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TROPE SHEAVES

A Topological Ontology of Tropes

CONTENTS

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1. Introduction

In this paper I want to show that topology has a bearing on the theory of tropes. More precisely, I propose a topological ontology of tropes. This is to be understood as follows: trope ontology is a „one-category"-ontology countenancing only *one* kind of basic entities, to wit, tropes.[1] Hence, individuals, properties, relations, etc. are to be constructed from tropes.[2] However, the world can't be considered as a mere *set* of tropes. Tropes do not come as a heap, so to speak, rather, the set of tropes has to have some *structure*. In this paper I want to deal with the problem of what kinds of structures are necessary to make trope theory work. The example of geometry may explain how the role of structure is to be understood here: The starting point of geometry as the theory of space is not an unstructured set of space-points, from a geometrical point of view space has to be conceived of as a *structured* set of space points. Similarly, trope theory is to be considered as a theory of *structured* sets of tropes. This proposal is, of course, not new. From the very beginning of trope theory, philosophers have realized that tropes as „the very alphabet of being" (Williams (1953:7)) do not suffice. We need a kind of syntax, at least. Thus, the set of tropes has to be endowed with some further structure. Williams and others proposed two „complementary" relations I want to call in the following the relation of compresence and the relation of resemblance[3]. Often, it has not been quite clear what requirements those relations have to satisfy. The example of geometry should warn us that such an approach is bound not to be overly successful. Pondering on what tropes „really are" and remaining vague about their structural

[1] Tropes are abstract particulars such as „the redness of that billiard ball", „the charge of that electron" or „Socrates's wisdom". Abstract particulars are contrasted to abstract universals, e.g. „redness", „charge", or „wisdom" on the one hand and to concrete particulars such as „that billiard ball" or „Socrates" on the other hand.

[2] There are several accounts of one category ontologies, e.g. Stout's theory of individual qualities, Campbell's abstract particulars, Armstrong's ontology based on states of affairs, or Puntel's „primary states of affairs". Of course, all these theories differ in important respects. However, in this paper I'm not interested in the differences that sometimes are difficult to nail down. Rather I'd like to contribute to the structural theory underlying all these accounts.

[3] Terminology in this field widely varies: other names of compresence are „(co)location , „coinherence , „concrescence , „togetherness , or „concurrence. The relation of resemblance also appears under the names of „similarity and „exact resemblance (cf. Williams (1953:8).

relations corresponds to a conception of geometry that concentrates on what points and lines „really are" while ignoring their structural relationships. This amounts to a rather outdated style of geometric theorizing. I contend that an analogous assertion holds for trope theory. For this reason, I propose to pursue trope theory along similar lines as Hilbert did for geometry, i.e. considering trope theory as a *structural* theory of tropes. Fine recent examples of this line of research can be found in Bacon (1987), Bacon (1988) and Fuhrmann (1991).[4] Although the spirit of my approach resembles theirs, I chose a different line of investigation: I intend to deal with the hitherto unnoticed topological aspects of trope structures. More precisely, I want to show that the appropriate formal framework for trope theory is provided by the mathematical theory of sheaves (cf. Tennison (1975), MacLane/Moerdijk (1992)).

The outline of this paper is as follows: In the next section we recall and formalize the basic concepts of trope theory. As starting point I take the concept of a *trope space*, i.e. a set of tropes endowed with the relations of compresence and of resemblance. Traditionally, both of these relations are taken as equivalence relations. In this paper I want to show that a more convenient choice for resemblance is to conceptualize it topologically. As a first step of this task, I show that the resemblance relation need not be assumed to be an equivalence relation. Rather, it is sufficient to consider it as a similarity relation. This leads to the replacement of trope spaces by generalized trope spaces. (Generalized) trope spaces give rise to bundles in the sense of mathematical theory of bundles (cf. Goldblatt (1978), MacLane/Moerdijk (1992)).[5] In section 3 the topological properties of trope bundles are elucidated. We get that trope bundles give rise to sheaves in the sense of mathematical theory of sheaves (cf. Goldblatt (1977), Tennison (1975), MacLane/Moerdijk 1993). In section 4 we introduce continuous sections of trope sheaves as surrogates of universal properties. The relation between section properties and individual properties is investigated. As it turns out, it is just the mathematically well-known relation of a sheaf space to its presheaf of sections. We show that if there exist universals the sec-

[4] Bacon (1987) is concerned with modal aspects of trope theory, and in Bacon (1988) the problem of higher-order tropes is treated, Fuhrmann (1991) deals with mereological refinements of the compresence relation and a theory of laws based on tropes.

[5] The term bundle in this paper is used in a somewhat different sense than in the variegated versions of current metaphysical bundle theories (cf. Armstrong 1989). However, both meanings are related to each other. Bluntly spoken, trope bundles in the sense of mathematical bundle theory are bundles of bundles of tropes in the sense of the metaphysical theories.

tion properties can be identified with them. In section 5 we close with some remarks on what is to be understood by a general topological trope ontology.

2. Spaces and Bundles

Structural considerations in trope theory have concentrated on studying the relations of compresence and resemblance. Usually these relations have been assumed to be equivalence relations. At first look, this appears quite plausible. For instance, equivalence classes of the compresence relation lead to a simple and satisfying constitution of concrete individuals (cf. Campbell (1981:483))[6]. Analogously, properties could be constituted as equivalence classes of the resemblance relation (cf. Bacon 1988). However, this approach has some disadvantages. The assumption that resemblance is an equivalence relation gives trope theory a strongly antinominalist orientation toward a realist understanding of properties. It seems as though trope theory is committed to (ersatz)universals (cf. Armstrong 1989). Maybe realism about universals is right, but I think it's not necessary to incorporate this claim into the basic assumptions of trope theory. The perspective of trope theory should be more general, it should provide a general and flexible framework wherein different ontological currents can be discussed. Moreover, the assumption that compresence and resemblance both are equivalence relations renders universals to be formally on a par with concrete individuals. Both are constituted as equivalence classes of equivalence relations. In this way, trope theory gets entangled in what has been called Ramsey's problem (cf. Ramsey 1925 (1990), Armstrong (1989:44)), namely, what marks off individuals from properties in the formal scheme of trope theory. In the following I propose a version of trope theory that sidesteps Ramsey's problem by not assuming that compresence and resemblance are formally on a par. I propose a solution of what Campbell has called „the problem of universals" or „the problem of Resemblance", to wit, I offer a topological theory of the nature of properties that is ontologically cheaper and more general than the standard account leading to ersatz universals. Moreover, it appears

[6] Campbell considers the trope approach as quite successful in this respect. For him the constitution of concrete individuals as equivalence classes of the compresence relation provides the „solution of the problem of individuals" (Campbell (1981:482). A more sophisticated constitution of individuals is to be found in Fuhrmann (1991). Campbell is not so optimistic whether a trope approach is able to solve the corresponding „problem of universals". Others judge the score of trope theory differently: Simons argues that trope theory scores quite well with respect to the problem of universals, but not with the problem of individuals (cf. Simons (1994)).

to be more flexible and realistic in so far as it does not lead to the assertion that there is one privileged system of universal properties. My solution depends on the assumption that the problem of individuals is solved: all I have to say about the constitution of individuals is that individuals are equivalence classes of the compresence relation.[7] Thus, I subscribe to a version of trope theory where tropes and concrete individuals are already available. What is left, is the task of constructing global properties, i.e. properties that may be ascribed to many concrete individuals. My proposed solution is the following: properties are maximal continuous sections of trope sheaves, i.e. properties are certain structure-preserving functions from concrete individuals to tropes. What this exactly means will be explained in the following. To fix notation, let us start with the following definition:

(2.1) Definition. *A **trope space**[8] is a triple $\langle E, =_c, =_r \rangle$ having the following components:*

(i) *E is a set of tropes.*

(ii) *$=_c$ is the equivalence relation of compresence whose equivalence classes are to be considered as individuals.*

(iii) *$=_r$ is the equivalence relation of resemblance whose equivalence classes are to be considered as universals or „ersatz universals"* (cf. Armstrong (1989)).

(iv) *The relations $=_c$ and $=_r$ are orthogonal:* $\forall_{e,e'}(e =_c e' \;\&\; e =_r e' \Leftrightarrow e = e')$. *Intuitively, this means that two tropes whose individual and property equivalence classes coincide, are identical.*

The first main technical aim of this paper is to show that the relation of resemblance need *not* be assumed to be an equivalence relation. It suffices to assume it to be a similarity relation. This results in a thorough-going particularization of trope theory: the trope account no longer is committed to a unique set of universal properties. Or, more precisely, it does not fol-

[7] Some authors argue that the relation of compresence cannot fulfill the task of constituting individuals. Rather, we need a real being to account for the constitution of concrete individuals from tropes (cf. LaBossiere (1994:365). As will be shown in the following, the structural approach to be developed in this paper remains neutral in this dispute.

[8] A trope space is part of what Bacon calls a monadic trope model (cf. Bacon (1987:96). More precisely, a trope space is that part of a trope model that corresponds to the actual (or some other possible) world in Bacon's approach. The concept of a trope spaces captures William's definition of a possible world: „Any possible world, and hence, of course, this one, is completely constituted by its tropes and their connections of location and similarity, and any others there may be." Williams (1953:80).

low from the basic axioms of trope theory that there are universals. Rather, generally, trope theory only subscribes to the existence of particular properties or properties strictly in re. It may be that certain specializations lead to universals in the traditional sense. Thereby the embarrassing symmetry between the relation of compresence and of resemblance is broken. Nevertheless we'll be able to constitute systems of global properties that enjoy most of the features of the traditional universals. Let us start with the following preparatory definition:

(2.2) Definition. *Let X be a set. A similarity relation $\approx$ on X is a binary relation that is reflexive and symmetric, i.e.*

(i) $\forall_x x \approx x$ *and*

(ii) $\forall_{x,y}(x \approx y \Leftrightarrow y \approx x)$ holds.

X endowed with a similarity relation is denoted by $\langle X, \approx\rangle$ and called a ***similarity structure.***

A subset $Y \subseteq X$ is a ***similarity circle*** *iff in satisfies the following two conditions:*

(iii)$_1$ $\forall_{x,y}(x, y \in Y \Rightarrow x \approx y)$.

(iii)$_2$ $\forall x(x \notin Y \Rightarrow \exists_y(y \in Y \;\&\; x \not\approx y)$.

The class of similarity circles of $\langle X, \approx\rangle$ is denoted by $SC(X)$.

As is easily seen, if the similarity relation $\approx$ happens to be an equivalence relation a similarity class is just an equivalence class. In contrast to equivalence classes similarity circles may overlap. However, similarity classes do not properly include each other. After these preliminaries we are able to define the central notion of this section as follows:

(2.3) Definition. *A* ***generalized trope space*** *is a triple $\langle E, =_c, \approx_r\rangle$* having the following components:

(i) *E is a set of tropes.*

(ii) *$=_c$ is the equivalence relation of compresence whose equivalence classes are to be interpreted as concrete individuals, i.e. $B := E/_{=_c}$.*

(iii) *$\approx_r$ is the similarity relation of resemblance. It is assumed to be reflexive and symmetric but we no longer assume it to be transitive.*

(iv) *The relations $=_c$ and $\approx_r$ are orthogonal, i.e.*
$\forall_{e,e'}(e \cong_c e' \;\&\; e \approx_r e' \Leftrightarrow e = e')$.

The replacement of an equivalence relation by a similarity relation as resemblance structure is only the first step on the road of formulating a general topological ontology of tropes. Later I will show that the essence of resemblance is best captured in topological terms, namely, in the concept of a topological sheaf. In order to explain what this and other generalizations amount to, I'd like to reformulate the concept of a (generalized) trope space in terms of the mathematical theory of bundles (cf. Goldblatt 1978, MacLane/Moerdijk 1992)). This leads to a new perspective on (generalized) tropes spaces bringing to the fore their topological features:

(2.4) Definition. (i) *A* ***bundle*** *is a triple* $\langle E, p, B\rangle$, *where* E *and* B *are sets and* $p : E \to B$ *is a map.* E *is called the total space of the bundle,* B *is called the base space, and the map* p *is called the projection of the bundle. For* $b \in B$, *the set* $p^{-1}(b)$ *is called the fibre of the bundle over* b. (ii) *If* E *and* B *are topological spaces, and* p *is a continuous map the bundle* $\langle E, p, B\rangle$ *is called a* ***topological bundle****.*

Thus, a (topological) bundle is essentially just a (continuous) map (cf. Goldblatt 1978: 90). However, as I want to show in the following, the concept of a (topological) bundle offers a new conceptual perspective on trope spaces. First we treat the case of set theoretical bundles, later we will show that trope spaces actually give rise to topological bundles.

(2.5) Lemma. *Let* $\langle E, =_c, \approx_r\rangle$ *be a generalized trope space. It defines a bundle* $\langle\langle E, \approx_r\rangle, p, B\rangle$ *in the following way:*

(i) $\langle E, p, B\rangle$ *is a set theoretical bundle with base space* $B := E/_{=_c}$. *The projection* $p : E \to B$ *is the canonical function mapping a trope onto its compresence equivalence class.*[9]

(ii) $\langle E, \approx_r\rangle$ *is a similarity structure,* $\approx_r$ *being the resemblance relation.*

(iii) *The relation* $\approx_r$ *is trivial on the fibres:*
$\forall_{e,e'}(e \approx_r e' \;\&\; p(e) = p(e') \Rightarrow e = e')$.[10]
$\langle\langle E, \approx_r\rangle, p, B\rangle$ *is to be called the* ***trope bundle*** *of the generalized trope space* $\langle E, =_c, \approx_r\rangle$. *If* $\langle E, =_c, =_r\rangle$ *is a traditional trope space its bundle is denoted by* $\langle\langle E, =_r, \rangle, p, B\rangle$. *If the similarity relation* $\approx_r$ *is to be understood a trope bundle is simply denoted by* $\langle E, p, B\rangle$.

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[9] If one is unhappy with the compresence relation as the method of constitution of concrete individuals (cf. LaBossiere (1994)) the base B may equally well be interpreted as the class of substrata. In this case, the projection is to be interpreted as the function that maps a particular property to its underlying substratum. From this point of view, the projection p defines a good „ersatz-compresence": the tropes e, e' stand in the relation of „ersatz-compresence" iff $p(e) = p(e')$, i.e. iff they are grounded in the same substratum b.

[10] Obviously, this requirement corresponds to the condition that compresence, defined by the projection p, and resemblance are orthogonal (cf. (2.4)(iv)).

Obviously, a trope bundle can be defined from its generalized trope space and vice versa. The same holds for traditional trope bundles and traditional trope spaces. Thus, in both cases, the „space-concept" and the „bundle--concept" are strictly equivalent. However, as we shall see in following, the crucial topological concepts we are going to introduce „live" on trope bundles rather than on trope spaces.

In the rest of this paper I want to render plausible the claim that trope bundles are to be considered as the basic unit of a general trope theory. Before we go on to elucidate the structure of trope bundles in more detail let us make some preliminary remarks on the trope bundles in general which show that generalized trope bundles can better cope with some notorious problems than the traditional account based on traditional trope bundles (or spaces). These problems are (i) Ramsey's Problem, (ii) the Swapping Problem, and (iii) the Problem of Irreducibility (cf. Armstrong (1989a: 136f).

ad (i) Ramsey's Problem: For generalized trope bundles „Ramsey's problem" no longer occurs (cf. Armstrong 1989:44): compresence and resemblance no longer are formally on a par. The disappearance of Ramsey's problem will become even more evident when we introduce fully general trope sheaves as the basic concept of trope theory.

ad (ii) The Swapping Problem: Likewise, Armstrong's problem of the possibility of „swapping exactly resembling tropes" disappears: according to the definition of generalized trope bundles there are no exactly resembling tropes but only resembling ones. (cf. Armstrong 1989a:136).

ad (iii) The Problem of Irreducibility: By giving up transitivity of the resemblance relation, Armstrong's problem of irreducibility is resolved. It consists in the fact that the axioms of resemblance, to wit, reflexivity, symmetry, and transitivity, can be derived and explained most naturally if resemblance is analysed in terms of universals, i.e., iff two tropes are resemblant iff they have a universal in common. According to Armstrong, trope theory must treat this as a mere metaphysical coincidence between the properties of resemblance and the properties of identity: „It is a serious difficulty for any resemblance analysis that the irreducibility of resemblance is so implausible an irreducibility." (ibidem: 137). A trope approach based on the concept of a generalized trope bundle is not afflicted by this difficulty: according to it, a theory committed to universals is a special case of a more general resemblance account. Thus, the commitment to universals or at least to „ersatz universals" (cf. Armstrong (1989: 122)) turns out to be a (possibly not justified) strengthening of the more general resemblance approach.

The (re)solutions of these problems, however, may appear as cheap achievements as long as we aren't able to proffer a theory of (ersatz)universals.

This will be achieved in the following. Before we can tackle this problem, however, let us scrutinize in some more detail what the existence of universals amounts to in terms of bundle theory. Let $\langle E, p, B \rangle$ be a trope bundle. The elements of the fibre $p^{-1}(b)$ are to be interpreted as individual properties of b. Individual properties are properties strictly in re. Different elements b, b' cannot have the same individual properties, i.e. the same tropes. The map p can be interpreted as a founding or grounding map which provides a common basis for the compresent tropes. If we switch to the states of affairs interpretation of tropes the elements of the fibre $p^{-1}(b)$ over b are the states of affairs where the individual b occurs. Since the resemblance relation $\approx_r$ is no longer an equivalence relation we can't speak of the „same" property of different elements b and b'. We only have the weaker relation of similarity between the elements of different fibres. A fortiori, a trope space $\langle E, =_c, =_r \rangle$ defines a bundle $\langle E, p, B \rangle$. Such a bundle, however, has some special features not had by a generalized trope bundle. To explain this we need the following definition:

(2.6) Definition. *Let $X \times Y$ be a Cartesian product. Then the projection $p_X : X \times Y \to X$ is a bundle. It is called the **trivial bundle** over X with fibre Y.*

(2.7) Lemma. *Let $\langle E, =_c, =_r \rangle$ be a trope space, denote the set of equivalence classes $E/_{=_r}$ by U. It is to be interpreted as the set of global properties. For any trope space $\langle E, =_c, =_r \rangle$ we have the trivial bundle $\langle B \times U, p, B \rangle$. This trivial bundle is to be called the property bundle of the trope space $\langle E, =_c, =_r \rangle$. The trope bundle $\langle E, p, B \rangle$ of a trope space $\langle E, =_c, =_r \rangle$ comes along with a ready-made embedding $i : \langle E, p, B \rangle \to \langle B \times U, p, B \rangle$ into the trivial property bundle $\langle B \times U, p, B \rangle$ defined by $i(e) := \langle [e]_{=_r}, [e]_{=_c} \rangle$. Due to the fact that compresence and resemblance are orthogonal to each other (cf. (2.1)) the map i indeed is an embedding, i.e., i is injective.*

For generalized trope spaces such an embedding is not available. Thus, it seems, we cannot define global properties anymore. In the following I want to show that we actually don't need it for the trope theoretical constitution of (global) properties. The bundle approach offers a new „intrinsic" approach to properties that does not depend on the existence of an „extrinsic" property bundle. Its basic idea can be explained as follows:

(2.8) Definition. *Let $\langle E, p, B \rangle$ be a bundle, $V \subseteq B$. A **global property defined on V** is a section of $\langle E, p, B \rangle$, i.e. a map $s : V \to E$ with $p \circ s = \mathrm{id}_V$. For $b \in V$, the individual property $s(b)$ of b is said to instantiate the global property s.*

To give an intuitive idea[11] of how the section approach to global properties works let us consider the following example: according to the section approach the trope „Nero's viciousness" is parsed as a function s_v (the „vicious"-section) from an appropriate subset V of the set of concrete individuals B to the set E of individual properties. „Nero" belongs to the domain V of s_v and the value s_v(Nero) is the individual property „Nero's viciousness". The section approach of global properties is not restricted to a particular trope account. It works for states of affairs as well. In this case the elements of total space E are to be interpreted as states of affairs.[12] Then the section approach to global properties (universals) amounts to something like this: „vicious" is a continuous function defined on some subset of concrete particulars, to which „Nero" belongs. The states of affairs „Nero is vicious" is the value of „vicious" for the argument „Nero". The pleasant thing to note is that the universal „vicious", as a function, does *not* occur as a constituent of its value „Nero is vicious". The projection function p can be interpreted as a constitution or grounding function mapping states of affairs to their constituting concrete particulars. Thus, the section approach of properties does not get involved into the well-known problems how individuals and universals are „added" or „related" to become a state of affair (cf. van Cleve (1994:589). Considering universals (global properties) as sections, the problem of „individuation" or „particularization of universals" (cf. Puntel (1993:130)) disappears. We no longer need to assume that a universal has to occur as a constituent in an individual since a function $f : X \to Y$ need not be thought as a constituent of some of its values $f(x) \in Y$. Thus, the idea that an individual's having a property P has to be conceived of as a relation of this individual and a universal can be dismissed. Thereby we avoid some of the notorious metaphysical difficulties of this account. The section approach also has some appeal for the metaphysics of states of affairs. We no longer need to assume that universals are (somehow) „parts" of states of affairs. Rather, states of affairs are grounded in individuals, universals are functions from individuals to states of affairs.

Admittedly, until now, the assertion that global properties are to be considered as sections of a bundle $\langle E, p, B\rangle$ is a rather bald and unspecific

[11] The idea that properties are to be conceived of as sections might be traced back to Frege's account of concepts as functions.

[12] Whether states of affairs and tropes are the same, or at least closely related concepts, is a matter of dispute among the experts. Bacon succinctly affirms: „... states of affairs are tropes" (Bacon (1987:112). Armstrong disagrees. Obviously, this problem need not concern us here. The structural approach to be developed in this paper remains neutral to these differences.

one. To improve the appeal of the section account of properties one should be more specific what kind of functions global properties are to be. Not all kinds of functions will do. I consider it as an important virtue of the sheaf approach that it will provide us with a quite precise answer to this question. However, before we can get this result we have to do some more work. The following example shows that the section approach to (global) properties satisfies at least a minimal adequacy condition:

(2.9) Lemma. *Let $\langle\langle E, =_r\rangle, p, B\rangle$ a traditional trope bundle and $\langle B \times U,$ $p, B\rangle$ its (trivial) property bundle. Then each $u \in U$ defines a section $s_u :$ $V_u \to E$ in the sense of (2.8).*

Proof: Recall that $u \in U$, being an equivalence class of the resemblance relation $=_r$ is a subset of E, i.e. $u \subseteq E$. Define $V_u := \{b \mid p^{-1}(b) \cap u \neq \emptyset\}$. Then the section $s_u : V \to E$ is defined by $s_u(b) := e$, $e \in p^{-1}(b) \cap u$. As is easily seen s_u is well-defined, since for $b \in V_u$ the set $p^{-1}(b) \cap u$ is always a singleton.

I hasten to add that (2.8) is *not* my final definition of a global property. Definition (2.8) as it stands is far too liberal. It allows too many „properties". This can be seen if we take a traditional trope bundle $\langle E, p, B\rangle$ with property bundle $\langle B \times U, p, B\rangle$ as a test case. Then, obviously, beyond the sections s_u, $u \in U$, (2.8) admits many more contrived „properties" only a hard boiled nominalist would be prepared to accept as properties. Actually, (2.8) cannot be considered as a satisfying bundle theoretic explication of properties. We have to work somewhat harder to make the section account of properties work. However, we are on the right track. What we have to do is to replace the set theoretical bundle by a structured bundle. More precisely, in the case of trope bundles we have to bring to work the topological structure built into this kind of bundles. This amounts to the reconceptualization of the bundles $\langle E, p, B\rangle$ as sheaves (cf. MacLane/Moerdijk (1992), Tennison (1975)). This will enable us to structurally restrict the profusion of „properties" in the sense of (2.8) in such a way that the section approach to properties becomes a viable particularist theory of properties. Before we come to the details in the next section, first observe that the similarity relation on E induces a similarity relation on B in the following way:

(2.10) Lemma and Definition. *Let $\langle\langle E, \approx_r\rangle, p, B\rangle$ be a trope bundle. The relation $\approx_r$ induces a similarity relation $\sim$ on B by the definition: $b \sim b' \Leftrightarrow \exists_{e,e'}(p(e) = b \;\&\; p(e') = b' \;\&\; e \approx_r e')$. From now on let us assume that the the base space B of a trope bundle $\langle\langle E, \approx_r\rangle, p, B\rangle$ is endowed with* *this similarity relation.*[13] *It is denoted by* $\langle\langle E, \approx_r\rangle, p, \langle B, \sim\rangle\rangle$ *or simply by* $\langle E, p, B\rangle$.

In the case of a traditional trope bundle $\langle\langle E, =_r\rangle, p, B\rangle$ the similarity relation $\sim$ defined on B has a very natural meaning: $b \sim b'$ holds iff b and b' share a universal. For generalized trope spaces there are no universals anymore, in this case the relation $\sim$ can be interpreted as Carnap's relation of part similarity (cf. Carnap (1928)): two concrete individuals are resemblant iff they have at least one resemblant „quasiproperty".

Now, the main formal contention of this paper is that trope bundles are the desired generalization of trope spaces that can serve as the basic concept of a truly general trope theory not committed to the assumption that the resemblance relation is transitive. In order to render this claim plausible we show in the next section that trope bundles actually are topological bundles, i.e., they can be endowed with topological structures in a natural way.This topological structure will enable us to define global properties that mimic the essential features of the universal properties of traditional trope spaces.

3. Topological Considerations

In this section we show that trope bundles can be endowed with topological structures in such a way that one can define global properties that possess the essential features of the global properties of traditional trope spaces. For this task we rely on the similarity relations defined on E and B. More precisely, first we show that a trope bundle $\langle E, p, B\rangle$ can be considered as a special *topological* bundle, to wit, sheaves. First let us briefly recall some basic definitions of a topology. The introduction of topological structures serves two purposes: (i) they can be used to explicate concepts such as neighborhood, nearness, degrees of resemblance and the like; (ii) topological structures are essential for the explication of continuity:

(3.1) Definition. *Let* X *be a set. A topological structure or topology on* X *is a subset* $O(X)$ *of the power set* $P(X)$ *of* X *which satisfies the following properties: (i)* $\emptyset, X \in O(X)$*; (ii) if* $Y, Y' \in O(X)$ *then* $Y \cap Y' \in O(X)$*; (iii) if* $Y_\lambda \in O(X)$, $\lambda \in \Lambda$, $\bigcup\{Y_\lambda \mid \lambda \in \Lambda\} \in O(X)$.

The elements of $O(X)$ are called open subsets of X (with respect to $O(X)$). X (or more precisely, $\langle X, O(X)\rangle$ is called a topological space. An

[13] It should be noted that in general the relation $\sim$ is not an equivalence relation even if $\approx_r$ happens to be one.

open subset U such that $x \in U$ is called an open neighborhood of x. A topology $O(X)$ is finer than a topology $O'(X)$ iff $O'(X) \subseteq O(X)$. In this case, $O'(X)$ is said to be coarser than $O(X)$. The finest topology on X is the discrete topology $O(X) = P(X)$. The coarsest topology on X is the indiscrete topology $O(X) = \{\emptyset, X\}$.

(3.2) Definition. *Let $\langle X, O(X)\rangle$ and $\langle Z, O(Z)\rangle$ be topological spaces. A map $f : X \to Z$ from the topological space X to the topological space Z is continuous iff for all $W \in O(Z)$, $f^{-1}(W) \in O(X)$.*

(3.3) Definition. *Let $\langle X, \sim\rangle$ be a similarity structure. For $x \in X$ the similarity neighborhood $co(x)$ of x is defined to be the set $co(x) := \{y \mid x \sim y\}$. A preorder, i.e. a reflexive and transitive relation on X is defined in the following way: $x \leq y := co(x) \subseteq co(y)$. With the aid of the preorder we can define a topology $O(X)$ for a similarity structure $\langle X, \sim\rangle$ as follows:*

$$Y \in O(X) := \forall_{x,y}(x \in Y \;\&\; x \leq y \Rightarrow y \in Y).$$

This topological structure is called the ***order topology****.*

(3.4) Lemma. *Let $\langle X, \sim\rangle$ be a similarity structure endowed with the order topology. If T is a similarity circle, i.e. $T \in SC(X)$, then T is open.*

Proof: Let $x \in T \in SC(X)$, and $x \leq y$. One has to show $y \in T$. $x \leq y$ iff $co(x) \subseteq co(y)$. Since T is a similarity circle we have $T \subseteq co(x) \subseteq co(y)$. Hence $y \in T$ by the very definition of a similarity circle.

(3.5) Definition. *Let $f : X \to Y$ be a continuous map between the topological space X and Y. f is a* ***local homeomorphism*** *iff the following holds: for each $x \in X$ there is a open neighborhood U_x of x that is mapped by f onto the open set $f(U_x)$ in a 1–1-fashion.*

From now on we assume that the base space B of a trope bundle $\langle\langle E, \approx\rangle, p, \langle B, \sim\rangle\rangle$ is endowed with the order (or Aleksandrov) topology induced by the similarity structure that is induced by the similarity relation of resemblance defined on the total space E. Thus, if we define a topology on the total set E too one may ask whether the projection map p is continuous or not. In the case of trope bundles we get the following pleasing result:

(3.6) Proposition. *Let $\langle\langle E, \approx\rangle, p, \langle B, \sim\rangle\rangle$ be an bundle. Then it can be rendered a topological bundle such that the projection p is a local homeomorphism. Moreover, the similarity circles $T \in SC(E)$ are mapped homeomorphically onto $p(T)$.*

Proof: Let $T \in SC(E)$. Since the similarity relation $\sim$ is trivial on the fibres of p. T is mapped in a 1–1-fashion onto $p(T)$. Now define the topology $O(B)$ on B as the coarsest refinement of the order topology such that $p(T)$ is open in B for all $T \in SC(E)$. Endow E with the coarsest topology $O(E)$ such that p is continuous and the elements of $SC(E)$ are open. Then p is a local homeomorphism on T.

Topological bundles whose projections are local homeomorphisms are well-known mathematical entities called sheaves (cf. Goldblatt (1978), Tennison (1975), MacLane/Moerdijk (1992)). Thus we may reformulate (3.1) as follows:

(3.7) Definition. *Let $\langle E, p, B\rangle$ be an trope bundle. If the total space E and the base space B are endowed with the topologies defined in (3.6) the resulting sheaf is called a trope sheaf. It is also denoted by $\langle E, p, B\rangle$.*

Sheaves abound. They occur in topology, algebraic geometry, algebra, logic complex function theory, etc. (cf. MacLane/Moerdijk 1992). Thus, proposing sheaves as the „correct" generalization of trope spaces casts trope theory in a well established and versatile mathematical framework.

Before we go on to the technicalities necessary for defining global properties of trope sheaves, let us make the following general remark. Trope sheaves are genuinely particularist entities. For their definition we don't need any global concepts. In the next section we show how to construct for trope sheaves global properties that for traditional trope spaces boil down to the classical (ersatz) universals.

4. Sections and Global Properties

In this section we are going to characterize global properties, i.e. properties that may be instantiated by many particulars, as continuous sections of a trope bundle $\langle E, p, B\rangle$. As it should be, in the case of traditional trope bundles our construction yields the familiar universal properties. First some further definitions from bundle theory:

(4.1) Definition. *Let $\langle E, p, B\rangle$ be a bundle, and $V \in O(B)$. A section defined on V is a continuous map $s : V \to E$ such that $p|_V \circ s = \mathrm{id}_V$. The set of sections defined on V is denoted by $\Gamma(E, V)$. If $V \subseteq V'$ there is a natural restriction map $\rho_{V'V} : \Gamma(V', E) \to \Gamma(V, E)$ defined by $\rho_{V'V}(s) := s|_V$. A section $s \in \Gamma(E, V)$ is **maximal** if it is not the restriction of a section* *defined on a connected open set V' that properly includes V, i.e. if it is not the image $\rho_{V'V}(s')$ of some $s' \in \Gamma(V', E)$ with $V' \supset V$. The set of maximal sections of $\langle E, p, B\rangle$ is denoted by $\Gamma_{\max}E$.*

After these preparatory definition now we are ready to define global properties as the desired particularist surrogates of universals. The similarity structures defined on E and B will play an essential role in this task :

(4.2) Definition. *Let $\langle\langle E, \approx\rangle, p, \langle B, \sim\rangle\rangle$ be a trope bundle. Let $V \in SC(B)$ be an open subset of B. A global property for $\langle\langle E, \approx\rangle, p, \langle B, \sim\rangle\rangle$ is a maximal continuous equivariant*[14] *section $s_V : V \to E$. Elements $b, b' \in B$ are said to share the global property s_V if and only if they have individual properties e and e', respectively, such that $s_V(b) = e$ and $s_V(b') = e'$. The individual properties e and e' are said to be instantiations of the global property s_V.*

The following proposition is the main result of this section and characterizes global properties as *continuous* sections. It should be compared with the provisional definition of global properties put forward in (2.8). The crucial difference is that in the improved definition the *structure* of the bundle plays an essential role:

(4.3) Proposition. *Let $\langle E, p, B\rangle$ be a trope sheaf. The similarity circles $T \in SC(E)$ correspond in a 1–1-fashion to the global properties of the elements of the base space B.*

Proof: Let $T \in SC(E)$ be a similarity circle. The projection p maps T in a 1–1-way onto an open subset $p(T)$ of B. Hence we may define an equivariant continuous section as follows: $s_T : p(T) \to T$. The set $p(T)$ is a subset of a similarity circle C of $\langle B, \sim\rangle$, i.e. $p(T) \subseteq C \in SC(B)$. We have to show that s_T is maximal, i.e. that it cannot be expanded beyond $p(T)$. Let us assume the contrary, i.e. there is an element $b \notin p(T)$ such that there is an equivariant continuous section s'_T defined on $p(T) \cup \{b\}$. Since s'_T is equivariant $s'_T(b)$ must be similar to all elements of T. Since T is a similarity circle we get $s'_T(b) \in T$. This is a contradiction. Hence s_T is already maximal.

Let $s : V \to T$ be a maximal continuous equivariant section of $\langle E, p, B\rangle$. Since s is equivariant there is a similarity circle $T \in SC(E)$ such that $s(V) \subseteq T$. Since s is maximal we may assume that $s(V) = T$. Otherwise we could expand s to a section defined on $p(T)$ which would be strictly larger than V. Hence s coincides with s_T, i.e., a section defined by an element of $SC(E)$.

[14] A map s_v is equivariant iff $\forall_{x,x'}(x \sim x' \Rightarrow s_v(x) \approx s_v(x'))$.

(4.4) Corollary. *Let $\langle\langle E, =\rangle, p, \langle B, \sim\rangle\rangle$ be a traditional trope sheaf with property bundle $B \times U$. Then the universal properties $u \in U$ correspond in a 1–1-fashion to maximal equivariant continuous sections of the bundle $\langle E, p, B\rangle$.*

Now we go on to show that in the general case of a trope sheaf $\langle E, p, B\rangle$ the set $\Gamma_{\max}E$ can be considered as the starting point for well-behaved systems of global „ersatz-properties". First note that for any $b \in V \in O(B)$ we have the canonical restriction map $\rho_b : \Gamma(V, E) \to p^{-1}(b)$ defined by: $\rho_b(s) := s(b)$. As is well-known, this map can be used to realizing any individual property $e \in p^{-1}(b)$ by a global property in the following sense (cf. Tennison (1975)):

(4.5) Lemma. *For any $e \in p^{-1}(b)$ there is a section s_e defined on an open neighborhood W of b such that $\rho_b(s_e) = e$.*

Thus, expanding the section of (4.5) in an appropriate way each individual property e of $p(e)$ can be realized as the instantiation of a global section property $s_e \in \Gamma_{\max}E$. This expansion is *not* arbitrary, in contrast to the purely set theoretical properties in the sense of (2.8). However, in contrast to the traditional ersatz universals it is not unique, i.e. it might happen that there are different global sections s_e, s'_e with $s_e(b) = s'_e(b) = e$. This phenomenon will be discussed later in more detail. Thus the set $\Gamma_{\max}E$ can be considered as a system of global properties which gathers the individual properties in a way that more or less resembles that of universals in the case of traditional trope bundles. Hence, one might be tempted to consider $\Gamma_{\max}E$ as a surrogate for the set of (ersatz-)universals U of a traditional trope space. This is basically correct. However, on closer inspection it is revealed that $\Gamma_{\max}E$ needs some further enhancement and regimentation for being acceptable as a good system of global properties (SGP). This can be seen when we spell out some plausible adequacy conditions such a system should satisfy.

(i) A good SGP should be complete in the following sense: if b and b' are resemblant individuals there should be a global property, i.e. a section defined for b and b which realizes this resemblance.

(ii) A complementary requirement a good SGP should satisfy that it should not feign a resemblance where there is none: that means, if b and b' are not resemblant by individual properties, there should be no global property s rendering b and b' resemblant.

(iii) Finally, a good SGP for a trope sheaf should satisfy some axiom of economy according to which it should not contain superfluous global properties.

Other requirements can be formulated, and the ones mentioned can be rendered precise in different ways. Thus the following definition does not claim to definitely characterize good systems of global properties. Rather, it intends to be one reasonable proposal among other possible ones:

(4.6) Definition. *Let $\langle\langle E, \approx\rangle, p, \langle B, \sim\rangle\rangle$ be a trope sheaf. A good system of global properties $SGP(E)$ for $\langle E, p, B\rangle$ is a class of sections $\{s_V \mid s_V \in \Gamma_{\max}E\}$ which satisfies the following conditions:*

(i) $\forall_{b,b'}(b \sim b' \Rightarrow \exists_{s_V}(s_V \in SGP(E)\ \&\ b, b' \in V))$

(ii) $\forall_{s_V}(s_V \in SGP(E) \Rightarrow V \in SC(B))$

(iii) $\forall_{b,b'}(co(b) = co(b') \Rightarrow \forall_{s_V}(s_V \in SGP(E) \Rightarrow (b \in V \Leftrightarrow b' \in V)))$

(iv) *There is no section $s_V \in SGP(E)$ that can be removed such that for the resulting system (i)–(iii) are still satisfied.*

The condition (i) captures the informal requirement of completeness, i.e. if two concrete individuals b and b' resemble each other by some resembling particular properties this resemblance can be realized by some global property $s_V \in SGP(E)$. The second condition (ii) renders precise the concept of correctness, and (iv) is a requirement of economy. The third condition (iii) also is some kind of parsimony requirement but somewhat more complicated. It intends to capture the intuition that individuals that are resemblant to the same class of individuals should have the same global properties.

To be honest, the definition (4.6) of a good property system is not new. It can be traced back to an unpublished paper of Carnap's written some seventy years ago (cf. Carnap (1923)). A good $SGP(E)$ for a sheaf $\langle E, p, B\rangle$ is a straightforward generalization of what Carnap called a quasianalysis of a similarity structure (cf. Mormann (1994)). I do not want to go into the details of the method of quasianalysis here. For the purposes of this paper it suffices to describe its aims as follows: for a given similarity structure $\langle E, \sim\rangle$, e.g. a set of tropes endowed with a resemblance relation, we have to find a system of global properties, called by Carnap „quasiproperties", which fits the given similarity relation. According to Carnap, the requirements for this fitting can be rendered precise as follows:

(4.7) Definition (Carnap's Adequacy Conditions for Quasiproperties). *Let $\langle X, \sim \rangle$ be a similarity structure. A quasianalysis of $\langle X, \sim \rangle$ attributes to each element of X certain quasiproperties or quasiconstituents such that the following conditions are satisfied:*

(C1) *If two elements are resemblant they share at least one quasiproperty.*

(C2) *If two elements are not resemblant they do not share any quasiproperty.*

(C3) *If two elements are resemblant to exactly the same elements they have the same quasiproperties.*

(C4) *There is no quasiproperty which can be removed such that the conditions (C1)–(C3) are still satisfied.*

A similarity structure $\langle E, \sim \rangle$ defines a rather trivial sheaf $\langle E, \mathrm{id}, E \rangle$. Then the global section properties of this sheaf are just the inclusions $i_T : T \to E$ of similarity circles $T \in SC(E)$ that can be identified with the elements of $SC(E)$. As is easily seen $SC(E)$ as a system of global properties satisfies the conditions (4.6) (i)–(iv) and (4.7) (C1)–(C4). Hence, for $\langle E, \mathrm{id}, E \rangle$ the set $SC(E)$ (or some appropriate subset) provides a good system of global properties. Now, the natural question arises whether such a good system of global properties for sheaves $\langle E, \mathrm{id}, E \rangle$ is unique or not. The answer is No (cf. Mormann (1994:100)). There are similarity structures, i.e. sheaves $\langle E, \mathrm{id}, E \rangle$ that possess more than one good system of global properties. A fortiori, generally, trope sheaves $\langle E, p, B \rangle$ will possess different good systems of global properties.[15]

Finally let us mention a far reaching generalization of the trope sheaf approach as it has been developed till now. It is possible, and moreover intuitively appealing, to replace the resemblance relation $\approx$ by a general topological structure. This generalization keeps, if not letter but the spirit of the resemblance approach. The essential feature of the resemblance relation $\approx$ we put to work has been its topological structure, i.e., the order topology. It is perfectly natural to free ourselves from this restriction and to consider general topological bundles. The only assumption we shall retain will be a topological form of the orthogonality requirement (2.4) (iv) which is just the condition that the projection p is a local homeomorphism.

This results in a thorough-going concretization of the trope sheaf approach. The conceptualization of resemblance as a similarity relation amounts to a rather strong idealization. According to it, resemblance is a

[15] As is easily seen a traditional trope sheaf has only one good system of global properties, to wit, its system of universal properties.

yes-or-no affair: two tropes are resemblant or they are not. A more realistic account of resemblance would take it as a matter of degree (cf. Armstrong (1989:40): two entities resemble each other to a degree. This generalization can be conceptualized topologically in various ways. I want to mention just the most general one.

Let X be a topological space. For $x \in X$, the elements of X resembling x to at least a particular degree D may be considered as an open neighborhood $U_D(x)$. If $U_{D'}(x) \subseteq U_D(x)$ this is to be interpreted that the D'-degree of resemblance is at least as strong as the D-degree of resemblance.[16] In this way degrees of resemblance can be conceptualized as partially ordered by inclusion. They form a „directed system" (cf. Tennison (1975).[17] Thus, conceptualizing the set of tropes as a topological space amounts to the assumption that we can speak of a graded resemblance of tropes. Accordingly, we may envisage a general theory of tropes based on the concept of a (general) topological trope sheaf $\langle E, p, B\rangle$. The interesting mathematical fact to note is that also for this case, there is an intimate relation between the total space E and the sets $\Gamma(E, V)$ of (continuous) sections. In a quite precise sense, a sheaf can be identified with the limit of the system $\{\Gamma(E, V) \mid V \in O(B)\} := \Gamma E$. Systems such as ΓE are called presheaves (cf. Tennison 1975, MacLane/Moerdijk 1992). That is to say, given a trope sheaf $\langle E, p, B\rangle$ its corresponding presheaf ΓE provides a frame for defining appropriate systems of global properties. More precisely, the task is to select a subset $\Pi E \subseteq \Gamma E$ as a good system of global properties satisfying some appropriate adequacy requirements. As the example of Carnap's quasiproperties show usually such a ΠE will not be unique. Thus the question arises how should we deal with this ambiguity? Which of several good systems of global properties should we chose? Or should we take an „ecumenical" stance considering any good system of global properties as acceptable. I do not want to tackle these problems in this paper. Be it sufficient to state that different systems of global properties may be conceived of as different theories about the world. In this way questions about the epistemological status of systems of properties can be related to familiar problems of the underdetermination of theories.[18]

[16] Obviously, the D-degree of resemblance depends on x. It would be nice if we could compare degrees of resemblance for different x, x'. This amounts to the comparison of open neighborhoods $U_D(x)$ and $U_{D'}(x')$ for different elements $x, x' \in X$. As is well-known this can be done if we assume that E is a *uniform* topological space.

[17] A special case of a directed system are Lewis's centered systems of „similarity spheres (cf. Lewis (1986:14f)).

[18] As has been explained above the good systems of universal properties of traditional trope spaces are neither ambiguous nor do they show the phenomenon of branching. Hence,

5. Concluding Remarks

The sheaf account of trope theory offers a general framework for various accounts of trope theories.[19] According to it the world is to be conceived as a trope sheaf $\langle E, p, B\rangle$. Global properties (universals) are maximal equivariant continuous sections of $\langle E, p, B\rangle$. This conceptualization of global properties provides a particularist solution of the so called „problem of universals": It is an empirical question whether the world-sheaf $\langle E, p, B\rangle$ has an embedding into a trivial sheaf $\langle B \times U, p, B\rangle$ or not. This question has to decided by science (if at all), not by philosophy. If there is such an embedding, we get a unique family of universal properties in the traditional sense. If not, there may be several rival systems of global properties. Then, it is a largely pragmatic question to decide which of them serves our purposes best.

The inherent ambiguity of the constituted systems of global properties should not be considered as an objection against the trope sheaf ontology. Quite the contrary, it should be appreciated as a virtue of this approach, since it leaves sufficient leeway for empirical research that should not be restricted in advance by philosophical apriori considerations. The sheaf approach can be considered as a neutral framework wherein different ontological accounts for quite a lot of trope theories (or more generally of „layer--cake" theories (cf. Armstrong (1989:38)) can be pursued. Two extremal examples may be mentioned: if we endow $\langle E, p, B\rangle$ with the discrete topology (cf. (3.1) (v)) we arrive at a (not very attractive) nominalism according to which just any collection of individual properties counts as a global property. A global property is just a conventional gathering of individual properties. At the other end of the specter we find strongly realist ontologies maintaining that the world sheaf $\langle E, p, B\rangle$ can be embedded in a trivial bundle of universal properties $\langle B \times U, p, B\rangle$. Intermediate accounts are based on more or less strong assumptions on the topological structure of the sheaf $\langle E, p, B\rangle$

realism about universals corresponds to the strong metaphysical thesis that there is one and only one true theory of the world.

[19] It may be noted that the sheaf account is not committed to set theory as it might appear at first look. Actually, the theory of sheaves can be developed in the more general framework of category theory without any recourse to set theory (cf. MacLane/Moerdijk 1992). Nothing of the deliberations of this paper depends on the fact that the bundles we considered have been conceived as trope bundles rather than, say, as bundles of states of affairs in the sense of Armstrong (cf. Armstrong 1989): whether the trope of snow's whiteness is the same thing or different from the state of affairs that snow is white, for both kinds of entities we can define reasonable relations of compresence and resemblance in such a way that the topological theory of tropes can be applied (cf. Bacon (1988:151)).

leading to the constitution of global properties that show a certain amount of ambiguity or underdetermination.

This neutrality of the ontology of trope sheaves corresponds to the neutral character of modern geometry that is no longer concerned with the problem which geometric axioms are the correct ones. Rather, geometry in the modern sense is to be considered as a structural theory about geometries. Similarly, a sheaf theoretical theory of tropes should be considered as a general framework for many different ontological accounts.

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