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# CONTINUA WITHOUT SETS

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# I. MOTIVATION AND PURPOSE

## 1. From the phenomenology of the continuum

Initially, we perceive an indefinite extension imprecisely, a spread $C$; this perception can be visual, aural, or tactile. Next we perceive in $C$ an uncertain number of aspects $C^r, C^s, \ldots$; these aspects provide $C$ with the beginning of a structure. Then in each of these aspects we perceive regions $x^r, y^r, x^s, \ldots$, which give the aspects their internal composition. Yet it would be an error to assume that because we label these regions with distinct letters, each is a fully separated entity. To think thus is to succumb to the insidious atomism induced by written language, a prejudice we must avoid at all costs if we are ever to understand the nature of a concrete continuum as we really perceive it, that is, free of the characteristics injected by our inherited preconceptions, preconceptions as prevalent as they are inadequate. We need a binary predicate of distinguishability to represent the fact that $x^r$ is distinguishable from $y^s$ but not necessarily vice versa: each distinction implies a direction. Subsequently, we perceive that some regions and aspects are united to one another; some not. Then we see that some regions and aspects are part of other regions and aspects; some not. And finally, we perceive that some regions and aspects lie between other regions and aspects, which represents a fundamental ternary relation of betweenness that is — together with the two binary relations of unity and being a part of — essential to the constitution of any concrete continuum and especially to its topological structure.

## 2. The inadequacy of continua built on set theory

The current mathematical theory of the continuum is Dedekind's, based on sets. In this theory, the continuum is built from the bottom up, with deceiving looseness. In fact, it is so weakly put together that using nonstandard analysis it is not difficult to totally unglue it and fill its uncountable holes with equally uncountable infinitesimals and infinities. These so easily created openings in the continuum then become interspersed with an unlimited number of variably sized monads, that is, monads, and monads within monads, and so on ad infinitum, all of which makes the original continuum impossible to recapture with further Dedekind cuts. In themselves, these additions and interpolations are a positive feature in that, phenomenologically speaking, many continua are indeed soft, penetrable, and amenable to unlimited interspersion. The key shortcomings of Dedekind's theory are unity's feeble role among the parts as well as the way the theory overlooks the fact that continua are genetically prior to sets, prior to any abstract gathering and separating. A surprising failure given the historical fact that there was geometry for centuries before there were sets.

The irony of Dedekind's approach lies in its characterization of the continuum through the least absolute and least important of its properties, divisibility, when actually what characterizes the continuum is its indivisibility, the encroachment of parts on one another, and the unbreakable linkages. This is why Dedekind's continuum is so unreal, and why it cannot account for the continuum's most essential properties: (i) that it has distinguishable aspects, not cuts; (ii) that some of its regions form a unity while others do not, union being a true or false predicate and not a set-theoretic operation that has no exceptions; (iii) that it has no holes — no empty regions — and itself is never a universal depository of aspects and regions but is always open to endless downward analysis and upward synthesis; (iv) that some of its regions interpenetrate mutually while others do not; and finally (v) that there are always regions and more regions between regions.

## 3. What is a continuum?

A continuum is an extension with distinguishable and often overlapping aspects, aspects in which, in turn, overlapping and nonoverlapping regions can be distinguished. In this extension each distinction discloses an in-between: every division engenders its own bridge. A continuum that is absolutely separable is not a continuum at all. The continuum mends its own tearings like flesh that heals its own wounds. In the continuum, holes are abstract illusions, somewhat like the black holes that current cosmology interprets as not really holes but as a kind of in-between, a link between two worlds that establishes the continuity of one into the other. When we consider an aspect as an intermediate — provisional — universal continuum all by itself, it becomes a "black hole" through which to travel from one "universal" level to another.

A theory for a concrete continuum — a continuous continuum — must reflect several fundamental characteristics: (i) its primitiveness — the fact that the continuum is an ultimate aspect of reality to be described, not a building to be formally erected on a base rooted in set theory; (ii) the fact that the continuum does not gather regions in a set-theoretic fashion — an axiom of comprehension would distort the unlimited openness, both upward and downward, with which a continuum presents itself in the middle of a vast expanse; (iii) the fact that each "universal" continuum is only an aspect temporarily taken as a universe, an "intermediate universal continuum", to use technically the paradoxical expression already employed (this expression well conveys the antinomic situation in which we find ourselves, perceiving as we must only a limited portion of reality that fades indefinitely toward the very big and the very small, despite our obsessive attempts to describe the world in its entirety as though it were a closed aggregate).

## 4. Mingling versus gathering

Dedekind's continuum, being an upwardly directed construction, misses the point that in the real world wholes and parts exchange roles more freely than our habit of "collecting into a whole" allows us to perceive. This tendency to gather, to collect and enclose, is a major source of fallacious descriptions of reality. Wholes do not necessarily gather their parts; instead, the relations between parts and wholes are often like a mingling together. Our technical use of the word mingling will convey that a part often extends beyond many of its wholes. Mingling is of course compatible with gathering, but it generally transcends gathering and erases all stratifications of the part-whole relation. The predicate "being a part of" will be used technically in this sense of mingling, not in that of a set-theoretic gathering of elements or of an inclusion of one set into another.

## 5. In the middle of things

Franz Kafka noted that things never present themselves "by their roots" — nor in their totality, we must add — "but by some point or other situated toward the middle". This is an accurate description of man's perceptive predicament. We can only approach reality at its middle, a reality which in effect may well have no roots and be utterly uncollectable in its entirety.

The intermediate universal continuum $C$ is at the middle of all its aspects, and each aspect $C^r$ is at the middle of all its regions $x^r, y^r, z^r, \ldots$. Only set-theoretic preconceptions make us fail to see haw entities are observed in direct experience. Yet $C$ itself is only an aspect of a vaster expanse, an aspect that can itself be "horizontally"analyzed into further aspects and regions that are at a lower level than the vaster expanse. This level structure is reminiscent of Hao Wang's theory of positive and negative types — which also excludes initial levels — except for two considerations: (i) types are closed to one another, while levels share the same aspects, taken alternatively as aspects or universes, and (ii) types are arranged in linear succession, whereas levels ramify nonlinearly upward and downward.

## 6. Union as a predicate

Some objects cannot unite even if one puts them in the same box in perpetuity. The moment that uniting entities becomes synonymous with collecting them into the same set, unification loses its concrete meaning and becomes an abstract, trivialized operation. In that follows, union is to be considered a binary predicate that holds if and only if two continua are truly unified, that is, if there is a definite coalescence between them, be they continguous or mediated by other continua. Thus the intermediate universal continuum $C$ coalesces with each of its aspects $C^r, C^s, \ldots$, unifies with each of them — which is why they are *its* aspects, not because of any membership relation. Similarly, each aspect $C^r$ coalesces with each of its regions $x^r, y^r, z^r, \ldots$, unifies with each of its regions — what is why they are *its* regions. $C$ and its aspects $C^r, C^s, \ldots$ are therefore essentially amalgamated.

## 7. Betweenness

The intermediate universal continuum $C$ interjects itself between any two of its aspects; in symbols, $\mathcal{B}(C^r, C, C^s)$. Similarly, each aspect interjects itself between any two of its regions, $\mathcal{B}(x^r, C^r, y^r)$. Betweenness is not an external relation, for $\mathcal{B}(C^r, C, C^r)$ and $\mathcal{B}(x^r, C^r, x^r)$ are valid special cases of the two symbolic expressions just given; here, betweenness parallels the intimate coalescence of unification, as well as the intimate meaning of presence that "being a part of" provides. Further, because between two different regions of a given aspect a third can always be interjected, betweenness satisfies a density property similar to the one that rational numbers satisfy in their usual ordering. For this reason it is impossible to produce finite models of continua. Finally, betweenness — like distinguishabity — is a one-directional relation; i.e., $y^r$ may lie between $x^r$ and $z^r$ but not between $z^r$ and $x^r$.

## 8. Places versus points

Our ordinary conception of the space-time continuum, even the relativistic one with its overlapping points of view, is a misleading model of the real continuum: it still relies on points as ultimates and converts separation into an absolute. Actually, there are no perceived points, only aspects and regions, and these can overlap just as the different perspectives provided by each system of reference overlap, which is impossible for points. To say that there are no points but only aspects and regions means that the continuum has no ultimate components, neither points nor elements of any kind — in particular, no atomic regions. The terms for all predicate formulas, then, must be places, regions, or aspects but never points, places that are provisional units of discourse in the same way that any partially disengaged object of our perception is a unit of discourse. Given that there is not the slightest evidence to support the existence of a bottom or a top in our cosmos, we shall operate on the more adequate assumption that there is no truly indivisible atom, just as there is no complete universe, and that, further, it is in the nature of things to be irremediably caught in the middle; i.e. *there is only the middle* — an ontological as well as a perceptual fact, in accordance with the principle that places are concrete while points are incurably unreal.

Aspects differ from regions in that they themselves can be temporarily taken as universes; region cannot. Also, aspects are in the middle of each of their regions as well as between any two of them, that is, aspects are a kind of floating medium for their regions; yet a region is not necessarily in the middle of any two other regions. Aspects, like phases, are snapshots of a transcendent reality, the temporary visitation of a far-reaching entity, i.e., a higher-level continuum whose origin is perhaps remotely located — just as light visits us daily from afar. This unfolding of aspects within an aspect takes place at all levels in the never-ending upward and downward ramifications of continua, each aspect always relating to higher and lower aspects. This parallels the macroscopic amplification of microscopic structures and processes that Pascual Jordan considered the essence of life. It also parallels the unlimited self-similarity of fractals without regard for the up or down scale in which they are considered, that is, the fractal's entire geometric pattern reappears without change whatever the size and scope of observation; Benoit Mandelbrot found this characteristic dominant in most natural forms and processes. Regions, since they are patches of aspects, do not have this ability to move from one level to another.

## 9. Relations of the primitive predicates

Since unity, being a part of, and betweenness are independently introduced predicates, it is desirable to differentiate between the following cases. (i) Each aspect coalesces with every one of its regions and is at the middle of any of them. (ii) Many regions intersect because they have another region as a common part without the latter coalescing with any of the former. (iii) Regions are always mediated by other regions but none of these regions need necessarily coalesce. (iv) An aspect is part of many of its regions without their necessarily coalescing. Whenever a region is a part of an aspect and vice versa, we say that the whole is part of the part. To see a family trait in a face is an example of an aspect being part of a region, as is the case when parsley gives flavor to a taste — parsley giving something of itself but not all of itself, staying on the outside yet inside. The obstacle to perceiving these standard facts is that our thinking is overwhelmingly controlled by atomism, a prejudice that is exceedingly difficult to escape because the moment we put down symbols on paper we are already automatically committing ourselves to simple location; and yet, we constantly perceive regions that are "larger" than the aspects of which they are a region.

## 10. The incompleteness of any analysis of the continuum

We must differentiate between the following: (i) distinguishability, i.e., given two regions, a third one is unified with the first but not with the second, a nonsymmetric fact; (ii) nonintersection, i.e., two regions have no third region as a common part; (iii) disunity, i.e., two regions do not coalesce; plus (iv) combinations of all of these. It is possible for a region to be part of another without either one being distinguishable from the other. Further, unity is a form of connection even for nonintersecting regions, regions seemingly disconnected at first sight; indeed, it is common for regions to be united despite a wide breadth of in-between — painters demonstrate this when they distribute color on a canvass. Finally, disunited regions have parts in common.

Perceptively we may concentrate on an aspect and exclude its source, but this should not make us forget that any continuum can become a stage at a given level, a phase. Parts II and III will present a horizontal analysis of such stages, an analysis that can be repeated at all levels, keeping in mind that these levels have no linear ordering and no beginning and no end. Because of the infinite ramification of levels up and down at whatever aspect our mind starts, no description of the continuum can ever be complete nor be completed without violating its true nature.

The intermediate universal continuum $C$, then, not only is not a universal class but also is not a product graph or a complete universal-algebraic object. Rather, it is a universal-algebraic fragment taken at the middle of things, as it must be; as such, it contains all the internal transformations relative to $C$ — the tearings and joinings between its aspects, for example. And, again, $C$ can only be a fragment because it is itself an aspect of many higher-level continua, just as each of $C$'s aspects can in turn be taken as a universe: this is the inevitable principle of the relativity of all continua. When attempting to comprehend this fragmentation, this purely relative articulation of any intermediate universal continuum, the built-in atomism of set theory is a serious obstacle; in fact, it destroys our ability to conceive continuity correctly.

## 11. Topology of continua without set

Ordinary topology is so dependent on one-directional part-whole relationships, and it so weakens the role of the relations of unity and betweenness in the structure of extension, that is not capable of dealing with the phenomenology of space. Coalescence, two-way part-whole relationships, and mediation of regions and aspects are the essential features of continua. We need a topology based on direct perception, not on abstraction, that is, not on our tendency to collect isolated items. We must always bear in mind that wholes do not necessarily gather parts, and that they often mingle with their parts.

In ordinary topology, closed simple boundaries enclose a region. Concrete boundaries, however, have direction, and may be impenetrable in one direction but open in another. Even Gestalt psychology commits the mistake of holding that a "good Gestalt" is necessarily a form that implies a closed boundary. But an "unfinished" driving made with an open line should not be judged as demanding that we imagine the completion of a closed contour for our theoretical satisfaction. We can see faces in a cloud surrounded by unfinished, soft, penetrable contours, and these diffuse properties are essential to the face we perceive, inseparable from the perception's quality; in other words, the contours are not only open but must be so described theoretically.

The existence of soft, one-directional boundaries implies that not every simple closed boundary divides an inside from an outside. Actually, some boundaries are their own inside. Only our obsession with univocity makes this difficult to accept, even if it is a usual characteristic of perception. We believe that we draw a line made of points even if we know that we can neither draw nor perceive a point. And yet, that boundaries can be their own interior should not be surprising when we know that Peano curves fill a plane, that Lebesgue curves have positive area, and that there are surfaces with positive volume.

In this "Topology without points" Karl Menger outlines a system in which the ultimate objects are "lumps" rather then points. He ends the paper with the following recommendation: "By a lump, we mean something with a well defined boundary. But well defined boundaries are themselves results of limiting processes rather than objects of direct observation. Thus, instead of lumps, we might use at the start something still more vague — something perhaps which has various degrees of density or at least admits a gradual transition to its complement. Such a theory might be of use for wave mechanics" ([2], p. 107). But what is a curve in a topology of lumps? In Part III we shall offer an answer to this question, "lumps" for us meaning either regions or aspects, regions $x^r, y^r, \ldots$ and aspects $C^r, C^s, \ldots$ being local continua, and the intermediate universal continua $C', C'', \ldots$ being global ones — a purely relative distinction.

If we can manage not to succumb to prevalent formal-ontologic prejudices, it is easy to see that a boundary does not provide a razor-sharp division, that it often overlaps both its inside and outside — if it has them — and is frequently a kind of no-man's-land between, and encroaching upon, the regions immediately beside it.

## 12. In synthesis

To recapitulate briefly, a multiplicity of regions and aspects — one constituting a boundary, say — cannot be gathered in a set-theoretic manner. We distinguish continua partially before we can divide them, and even then new regions keep appearing between regions. Hence, to distinguish is to add; to divide is to multiply the number of viewpoints, to create further possible interpenetrations and coalescences: each act of distinction reveals an in-between. To serve is too final to be realistically concrete, for there is often a gradual transition from a region to its complements. Distinguishability of entities, again, implies their giving something of themselves but not all.

There is a never ending descent and a never ending ascent in the analysis of the continuum. There is no empty continuum, and there is no absolutely universal continuum. Further, the distinction between local and global transformations is blurred. Every local change has global implications, and every global change affects local relations.

Unity, being a part of, and betweenness are primitive and irreducible relations, each contributing differently to the constitution of continua. It is a mistake to think that this proliferation of predicates makes the continuum a less exact structure. Although Whitehead said that "exactness is an ideal of thought and is only realized in experience by the selection of a route of approximation", it is misleading to identify exactness with clear-cut, razor-sharp definition. There is a stage in the description of reality beyond which sharpness becomes synonymous with inexactness. In a realm of diffuse entities, exactness demands diffuseness. This is a routine paradox of scientific thinking that we must simply accept. Exactness in the sense of full adequacy often requires moving away from our most cherished, most habitsoothing notions, notions which — like whose of set and membership — only function as barriers after a point, preventing us from grasping the concrete structure of space as it really is.

# II. A FORMAL LANGUAGE

## 13. The primitive symbols

A three-sorted classical predicate calculus is assumed. The three sorts of individual variables are

I. $C, C', C'', \ldots$ to range over intermediate universal continua: i.e., aspects taken provisionally as universes (multiverses would be perhaps more appropriate, since $C$ is not a closed whole of all its aspects).

II. $C^r, C^s, C^t, \ldots$ to represent aspects of an intermediate universal continuum $C$; i.e., partial continua of universal continuum (these aspects are not to be taken as elements of "finished" products of any kind, for each of them may itself be taken as a universe). It is of course possible for the variable superindex $r$ to be identical to $s$, i.e., for $C^r$ to be $C^s$.

III. $x^r, y^r, z^r, \ldots$, regions of an aspects $C^r$, also occasionally denoted $\ldots, x^r_{-1}, x^r_0, x^r_1, x^r_2, \ldots$ for convenience.

We take an aspect — any aspect at the midle of things — and consider it provisionally as a universe $C$. Then we find that $C$, in turn, displays aspects $C^r, C^s, \ldots$, and that each of these aspects then displays regions $x^r, y^r, \ldots, x^s, y^s, \ldots$. This constitutes the horizontal analysis of $C$'s structure. But we can take any universe $C$ and observe that it is merely as aspect of a more sweeping universe $C'$, actually, of many universes $C', C'', C''', \ldots$; that is, $C$ can be taken as a $C'^r$, a $C''^s$, a $C'''^t$, etc.; this is analysis in the upward direction. Or still, we can take an aspect $C^r$ of a given universe $C$, and consider it, in turn, as a universe $C'$ in itself, with its own aspects $C'^r, C'^s, \ldots$; this is downward analysis. The upward and downward directions of analysis can be pursued limitlessly along all their ramified paths; i.e., we are unavoidably "in the middle of things". There is no way to reach a bottom or a ceiling because neither exists; there is no empty continuum, no absolutely universal continuum. We cannot collect all aspects but can only analyze their structure in a fragmentary way, that is, at a given horizontal level or, in a limited way, at some of their neighboring levels. There is, then, no essential difference between $C$ and $C^r$ insofar as their structure is concerned; $C$ is merely the irreducible multiplicity of all its aspects. Whereas Cantor sets collect elements, universal continua spread aspects. The $C^r, C^s, \ldots$ are the internal composition of $C$. The intermediate universal continua $C', C'', \ldots$, of which $C$ is an aspect, establish the external relations of $C$.

In addition we have:

IV. Three undefined binary predicates: $\mathcal{U}$ ("being united with"), $\mathcal{P}$ ("being a part of") and $\mathcal{N}$ ("being a neighborhood of").

V. A ternary predicate: $\mathcal{B}$ ("betweenness").

Intermediate universal continua, aspects and regions are all continua; they differ in some of their formal properties as specified by the axioms that follow. Intuitively, these various continua emerge in the order of their successive discernibility. As we concentrate on an aspect $C$, arbitrarily chosen, and take it as a world in itself, we should not have the illusion that it is anything more than a patch in the middle of vaster spreads, just as $C$'s aspects and regions are patches in $C$'s midst.

## 14. Four defined predicates

**1. Equality $=$.**

**Definition 14.1.** $x^r = y^r$ *iff* $\forall z^r(\mathcal{U}(x^r, z^r) \Leftrightarrow \mathcal{U}(y^r, z^r)) \wedge \forall z^r(\mathcal{P}(z^r, x^r) \Leftrightarrow \mathcal{P}(z^r, y^r)) \wedge \forall z^r(\mathcal{P}(x^r, z^r) \Leftrightarrow \mathcal{P}(y^r, z^r))$.

**Definition 14.2.** $x^r = y^s$ *iff* $\forall u^r \forall v^s((\mathcal{U}(x^r, u^r) \Leftrightarrow \mathcal{U}(y^s, u^r)) \wedge (\mathcal{U}(x^r, v^s) \Leftrightarrow \mathcal{U}(y^s, v^s))) \wedge \forall u^r((\mathcal{P}(u^r, x^r) \Leftrightarrow \mathcal{P}(u^r, y^s)) \wedge (\mathcal{P}(x^r, u^r) \Leftrightarrow \mathcal{P}(y^s, u^r))) \wedge \forall v^s((\mathcal{P}(v^s, x^r) \Leftrightarrow \mathcal{P}(v^s, y^s)) \wedge (\mathcal{P}(x^r, v^s) \Leftrightarrow \mathcal{P}(y^s, v^s)))$.

Since $x^r$, $C^r$ and $C$ are different sorts of variables, $x^r = C^r$, $x^r = C$ and $C^r = C$ are never the case. However, regions of one aspect can be part of other aspects. Note further that $x^r \neq y^r$ implies $\exists z^r\ ((\mathcal{U}(x^r, z^r) \wedge \neg\mathcal{U}(x^r, z^r)) \vee (\neg\mathcal{U}(x^r, z^r) \wedge \mathcal{U}(y^r, z^r))) \vee \exists z^r((\mathcal{P}(z^r, x^r) \wedge \neg\mathcal{P}(z^r, y^r)) \vee (\neg\mathcal{P}(z^r, x^r) \vee \mathcal{P}(z^r, y^r))) \vee \exists z^r((\mathcal{P}(x^r, z^r) \wedge \neg\mathcal{P}(y^r, z^r)) \vee (\neg\mathcal{P}(x^r, z^r) \wedge \mathcal{P}(y^r, z^r)))$.

**2. Two complements, $\mathcal{U}$-comp and $\mathcal{P}$-comp.**

**Definition 14.3.** $\mathcal{U}$-comp$(x^r, y^r)$ *iff* $\forall z^r\ (\mathcal{U}(z^r, y^r) \Leftrightarrow \neg\,\mathcal{U}(z^r, x^r))$.

**Definition 14.4.** $\mathcal{P}$-comp$(x^r, y^r)$ *iff* $\forall z^r\ (\mathcal{P}(z^r, y^r) \Leftrightarrow \neg\,\mathcal{P}(z^r, x^r)) \wedge \forall z^r\ (\mathcal{P}(y^r, z^r) \Leftrightarrow \neg\,\mathcal{P}(x^r, z^r))$.

Both complements are relative to a given aspect $C^r$, but neither gathers or fills regions.

**3. Distinguishability $\mathcal{D}$**

**Definition 14.5.** $\mathcal{D}(x^r, y^s)$ *iff* $\exists z^t(\mathcal{U}(z^t, x^r) \wedge \neg\,\mathcal{U}(z^t, y^s))$. *The definition extends to aspects as follows:* $\mathcal{D}(C^r, C^s)$ *iff* $\exists z^t(\mathcal{U}(z^t, C^r) \wedge \neg\,\mathcal{U}(z^t, C^s))$.

$\mathcal{D}$ is not symmetric.

## 15. Axioms for "being united with"

$\mathcal{U}(x^r, y^r)$ reads "$x^r$ is united with $y^r$".

(A15.1) $$\forall x^r\ \mathcal{U}(x^r, x^r)$$

(A15.2) $$\forall x^r \forall y^s\ (\mathcal{U}(x^r, y^s) \Rightarrow \mathcal{U}(y^s, x^r))$$

True in particular if $r = s$. "Being united" is not necessarily transitive.

(A15.3) $$\forall x^r\ \mathcal{U}(x^r, C^r)$$

(A15.4) $$\forall C^r\ \mathcal{U}(C^r, C^r)$$

Every aspect is united with all its regions and the intermediate universal continuum is united with all its aspects. This is the continuum's first basic structural principle.

(A15.5) $$\forall x^r \exists y^r\ \neg\,\mathcal{U}(x^r, y^r)$$

Unity of regions is not universal.

Axiom scheme: For each $i = 2, 3, \ldots$ axiom A15.6.i means

(A15.6.i) $$\begin{aligned}\forall x_1^r \forall x_2^r \ldots \forall x_i^r (x_1^r \neq x_2^r \wedge x_1^r \neq x_3^r \wedge \ldots \wedge x_{i-1}^r \neq x_i^r \Rightarrow\\ \exists x_{i+1}^r (x_1^r \neq x_{i+1}^r \wedge \ldots \wedge x_i^r \neq x_{i+1}^r\\ \wedge\,\mathcal{U}(x_1^r, x_{i+1}^r) \wedge \ldots \wedge \mathcal{U}(x_1^r, x_{i+1}^r)))\end{aligned}$$

This axiom implies the existence of an infinity of regions provided that two distinct regions exist.

Note that two aspects $C^r$ and $C^s$ may not be united, and that the intermediate universal continuum $C$ and a region $x^r$ may not be united either. Further, unity is independent of simple location; regions which may not be considered contiguous or even near to one another in a superficial sense may form a union.

## 16. Axioms for "being a part of"

$\mathcal{P}(x^r, y^r)$ reads "$x^r$ is a part of $y^r$" or, alternatively, "$y^r$ is a whole of $x^r$".

(A16.1) $$\forall x^r\, \mathcal{P}(x^r, x^r)$$

Every region is a part of itself.

(A16.2) $$\forall x^r \forall y^r \forall z^r (\mathcal{P}(x^r, y^r) \wedge \mathcal{P}(y^r, z^r) \Rightarrow \mathcal{P}(x^r, z^r))$$

Transitivity of $\mathcal{P}$; $\mathcal{P}$ is neither necessarily symmetric, nor necessarily asymmetric or antisymmetric, i. e., it is possible to have $x^r \neq y^r$ and $\mathcal{P}(x^r, y^r) \wedge \mathcal{P}(y^r, x^r)$. Being a part of is a form of presence, not an inclusion.

(A16.3) $$\forall x^r \exists y^r (x^r \neq y^r \wedge \mathcal{P}(y^r, x^r))$$

(A16.4) $$\forall x^r \exists y^r (x^r \neq y^r \wedge \mathcal{P}(x^r, y^r))$$

The infinite descent and infinite ascent implied by these two axioms means that each region is the whole of other parts and a part of other wholes. No region is an atom or a complete whole. This property of being unlimitedly analyzable and synthesizable is the continuum's second basic structural principle. The continuum has no points, only places that have places, and it cannot be boxed into a set.

(A16.5) $$\forall x^r (\mathcal{P}(x^r, C^r) \vee \mathcal{P}(C^r, x^r))$$

This is nonexclusive disjunction. Further, $\mathcal{P}(C^r, C^s)$ is possible, aspects can be part of other aspects; also, $\mathcal{P}(C^r, C^r)$, $\mathcal{P}(C^r, x^s)$, $\mathcal{P}(x^s, C^r)$, $\mathcal{P}(x^r, y^s)$,

$\mathcal{P}(C, x^r)$, $\mathcal{P}(x^r, C)$, $\mathcal{P}(C^r, C)$ and $\mathcal{P}(C, C^r)$ are all possible (examples are given in Part IV).

(A16.6) $$\forall x^r \forall y^r \exists z^r (x^r \neq z^r \wedge y^r \neq z^r \wedge \mathcal{P}(x^r, z^r) \wedge \mathcal{P}(y^r, z^r))$$

## 17. Axioms for betweenness

(A17.1) $$\forall x^r \, \forall y^r \, \neg \mathcal{B}(x^r, y^r, x^r)$$

This is a form of irreflexivity: no region stands between any region $x^r$ and $x^r$.

(A17.2) $$\forall x^r \forall y^r (\neg \mathcal{B}(x^r, x^r, y^r) \wedge \neg \mathcal{B}(x^r, y^r, y^r))$$

A third region is necessary for linkage between regions.

(A17.3) $$\mathcal{B}(x^r, y^s, z^t) \wedge \mathcal{B}(z^t, u^m, v^m) \wedge y^s \neq u^m \Rightarrow \mathcal{B}(y^s, z^t, u^m)$$

The only transitivity of betweenness, which holds in particular if $r = s = t = m = n$.

(A17.4) $$x^r = u^s \wedge y^r = v^s \wedge z^r = w^s \wedge \mathcal{B}(x^r, y^r, z^r) \Rightarrow \mathcal{B}(u^s, v^s, w^s)$$

Substitutivity of betweenness — invariance with respect to aspects.

(A17.5) $$\forall x^r \forall y^r \; (x^r \neq y^r \Rightarrow \exists z^r \; \mathcal{B}(x^r, z^r, y^r))$$

Density of betweenness in the regions of a given aspect, even if $\mathcal{P}(x^r, y^r)$, or $\mathcal{U}(x^r, y^r)$, or both. This is the continuum's third basic structural principle.

(A17.6) $$\forall x^r \forall y^r \; \mathcal{B}(x^r, C^r, y^r)$$

Each aspect is at the middle of all its regions, including $\mathcal{B}(x^r, C^r, x^r)$.

(A17.7) $$\forall C^r \forall C^s \; \mathcal{B}(C^r, C, C^s)$$

$C$ is not a container but rather an intermediate between all its aspects, as well as the center of each of them, that is $\mathcal{B}(C^r, C, C^r)$.

(A17.8) $$\forall x^r \forall y^s \; \mathcal{B}(x^r, C, y^s)$$

$C$ is the intermediate of all its regions; in particular, $\mathcal{B}(x^r, C, x^r)$.

That each aspect $C^r$ is at the centre of all its regions and $C$ is at the centre of all its aspects and all its regions is a form of multiple location which betweenness provides. Note also that $\mathcal{B}(x^r, y^r, z^r)$ is compatible with $\neg\mathcal{B}(z^r, y^r, x^r)$; betweenness is not symmetric, and has to be thought of as having direction. Further, $\mathcal{B}(C^r, C^s, C^t)$ is possible, and so are $\mathcal{B}(C^r, x^s, C^t)$, $\mathcal{B}(C^r, x^s, x^t)$, $\mathcal{B}(x^r, x^s, C^t)$, $\mathcal{B}(x^r, C^s, x^t)$ and $\mathcal{B}(x^r, x^s, x^t)$. If $\mathcal{B}(x^r, C^s, x^t)$, then $C^s$ contributes to the external continuity of $x^r$ and $x^t$. Aspects, therefore, are not necessarily fully separated: they can link and be linked by regions and other aspects, and are always linked by the intermediate universal continuum.

## 18. Axioms for complementation

(A18.1) $$\forall x^r \exists y^r\, \mathcal{U}\text{-comp}(x^r, y^r)$$

(A18.2) $$\forall x^r \exists y^r\, \mathcal{P}\text{-comp}(x^r, y^r)$$

$\mathcal{P}$-comp$(x^r, y^r)$ is compatible with $\mathcal{U}(x^r, y^r)$ and $\mathcal{U}$-comp$(x^r, y^r)$ is compatible with $\mathcal{P}(z^r, x^r) \wedge \mathcal{P}(z^r, y^r)$, $\mathcal{P}(x^r, z^r) \wedge \mathcal{P}(y^r, z^r)$, $\mathcal{P}(x^r, C^r) \wedge \mathcal{P}(y^r, C^r)$ and $\mathcal{P}(C^r, x^r) \wedge \mathcal{P}(C^r, y^r)$; i.e., there can be a gradual transition from a region to each of its complements, a transition that Karl Menger mentioned as desirable. Actually, more than one region may be a part of $x^r$ and its $\mathcal{U}$-complement; to complement is to fill up even if it involves spilling over.

## 19. Axioms for distinguishability

$\mathcal{D}$ is clearly irreflexive, nonsymmetric and nontransitive.

(A19.1) $$\forall x^r\, \mathcal{D}(C^r, x^r)$$

An aspect is distinguishable from any of its regions even though regions are indistinguishable from their aspect. Regions may also be distinguishable from the intermediate universal continuum.

(A19.2) $$\forall x^r \exists y^r\, \mathcal{D}(x^r, y^r)$$

If $\mathcal{D}(x^r, y^r)$ and $\mathcal{P}(x^r, y^r)$ then we say that $x^r$ *is distinguishable in* $y^r$. Distinguishability is not a clear-cut separation, for $\mathcal{D}(x^r, y^r)$ is compatible with $\mathcal{D}(y^r, z^r)$ and $\neg\mathcal{D}(x^r, z^r)$.

(A19.3) $$\mathcal{D}(C^r, C)$$

For each aspect $C^r$ there exists at least a region $x^s$ united with $C^r$ but not with $C$.

(A19.4) $$\mathcal{D}(C, C^r)$$

No aspect $C^r$ is united with all the regions of all the other aspects of $C$.

(A19.5) $$\forall C^r \forall C^s\, \mathcal{D}(C^r, C^s)$$

Regions behave as all aspects do whenever $\mathcal{D}(x^r, x^s)$. Further, $\mathcal{B}(x^r, y^s, z^t)$ is consistent with $\neg\mathcal{U}(x^r, y^s)$, $\neg\mathcal{U}(y^s, z^t)$ and $\neg\mathcal{U}(x^r, z^t)$, and therefore consistent with $\mathcal{D}(x^r, y^s)$, $\mathcal{D}(y^s, z^t)$, $\mathcal{D}(x^r, z^t)$ or their negations. In other words, betweenness is independent of unity.

## 20. Relational axioms

(A20.1) $$\forall x^r \exists y^r \forall z^r (\mathcal{U}(x^r, z^r) \Rightarrow \mathcal{P}(z^r, y^r) \vee \mathcal{P}(y^r, z^r))$$

Mingling of all regions united to a given region.

(A20.2) $$\forall x^r \exists y^r \forall z^r (\mathcal{P}(z^r, x^r) \vee \mathcal{P}(x^r, z^r) \Rightarrow \mathcal{U}(z^r, y^r))$$

$\mathcal{U}$-gathering of all the parts and wholes of a given region.

(A20.3) $$\mathcal{P}(x^r, y^r) \wedge \mathcal{P}(x^r, z^r) \wedge x^r \neq y^r \wedge x^r \neq z^r \wedge y^r \neq z^r \Rightarrow \mathcal{B}(y^r, x^r, z^r)$$

A part of mediates between the wholes it is a part of.

(A20.4) $$\mathcal{P}(x^r, y^r) \wedge \mathcal{P}(z^r, y^r) \wedge x^r \neq y^r \wedge z^r \neq y^r \wedge x^r \neq z^r \Rightarrow \mathcal{B}(x^r, y^r, z^r)$$

A whole mediates between any two of its parts.

Note that regions — $x^r$ and $y^r$, and $y^r$ and $z^r$, say — can be continuous in the sense of $\mathcal{B}(x^r, y^r, z^r)$ without $x^r$ and $y^r$ on the one hand, and $y^r$ and $z^r$ on the other, having any part in common.

(A20.5) $$\begin{aligned}&\forall x^r \forall y^r \exists z^r \forall w^r (\mathcal{P}(w^r, z^r) \vee \mathcal{P}(z^r, w^r) \Leftrightarrow \\ &\qquad (\mathcal{P}(w^r, x^r) \vee \mathcal{P}(x^r, w^r)) \wedge \mathcal{U}(w^r, y^r))\end{aligned}$$

Axiom of $\mathcal{U}$-comprehension: $z^r$ is the mingling of all parts and all wholes of $x^r$ united to $y^r$.

The following additional definitions are now in order to differentiate between the various kinds of separation obtainable.

**Definition 20.1.** *Two regions $x^r$ and $y^r$ are called cohesive iff* $\mathcal{U}(x^r, y^r) \wedge (\mathcal{P}(x^r, y^r) \vee \mathcal{P}(y^r, x^r))$.

**Definition 20.2.** *$x^r$ and $y^r$ are called disjoint iff* $\mathcal{U}(x^r, y^r) \wedge \neg(\mathcal{P}(x^r, y^r) \vee \mathcal{P}(y^r, x^r))$.

**Definition 20.3.** *$x^r$ and $y^r$ are called detached iff* $\neg\mathcal{U}(x^r, y^r) \wedge (\mathcal{P}(x^r, y^r) \vee \mathcal{P}(y^r, x^r))$.

**Definition 20.4.** *$x^r$ and $y^r$ are called severed iff* $\neg\mathcal{U}(x^r, y^r) \wedge \neg(\mathcal{P}(x^r, y^r) \vee \mathcal{P}(y^r, x^r))$.

Note that even severance is not complete because it is compatible with the existence of a common part $z^r$, i.e., $\mathcal{P}(z^r, x^r) \wedge \mathcal{P}(z^r, y^r)$.

**Definition 20.5.** *$x^r$ and $y^r$ intersect — in symbols, $\mathcal{I}(x^r, y^r)$ — iff* $\exists z^r\ (\mathcal{P}(z^r, x^r) \wedge \mathcal{P}(z^r, y^r))$.

**Definition 20.6.** *$x^r$ and $y^r$ form a gap iff* $\mathcal{U}(x^r, y^r) \wedge \neg\mathcal{I}(x^r, y^r)$.

**Definition 20.7.** *$y^r$ is wound between $x^r$ and $z^r$ iff* $\mathcal{B}(x^r, y^r, z^r) \wedge \neg(\mathcal{U}(x^r, y^r) \vee \mathcal{U}(y^r, z^r))$.

**Definition 20.8.** *$y^r$ is an incision between $x^r$ and $z^r$ iff* $\mathcal{B}(x^r, y^r, z^r) \wedge \neg(\mathcal{P}(y^r, x^r) \vee \mathcal{P}(x^r, y^r) \vee \mathcal{P}(y^r, z^r) \vee \mathcal{P}(z^r, y^r))$.

**Definition 20.9.** *$y^r$ is an internal immediate constituent of $x^r$ iff* $\exists z^r\ (\mathcal{B}(x^r, y^r, z^r) \wedge \mathcal{I}(x^r, y^r))$ *(or alternatively of $z^r$ iff* $\exists x^r\, \mathcal{B}(x^r, y^r, z^r) \wedge \mathcal{I}(y^r, z^r))$*)*.

**Definition 20.10.** *$y^s$ is an external immediate constituent of $x^r$ iff* $r \neq s \wedge \exists z^t (\mathcal{B}(x^r, y^s, z^t) \wedge \mathcal{I}(x^r, y^s))$ *(or alternatively of $z^t$ iff* $s \neq t \wedge \exists x^r (\mathcal{B}(x^r, y^s, z^t) \wedge \mathcal{I}(y^s, z^t))$*)*.

**Definition 20.11.** *$z^r$ has multiple $\mathcal{U}$-location in $x^r$ and $y^r$ iff* $\neg\mathcal{I}(x^r, y^r) \wedge \mathcal{U}(z^r, x^r) \wedge \mathcal{U}(z^r, y^r)$.

**Definition 20.12.** *$z^r$ has multiple $\mathcal{P}$-location in $x^r$ and $y^r$ iff* $\neg\mathcal{U}(x^r, y^r) \wedge \mathcal{P}(z^r, x^r) \wedge \mathcal{P}(z^r, y^r)$.

Multiple location has two meanings: (i) that of a region being united with nonintersecting regions, and (ii) that of a region being part of disunited regions.

# III. TOPOLOGY WITHOUT SETS

## 21. Neighborhoods

Associated with each region $x^r$ there are other regions $y^r$ called $x^r$-neighborhoods; the latter are not necessarily wholes or parts of $x^r$. This relation is denoted $\mathcal{N}(x^r, y^r)$ (reads "$y^r$ is a neighborhoods of $x^r$").

**Definition 21.1.** *An aspect $C^r$ is called a topological continuum iff the following six axioms are satisfied:*

$$\forall x^r\, \mathcal{N}(x^r, x^r) \tag{A21.1}$$

*Every region is a neighborhood of itself.*

$$\forall x^r \exists y^r (x^r \neq y^r \wedge \mathcal{N}(x^r, y^r)) \tag{A21.2}$$

*Each region has at least one neighborhood other then itself.*

$$\forall x^r \forall y^r (\mathcal{N}(x^r, y^r) \Rightarrow \mathcal{U}(x^r, y^r)) \tag{A21.3}$$

*Each region is united to all its neighborhoods.*

$$\begin{aligned}&\mathcal{N}(x^r, y^r) \wedge \mathcal{N}(x^r, z^r) \Rightarrow \exists w^r (w^r \neq y^r \wedge w^r \neq z^r \wedge \\ &\mathcal{N}(x^r, w^r) \wedge (\mathcal{P}(w^r, y^r) \vee \mathcal{P}(y^r, w^r)) \wedge (\mathcal{P}(w^r, z^r) \vee \mathcal{P}(z^r, w^r)))\end{aligned} \tag{A21.4}$$

*Given two neighborhoods of the same region, there is always a third that mingles with other two.*

$$x^r \neq y^r \Rightarrow \exists z^r (\mathcal{N}(x^r, z^r) \wedge \neg \mathcal{N}(y^r, z^r)) \tag{A21.5}$$

*The first separation property.*

$$\begin{aligned}\mathcal{N}(x^r, y^r) \Rightarrow &(\mathcal{P}(z^r, y^r) \Rightarrow \exists w^r (\mathcal{N}(z^r, w^r) \wedge \mathcal{N}(w^r, y^r))) \\ &\wedge (\mathcal{P}(y^r, z^r) \Rightarrow \exists w^r (\mathcal{N}(w^r, z^r) \wedge \mathcal{N}(y^r, w^r))).\end{aligned} \tag{A21.6}$$

**Definition 21.2.** *A region $x^r$ is a topological continuum iff the six preceding axioms obtain relativized to all the parts and wholes of $x^r$.*

Note that, in contrast to ordinary set-theoretic topology, there is no empty neighborhood (axiom A 21.4 involves a mingling, not an intersection of neighborhoods). Note also that given that an aspect can be part of some of its neighborhoods (as can a topological region $x^r$ to which we relativize Definition 21.1), we cannot talk about $C^r$ (or respectively $x^r$) as being the "entire" topological space or the "entire" topological continuum. Even topologically speaking, topological continua are "at the middle of things". Further, note once more that $\mathcal{N}(x^r, y^r)$ is compatible with $\neg\mathcal{P}(x^r, y^r) \wedge \neg\mathcal{P}(y^r, x^r)$ — a neighborhood is not necessarily contiguous in the part-whole sense to the region for which it is a neighborhood.

## 22. Separation properties

In addition to axiom A 21.5, some topological continua may have one or both of the following separation properties.

$$\text{(A22.1)} \qquad \neg\, \mathcal{U}(x^r, y^r) \Rightarrow \exists z^r \exists w^r (z^r \neq w^r \wedge \mathcal{N}(x^r, z^r) \wedge \mathcal{N}(y^r, w^r) \wedge \neg \mathcal{I}(z^r, w^r)).$$

$$\text{(A22.2)} \qquad \mathcal{N}(x^r, y^r) \wedge \mathcal{P}(z^r, y^r) \wedge z^r \neq x^r \Rightarrow \exists w^r (\mathcal{N}(x^r, w^r) \wedge \neg(\mathcal{P}(z^r, w^r) \vee \mathcal{P}(w^r, z^r))).$$

Note that in keeping with the general motivation, the separation properties given by axioms A 21.5, A 22.1 and A 22.2 do not imply absolute division: two united regions may be separated by distinct neighborhoods just as two disunited regions may by separated by distinct neighborhoods, and yet in both cases one region may be part of the other. There is, then, no assurance of complete divisibility in the sense of a Dedekind cut — which actually cuts nothing that was not already cut to begin with.

## 23. Cluster regions: interior and exterior

**Definition 23.1.** *A region $y^r$ is a cluster region of $x^r$ — in symbols $\mathcal{C}(x^r, y^r)$ — iff $\forall z^r (\mathcal{N}(y^r, z^r) \Rightarrow (\exists w^r (\mathcal{P}(w^r, z^r) \wedge (\mathcal{P}(w^r, x^r) \vee \mathcal{P}(x^r, w^r)) \wedge \neg\, \mathcal{P}(w^r, y^r)) \vee \exists w^r (\mathcal{P}(z^r, w^r) \wedge (\mathcal{P}(w^r, x^r) \vee \mathcal{P}(x^r, w^r)) \wedge \neg\, \mathcal{P}(y^r, w^r)))$.*

That is, given a neighborhood of a cluster region $y^r$ of $x^r$, either (i) this neighborhood has a part that is either a part or a whole of $x^r$ but is not a part of the cluster region itself, or (ii) it is the part of region $w^r$ that is either a part or a whole of $x^r$ but is not itself a whole of the cluster region $y^r$.

**Definition 23.2.** *A region $x^r$ is open iff every cluster region of $x^r$ is a part of $x^r$.*

**Definition 23.3.** *A region $x^r$ is closed iff every cluster region of $x^r$ is a whole of $x^r$.*

A region can be both open and closed because regions can be both part and whole of another region.

**Definition 23.4.** *A region $x^r$ is interior to $y^r$ iff $\forall z^r\ (\mathcal{N}(x^r, z^r) \Rightarrow \mathcal{P}(z^r, y^r))$.*

**Definition 23.5.** *A region $x^r$ is exterior to $y^r$ iff $\forall z^r(\mathcal{N}(z^r, x^r) \Rightarrow \mathcal{P}(y^r, z^r))$.*

A region can be both interior and exterior to another one.

## 24. Boundaries

Since we have no points, the topology that is being described here is in line with Karl Menger's geometry of lumps. An obvious question ensues: What is a line in a geometry of lumps? The following definition answers this question.

**Definition 24.1.** *A sequence of regions*

$$\ldots, x^r_{-k}, \ldots, x^r_{-1}, x^r_0, x^r_1, \ldots, x^r_k, \ldots$$

*is called a one-sided line iff all the following expressions hold:*

$$\ldots, \mathcal{B}(x^r_{-k}, x^r_{-k+1}, x^r_{-k+2}), \ldots, \mathcal{B}(x^r_0, x^r_1, x^r_2), \ldots, \mathcal{B}(x^r_{k-1}, x^r_k, x^r_{k+1}), \ldots.$$

Lines have direction and can of course have two directions (two-sided lines).

**Definition 24.2.** *A sequence of regions $x^r_1, x^r_2, \ldots, x^r_n, x^r_1$ is called a one-sided closed line iff the following expression holds:*

$$\mathcal{B}(x^r_1, x^r_2, x^r_3) \wedge \ldots \wedge \mathcal{B}(x^r_{n-1}, x^r_n, x^r_1) \wedge \mathcal{B}(x^r_n, x^r_1, x^r_2).$$

**Definition 24.3.** *The sequence* $x_1^r, x_2^r, \ldots, x_n^r, x_1^r$ *is called a two-sided closed line iff it is a one-sided closed line and* $x_1^r, x_n^r, x_{n-1}^r, \ldots, x_1^r$ *is also a one-sided closed line.*

Note that a finite number of regions suffices to determine a closed line, although, given the density of betweenness, between any two regions in a line an infinity of regions is always interspersed. One-sided closed lines which are not two-sided are similar to Möbius strips in that they can never encircle an inside; only two-sided lines can (see the following definitions). These one-sided closed lines are "soft" in the sense that they can be travelled in one direction but have wholes in the other direction. In contrast, a two-sided closed line can even be its own inside.

**Definition 24.4.** *The region* $y^r$ *constitutes by itself a one-sided boundary of* $x^r$ *iff* $\forall z^r(\mathcal{B}(x^r, w^r, z^r) \Rightarrow w^r = y^r)$*. If, in addition,* $\forall z^r(\mathcal{B}(z^r, w^r, x^r) \Rightarrow w^r = y^r)$*, then* $y^r$ *is called a boundary of* $x^r$*.*

**Definition 24.5.** *Let* $x_1^r, \ldots, x_n^r, x_1^r$ *be a two-sided closed line. This line is called a boundary of* $z^r$ *iff* $\exists y_1^r\ \mathcal{B}(z_1^r, x_1^r, y_1^r) \wedge \exists y_2^r\ \mathcal{B}(z_2^r, x_2^r, y_2^r) \wedge \ldots \wedge \exists y_n^r\ \mathcal{B}(z^r, x_n^r, y_n^r)$*.*
*The region* $z^r$ *is called the inside of the boundary. If there is no region* $z^r$ *which is the inside of the closed line* $x_1^r, \ldots, x_n^r, x_1^r$*, and* $\exists y^r(\mathcal{B}(x_1^r, x_2^r, y^r) \wedge \mathcal{B}(x_2^r, x_3^r, y^r) \wedge \ldots \wedge \mathcal{B}(x_{n-1}^r, x_n^r, y^r) \wedge \mathcal{B}(x_n^r, x_1^r, y^r))$*, then we say that the closed line is its own inside, even if some or all the regions of the closed line are the inside of other boundaries.*

**Definition 24.6.** *A region* $x^r$ *is called a medium of a one-sided line iff*

$$\ldots, \mathcal{B}(y_{-k}^r, x^r, y_{-k+1}^r, \ldots, \mathcal{B}(y_0^r, x^r, y_1^r), \ldots, \mathcal{B}(y_k^r, x^r, y_{k+1}^r), \ldots$$

*all hold. In this case, the regions of the one-sided line are "immersed" in* $x^r$*.*

**Definition 24.7.** *A boundary* $x^r, \ldots, x_n^r, x_1^r$ *is universal boundary iff it is a boundary of every region* $z^r$ *of* $C^r$*.*

Note that a kind of compactness, a Heine-Borel type of property, obtains in that despite the density of betweenness a boundary is determined by a finite number of regions.

**Jordan's Theorem revisited.** Every boundary of a region $z^r$ articulates the topological continuum $C^r$ into four domains: (i) the inside $z^r$ of the boundary, (ii) the boundary itself, (iii) the $\mathcal{U}$-complement of $z^r$, and (iv) the $\mathcal{P}$-complement of $z^r$, the latter two being the two corresponding outsides of the boundary. If the boundary is its own inside, then it has no outside. The four domains may overlap; further, a region $z^r$ may have more than one boundary, and a boundary more than one inside.

## 25. Degrees of Transition

Let $\mathcal{U}$-comp$(x^r, y^r)$ or $\mathcal{P}$-comp$(x^r, y^r)$. According to the density property of betweenness, $\exists z^r \mathcal{B}(x^r, z^r, y^r)$ in either case; this region $z^r$ provides what Menger calls "a gradual transition" of a region to its complement. Because of the density of betweenness, such transition is always infinitely gradual.

In general, we can now define degrees of transition in terms of neighborhoods as follows.

**Definition 25.1.** *The degree of transaction from a region $x^r$ to a region $y^r$ is finite and equal to $k$ iff a finite number of $x^r$-neighborhoods $z^r_1, \ldots, z^r_k$ satisfy $\mathcal{B}(x^r, z^r_1, z^r_2) \wedge \mathcal{B}(z^r_1, z^r_2, z^r_3) \wedge \ldots \wedge \mathcal{B}(z^r_{k-1}, z^r_k, y^r)$, and $k$ is the least number of neighborhoods with such property.*

The degree of transition from $y^r$ to $x^r$ may be different or nonexistent.

## 26. Tearings and joinings

**Definition 26.1.** $\mathcal{T}(x^r, y^s, z^s)$ *— read "$y^s$, $z^s$ are a tearing of $x^r$" — iff* $\mathcal{U}(x^r, y^s) \wedge \mathcal{U}(x^r, z^s) \wedge \neg\, \mathcal{U}(y^s, z^s) \wedge \neg\mathcal{I}(y^s, z^s)$.

**Definition 26.2.** $\mathcal{J}(x^r, y^r, z^s)$ *— read "$z^s$ is a joining of $x^r$ and $y^r$" — iff* $\neg\,\mathcal{U}(x^r, y^r) \wedge \neg\mathcal{I}(x^r, y^r) \wedge \mathcal{U}(x^r, z^s) \wedge \mathcal{U}(y^r, z^s)$.

Tearings and joinings take place inter and intra aspects — the latter when $r = s$ — and for each region and a pair of regions there are, respectively, at least as many tearings and joinings as aspects, as the following axioms state.

$$\forall x^r \forall C^s \exists y^s \exists z^s\ \mathcal{T}(x^r, y^s, z^s). \tag{A26.1}$$

$$\forall x^r \forall y^r \forall C^s \exists z^s\ \mathcal{J}(x^r, y^r, z^s). \tag{A26.2}$$

## 27. Homeomorphisms

**Definition 27.1.** *A function between two aspects or on an aspect into itself is a relation* $F(x^r, y^s)$ *that satisfies* $F(x^r, y^s) \wedge F(x^r, z^s) \Rightarrow y^s = z^s$.

Note that $F(x^r, y^s)$ is compatible with $F(x^r, y^t)$ provided that $s \neq t$ and $y^s \neq y^t$. A function can have many values but no more than one distinct one for each aspect.

**Definition 27.2.** *A binivocal function between* $C^r$ *and* $C^s$ *is a function* $F(x^r, y^s)$ *that satisfies* $F(x^r, y^s) \wedge F(z^r, y^s) \Rightarrow x^r = z^r$.

**Definition 27.3.** *Let us indicate by* $x^s$, $y^s$,... *the respective images of* $x^r$, $y^r$, ... *under a function* $F(x^r, x^s)$. *Given a biunivocal function* $F(x^r, x^s)$, *this function is a homeomorphism from* $x^r$ *to* $x^s$ *iff* $\forall z^r\, (\mathcal{N}(x^r, z^r) \Leftrightarrow \mathcal{N}(x^s, z^s)) \wedge \forall z^r\, (\mathcal{C}(x^r, z^r) \Leftrightarrow \mathcal{C}(x^s, z^s)) \wedge \forall z^r \forall y^r\, (\mathcal{B}(x^r, y^r, z^r) \Leftrightarrow \mathcal{B}(x^s, y^s, z^s)) \wedge \forall z^r \forall y^r\, (\mathcal{B}(y^r, x^r, z^r) \Leftrightarrow \mathcal{B}(y^s, x^s, z^s)) \wedge \forall z^r \forall y^r\, (\mathcal{B}(y^r, z^r, x^r) = \mathcal{B}(y^s, z^s, x^s))$.

Homeomorphisms preserve the neighborhoods and cluster regions of $x^r$ as well as its relations of betweenness; $x^r$ and $x^s$ are then called homeomorphic regions.

Note that homeomorphic regions are compatible with their tearing and joining; not surprising, since tearings are not absolute separations and joinings are not complete fusions. Tearings and joinings may not alter the topological structure of continua, this structure being determined by the neighborhood systems, the cluster regions, and the relations of betweenness. The topological properties of a topological continuum — the ones preserved by homeomorphisms — are those of boundary and inside, for example, not the complements. Rather than studying those qualitative, intrinsic properties of space invariant under "stretchings and bendings without tearings or joinings", topology in the concrete sense that are giving it deals with properties of general location that are compatible with some tearings and joinings, properties such as being the inside of a boundary, being a boundary, being a neighborhood, being a cluster region, and, most fundamental of all, being in between. Topology in this sense does not necessarily imply the preservation of unions or part-whole relationships.

## 28. Dimension

**Definition 28.1.** *The dimension of a boundary is the number of its regions.*

**Definition 28.2.** *The inner dimension of an aspect $C^r$ at a region $x^r$ is the least dimension (greater than one) of all boundaries of which $x^r$ is an inside. Inner dimension is a local property of aspects, and varies form region to region.*

**Definition 28.3.** *The outer dimension of $C^r$ is the supremum of all its inner dimensions at each of its regions, if such supremum (a natural number) exists.*

**Definition 28.4.** *If all aspects $C^r, C^s, \ldots$ of $C$ have an outer dimension, the supremum of these dimensions, if it exists, is the dimension of $C$.*

**Definition 28.5.** *The dimension of a region $x^r$ is the degree of transition of $x^r$ to its $\mathcal{U}$-complement.*

Except for this last definition, all other dimensions are defined in terms of the $\mathcal{B}$-predicate, and are, therefore, homeomorphic invariants. Note also that inner and outer do not bear the usual connotations of simple location. This is in line with the fact that $\mathcal{U}$-complement of a region $x^r$, for example, may have as parts, parts of $x^r$. The difficulty in comprehending all this originates in the visual fallacy that a boundary absolutely severs a two-dimensional continuum, a fallacy induced, say, by simply drawing a line on paper. This leads us to believe unconsciously that any line, closed or not, stands out as if detached, figure against a clearly and absolutely separated background. For this reason, the graphic structure of writing — which is the way formal logic is usually presented — immediately prejudices our thinking and deforms our perception to make it fit the nature of our visual symbols and the well-ordered theories derived from them. Thus, we are always surprised by the stubborn resistance that reality has to being described in neat linear order. Like it or not, a boundary $x_1^r, \ldots, x_n^r, x_1^r$ is compatible with $\mathcal{B}(x_2^r, x_1^r, x_3^r)$ and $\mathcal{B}(x_4^r, x_1^r, x_3^r)$ — a fact of nature.

# IV. TOWARD A CONTINUOUS GRAMMAR

## 29. Interpretations and models

Although interpretations are syntactic when based on functions that map the objects of one formal theory into those of another, they still have an essentially semantic character because the values of the function — the formal objects of the second theory — give meaning to the formal objects of the first theory. This is the case also when a concrete object of any nature — a road sign, a souvenir trinket — refers to another concrete object, the latter providing the meaning of the former. These and other examples should be sufficient reminders that logic's current set-theoretic approach to interpretations and models is not only relatively recent but of a special kind as well. The set-theoretic approach imposes grave restrictions on the nature and structure of models, and makes semantics too much a servant of set theory, hence far removed from the infinite variety of concrete meanings, meanings always imbued with overlappings, inconclusiveness, gradual fadings, multiple locations, and continuous connotations. The following interpretations and models are continua as specified by the definitions of this section. (Note that the standard logical definition of satisfiability does not necessarily require that the values of the interpreted formal variables be members of a set.)

Let $\boldsymbol{C}$ be a concrete intermediate universal continuum having aspects $\boldsymbol{C}^r$, $\boldsymbol{C}^s$, ..., which in turn have regions $\boldsymbol{x}^r$, $\boldsymbol{y}^s$, ..., all satisfying the axioms of Part II, i.e., $\mathcal{U}$, $\mathcal{P}$ and $\mathcal{B}$ each refers to a specific relation between the entities of $\boldsymbol{C}$. Given the correct number of appropriate entities of $\boldsymbol{C}$ for each predicate, the concrete relations $\mathcal{U}$, $\mathcal{P}$ and $\mathcal{B}$ always either hold or do not hold for such entities — not both — and this holding property is given as composing the structure of $\boldsymbol{C}$, and denoted by $\models$. (Again, there is no reason to attach to "holding" the meaning of an $n$-tuple of individuals that belongs to any $n$-ary relation. Holding is a primitive correspondence that obtains for some interpreted formal entities in some structures $\boldsymbol{C}$ and not in others).

Further, let us agree to distinguish the formal italicized symbols of the language in Part II from the corresponding interpretative entities of $\boldsymbol{C}$ by writing the latter in bold face. Thus, $(x^r, \boldsymbol{x}^r)$, $(C^r, \boldsymbol{C}^r)$, $(C, \boldsymbol{C})$ indicate a specific valuation of the formal symbols $x^r$, $C^r$ and $C$, a valuation which assigns to the latter the concrete continua $\boldsymbol{x}^r$, $\boldsymbol{C}^r$ and $\boldsymbol{C}$ respectively.

**Definition 29.1.** *Given a concrete intermediate universal continuum* $\boldsymbol{C}$ *and a valuation* $(x_i^r, \boldsymbol{x}_i^r)$, $(C^r, \boldsymbol{C}^r)$, *then we say that this valuation satisfies the formula* $\mathcal{P}(x_1^r, x_2^s)$ *iff* $\mathcal{P}(\boldsymbol{x}_1^r, \boldsymbol{x}_2^s)$ *holds in* $\boldsymbol{C}$. *In symbols* $(x_1^r, \boldsymbol{x}_1^r)$, $(x_2^s, \boldsymbol{x}_2^s) \models \mathcal{P}(x_1^r, x_2^s)$.

Similarly, $(C^r, \boldsymbol{C}^r)$, $(x_1^s, \boldsymbol{x}_1^s) \models \mathcal{P}(C^r, x_1^s)$ iff $\mathcal{P}(\boldsymbol{C}^r, \boldsymbol{x}_1^r)$ holds in $\boldsymbol{C}$. $(x_1^r, \boldsymbol{x}_1^r)$, $(x_2^s, \boldsymbol{x}_2^s) \models \mathcal{U}(x_1^r, x_2^s)$ iff $\mathcal{U}(\boldsymbol{x}_1^r, \boldsymbol{x}_2^s)$ holds in $\boldsymbol{C}$. $(x_1^r, \boldsymbol{x}_1^r), (x_2^s, \boldsymbol{x}_2^s), (x_3^t, \boldsymbol{x}_3^t) \models \mathcal{B}(x_1^r, x_2^s, x_3^t)$ iff $\mathcal{B}(\boldsymbol{x}_1^r, \boldsymbol{x}_2^r, \boldsymbol{x}_3^t)$ holds in $\boldsymbol{C}$. For compound expressions, satisfiability is defined as usual in terms of a valuation $(x_i^r, \boldsymbol{x}_i^r)$, $(C^r, \boldsymbol{C}^r)$ where for a given $x_i^r$, $\boldsymbol{x}_i^r$ is a fixed region of $\boldsymbol{C}^r$, and for a given $C^r$, $\boldsymbol{C}^r$ is a fixed aspect of $\boldsymbol{C}$.

**Definition 29.2.** *We call a concrete intermediate universal continuum* $\boldsymbol{C}$ *a model formula* $\phi$ *in a formal language of Part II iff for all valuations of the variables that occur in* $\phi$, $\phi$ *is satisfied by each of these valuations.*

All models of formulas with continua as variables must, of course, be infinite. The density property of betweenness requires this.

## 30. Continuity of concrete languages vs. discreteness of abstract ones

There are aspects of a drawing which cannot be fully conveyed by words — the gaze in the portrait, say, or the figure's attitude. There are also aspects of reality that cannot be conveyed by either a drawing or by wards — the atmosphere of a situation, a scent, and the many other nonvisual qualities that lie close to the boundary of our senses. Even making allowances for all this, our usual conception of language as a tool continues to give us an exceedingly distorted and limited view of how language really functions. We must eradicate our firmly established misconception that words are merely finite strings of separated symbols ready-made to be fed to a digital computer; these rigid strings are only the words' skeleton. Karl Bühler and Jost Trier, examining the semantic aspects of language, introduced the expression "field of a word", a continuous field of meaning which overlaps other words' fields and which varies substantially according to the company that a given word keeps. Parallel, then, the conventionally conceived syntactic object — the word, the sentence, the paragraph — we have, concretely speaking, a realm of continuous realities of various kinds. Structural semanticists present the following essential characteristics of a semantic field: totality, orderliness (ramified, not linear), reciprocal determination of its parts, absence of gaps, incomplete distinguishability. In the words of H. Schwarz: "The relations of concepts within a field can be of different types: one must consider subordinations, supraordinations and coordinations, as well as, of course, the interferences of conceptual spheres (even up to multiple superposition). Frequently, it is much less important to determine the exact external limits of a field … than to establish the centres of gravity and their reciprocal disposition" ([1]). A word as a syntactic object is interpreted by a continuous region of meaning; a sentence by an aspect; and a paragraph by either an aspect or an intermediate universal continuum, depending on the paragraph's context — or lack of it. Semantic reality, then, consists of continua of various kinds, continua subject to a variety of "subordinations, supraordinations and coordinations". It is this reality we shall now examine, keeping in mind that a discrete syntax is only an approximation of the kind of continuous syntax that concrete continuous semantics demands.

To understand that semantically words are regions in the continuum of meaning is the first step in building a grammar close to the actual use of language. Discrete written language is a crystallization of continuous regions of thought, not thought the product of discontinuous language. But the moment we become aware that the concrete sentence is not the printed one, all our semantic models necessarily become infinite; words can no longer be interpreted by isolated objects because words are an inextricable part of the sentences in which they occur, sentences which constantly modify the words' field of meaning: the sentence is routinely part of the word. This coordination of continuous semantic realities opens our mind to simple facts that otherwise would pass unrecorded; in particular, that a concept which emerges in the middle of a text incorporates the sequence of sentences that converge on it, as we shall now make clear.

### 31. Rickert's theory of definition

In his "Zur Lehre von der Definition", Heinrich Rickert provides a cogent look into a concrete nature of the concept as a semantic entity. Some quotations are appropriate here to do justice to this disregarded but important work. Rickets points out that a concept depends on the thought processes that precede it. "Ordinarily the concept is considered as a preliminary stage to thought, and a judgment as a relation between two concepts". Yet "the content of a concept ... is a series of judgments. We do not realize this very clearly because we never have occasion to complete verbally such act of concept formation, expressing it in a sentence .... We can then compare the content of our knowledge with a spread of threads in which nodal fixed points are the concepts, while the threads that go from one node to another would represent the relations between concepts, that is, the judgments. If we conceive the threads in their direction toward the nodes, we have an analogy of the synthetic definition, for here the judgments meet in the concept". "The concept divides into its judgments". "In a strict sense, thought only moves ... in the level of judgments, and this fact throws light on the theory of the concept" ([3], all quotations from Chapter III).

This approach has important logical and phenomenological consequences. To begin with it is concrete, for it conveys the true facts of the mind, and as Rickert says, "the concept of gravitation is identical with the law of gravitation; and laws are always judgments". To consider concepts as composed of judgments — sentences — is a more realistic and promising phenomenological point of departure than the usual one of seeing in the concept the incarnation of a single Platonic Idea forever identical to itself. We should not "look in a word for the "essence" of a thing which the concept must express" ([3]).

Words are devices to express complexes of judgments taken as aspects present in a semantic region. A concept is more than the limit of a convergent sequence of sentences in the manner of a Kantian idea; in effect, the concept has as parts the sentences that converge on it, in accordance with the principle that in any field the whole is a part of the part. Concepts are independent only to the extent to which they are composed of different streams of sentences.

Rickert's approach to the way concepts are constituted in the mind also forces us to realize that essentialism is a crippling phenomenological error, an antiphenomenalistic error that obscures the way meaning evolves in our consciousness. Just as regions can be part of aspects and vice versa, concepts are the mingling of sentences. A concept emerges from thoughts that it preserves as parts, and then becomes the constituent of other thoughts. Terms absorb a continuous stream of sentences to become part of new sentences.

This is to some extent Tarski's approach in his paper "Methodological investigation on the definability of concepts". Here concepts are defined in relation to two collections of sentences, the first being an immediate part of the explicit definition of the concept and the second a broader collection that provides a general frame of reference within which the definition is to function. No concept can be defined logically without sentences being given prior to the definition. In this, logic reflects the concrete fact that concepts are sentences, sentences that routinely converge on a nodal point where they then change semantic direction. Hence, concepts can even be inconsistent, when they embrace contradictory sentences, just as sentences can be inconsistent when they embrace contradictory paragraphs. The truth and falsity of opposing sentences is simply projected into the antinomic concept that absorbs and coordinates them, with the result that the field of meaning acquires the richest possible polarisation.

## 32. Continuous syntax

Not only do concepts incorporate sentences and sentences paragraphs — or even entire volumes — but sentences can also be part of one another. If these concrete relationships are to be explicitly described by grammar, we must find a way to blend the formal symbols with the area in which they are placed, with their specific neighborhoods. This area would become the immediate syntactic context of the symbol, part of the symbol, and a place where alien meanings could occur and which could accommodate the overspill of neighboring as well as distant symbols.

Symbols, then, are inevitably variable; they are a function of context and, as such, continuous linguistic realities despite their detached appearance. Every word is many words, for a symbol is not complete until the syntactic whole in which it is inserted has been incorporated by it. Specifically, the blank that appears between the letters of a word, or between the words of a sentence, is the channel in which currents of meaning — Ricket's "threads" — move constantly along. There is a syntactic continuum in grammar that must be emphasized if the semantic continuum is to be captured and systematically registered. In this syntactic continuum, to read a word means: I discern a complex of sentences converging toward the particular area in which the word is placed, an area in which neighboring statements dominate but where carry-overs from previous reading and thinking are freely present. Hence, discernment is a process, usually subconscious and normally submerged or even superseded by further reading and thinking, for often "the resolution of concepts into judgments cannot be continued indefinitely and ..., therefore, not all judgments have subjects and predicates consisting of defined concepts, that is, of judgments" ([3]). Therefore, the analysis of presences in the context area of a word should distinguish between the undefined concept — a region without any aspects — and the defined one. In the latter case, the analysis of the convergence of aspects into the concept being discerned is like the computation of the sum of an infinite series. However, the key point with either the defined or undefined concept is that, syntactically speaking, the blank surrounding a word is not a blank but a failure to perceive the inbetween. To see the blank as a blank only reveals a blank in our thoughts.

A major obstacle in developing the rules of a continuous syntax lies, once more, in our linguistic habit — also the habit of essentialist phenomenology — of not being able to articulate theoretically that the whole is part of the part, a factor we must consistently learn to recognize. We keep looking for essences where there is only process, expecting to identify the Idea where we should be perceiving the sum of a sequence of minglings.

## 33. Final remarks

One advantage of continua without sets is that they allow us to avoid the post-Cantor obsession with the principle of comprehension, that is, the set-theoretic compulsion to collect and seal, to gather, always gather (an "anal obsession", Freud would claim). From the viewpoint of continua without sets it is impossible to collect all regions which satisfy a given property, for something is always left out in any act of collecting. From the same viewpoint, it is easy to understand that no continuous sentence can be interpreted forever by a fixed aspect — have a fixed meaning. Even in scientific writing concretely understood, the most one can say is that the interpretations of a sentence converge toward a clearly discernable cluster of aspects. Thus, the true model of all expression is the unfinished work that remains forever uncoagulated and flowing in open-ended continuity. This means that even the position of a word is never a matter of simple location, since each word spreads over the entire page, without regard for distance. Regrettably, an expert linguist like A. J. Greimas gives distance a dominant role in his topological studies of grammar, ignoring nonmetrical properties. It is not distance that matters when one is trying to express something, but rather unity, mingling, and betweenness. Given the ceaseless cross-references to which thinking — articulate and inarticulate — is constantly subject, meaning is always meaning *in statu nascendi*. And yet there *is* truth, as there is truth in saying that a portrait expresses a person despite our changing perceptions. Atomism, which hides continuity and cuts the threads that are the flesh and blood of language, is the greatest obstacle to our understanding of this fluid truth.

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