_2)$).

Axioms 1 and 2 are ambiguous in that the kind of membership of x to $\mathcal{C}_1$ (or to $\mathcal{C}_2$) is not uniquely determined, making $\mathcal{C}_1$ (and $\mathcal{C}_2$) not uniquely determined either. The situation is similar to the one which arose in connection with the separation axiom. Therefore, in order to determine the choice sets $\mathcal{C}_1$ and $\mathcal{C}_2$ more specifically, we must state the following. Note that x and S, either one or both, can be well-ordered (or nonmediate), or the opposite.

Axiom 1′. $\mathsf{wo}(\mathcal{F}) \Rightarrow \exists_{\mathcal{C}_1'}(\forall_x(x \in \mathcal{C}_1' \Leftrightarrow \exists_S(S \epsilon \mathcal{F} \wedge x \in S \wedge \forall_y(y \in \mathcal{C}_1' \wedge y \in S \Rightarrow x \doteq y)) \wedge \forall_u(u \,\epsilon\!\!\notin \mathcal{C}_1' \Leftrightarrow \exists_S(S \epsilon \mathcal{F} \wedge u \,\epsilon\!\!\notin S \wedge \forall_v(v \,\epsilon\!\!\notin \mathcal{C}_1' \wedge v \,\epsilon\!\!\notin S \Rightarrow u \doteq v)) \wedge \forall_r \forall_s(r \epsilon \mathcal{C}_1' \wedge r \epsilon S \wedge s \epsilon \mathcal{C}_1' \wedge s \epsilon S \Rightarrow r \doteq s))$.

Axiom 2′. With $\neg\mathsf{m}(\mathcal{F})$ as a premise, same conclusion as in Axiom 1′ replacing $\mathcal{C}_1'$ by $\mathcal{C}_2'$.

Since in the four preceding axioms the major implication goes only in one direction, the existence of a choice set does not mean that $\mathcal{F}$ must be well-ordered or nonmediate; in fact, $\mathcal{C}_1'$ and $\mathcal{C}_2'$ themselves can be non--well-ordered or mediate respectively. If $\mathcal{F}$ is strictly mediate, the bijection $F : S \to x$ on $\mathcal{F}$ onto $\mathcal{C}_2'$ makes the latter strictly mediate, as the following metamathematical reasoning shows. If $\mathcal{C}_2'$ were antinomically mediate, then a bijection G would exist that maps $\mathcal{C}_2'$ onto a proper subset of itself y; however, such mapping must have at least a pair $\langle u, v\rangle$ such that $u \epsilon \mathcal{C}_2'$, $v \epsilon y$, and $\langle u, v\rangle \,\epsilon\!\!\notin G$, since $\mathcal{C}_2'$ is both reflexive and nonreflexive. The composition of the three mappings F, G, and F^{-1} in this order is a bijection on $\mathcal{F}$ onto a proper subset of itself, where F^{-1} is the inverse of F. But then $\mathcal{F}$ would be both strictly mediate and antinomically mediate, which is a metamathematical impossibility. Of course, as mentioned above, $\mathcal{F}$ could be both well-ordered and nonmediate, and the bijection $F : S \to x$ in each case would yield the necessary function to also make $\mathcal{C}_1'$ and $\mathcal{C}_2'$ well-ordered and nonmediate, respectively.

Other set-theoretic properties could be used to make room for a subuniverse in which the axiom of choice holds and in whose complement it does not necessarily hold. For example, since using choice it cannot be proved that every set is similar to an ordinal, once the ordinals are introduced one could use $\mathsf{Count}(x)$, "x is similar to an ordinal", as a substitute for either $\mathsf{wo}(x)$ or $\neg\mathsf{m}(x)$, which would lead to an alternative version of Axioms 1 and 2. The same applies to other principles usually given as equivalents of the axiom of choice. We shall not pursue this matter here.

Let us finally link well-ordered sets and nonmediate sets with the following.

Definition 2. $w_o = \{x : x \epsilon w \wedge \neg\mathsf{m}(x)\}$, w_o is the nonmediate subuniverse.

Axiom 3. $\mathsf{wo}(x) \Rightarrow x \epsilon w_o$, every well-ordered set is nonmediate.

§ 22. Ordinals

We now represent the sequence $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, ... by 0, 1, 2, ..., the finite ordinals or natural numbers. Using the axiom of infinity, let us call ω the intersection of all sets that have 0, 1, 2, ..., as members. $\mathsf{S}\omega = \omega \cup \{\omega\}$ will be denoted by $\omega + 1$, etc. We then define the sequence of ordinals *Ord* in the usual way:

Definition 3. $\alpha \in \mathit{Ord}$ iff (i) $x \in y \in \alpha \Rightarrow x \in \alpha$ and (ii) $\forall_z (z \in \alpha \Rightarrow (u \in v \in z \Rightarrow u \in z))$.

Ordinals will be represented by Greek letters except for the class of all ordinals *Ord*, which is also an ordinal, and belongs and does not belong to itself. For ordinals α and β we have the following:

Axiom 4. $\forall_\alpha \forall_\beta (\alpha \in \beta \vee \alpha \doteq \beta \vee \beta \in \alpha)$. In addition, $\alpha \in \mathit{Ord} \Rightarrow \mathsf{wo}_{\in}(\alpha)$ and $\alpha \epsilon \mathit{Ord} \Rightarrow \neg \mathsf{m}(\alpha)$, every ordinal is well-ordered by $\in$ and no ordinal is mediate. Finally, *Ord* is well-ordered by $\in$.

Essentially, ordinals behave like their standard counterparts, although they can be antinomic members of sets which are not ordinals. Addition, multiplication, and exponentiation of ordinals can be defined inductively in the usual way, and the necessary theorems postulated whenever their classical proofs include negative formulas; such theorems include the principle of transfinite induction, the statement that ω is the smallest limit ordinal, the uniqueness of ordinal operations, etc.

Extending to set theory an idea introduced in a previous paper for formal arithmetic (cf. [5]) let us now add strict order as a primitive antinomic predicate.

Axiom 5. $\forall_\alpha \forall_\beta \forall_\gamma ((\alpha < \beta \wedge \beta < \gamma \Rightarrow \alpha < \gamma) \wedge (\neg \mathsf{Nat}(\alpha) \wedge \neg \mathsf{Nat}(\beta) \Rightarrow \alpha < \beta \wedge \beta < \alpha) \wedge \alpha < \mathit{Ord} \wedge \mathit{Ord} < \alpha))$. Transitivity of $<$ in *Ord*, and symmetry (hence reflexivity) of $<$ for all nonfinite ordinals. Each nonfinite ordinal is greater and less than all other ordinals including itself, that is, every geometric representation of $<$ requires bilocation. Whereas the order type of the $\in$-ordering of *Ord* is $\overline{\underline{\mathit{Ord}}}$, the usual one, the $<$-ordering of *Ord* has the following order type: $1 + \overline{\underline{\mathit{Ord}}}^\star + \omega + \overline{\underline{\mathit{Ord}}} + 1$, '1' being the order type of *Ord* itself, the greatest and hence the least ordinal, $\overline{\underline{\mathit{Ord}}}^\star$ is the mirror image of $\overline{\underline{\mathit{Ord}}}$, the last being the standard type of the set of nonfinite ordinals, and ω the order type of the set of finite ordinals placed in the middle of any model. Each finite ordinal has simple location, and each nonfinite ordinal has one location to the right and another to the left of the fragment of all finite ordinals, that is, one in the segment of type $1 + \underline{\overline{Ord}}^{\star}$ and another in the segment of type $\underline{\overline{Ord}} + 1$.

§ 23. A mediate continuum hypothesis

Although by Axiom 3 every well-ordered set x is nonmediate, as already mentioned x is not necessarily similar to an ordinal; Gödel's indirect proof that every well-ordered set is similar to an ordinal cannot be carried out in AS_1 (cf. [19], p. 8). Further, the converse of Axiom 3 does not hold, as the following classical example shows: if μ is strictly mediate, $\mathscr{P}\mu$ is either strictly mediate or strictly nonmediate, and $\mathscr{P}\mathscr{P}\mu$ is strictly nonmediate, yet although reflexive, the latter is not well-ordered since it contains as a subset a replica of μ. In other words, some nonmediate sets have mediate subsets, whereas all mediate sets have nonmediate subsets.

The cardinal number $\mathsf{Card}x$ of a set x was defined in § 16 as the equivalence class of all sets equinumerous to the set x; $\mathsf{Card}x$ is a subset of the universe w and is relative to that universe. Each set $x \mathrel{\epsilon} w$, then, has a cardinal number $\mathsf{Card}x$ regardless of the kind of order it may have, and whether or not x is a mediate set.

The alephs can now be defined as follows.

Definition 4. $\aleph_\alpha$ is the cardinal number of a given nonfinite ordinal γ. The class of all alephs is well-ordered as follows: $\aleph_\alpha \leqslant \aleph_\beta$ iff $\gamma_1 \in \gamma_2 \vee \gamma_1 \doteq \gamma_2$, where γ_1 and γ_2 are any ordinals such that $\gamma_1 \mathrel{\epsilon} \aleph_\alpha$ and $\gamma_2 \mathrel{\epsilon} \aleph_\beta$.

Axiom 6. For every ordinal α there exists a cardinal number $\aleph_\alpha$. The class of all alephs is not only well-ordered but it is also similar to *Ord*. Since mediate sets are members of w, not every nonfinite set has an aleph for its cardinal number.

Cardinal arithmetic can be defined as follows. Let us symbolize cardinal numbers with bold face letters $\mathbf{m}$, $\mathbf{n}$, ..., and let m, n, ..., be any representative of the classes $\mathbf{m}$, $\mathbf{n}$, ..., respectively; $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$, ..., are $\mathsf{Card}\,1$, $\mathsf{Card}\,2$, $\mathsf{Card}\,3$, ...

Definition 5. (i) $\mathbf{m} + \mathbf{n}$ is the cardinal number of the disjoint union of m and n;

(ii) $\mathbf{m} \bullet \mathbf{n}$ is the cardinal number of the Cartesian product $m \times n$;

(iii) $\mathbf{m}^{\mathbf{n}}$ is the cardinal number of the set of all functions on m into n.

The antinomicity of some of the entities involved in (i)–(iii) does not affect the uniqueness of the operations defined.

The beth numbers are defined as follows.

Definition 6. $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = \mathbf{2}^{\beth_\alpha}$.

Assuming the generalized continuum hypothesis (GCH), $\beth_\alpha = \aleph_\alpha$, and the beth notation becomes superfluous. GCH is not assumed here, and the relation between the alephs and the beths is left undetermined. In addition to these two kinds of nonfinite cardinals, now we need to introduce two more, given that not every nonfinite cardinal is an aleph or a beth.

Definition 7. If $\boldsymbol{\mu} = \mathsf{Card}\,\mu$ is the mediate cardinal of a mediate set μ, then $\mathbf{2}^{\boldsymbol{\mu}} = \gimel_\mu$ is a gimel number indexed by μ to indicate it provenance. Gimel numbers can be mediate or nonmediate, and only some mediate numbers are gimel numbers.

Definition 8. If $\boldsymbol{\mu} = \mathsf{Card}\,\mu$ is the mediate cardinal of a mediate set μ, then $\mathbf{2}^{\mathbf{2}^{\boldsymbol{\mu}}} = \daleth_\mu$ is a daleth number indexed by μ to indicate its provenance. Daleth numbers are nonmediate. Whether $\daleth_\mu$ is an aleph, a beth, or another yet undefined kind of nonfinite reflexive cardinal is left as an open question. The relation between daleths and gimels is given by Axiom 8 below.

Axiom 7. $\mathsf{Card}\,\mu \neq \mathsf{Card}\,\mu' \Rightarrow (\gimel_\mu \neq \gimel_{\mu'} \wedge \daleth_\mu \neq \daleth_{\mu'})$.

The gimel and daleth numbers are not linearly ordered, and even if a gimel number is nonmediate, it is not necessarily equal to a daleth, a beth, or an aleph. Further, if the daleths were beth or aleph numbers, they would be well-ordered by the ordinals, thus inducing a well-ordering of the mediate sets. However, we shall postulate the following mediate continuum hypothesis (MCH).

Axiom 8. $\mathsf{m}(\mu) \Rightarrow \exists_{\mu'}(\mathsf{m}(\mu') \wedge \daleth_\mu = \gimel_{\mu'})$. Every daleth equals a gimel number, i.e., the cardinal number of the power set of the power set of a mediate set is the cardinal number of the power set of some mediate set.

From the viewpoint of the Foundations of Mathematics, Axiom 2 has the advantage over Axiom 1 of making the choice operation independent of order, for there is indeed something more basic about choice than any kind of ordering that one might attach to a set. But as mentioned, the alternative of taking $\mathcal{F}$ as nonmediate to guarantee the existence of a choice set is not indispensable either: $\mathcal{F}$ could be merely nonamorphous, in which case some mediate families $\mathcal{F}$ could also yield a choice set. However, it seems rather forced to extrapolate the well-ordering principle from the set of natural numbers to all unimaginable sets simply to be able to single out a definite individual from every nonempty set. And it seems just as forced to identify infinity with Dedekind infinity since, for example, it is shortsighted to assume that nonfinite nonreflexive sets are useless because we have not yet found any use for them. In contrast, the operation of choice is itself truly primitive and intuitively natural whenever it is applicable. Although not always feasible, it is essential even for selecting the very first symbol to put on paper. Indeed, choice is as indispensable from a mathematical point of view as the equally primitive operation of comprehension, i.e., the gathering of individuals that share in a given property. Still other antinomic versions of the axiom of choice should yield new foundational approaches as well as new structural understanding of these two fundamental mathematical operations of choosing and gathering.

§ 24. Axioms of choice for AS_3

Axioms of choice for AS_2 and AS_3 parallel those proposed for AS_1. Let us look briefly at the case of AS_3.

Axiom 1. $\mathsf{wo}(\mathcal{F}) \Rightarrow \exists_{\mathcal{C}}(\forall_x(\mathsf{U}(x,\mathcal{C}) \Leftrightarrow \exists_S(\mathsf{U}(S,\mathcal{F}) \wedge \mathsf{U}(x,S) \wedge \forall_y(\mathsf{U}(y,\mathcal{C}) \wedge \mathsf{U}(y,S) \Rightarrow x \doteq y)) \wedge \forall_u(\mathsf{u}\neg\mathsf{u}(u,\mathcal{C}) \Leftrightarrow \exists_S(\mathsf{u}\neg\mathsf{u}(S,\mathcal{F}) \wedge \mathsf{u}\neg\mathsf{u}(u,S) \wedge \forall_v(\mathsf{u}\neg\mathsf{u}(v,\mathcal{C}) \wedge \mathsf{u}\neg\mathsf{u}(v,S) \Rightarrow u \doteq v))) \wedge \forall_r\forall_s(\mathsf{u}(r,\mathcal{C}) \wedge \mathsf{u}(r,S) \wedge \mathsf{u}(s,\mathcal{C}) \wedge \mathsf{u}(s,S) \Rightarrow r \doteq s)$. The predicate $\mathsf{wo}(z)$ was defined in § 20.

As is the case with AS_1, premises other than $\mathsf{wo}(\mathcal{F})$ may condition the existence of choice set $\mathcal{C}$; for example, we may gather all the sets generated by applications of separation scheme 3 given in § 20, as shown in the following definition:

Definition 1. $\mathsf{Comp}(z)$ means z exists by virtue of Axiom scheme 3, § 20, i.e., there is a well-formed formula $A(x)$ in the language of AS_3 which gathers z. If the language of AS_3 is uncountable, there would be an uncountable number of such formulas, and potentially an uncountable number of sets z satisfying $\mathsf{Comp}(z)$.

Axiom 2. With $\mathsf{Comp}(\mathcal{F})$ as a premise, same conclusion as in Axiom 1 above. Again, not only is the existence of a choice set not equivalent to the well-ordering of $\mathcal{F}$ but also is not equivalent to the "predicability" of $\mathcal{F}$ as given in the definition of $\mathsf{Comp}(z)$ just proposed. (It is ironic that here choice depends on predicability even if it is nonconstructive.)

§ 25. A final remark

The logic on which the set theories AS_1, AS_2, AS_3 are based is obviously a limit one in that, apart from its positive fragment, it is built semantically, posing negative formulas true, false, or both when desired, then postulating the true and antinomic ones as axiom-theorems — syntax following semantics except for some metamathematical reasonings. At the level of formulas, this is not unlike the device of adding an uncountable number of constant symbols to the language of a theory in order to use them syntactically in the formation of terms and formulas. These symbols provide a name for each individual in the universe of a given structure, thus producing an uncountable number of formal atomic sentences from which to gather those which are true in the structure. The notion of diagram introduced by A. Robinson employs these constants and is the set of atomic sentences true in the given structure. This diagram constitutes a ready-made complete theory.[3] Here, the structure is not given in advance, and negative formulas are successively incorporated as true or antinomic in the development of what we may call an "open diagram", a progressing diagram that keeps adding determining characteristics and entities to the models of the true and antinomic formulas previously posited. The purpose is not to obtain a syntactically complete theory but to establish the existence of desirable entities or to modify those already introduced. The next step in the evolution of this and other chapters of antinomic mathematics should move from this limit position toward one more proof-theoretically balanced. How far it is possible to go in this direction and how advantageous it would be to do so are open questions. Yet, the effort involved cannot fail to throw valuable light on the foundational problems that have been touched upon here.

Appendix

Since the referee queries may well occur to the reader, at the suggestion of Professor Perzanowski it seems worthwhile to append them followed by the author's answers.

Referee: What does it mean to say that an assertion is in opposition or disagreement with a given sentence as used in the definition of negation?

[3] [18], p. 24. Robinson's definitions of "positive" and "negative" diagram are different from the ones given for the same expressions in § 14 above in connection with Axiom 8.

Answer: The reference to "opposition" is only to informally characterize negation the way Cantor characterizes the notion of set as "a multiplicity taken as a unit" without expanding on what multiplicity and unity mean. The reference does not involve a definition in the proper sense, it is only a way to make the broadening of negation's meaning more intuitive. The objective is to prepare the reader for the separation of assertion and negation from truth and falsity. I could add an example to clarify the reference saying that although "A less than B" can be considered to be the negation of "A greater than B", if "less than" and "greater than" are taken as primitive predicates, neither one is the negation of the other in the usual sense, but each is the negation of the other in the expanded sense of each being the opposite of the other. But then, "A greater-then-and-less-than B" still is an antinomic combination of both.

Referee: The whole paper contains only definitions and no theorems or proofs of theorems. Thus it is not clear what do the proposed new set theories give us? Why should they be introduced? Why are they better than the set theories developed so far?

Answer: The positive fragment of the antinomic logic proposed has all the corresponding classical theorems, and its metatheory is the classical one. The negative fragment, on the other hand, has no rules of inference, therefore no theorems can be proved: the theorems are all axioms, that is, sentences selected to achieve a definite mathematical purpose. The three set theories in the paper are based on this logic and are introduced as mathematical applications of various alternative antinomic predicates. None of these predicates is necessarily the best, but each improves on current set theories in freeing the collecting of individuals into a whole from the usual, rigid gathering of elements into a set only by means of the nonantinomic predicate of membership.

Referee: In meta-axiom 8 we have a vicious circle. One cannot define axioms of a theory by saying that a sentence is an axiom if an only if it is valid in all models of the theory being just defined!

Answer: Not only is the completeness theorem not available, but also we cannot expect the sentences that are true in all models of axioms 1 to 7 to be only those axioms. Let us then call AS_1 the collection of all sentences that are true in all models of axioms 1 to 7, understanding by model of a sentence any fixed predetermined type of universal-algebraic structure (not necessarily set-theoretical) in which a sentence is true. Since the positive side of the antinomic logic described has both a syntax and a semantics, we extend that syntax to the negative fragments of logic and set theory by Axiom 8, whose application is relative to the fixed type of structure previously chosen.

Referee: There is no information about the relations between the three proposed set theories.

Answer: The various systems are proposed to show that membership is not the only possible antinomic predicate on which to base set theory. They stand in contrast of one another, like Euclidean and Non-Euclidean geometries. No special relation exists between them except their being alternative options.

Referee: The paper does not contain any information on the metatheoretical properties of the considered logic and set theories. Are they consistent for example?

Answer: Since the object theory is left open to additions, so also is the metatheory. There is no reason for the metatheory not to be antinomic as well, except that the metalinguistic "not" is explicitly excluded from antinomicity as far as a sentence cannot be true and not true, or false and not false.

Perhaps I should reemphasize that the intention of the paper is to open new avenues, not to give a univocal answer to the question of which is the way to broaden mathematics by incorporating antinomies. Finally, I should like to thank the referee for spotting an error (now eliminated) in § 5.

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