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EXISTENCE,

the Square of Opposites, and Two-Dimensional Logic

CONTENTS

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1. The Starting Point

Ontological commitments and other problems concerning existence arise in connection with various aspects of logical theories. The semantics of quantification theory is usually formulated in such a manner that theorems are all and only those formulae which come out true under all interpretations in all *non-empty domains*. There are several approaches to *include the empty domain*. Paradoxically this apparent semantic extension means surrendering several formulae which are valid and intuitively plausible.

Outside of these questions concerning *general existence* there are a lot of difficulties regarding *singular existence*: Is it a tacit presupposition that any singular term has a meaning/denotes an entity (Frege)? Does quantifying singular terms commit us to acknowledge singular existence (Quine)? Is Existence a first order predicate, a predicate at all? If so, how is it do define? $\exists!x \overset{\mathrm{df}}{=} \exists xFx$ or $\exists!x \overset{\mathrm{df}}{=} \exists x(x = a)$ or $x = x$?

In this paper I am going to discuss questions of existence connected with the traditional square of opposites. There are several approaches to explicate the logical connection. But solutions, if any, are heterogeneous, incomplete or do not take into consideration the empty domain. After introducing my two-dimensional system $\mathcal{Q}$ I want will demonstrate the expressive power of such a framework by offering a new explication of the square of opposites. Two-dimensionality allows the syntactical formalization of existence conditions explicitly. Negation can be understood as denying only the explicitly asserted but not the implicitly presumed (existence) part. Regarding categorical inferences both assertion and existence presupposition can be relevant. A unique two-dimensional translation of the traditional square of opposites covers all interesting cases.

2. The Two-Dimensional System $\mathcal{Q}$

In this section I introduce the formal background for my discussion of several two-dimensional ways of representing *existence* and some interesting *variable quantifiers*.

2.1. Primitive symbols

1. Propositional variables: p, q, r, s, p_1,...
2. Classical functors: $\sim$, $\wedge$, $\vee$, $\supset$, $\equiv$, $\not\equiv$.

3. Individual variables: x, y, z, x_1,...
4. Individual constants: a, b, c, a_1, ...
5. Infinite list of singular, binary etc. functional variables: F^1, G^1, H^1, F^1_1,..., F^2, G^2, H^2, F^2_1,...
6. Quantifiers: $\forall$, $\exists$
7. Equality: $=$.
8. Operator forming **pairs** of classical formulae: $\Big[\ \Big]$.
9. Form of 1-placed, 2-placed variable functors: V^1_i $(1 \leqslant i \leqslant 4)$, V^2_j $(1 \leqslant j \leqslant 4^{4^n})$.
10. Form of variable quantifiers: $\forall_i$, $\exists_j$.
11. Parentheses: ( ) $\Big(\ \Big)$.

2.2. Formation rules

1. A propositional variable standing alone is a formula of $\mathcal{Q}$.
2. If $\boldsymbol{f}$ is an n-ary functional variable and if $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ are individual variables or individual constants or both (not necessarily all different), then $\boldsymbol{f}(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n)$ and $\boldsymbol{a}_1 = \boldsymbol{a}_2$ are formulae of $\mathcal{Q}$.
3. If X and Y are formulae of $\mathcal{Q}$, then $\sim X$, $(X \wedge Y)$, $(X \vee Y)$, $(X \supset Y)$, $(X \equiv Y)$ and $(X \not\equiv Y)$ are formulae of $\mathcal{Q}$.
4. If X is a formula of $\mathcal{Q}$ and $\boldsymbol{a}$ is an individual variable, then $\forall \boldsymbol{x} X$ and $\exists \boldsymbol{x} X$ are formulae of $\mathcal{Q}$.
5. If A, B are formulae of $\mathcal{Q}$ formed without reference to the formation rules 5.–7. (i.e. usual classical formulae), then $\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]$ is a formula of $\mathcal{Q}$.
6. If X, Y are formulae of $\mathcal{Q}$, then V^1X, and V^2XY are formulae of $\mathcal{Q}$.
7. If X is a formula of $\mathcal{Q}$ and $\boldsymbol{x}$ is an individual variable, then $\forall_i\, \boldsymbol{x} X$ and $\exists_j\, \boldsymbol{x} X$ are formulae of $\mathcal{Q}$.
8. X is a formula of $\mathcal{Q}$ iff its being so follows from 1.–7.

2.3. Types of formulae

CL-FORMULAE A, B, C, D (i.e. classical formulae) are those formulae which were exclusively formed by means of formation rules 1.–4.

An E-FORMULA $\mathcal{E}$ (i.e. elementary formula) is a formula of the form $\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]$.

F-FORMULAE $\mathcal{F}$ are formulae of the forms: $V^1_i \left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]$, $V^1_i A$, $V^2_j \left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right] \left[\begin{smallmatrix} C \\ D \end{smallmatrix}\right]$, $V^2_j \left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right] C$, $V^2_j A \left[\begin{smallmatrix} B \\ C \end{smallmatrix}\right]$, $V^2_j AB$, $\forall_i\, \boldsymbol{x} \left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]$, $\exists_j\, \boldsymbol{x} \left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]$.

An NC-FORMULA $\mathcal{Z}$ (i.e. a non-classical formula) is a formula of $\mathcal{Q}$ which is neither a CL-formula nor an E-formula.

2.4. Reduction rules

How should reduction rules act in the system $\mathcal{Q}$? Roughly speaking, reduction rules should support a complete transformation of any non-classical formula $\mathcal{Z}$ to a formula of the form $\begin{bmatrix} A \\ B \end{bmatrix}$ (i.e. an *E-formula* of a special kind).

I use the following abbreviation of $X \Longrightarrow X[Y_1/Y_2]$: $Y_1 \Longrightarrow Y_2$. Both "$X \Longrightarrow X[Y_1/Y_2]$" and "$Y_1 \Longrightarrow Y_2$" are read as "From X to infer $X[Y_1/Y_2]$", with $X[Y_1/Y_2]$ we mean that formula which is the result of substituting any formula Y_2 for the formula Y_1 in all of its occurrences in X.

1. Reduction rules for classical functors and quantifiers:

(i) $\sim \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} \sim A \\ \sim B \end{bmatrix}$

(ii) $\begin{bmatrix} A \\ B \end{bmatrix} \wedge \begin{bmatrix} C \\ D \end{bmatrix} \Longrightarrow \begin{bmatrix} A \wedge C \\ B \wedge D \end{bmatrix}$

$\begin{bmatrix} A \\ B \end{bmatrix} \wedge C \Longrightarrow \begin{bmatrix} A \wedge C \\ B \wedge C \end{bmatrix} \qquad A \wedge \begin{bmatrix} B \\ C \end{bmatrix} \Longrightarrow \begin{bmatrix} A \wedge B \\ A \wedge C \end{bmatrix}$

(iii) Disjunction, implication, equivalence, and negequivalence (exclusive disjunction) as in 1.2.

(iv) $\forall \boldsymbol{x} \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} \forall \boldsymbol{x} A \\ \forall \boldsymbol{x} B \end{bmatrix} \qquad \exists \boldsymbol{x} \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} \exists \boldsymbol{x} A \\ \exists \boldsymbol{x} B \end{bmatrix}$

2. Reduction rules for variable functors

The *general form of substitution* is

SR $$X \Longrightarrow X[\mathcal{F}/\mathcal{E}],$$

where by $X[\mathcal{F}/\mathcal{E}]$ I mean the result of substituting the *E-formula* $\mathcal{E}$ for the *F-formula* $\mathcal{F}$ in all occurrences of $\mathcal{F}$ in X.

The special forms of V-SUBSTITUTION are

(i) $V^1 \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} \Phi^2 AB \\ \Psi^2 AB \end{bmatrix}$

(ii) $V^1 A \Longrightarrow \begin{bmatrix} \Phi^2 AA \\ \Psi^2 AA \end{bmatrix}$

(iii) $V^2 \begin{bmatrix} A \\ B \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} \Longrightarrow \begin{bmatrix} \Phi^4 ABCD \\ \Psi^4 ABCD \end{bmatrix}$

(iv) $V^2 \begin{bmatrix} A \\ B \end{bmatrix} C \Longrightarrow \begin{bmatrix} \Phi^4 ABCC \\ \Psi^4 ABCC \end{bmatrix}$

(v) $V^2 A \begin{bmatrix} B \\ C \end{bmatrix} \Longrightarrow \begin{bmatrix} \Phi^4 AABC \\ \Psi^4 AABC \end{bmatrix}$

(vi) $V^2 AB \Longrightarrow \begin{bmatrix} \Phi^4 AABB \\ \Psi^4 AABB \end{bmatrix}$

where Φ^2, Ψ^2 are 2-placed classical functors, and Φ^4, Ψ^4 are 4-placed classical functors definable by given functors (because of the truth-functional completeness of classical logic). It is immediately clear that V-substitution 2(ii) is a subcase of 2(i), and V-substitutions 2(iv), 2(v), and 2(vi) are subcases of 2(iii).

3. Reduction rules for variable quantifiers

The special forms of Q-substitution are

(i) $\forall_i \boldsymbol{x} \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} \forall \boldsymbol{x} \Phi^2 AB \\ \forall \boldsymbol{x} \Psi^2 AB \end{bmatrix}$ $\qquad \forall_i \boldsymbol{x} A \Longrightarrow \begin{bmatrix} \forall \boldsymbol{x} \Phi^2 AA \\ \forall \boldsymbol{x} \Psi^2 AA \end{bmatrix}$

(ii) $\exists_i \boldsymbol{x} \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} \exists \boldsymbol{x} \Phi^2 AB \\ \exists \boldsymbol{x} \Psi^2 AB \end{bmatrix}$ $\qquad \exists_i \boldsymbol{x} A \Longrightarrow \begin{bmatrix} \exists \boldsymbol{x} \Phi^2 AA \\ \exists \boldsymbol{x} \Psi^2 AA \end{bmatrix}$

Later special V-reduction and Q-reduction rules will be formulated in such a way, that every NC-formula can be transformed into an *E-formula* in a finite number of steps starting from the inside of a given NC-formula.

Example: Let $\neg$ be a variable functor and $\forall_1$ be a variable quantifier characterized by the following reduction rules

$$\neg \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} \sim A \\ B \end{bmatrix} \qquad \forall_1 \boldsymbol{x} \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} \forall \boldsymbol{x}(B \supset A) \\ \forall \boldsymbol{x}(B \supset A) \end{bmatrix}$$

and suppose the following NC-formula is given:

$$\neg\neg\forall_1 x \begin{bmatrix} Gx \\ Fx \end{bmatrix} \equiv \forall x(Fx \supset Gx).$$

Then we can definitely generate a corresponding E-formula using the following steps:

$$\neg\neg\begin{bmatrix}\forall x(Fx \supset Gx)\\ \forall x(Fx \supset Gx)\end{bmatrix} \equiv \forall x(Fx \supset Gx) \qquad (\forall_1\text{-REDUCTION})$$

$$\neg\begin{bmatrix}\sim \forall x(Fx \supset Gx)\\ \forall x(Fx \supset Gx)\end{bmatrix} \equiv \forall x(Fx \supset Gx) \qquad (\neg\text{-REDUCTION})$$

$$\begin{bmatrix}\sim\sim \forall x(Fx \supset Gx)\\ \forall x(Fx \supset Gx)\end{bmatrix} \equiv \forall x(Fx \supset Gx) \qquad (\neg\text{-REDUCTION})$$

$$\begin{bmatrix}\sim\sim \forall x(Fx \supset Gx) \equiv \forall x(Fx \supset Gx)\\ \forall x(Fx \supset Gx) \equiv \forall x(Fx \supset Gx)\end{bmatrix} \qquad (\equiv\text{-REDUCTION})$$

2.5. Semantics

Validity and inconsistency of CL-formulae. Let $\vdash A$ indicate that the classical formula A is CLASSICALLY VALID (is a TAUTOLOGY), i.e. valid in the usual classical sense.

Furthermore, let $\nvdash A$ indicate that the classical formula A is NOT CLASSICALLY VALID (is NOT A TAUTOLOGY).

Validity of E-formulae.

Definition 1. *The E-formula* $\begin{bmatrix}A\\B\end{bmatrix}$ *is* ***E-valid*** *(symb.:* $\models \begin{bmatrix}A\\B\end{bmatrix}$*) iff* $\vdash A$ *and* $\vdash B$.

Theorem 1. $\models \begin{bmatrix}A\\B\end{bmatrix}$ *iff* $\vdash (A \wedge B)$.

Since all *NC-formulae* can be reduced to *E-formulae* this theorem means that validity in $\mathcal{Q}$ is reducible to classical validity. Hence, what follows is the representation of existence in a classical style.

Validity of NC-formulae

Definition 2. *Let* $\mathcal{Z}$ *be any NC-formula and* $\begin{bmatrix}A_{\mathcal{Z}}\\B_{\mathcal{Z}}\end{bmatrix}$ *that E-formula which is the result of the complete reduction of* $\mathcal{Z}$*, i.e. that both all occurrences of variable functors and all occurrences of classical functors outside the scope of brackets are eliminated:*

$$\models \mathcal{Z} \quad \textit{iff} \quad \models \begin{bmatrix}A_{\mathcal{Z}}\\B_{\mathcal{Z}}\end{bmatrix}.$$

3. A New Representation of the Square of Opposites

It has been widely discussed that there is apparently no perfect formal representation of the traditional square of opposition by using modern formal translations. Which circumstances are responsible for this situation?

3.1. The traditional square of opposites

Let us begin with a short characterization of the *traditional* situation: Usually propositions not compounded of other propositions are called *categorical*. They have a *subject term* and a *predicate term*. In the famous example "All men are mortal", "men" is the subject term and "mortal" the predicate term. According to their QUANTITY, categorials can be subdivided into *universals* ("All men are mortal", "No men are mortal") and *particulars* ("Some men are mortal"), and according to their QUALITY, into *affirmatives* ("All men are mortal", "Some men are mortal") and *negatives* ("No men are mortal", Some men are not mortal").

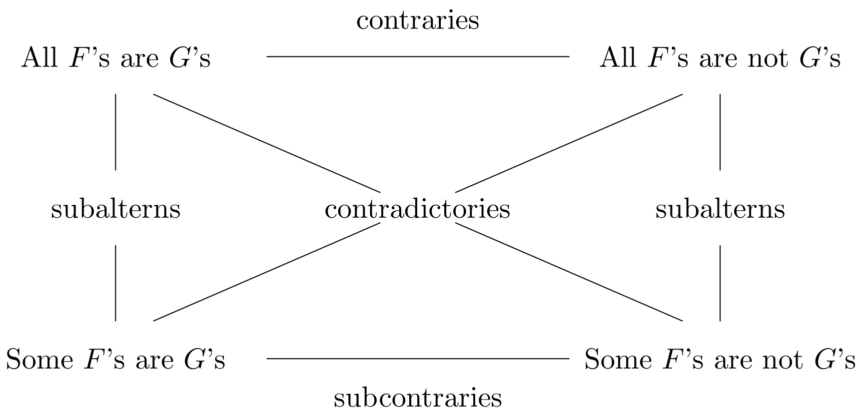
The traditional *four forms* of categorial propositions are the following:

A-propositions	universal affirmatives	All F's are G's
E-propositions	universal negatives	All F's are not G's
I-propositions	particular affirmatives	Some F's are G's
O-propositions	particular negatives	Some F's are not G's

The traditional name *square of opposites* goes back to the fact that propositions with the same terms in the same order may be *opposed* in several ways:

(a)	**A** and **O** are contradictories (cannot be jointly true or jointly false)	$\mathbf{A} \dashv\vdash \sim \mathbf{O}$
(b)	**E** and **I** are contradictories	$\mathbf{E} \dashv\vdash \sim \mathbf{I}$
(c)	**A** and **E** are contraries (cannot be jointly true)	$\vdash \sim (\mathbf{A} \wedge \mathbf{E})$, $\mathbf{A} \vdash \sim \mathbf{E}$
(d)	**I** and **O** are subcontraries (cannot be jointly false)	$\vdash \mathbf{I} \vee \mathbf{O}$, $\sim \mathbf{I} \vdash \mathbf{O}$
(e)	**A** and **I** are subalterns	$\mathbf{A} \vdash \mathbf{I}$ and $\mathbf{I} \nvdash \mathbf{A}$
(f)	**E** and **O** are subalterns	$\mathbf{E} \vdash \mathbf{O}$ and $\mathbf{O} \nvdash \mathbf{E}$

If we display this situation as a rectangle — with universals at the top, particulars at the bottom, affirmatives on the left, negatives on the right — we get the following picture:



3.2. Modern translations

Let us try the usual translation into a Russellian formalism:

A: $\forall x(Fx \supset Gx)$
E: $\forall x(Fx \supset\sim Gx)$
I: $\exists x(Fx \wedge Gx)$
O: $\exists x(Fx \wedge \sim Gx)$

With respect to this translation (a) and (b) are valid, but (c) to (f) are not valid.[1]

Leonard (cf. [4], 51) argues that the modern, classical logic is rich enough to explicate the *presupposition*.[2] Hence, we have to modify the translation for **A**- and **E**-expressions:

A: $\forall x(Fx \supset Gx) \wedge \exists x Fx$
E: $\forall x(Fx \supset\sim Gx) \wedge \exists x Fx$.

[1] (a) and (b) are valid *including* the *emty* domain, whereas (c) to (f) are not valid even in all non-empty domains.

[2] Following Frege's ([1], 40) terminology of "selbstverständliche Voraussetzung" with respect to singular terms Leonard uses the notion "tacit, unexpressed presupposition" with respect to general terms.

But now we are faced with a dilemma: on the one hand, we get the result that (c) to (f) are valid under this circumstances, on the other, we loose the validity of (a) and (b) and, therefore, the definability of one quantifier by using the other. Leonard argues that this two-faced solution is a "real advance":

> Must we say that the traditional logic was in error? Not at all! On the contrary, it may be held that the traditional logic was a quite correct abstract system of logic; but that it was set up and developed with a tacit, or unexpressed presupposition: namely, that its terms, S, P, etc., were terms having existent exemplars.
>
> At the same time, the modern logic marks a real advance. With its symbolism for quantifiers, it can represent the universal presupposition of the traditional logic. Hence, it can express its relevance when it is relevant (as in inference from **A** to **I**, express irrelevance (by omitting mention of it) when it is irrelevant (as in the inference from **A** to not $\sim$ **O** and even explore the consequence of its falsity. This gives the modern logic a much wider applicability then that enjoyed by the traditional logic. ([4], 51)

It remains an unsolved puzzle to give a *unique* translation corresponding to the whole square of opposites. If we take a universe of discourse (the F's) for granted we are able to offer an apparently simple solution:

A: $\forall x Gx$ **E**: $\forall x \sim Gx$ **I**: $\exists x Gx$ **O**: $\exists x \sim Gx$

It is easy to check the validity of (a) to (f) in the *non-empty* domains. Quine, among others (e.g. [6], [2] and [3]), offers an axomatization of the first-order predicate calculus called *inclusive quantification theory*, i.e. inclusive of the empty domain. But there is an easy test regarding the empty domain:

> An easy supplementary test enables us anyway, when we please to decide whether a formula holds for the emty domain. We have only to mark the universal quantifications as true and the existential ones as false, and apply truth-table considerations. ([7], 177)

Using this "supplementary test" we have to acknowledge that our last proposal does not work for (c) to (f) in the empty domain.

Strawson ([8], 170f.) offers a "formalistic solution" which is somehow crazy and gives cause for searching for a better "realistic solution" which "illuminates some general features of our ordinary speech"([8], 170). Here is Strawson's "formalistic solution" without further comment:

A: $\forall x(Fx \supset Gx) \wedge \exists xFx \wedge \exists x \sim Gx$
E: $\forall x(Fx \supset\sim Gx) \wedge \exists xFx \wedge \exists xGx$
I: $\exists x(Fx \wedge Gx) \vee \exists xFx \vee \sim \exists xGx$
E: $\exists x(Fx \wedge \sim Gx) \vee \exists xFx \vee \exists x \sim Gx.$

Any tested solution seems to force us to the following alternative: either to take existential preconditions for granted (tacit presuppositions), i.e. there is no need for a syntactic or even semantic representation of such preconditions, or to find a syntactic place and formalization of existence which seems to be possible only by adding a conjunct: $\ldots \wedge \exists xFx$.[3] But using logical conjunction forces us to give up the intuitive difference between explicit (asserted) and implicit (presupposed) meaning.

My thesis is that a two-dimensional framework allows to leave this pseudo-forcing alternative. I offer a unifying syntactic approach without conjunction at the beginning. The empty domain is also taken into consideration. But how does it work?

3.3. A two-dimensional representation of the square of opposites

I interpret the A in the E-expression $\begin{bmatrix} A \\ B \end{bmatrix}$ as ASSERTION, and the B as PRESUPPOSITION without further qualification.

PRESUPPOSITION-PRESERVING/INTERNAL NEGATION: $\neg \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} \sim A \\ B \end{bmatrix}$

Given this situation it is useful to introduce the following pseudo 3-valued reading of E-expressions:

1. $\begin{bmatrix} A \\ B \end{bmatrix} =$ TRUE iff $A = 1$ and $B = 1$,
2. $\begin{bmatrix} A \\ B \end{bmatrix} =$ FALSE iff $A = 0$ and $B = 1$,
3. $\begin{bmatrix} A \\ B \end{bmatrix} =$ INCORRECT iff $B = 0$.

Now it is clear that we should try the following translation of categorical expressions:

A: $\begin{bmatrix} \forall x(Fx \supset Gx) \\ \exists xFx \end{bmatrix}$

[3] Other candidates discussed in the literature are $x = x$ and $\exists x(x = a)$.

$$\textbf{E}: \begin{bmatrix} \forall x(Fx \supset\sim Gx) \\ \exists xFx \end{bmatrix}$$

$$\textbf{I}: \begin{bmatrix} \exists x(Fx \wedge Gx) \\ \exists xFx \end{bmatrix}$$

$$\textbf{O}: \begin{bmatrix} \exists x(Fx\wedge \sim Gx) \\ \exists xFx \end{bmatrix}.$$

It is sufficient to explicate *existence* by using $\exists xFx$ in any of the four cases (the B-part of E-expressions). The assertion part is identical with the Russellian translation above. But there is no conjunction-connection between *assertion*/first line and *existence presupposition*/second line.

The next step consists in interpreting any negation whose argument contains E-expressions as PRESUPPOSITION-PRESERVING/INTERNAL negations. This is in accordance with the idea that presuppositions are normally invariant regarding negation.

Finally, I introduce a variable functor with the following reduction property:

$$\begin{bmatrix} A \\ B \end{bmatrix} \to \begin{bmatrix} C \\ D \end{bmatrix} \Longrightarrow \begin{bmatrix} A \wedge B \supset C \\ A \wedge B \supset D \end{bmatrix}.$$

We can read the expression "$\models X \to Y$" as "From X follows Y". Inference in this sense allows the inclusion of implicit/presupposed premisses explicitly. Combining these points we obtain a re-formulation of the logical properties of the square of opposites:

(a)	**A** and **O** are contradictories	$\models \textbf{A} \to \neg\textbf{O}$ and $\models \neg\textbf{O} \to \textbf{A}$
(b)	**E** and **I** are contradictories	$\models \textbf{E} \to \neg\textbf{I}$ and $\models \neg\textbf{I} \to \textbf{E}$
(c)	**A** and **E** are contraries	$\models \textbf{A} \to \neg\textbf{E})$
(d)	**I** and **O** are subcontraries	$\models \neg\textbf{I} \to \textbf{O}$
(e)	**A** and **I** are subalterns	$\models \textbf{A} \to \textbf{I}$ and $\not\models \textbf{I} \to \textbf{A}$
(f)	**E** and **O** are subalterns	$\models \textbf{E} \to \textbf{O}$ and $\not\models \textbf{O} \to \textbf{E}$.

To give an impression how the apparatus work I show the translation for each case and the corresponding E-valid reductions:

(a) $$\models \begin{bmatrix} \forall x(Fx \supset Gx) \\ \exists xFx \end{bmatrix} \to \neg \begin{bmatrix} \exists x(Fx\wedge \sim Gx) \\ \exists xFx \end{bmatrix}$$

$$\begin{Bmatrix} \vdash \forall x(Fx \supset Gx) \wedge \exists xFx \supset\sim \exists x(Fx\wedge \sim Gx) \\ \vdash \forall x(Fx \supset Gx) \wedge \exists xFx \supset \exists xFx \end{Bmatrix}$$

$$\models \neg \begin{bmatrix} \exists x(Fx \wedge \sim Gx) \\ \exists xFx \end{bmatrix} \rightarrow \begin{bmatrix} \forall x(Fx \supset Gx) \\ \exists xFx \end{bmatrix}$$

$$\left\{ \begin{matrix} \vdash \sim \exists x(Fx \wedge \sim Gx) \wedge \exists xFx \supset \forall x(Fx \supset Gx) \\ \vdash \sim \exists x(Fx \wedge Gx) \wedge \exists xFx \supset \exists xFx \end{matrix} \right\}$$

(b) $$\models \begin{bmatrix} \forall x(Fx \supset \sim Gx) \\ \exists xFx \end{bmatrix} \rightarrow \neg \begin{bmatrix} \exists x(Fx \wedge Gx) \\ \exists xFx \end{bmatrix}$$

$$\left\{ \begin{matrix} \vdash \forall x(Fx \supset \sim Gx) \wedge \exists xFx \supset \exists x(Fx \wedge Gx) \\ \vdash \forall x(Fx \supset \sim Gx) \wedge \exists xFx \supset \exists xFx \end{matrix} \right\}$$

$$\models \neg \begin{bmatrix} \exists x(Fx \wedge Gx) \\ \exists xFx \end{bmatrix} \rightarrow \begin{bmatrix} \forall x(Fx \supset \sim Gx) \\ \exists xFx \end{bmatrix}$$

$$\left\{ \begin{matrix} \vdash \sim \exists x(Fx \wedge Gx) \wedge \exists xFx \supset \forall x(Fx \supset \sim Gx) \\ \vdash \sim \exists x(Fx \wedge Gx) \wedge \exists xFx \supset \exists xFx \end{matrix} \right\}$$

(c) $$\models \begin{bmatrix} \forall x(Fx \supset Gx) \\ \exists xFx \end{bmatrix} \rightarrow \neg \begin{bmatrix} \forall x(Fx \supset \sim Gx) \\ \exists xFx \end{bmatrix}$$

$$\left\{ \begin{matrix} \vdash \forall x(Fx \supset Gx) \wedge \exists xFx \supset \sim \forall x(Fx \supset \sim Gx) \\ \vdash \forall x(Fx \supset Gx) \wedge \exists xFx \supset \exists xFx \end{matrix} \right\}$$

(d) $$\models \neg \begin{bmatrix} \exists x(Fx \wedge Gx) \\ \exists xFx \end{bmatrix} \rightarrow \begin{bmatrix} \exists x(Fx \supset \sim Gx) \\ \exists xFx \end{bmatrix}$$

$$\left\{ \begin{matrix} \vdash \sim \exists x(Fx \wedge Gx) \wedge \exists xFx \supset \exists x(Fx \supset \sim Gx) \\ \vdash \sim \exists x(Fx \wedge Gx) \wedge \exists xFx \supset \exists xFx \end{matrix} \right\}$$

(e) $$\models \begin{bmatrix} \forall x(Fx \supset Gx) \\ \exists xFx \end{bmatrix} \rightarrow \begin{bmatrix} \exists x(Fx \wedge Gx) \\ \exists xFx \end{bmatrix}$$

$$\left\{ \begin{matrix} \vdash \forall x(Fx \supset Gx) \wedge \exists xFx \supset \exists x(Fx \wedge Gx) \\ \vdash \forall x(Fx \supset Gx) \wedge \exists xFx \supset \exists xFx \end{matrix} \right\}$$

$$\not\models \begin{bmatrix} \exists x(Fx \wedge Gx) \\ \exists xFx \end{bmatrix} \rightarrow \begin{bmatrix} \forall x(Fx \supset Gx) \\ \exists xFx \end{bmatrix}$$

$$\left\{ \begin{matrix} \nvdash \exists x(Fx \wedge Gx) \wedge \exists xFx \supset \forall x(Fx \supset Gx) \\ \vdash \exists x(Fx \wedge Gx) \wedge \exists xFx \supset \exists xFx \end{matrix} \right\}$$

(f) $$\models \begin{bmatrix} \forall x(Fx \supset \sim Gx) \\ \exists xFx \end{bmatrix} \rightarrow \begin{bmatrix} \exists x(Fx \wedge \sim Gx) \\ \exists xFx \end{bmatrix}$$

$$\left\{ \begin{matrix} \vdash \forall x(Fx \supset \sim Gx) \wedge \exists xFx \supset \exists x(Fx \wedge \sim Gx) \\ \vdash \forall x(Fx \supset \sim Gx) \wedge \exists xFx \supset \exists xFx \end{matrix} \right\}$$

$$\not\models \begin{bmatrix} \exists x(Fx \wedge \sim Gx) \\ \exists xFx \end{bmatrix} \rightarrow \begin{bmatrix} \forall x(Fx \supset \sim Gx) \\ \exists xFx \end{bmatrix}$$

$$\left\{\begin{array}{c} \nvdash \exists x(Fx\wedge \sim Gx)\wedge \exists xFx \supset \forall x(Fx \supset\sim Gx) \\ \vdash \exists x(Fx\wedge \sim Gx)\wedge \exists xFx \supset \exists xFx \end{array}\right\}.$$

We observe that in regarding the second (presupposition) line we always get an elimination of conjunction. In (a) and (b) the same happens regarding the first (assertion) line. But to prove the assertion in (c) to (f) we necessarily need the existence premise. Unlike Leonard's strategy we now have a unified approach to the traditional square of opposites. It depends only on the innerlogical structure whether we have to use the represented existential presupposition.

4. Outlook: Variable Quantifiers

Let us first recall the reduction rules for quantifiers in $\mathcal{Q}$:

$$\forall \boldsymbol{x} \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} \forall \boldsymbol{x} A \\ \forall \boldsymbol{x} B \end{bmatrix}$$

$$\exists \boldsymbol{x} \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} \exists \boldsymbol{x} A \\ \exists \boldsymbol{x} B \end{bmatrix}$$

In this way these quantifiers do not yield the usual form of restricted classical quantifiers: $\forall x(Fx \supset Gx)$, $\exists x(Fx \wedge Gx)$, due to

$$\not\models \forall x \begin{bmatrix} Gx \\ Fx \end{bmatrix} \equiv \forall x(Fx \supset Gx) \qquad \text{and} \qquad \not\models \exists x \begin{bmatrix} Gx \\ Fx \end{bmatrix} \equiv \exists x(Fx \wedge Gx).$$

We can take an **S5**-like variable functor into consideration (cp. [5]):

$$\Box \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} A \wedge B \\ A \wedge B \end{bmatrix}.$$

This allows the simulation of an aspect of quantifying–in difficulties:

$$\not\models \exists x \Box \begin{bmatrix} Gx \\ Fx \end{bmatrix} \equiv \Box \exists x \begin{bmatrix} Gx \\ Fx \end{bmatrix} \qquad \text{because of}$$

$$\nvdash (\exists xFx \wedge \exists xGx) \supset \exists x(Fx \wedge Gx).$$

It is, of course, possible to introduce variable quantifiers which act like classical quantifiers:

$$\forall_1 \boldsymbol{x} \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} \forall \boldsymbol{x}(B \supset A) \\ \forall \boldsymbol{x}(B \supset A) \end{bmatrix}$$

$$\exists_1 \boldsymbol{x} \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} \exists \boldsymbol{x}(B \wedge A) \\ \exists \boldsymbol{x}(B \wedge A) \end{bmatrix}.$$

Then, it holds that

$$\models \forall_1 x \begin{bmatrix} Gx \\ Fx \end{bmatrix} \equiv \forall x(Fx \supset Gx) \quad \text{and} \quad \models \exists_1 x \begin{bmatrix} Gx \\ Fx \end{bmatrix} \equiv \exists x(Fx \wedge Gx).$$

Despite the logical equivalence between an E-subformula and a CL-subformula we have to make a clear distinction between the classical/external negation (a functor) and the presupposition-preserving negation (a variable functor):

$$\models\sim \forall_1 x \neg \begin{bmatrix} Gx \\ Fx \end{bmatrix} \equiv \exists x(Fx \wedge Gx) \qquad \models\sim \forall_1 x \neg \begin{bmatrix} Gx \\ Fx \end{bmatrix} \equiv \exists_1 x \begin{bmatrix} Gx \\ Fx \end{bmatrix}$$

$$\models\sim \exists_1 x \neg \begin{bmatrix} Gx \\ Fx \end{bmatrix} \equiv \forall x(Fx \supset Gx) \qquad \models\sim \exists_1 x \neg \begin{bmatrix} Gx \\ Fx \end{bmatrix} \equiv \forall_1 x \begin{bmatrix} Gx \\ Fx \end{bmatrix}$$

Last but not least, it is worth noting that a little modification of the system $\mathcal{Q}$ gives the possibility of introducing *mixed variable quantifiers*:

$$\forall_\exists \boldsymbol{x} \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} \forall \boldsymbol{x}(B \supset A) \\ \exists \boldsymbol{x}(B \wedge A) \end{bmatrix}$$

$$\exists_\forall \boldsymbol{x} \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} \exists \boldsymbol{x}(B \wedge A) \\ \forall \boldsymbol{x}(B \supset A) \end{bmatrix}$$

It is a matter of further research whether special *variable quantifiers* can be interpreted as *generalized quantifiers*.

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