Logic and Logical Philosophy
Online First Articles (2026)
DOI: 10.12775/LLP.2026.011

Janusz Ciuciura

Connexivity and Logics of Formal Inconsistency

Abstract. This paper discusses connexive variants of the Logics of Formal Inconsistency (*LFI*s). Our primary focus is on these extensions of *LFI*s, which are obtained by expanding their axiomatic sets to include Aristotle's theses. We propose several non-trivial calculi, with particular emphasis on the connexive extensions of *Ci*. The calculi are defined over the positive fragment of classical propositional calculus. Additionally, we demonstrate that there is no non-trivial extension of *Cia* or *Cio* by the theses of Aristotle or Boethius. As a solution, we propose an intuitionistic variant of *Ci* that, when simultaneously extended by the so-called consistency propagation axioms and the theses of Aristotle or Boethius, remains non-trivial. A similar strategy is subsequently applied to maximal *LFI*s: *LFI1*, *LFI2* and P_1'. Since the calculi based on the intuitionistic variant of *Ci* no longer validate the principle of gentle explosion, a key conceptual question arises: can they still be regarded as *LFI*s once this fundamental principle is abandoned? This motivates revising the definition of *LFI*s and introducing the Logics of Formal quasi-Inconsistency (*LFqI*s) as an alternative to *LFI*s.

Keywords: paraconsistent logic; Logics of Formal Inconsistency; connexive logic; Aristotle's theses; Boethius' theses; consistency operator

1. Introduction

1.1. Basic terminology and definitions

Let Var denote the denumerable set of all propositional variables: p, q, r, $p_1, p_2, \ldots$. The set $\mathcal{F}$ of formulas is defined using propositional variables from Var and the symbols $\sim$, $\circ$, $\vee$, $\wedge$ and $\to$ for negation, consistency operator, disjunction, conjunction, and implication, respectively. The equivalence connective, $\alpha \leftrightarrow \beta$, is defined in the standard manner, i.e.

Received May 20, 2026. *Accepted* July 23, 2026. *Published online* July 29, 2026

as $(\alpha \to \beta) \wedge (\beta \to \alpha)$. Below, we will examine axiomatic paraconsistent calculi presented in a Hilbert-style formulation, where the sole rule of inference is *detachment* (MP) $\alpha \to \beta,\ \alpha\ /\ \beta$. For all $\alpha \in \mathcal{F}$ and $\Gamma \subseteq \mathcal{F}$, we say that α *is derivable from* Γ using (MP) iff there exists a finite sequence of formulas $\beta_1, \beta_2, \ldots, \beta_n$ such that $\beta_n = \alpha$ and for each $i \leqslant n$, either $\beta_i \in \Gamma$ or $\beta_k = \beta_j \to \beta_i$ for some $j, k \leqslant i$. A calculus $\mathcal{C}$, identified with the triple $\langle \mathcal{F}, A_{\mathcal{C}}, \vdash_{\mathcal{C}} \rangle$, is determined by its set of axioms $A_{\mathcal{C}}$, where $A_{\mathcal{C}} \subseteq \mathcal{F}$. For all $\alpha \in \mathcal{F}$ and $\Gamma \subseteq \mathcal{F}$, we say that α *is provable from* Γ *within* $\mathcal{C}$ (in symbols: $\Gamma \vdash_{\mathcal{C}} \alpha$) iff α is derivable from $\Gamma \cup A_{\mathcal{C}}$ using (MP). A formula α is a *thesis* of $\mathcal{C}$ (in symbols: $\alpha \in \mathrm{Th}(\mathcal{C})$) iff $\emptyset \vdash_{\mathcal{C}} \alpha$. For any calculi $\mathcal{C}$ and $\mathcal{C}^\star$ (in $\mathcal{F}$), we say that $\mathcal{C}$ is an extension (resp. subsystem) of $\mathcal{C}^\star$ iff $\mathrm{Th}(\mathcal{C}^\star) \subseteq \mathrm{Th}(\mathcal{C})$ (resp. $\mathrm{Th}(\mathcal{C}) \subseteq \mathrm{Th}(\mathcal{C}^\star)$). We say that $\mathcal{C}$ is a proper extension (resp. proper subsystem) of $\mathcal{C}^\star$ iff $\mathrm{Th}(\mathcal{C}^\star) \subseteq \mathrm{Th}(\mathcal{C})$ and $\mathrm{Th}(\mathcal{C}) \nsubseteq \mathrm{Th}(\mathcal{C}^\star)$ (resp. $\mathrm{Th}(\mathcal{C}) \subseteq \mathrm{Th}(\mathcal{C}^\star)$ and $\mathrm{Th}(\mathcal{C}^\star) \nsubseteq \mathrm{Th}(\mathcal{C})$).

Since (MP) is the only rule of inference for each calculus $\mathcal{C}$ discussed in this paper, and both $\alpha \to (\beta \to \alpha) \in \mathrm{Th}(\mathcal{C})$ and $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)) \in \mathrm{Th}(\mathcal{C})$, it follows that the deduction theorem holds. More precisely, for all $\Gamma \subseteq \mathcal{F}$ and $\alpha, \beta \in \mathcal{F}$, we have:

(DT) $\Gamma \cup \{\alpha\} \vdash_{\mathcal{C}} \beta$ iff $\Gamma \vdash_{\mathcal{C}} \alpha \to \beta$.

A calculus $\mathcal{C}$ is classified as *paraconsistent* iff the principle of explosion does not hold in $\mathcal{C}$, i.e., for some $\alpha, \beta \in \mathcal{F}$, we have $\{\alpha, \sim\alpha\} \nvdash_{\mathcal{C}} \beta$. We say that a calculus $\mathcal{C}$ is *non-trivial* iff there exists a formula α such that $\alpha \notin \mathrm{Th}(\mathcal{C})$. A formula β *trivializes* $\mathcal{C}$ iff $\Gamma \cup \{\beta\} \vdash_{\mathcal{C}} \alpha$, for every $\Gamma \subseteq \mathcal{F}$ and every $\alpha \in \mathcal{F}$. We say that $\mathcal{C}$ is a calculus with *inconsistent negation* iff $\mathcal{C}$ is non-trivial and there is a formula α of $\mathcal{C}$ such that both $\alpha \in \mathrm{Th}(\mathcal{C})$ and $\sim\alpha \in \mathrm{Th}(\mathcal{C})$. A calculus $\mathcal{C}$ is called *strongly paraconsistent* iff $\mathcal{C}$ is a paraconsistent calculus with inconsistent negation. The concept of paraconsistent calculi with inconsistent negation is not new in the literature. It has been examined by several logicians, including the authors of [20, 21, 18, 17, 24].

The axiomatic method is not the only approach for defining paraconsistent calculi. This paper will also employ matrix and non-deterministic bivaluational semantics. A matrix for $\mathcal{F}$ is a triple $\mathfrak{M} = \langle \mathbb{V}, \mathbb{D}, \mathbb{I} \rangle$, where $\mathbb{V}$ is a non-empty set of *truth-values*, $\mathbb{D}$ is a non-empty proper subset of $\mathbb{V}$, referred to as the set of *designated truth-values*, and $\mathbb{I} = \{I_\sharp, I_\ddagger\}$ is a set of *interpretation mappings* for used propositional connectives such that $I_\sharp\colon \mathbb{V}^2 \longrightarrow \mathbb{V}$, where $\sharp \in \{\wedge, \vee, \to, \leftrightarrow\}$, and $I_\ddagger\colon \mathbb{V} \longrightarrow \mathbb{V}$, where $\ddagger \in \{\sim, \circ\}$. These mappings correspond to truth-tables. A *valuation* in $\mathfrak{M}$ is any function $v\colon \mathcal{F} \longrightarrow \mathbb{V}$ such that for all $\alpha, \beta \in \mathcal{F}$, we have $v(\alpha \sharp \beta) = I_\sharp(v(\alpha), v(\beta))$, where $\sharp \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, and $v(\ddagger\alpha) = I_\ddagger(v(\alpha))$, where $\ddagger \in \{\sim, \circ\}$. For all $\alpha \in \mathcal{F}$ and $\Gamma \subseteq \mathcal{F}$, we say that α is a *semantic consequence* of Γ in $\mathfrak{M}$ (in symbols: $\Gamma \models_{\mathfrak{M}} \alpha$) iff for every valuation v: if $v(\beta) \in \mathbb{D}$ for every $\beta \in \Gamma$, then $v(\alpha) \in \mathbb{D}$. We say that α is *valid* in $\mathfrak{M}$ iff $\emptyset \models_{\mathfrak{M}} \alpha$. A calculus $\mathcal{C}$ is *sound* with respect to a matrix $\mathfrak{M}$ iff all theses of $\mathcal{C}$ are valid in $\mathfrak{M}$. A calculus $\mathcal{C}$ is *complete* with respect to the matrix $\mathfrak{M}$ iff all formulas valid in $\mathfrak{M}$ are theses of $\mathcal{C}$.

A $\mathcal{C}$-bivaluation is any function $v\colon \mathcal{F} \longrightarrow \{1, 0\}$ that satisfies some conditions for $\sim$, $\circ$, $\wedge$, $\vee$ and $\rightarrow$, which are suitable for $\mathcal{C}$. For all $\alpha \in \mathcal{F}$ and $\Gamma \subseteq \mathcal{F}$, α is a *semantic consequence of* Γ (in symbols: $\Gamma \models_{\mathcal{C}} \alpha$) iff for every $\mathcal{C}$-bivaluation v: if $v(\beta) = 1$ for every $\beta \in \Gamma$, then $v(\alpha) = 1$. A formula α is a $\mathcal{C}$-*tautology* iff $v(\alpha) = 1$, for every $\mathcal{C}$-bivaluation v.

1.2. Da Costa's calculus C_ω and Logics of Formal Inconsistency

The calculi presented in this paper contain all the axiom schemata of da Costa's calculus C_ω (see [13, p. 501], [23, pp. 586–587] and [22], for further details). This includes all instances of the following axioms:

$$\alpha \rightarrow (\beta \rightarrow \alpha) \tag{A1}$$
$$(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)) \tag{A2}$$
$$(\alpha \wedge \beta) \rightarrow \alpha \tag{A3}$$
$$(\alpha \wedge \beta) \rightarrow \beta \tag{A4}$$
$$\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta)) \tag{A5}$$
$$\alpha \rightarrow (\alpha \vee \beta) \tag{A6}$$
$$\beta \rightarrow (\alpha \vee \beta) \tag{A7}$$
$$(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \vee \beta \rightarrow \gamma)) \tag{A8}$$
$$\alpha \vee {\sim}\alpha \tag{ExM}$$
$${\sim}{\sim}\alpha \rightarrow \alpha \tag{NN1}$$

The sole inference rule is the detachment rule (MP).

Remark 1.1. The following formulas are provable in C_ω:

$$\alpha \rightarrow \alpha \tag{IL}$$
$$(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow (\alpha \rightarrow \gamma)) \tag{LC}$$
$$(\alpha \rightarrow (\beta \rightarrow (\gamma \rightarrow \delta))) \rightarrow (\alpha \rightarrow (\gamma \rightarrow (\beta \rightarrow \delta))) \tag{LC*}$$
$$(\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma)) \tag{HS}$$

$$(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta) \tag{C}$$

$$((\alpha \wedge \beta) \to \gamma) \to (\alpha \to (\beta \to \gamma)) \tag{LE1}$$

$$(\alpha \to (\beta \to \gamma)) \to ((\alpha \wedge \beta) \to \gamma) \tag{LE2}$$

$$(\alpha \to (\beta \to \gamma)) \to ((\beta \wedge \alpha) \to \gamma) \tag{LE2}$$

$$(\sim \alpha \to \alpha) \to \alpha \tag{CM1}$$

$$(\alpha \to \sim \alpha) \to \sim \alpha \tag{CM2}$$

The first step in obtaining the Logics of Formal Inconsistency (*LFI*s) is to expand the language of da Costa's C_ω by introducing a unary connective, denoted as '$\circ$'. This unary connective is intended to express the concept of consistency at the object language level. When the symbol $\circ$ is placed in front of a formula α, it signifies that α is consistent. Although *LFI*s are paraconsistent, they validate the principle of gentle explosion; i.e., for all $\alpha, \beta \in \mathcal{F}$, any β can be derived from the set of formulas: $\circ\,\alpha$, α, and $\sim \alpha$. Most *LFI*s are based on the extension of C_ω by Peirce's law:

$$((\alpha \to \beta) \to \alpha) \to \alpha \tag{PL}$$

Notable exceptions include the intuitionistic variants of *mbC*, *Cil*, *Cila*, and *Cilo* (see Section 5.1.4 of [2], and Sections 3 and 4 of [8]), and *LFI*s based on either Nelson's logic *N4*, or First-Degree Entailment (see [6, 5] for further details).

In the following discussion, we introduce some connexive extensions of *LFI*s. Our proposal carries certain philosophical implications, particularly from the perspective of da Costa's philosophy underlying the hierarchy of *C*-systems. Da Costa's approach involves a distinction within the object language between well-behaved and not well-behaved formulas — i.e., those for which the principle of explosion holds and those for which it fails. This idea was later generalized in *LFI*s, where the operator $\circ$ is assumed to express consistency, and the $\circ$-free fragment of *LFI*s is considered a subsystem of classical propositional calculus. In contrast, the approach presented here does not aim for the $\circ$-free fragment to coincide with classical propositional logic but rather with a connexive logic. Given that any connexive logic containing classical logic is trivial, da Costa's original idea does not apply in this context. From the standpoint we adopt, no philosophical assumptions about the meaning of the operator $\circ$ are initially implied. This particularly includes the absence of any assumption that the consistency operator recovers classical reasoning. The symbol '$\circ$' is simply a connective, i.e., an abstract object whose meaning is defined exclusively by axioms, rules, or semantics. We begin by adding Aristotle's theses to the axioms of *Ci* and subsequently expand the set of axioms for other, stronger *LFI*s. However, starting from Section 4, the main calculi discussed in the paper are no longer assumed to validate the principle of gentle explosion, which is formulated here as axiom (bcl) (see the next section for details). This axiom is replaced by its weaker variant (bcl~) (see Proposition 4.1 below). Consequently, the resulting calculi can no longer be considered original *LFI*s, at least not in the strict sense. To address this issue, we revise the definition of *LFI*s, proposing in Section 6 an alternative, more general set of requirements for a logic to be regarded as an *LFI*. The primary motivation for this revision is logical rather than philosophical, as each calculus is assumed to have da Costa's C_ω as a proper subsystem. Without this revision, some extensions of *LFI*s with connexive formulas result in trivial logic.

To improve clarity, an appendix has been provided at the end of the paper that contains a list of the calculi examined.

2. A partially connexive extension of *Ci*

The calculus *Ci* (see Section 3.5 of [4] and Definition 102 in [3]) is an extension of da Costa's C_ω by two axioms involving the consistency connective, viz.,

$$\circ\,\alpha \to (\alpha \to ({\sim}\,\alpha \to \beta)) \tag{bcl}$$

$$\sim\circ\,\alpha \to (\alpha \wedge {\sim}\,\alpha) \tag{ci}$$

Notice that Peirce's law (PL) is provable in *Ci*.

1.	$(\alpha \to \beta) \to \alpha$	(DT)
2.	$(\circ\,\alpha \wedge {\sim}\,\alpha) \to (\alpha \to \beta)$	(bcl), (LC*), (LE2), (MP)
3.	$(\circ\,\alpha \wedge {\sim}\,\alpha) \to \alpha$	(HS), 1, 2, (MP)
4.	$\circ\,\alpha \to ({\sim}\,\alpha \to \alpha)$	(LE1), 3, (MP)
5.	$\circ\,\alpha \to \alpha$	(CM1), (HS), 4, (MP)
6.	${\sim}\circ\,\alpha \to \alpha$	(ci), (A3), (HS), (MP)
7.	$\circ\,\alpha \vee {\sim}\circ\,\alpha$	(ExM)
8.	α	(A8), 5, 6, 7, (MP)
9.	$(\alpha \to \beta) \to \alpha \vdash_{Ci} \alpha$	1 to 8
10.	$((\alpha \to \beta) \to \alpha) \to \alpha$	(DT), 9.

From the axiomatization of C_ω, since Peirce's law (PL) belongs to *Ci*, the negation-free fragment of *Ci* corresponds to the positive fragment of classical propositional calculus. Before proceeding, we need to review some results related to *Ci*. Most of these results are well-established and evident; therefore, they will be presented without proofs. First, note that *Ci* is a finitary sentential calculus that is closed under substitution. Specifically, the following lemma holds:

Lemma 2.1. *For every $\Gamma, \Delta \subseteq \mathcal{F}$ and $\alpha, \beta \in \mathcal{F}$, we have:*

(REF) *If $\alpha \in \Gamma$, then $\Gamma \vdash_{Ci} \alpha$.*
(MON) *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash_{Ci} \alpha$, then $\Delta \vdash_{Ci} \alpha$.*
(TR) *If $\Gamma \cup \{\alpha\} \vdash_{Ci} \beta$ and $\Delta \vdash_{Ci} \alpha$, then $\Gamma \cup \Delta \vdash_{Ci} \beta$.*
(FIN) *$\Gamma \vdash_{Ci} \alpha$ iff for some finite $\Delta \subseteq \Gamma$, $\Delta \vdash_{Ci} \alpha$.*
(STR) *If $\Gamma \vdash_{Ci} \alpha$, then $e(\Gamma) \vdash_{Ci} e(\alpha)$, for all substitutions e.*

A bi-valuational semantics for *Ci* can be presented in the style of da Costa and Alves (see Section 2 of [14] for details). *Ci*-bivaluations satisfy the following conditions for all $\alpha, \beta \in \mathcal{F}$ (see Section 4.4 of [3] and Section 2 of [8]):

(v1) $v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$,
(v2) $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$,
(v3) $v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$,
(v4) if $v({\sim}\,\alpha) = 0$, then $v(\alpha) = 1$,
(v5) if $v({\sim}{\sim}\,\alpha) = 1$, then $v(\alpha) = 1$,
(v6) if $v(\circ\,\alpha) = 1$, then $v(\alpha) = 0$ or $v({\sim}\,\alpha) = 0$,
(v7) if $v({\sim}\circ\alpha) = 1$, then $v(\alpha) = 1 = v({\sim}\,\alpha)$.

Theorem 2.1 (4, §3.5). *For all $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$, $\Gamma \vdash_{Ci} \alpha$ iff $\Gamma \models_{Ci} \alpha$.*

We will examine some extensions of *Ci* below, particularly those that include the axioms known from connexive logic, namely,

$$\sim({\sim}\,\alpha \to \alpha) \tag{AT}$$

$$\sim(\alpha \to {\sim}\,\alpha) \tag{AT*}$$

$$(\alpha \to \beta) \to {\sim}(\alpha \to {\sim}\,\beta) \tag{BT}$$

$$(\alpha \to {\sim}\,\beta) \to {\sim}(\alpha \to \beta) \tag{BT*}$$

The formulas (AT) and (AT*) are referred to as Aristotle's theses, while (BT) and (BT*) as Boethius' theses. We call a calculus $\mathcal{C}$ *partially connexive* (or *subminimal connexive*) if it validates at least one, but not all, of the schemata: (AT), (AT*), (BT), (BT*), while also satisfying the condition of non-symmetry of implication, which says that the formula '$(p \to q) \to (q \to p)$' is not valid (cf. [25, p. 18] and [15, pp. 346–347]).

As suggested in [11, 12], the following formulas

$$\alpha \to \sim(\sim\alpha \to \alpha) \tag{ATa}$$

$$\sim\alpha \to \sim(\alpha \to \sim\alpha) \tag{ATb}$$

can be viewed as alternative formulations of the intuition underlying connexive implication, namely that 'no proposition implies its own negation' and 'no proposition is implied by its own negation' — phrases typically associated with the schemata (AT) and (AT*). From a logical standpoint, there is no difference between accepting the standard or alternative formulations of Aristotle's theses in *Ci*; more precisely, we have:

Proposition 2.1. *In Ci, the postulates* (AT) *and* (AT*) *are equivalent to* (ATa) *and* (ATb), *respectively.*

Proof. (AT) $\Rightarrow$ (ATa) (resp. (AT*) $\Rightarrow$ (ATb)): This follows directly from applying (A1), (AT) (resp. (AT*)) and (MP).

(ATa) $\Rightarrow$ (AT): From (ATa), (CM1), and (HS) by (MP), we obtain $(\sim\alpha \to \alpha) \to \sim(\sim\alpha \to \alpha)$; consequently, we derive $\sim(\sim\alpha \to \alpha)$ using (CM2) and (MP). (AT*) $\Rightarrow$ (ATb): Analogous to the previous case. ⊣

We can now define an extension of *Ci*, which we will henceforth refer to as *ACi*, by adding the axioms (ATa) and (ATb) to those of *Ci*.

Consider the matrix $\mathfrak{M}_1 = \langle\{1,2,0\},\{1,2\},\sim,\circ,\to,\vee,\wedge\rangle$. The connectives are defined by the following truth-tables:

	$\sim$	$\circ$
1	0	1
2	1	0
0	1	1

$\to$	1	2	0
1	1	2	0
2	2	2	0
0	2	2	1

$\vee$	1	2	0
1	1	1	1
2	1	2	2
0	1	2	0

$\wedge$	1	2	0
1	1	2	0
2	2	2	0
0	0	0	0

Notice that *ACi* is sound with respect to the matrix $\mathfrak{M}_1$. However, neither $(p \to q) \to (q \to p)$ nor the instances of (BT) and (BT*) for $\alpha = p$ and $\beta = q$ are provable in *ACi*. Indeed, the assignment $p/0$, $q/2$ in $\mathfrak{M}_1$ falsifies $(p \to q) \to (q \to p)$; the assignment $p/1$, $q/2$ in $\mathfrak{M}_1$ falsifies (BT) for $\alpha = p$ and $\beta = q$; the assignment $p/0$, $q/0$ in $\mathfrak{M}_1$ falsifies (BT*) for $\alpha = p$ and $\beta = q$.

With the aid of $\mathfrak{M}_1$, one can show that neither the principle of explosion holds nor is the formula $\circ((\alpha \to \sim\alpha) \to \sim(\alpha \to \sim\alpha))$ provable in *ACi*. This latter observation is particularly significant, given that *ACi* is not only partially connexive but also a strongly paraconsistent calculus, i.e., a calculus whose negation is explicitly inconsistent. In particular, we have:

$$(\alpha \to \sim\alpha) \to \sim(\alpha \to \sim\alpha) \in \mathrm{Th}(ACi),$$
$$\sim((\alpha \to \sim\alpha) \to \sim(\alpha \to \sim\alpha)) \in \mathrm{Th}(ACi).$$

Since the deduction theorem holds for ACi, and since (bcl) is among its axioms, we obtain, for all $\alpha, \beta \in \mathcal{F}$, that

$$\circ((\alpha \to \sim\alpha) \to \sim(\alpha \to \sim\alpha)) \vdash_{ACi} \beta.$$

This demonstrates that $\circ((\alpha \to \sim\alpha) \to \sim(\alpha \to \sim\alpha))$ trivializes ACi, implying that the symbol '$\circ$' should no longer be regarded as a consistency operator — at least not in the sense familiar from *LFI*s — but rather as a symbol denoting a connexive bottom particle. By a *bottom particle* in a calculus $\mathcal{C}$ we mean a formula α such that, for all $\Delta \subseteq \mathcal{F}$ and $\beta \in \mathcal{F}$, $\Delta \cup \alpha \vdash_{\mathcal{C}} \beta$.

Unfortunately, the interpretation does not extend to certain other formulas such as $\circ\circ\alpha$. Note that $\circ\circ\alpha$ is a thesis of *Ci*. Hence,

$$\circ\circ((\alpha \to \sim\alpha) \to \sim(\alpha \to \sim\alpha)) \in \mathrm{Th}(ACi),$$
$$\circ\circ\circ((\alpha \to \sim\alpha) \to \sim(\alpha \to \sim\alpha)) \in \mathrm{Th}(ACi).$$

By (DT), combined with axiom (bcl), it follows that

$$\sim\circ\circ((\alpha \to \sim\alpha) \to \sim(\alpha \to \sim\alpha)) \vdash_{ACi} \beta.$$

The additional difficulty in providing a philosophical interpretation of '$\circ$' is compounded by the existence of formulas in ACi that, although *not*-consistent, are nonetheless theses of the calculus. For example,

$$\sim\circ((\alpha \to \sim\alpha) \to \sim(\alpha \to \sim\alpha)) \in \mathrm{Th}(ACi).$$

Taken together, these observations reveal a peculiarity in the behavior of the consistency operator when applied to connexive formulas.

ACi-bivaluations satisfy, for all $\alpha, \beta \in \mathcal{F}$, the conditions (v1)–(v7) of *Ci*-bivaluations, in addition to the following clauses:

(v8) if $v(\alpha) = 1$, then $v(\sim(\sim\alpha \to \alpha)) = 1$,
(v9) if $v(\sim\alpha) = 1$, then $v(\sim(\alpha \to \sim\alpha)) = 1$.

THEOREM 2.2. *For all* $\Gamma \subseteq \mathcal{F}$ *and* $\alpha \in \mathcal{F}$, $\Gamma \vdash_{ACi} \alpha$ *iff* $\Gamma \vDash_{ACi} \alpha$.

The proof of soundness is conducted by induction on the length of a derivation in *ACi*. For the proof of completeness, we apply the method based on the notion of maximal non-trivial sets of formulas.

Definition 2.1. For any $\varDelta \subseteq \mathcal{F}$, we say that $\varDelta$ is a closed theory of $\mathcal{C}$ if, and only if for any $\beta \in \mathcal{F}$, we have $\varDelta \vdash_{\mathcal{C}} \beta$ iff $\beta \in \varDelta$. We say that $\varDelta$ is maximal non-trivial with respect to $\alpha \in \mathcal{F}$ in $\mathcal{C}$, if, and only if (1) $\varDelta \nvdash_{\mathcal{C}} \alpha$, and (2) for every $\beta \in \mathcal{F}$, if $\beta \notin \varDelta$ then $\varDelta \cup \{\beta\} \vdash_{\mathcal{C}} \alpha$.

Notice that Lindenbaum–Łoś theorem applies to any finitary calculus $\mathcal{C}$ [see 19, Theorem 3.31]. Thus, in particular, we have:

Lemma 2.2. *For all $\varGamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$ such that $\varGamma \nvdash_{ACi} \alpha$, there is a maximal non-trivial set $\varDelta$ with respect to α in ACi such that $\varGamma \subseteq \varDelta$.*

Lemma 2.3. *Every maximal non-trivial set with respect to some formula is a closed theory.*

Lemma 2.4. *For any maximal non-trivial set $\varDelta$ with respect to $\alpha \in \mathcal{F}$ in ACi the mapping $v_\varDelta \colon \mathcal{F} \longrightarrow \{1, 0\}$, defined for any $\delta \in \mathcal{F}$, as ($\star$): $v_\varDelta(\delta) = 1$ iff $\delta \in \varDelta$, is an ACi-bivaluation.*

Proof. The cases (v1)–(v7) are those from *Ci*, and are therefore omitted. For (v8): Suppose that $v_\varDelta(\beta) = 1$. Then, by ($\star$), $\beta \in \varDelta$. Thus, $\varDelta \vdash_{ACi} \beta$. By (ATa) and (DT), we derive $\{\beta\} \vdash_{ACi} {\sim}({\sim}\beta \to \beta)$, which entails $\varDelta \vdash_{ACi} {\sim}({\sim}\beta \to \beta)$. Hence, ${\sim}({\sim}\beta \to \beta) \in \varDelta$, and finally, $v_\varDelta({\sim}({\sim}\beta \to \beta)) = 1$. For (v9): Analogous to that of the case (v8). ⊣

Based on the above lemmas, we obtain the proof of completeness:

Proof. Suppose that $\varGamma \nvdash_{ACi} \alpha$. By Lemma 2.2, there is a maximal non-trivial set with respect to α in *ACi* such that $\varGamma \subseteq \varDelta$. For the bivaluation $v_\varDelta$ from the proof of Lemma 2.4, we find that $v_\varDelta(\alpha) = 0$ (since $\alpha \notin \varDelta$) and $v_\varDelta(\beta) = 1$, for any $\beta \in \varDelta$. This implies that $\varDelta \not\models_{ACi} \alpha$ and, in particular, $\varGamma \not\models_{ACi} \alpha$. ⊣

3. An extension of *ACi*

This section examines possible extensions of *ACi* obtained by introducing additional postulates that include the consistency connective into the axioms. The relevant postulates are presented below for $\sharp \in \{\wedge, \vee, \rightarrow\}$ (see [3, Definitions 105, 108, 114] and [4, §3.10]):

$$\sim\!(\alpha \wedge \sim\alpha) \rightarrow \circ\,\alpha \tag{cl}$$

$$\circ\,\alpha \wedge \circ\,\beta \rightarrow \circ(\alpha \,\sharp\, \beta) \tag{ca}$$

$$\circ\,\alpha \vee \circ\,\beta \rightarrow \circ(\alpha \,\sharp\, \beta) \tag{co}$$

It is worth mentioning that some postulates known from *LFI*s are provable within *ACi*. For example, the schema

$$\circ\,\alpha \vee (\alpha \wedge \sim\alpha) \tag{ciw}$$

or equivalently, $\circ\,\alpha \vee \alpha$ and $\circ\,\alpha \vee \sim\alpha$ [cf. 1, Remark 2], are theses of *ACi*. However, when the symbol '$\circ$' is interpreted as a connexive bottom-particle operator, certain postulates admissible in standard *LFI*s are no longer applicable. For instance, if one introduces $\circ\,\alpha$ as an axiom — alternative investigated in [2] — then, given that the intended purpose is to recover full classical propositional logic whenever required, any connexive variant of such *LFI*s collapses into triviality. This results from the fact that the connexive interpretation of '$\circ$' blocks the standard *LFI* mechanisms that prevent unrestricted propagation of inconsistency, thereby forcing the calculus to validate classical explosion in all contexts. Nevertheless, some connexive variants of *LFI*s remain non-trivial.

PROPOSITION 3.1. *Enriching ACi with* (cl) *does not trivialize ACi.*

PROOF. This follows from $\mathfrak{M}_2 = \langle\{1,2,0\},\{1,2\},\sim,\circ,\rightarrow,\vee,\wedge\rangle$, where $\sim$, $\circ$ and $\rightarrow$ are interpreted as in $\mathfrak{M}_1$, while $\wedge$ and $\vee$ are defined by the truth-tables:

$\vee$	1	2	0
1	1	1	1
2	1	1	1
0	1	1	0

$\wedge$	1	2	0
1	1	1	0
2	1	1	0
0	0	0	0

The axioms of *ACi*, (cl), (MP) are valid in $\mathfrak{M}_2$. However, $\mathfrak{M}_2$ validates neither $(p \rightarrow q) \rightarrow (q \rightarrow p)$ nor Boethius' theses for $\alpha = p$, $\beta = q$. ⊣

Let *ACil* denote the extension of *ACi* obtained by adding (cl). For all $\alpha, \beta \in \mathcal{F}$, for every ξ of the form $\alpha \rightarrow \sim\alpha$, we obtain

$$\sim\!((\xi \rightarrow \sim\xi) \wedge \sim(\xi \rightarrow \sim\xi)) \vdash_{ACil} \beta$$

since $\circ(\xi \to \sim\xi) \vdash_{ACil} \beta$ holds, by applying (cl), (DT) and (TR). This shows that $\sim((\xi \to \sim\xi) \wedge \sim(\xi \to \sim\xi))$ is not only a bottom particle in *ACil*, but also that, due to the monotonicity of the relation $\vdash_{ACil}$, certain variants of the principle of explosion hold in *ACil*. Specifically, for every ξ of the form $\alpha \to \sim\alpha$, and for all $\alpha, \beta \in \mathcal{F}$, we have

$$\{\sim((\xi \to \sim\xi) \wedge \sim(\xi \to \sim\xi)), (\xi \to \sim\xi) \wedge \sim(\xi \to \sim\xi)\} \vdash_{ACil} \beta$$

and, analogously to the case of *ACi*, $\{\circ(\xi \to \sim\xi), \xi, \sim\xi\} \vdash_{ACil} \beta$.

ACil-bivaluations satisfy all the conditions of *ACi*-bivaluations for all $\alpha, \beta \in \mathcal{F}$, in addition to the following clause:

(v10) if $v(\sim(\alpha \wedge \sim\alpha)) = 1$, then $v(\circ\alpha) = 1$.

THEOREM 3.1. *For all $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$, $\Gamma \vdash_{ACil} \alpha$ iff $\Gamma \vDash_{ACil} \alpha$.*

The proof is analogous to the proof of Theorem 2.2. The key point is to demonstrate that the following lemma holds for *ACil*:

LEMMA 3.1. *For any maximal non-trivial set Δ with respect to $\alpha \in \mathcal{F}$ in ACil the mapping $v_\Delta : \mathcal{F} \longrightarrow \{1, 0\}$, defined for any $\delta \in \mathcal{F}$, as ($\star$): $v_\Delta(\delta) = 1$ iff $\delta \in \Delta$, is an ACil-bivaluation.*

PROOF. For (v1)–(v9): Proceed as in *ACi*. For (v10): Apply (cl). $\dashv$

ACil is the top partially connexive, strongly paraconsistent extension proposed thus far. This naturally raises the question of whether an analogous phenomenon can be observed for *ACi* when it is extended by the so-called propagation axioms, namely (ca) and (co).

PROPOSITION 3.2. *ACi+(ca) and ACi+(co) are trivial logics.*

PROOF. For *ACi*+(ca), when (AT) is used in the proof:

1.	$\circ(\sim\sim p \to \sim p) \to ((\sim\sim p \to \sim p) \to (\sim(\sim\sim p \to \sim p) \to \sim\circ\sim p))$	(bcl)
2.	$\sim(\sim\sim p \to \sim p) \to (\circ(\sim\sim p \to \sim p) \to ((\sim\sim p \to \sim p) \to \sim\circ\sim p)))$	(LC*), (LC), 1, (MP)
3.	$\sim(\sim\sim p \to \sim p)$	(AT)
4.	$\circ(\sim\sim p \to \sim p) \to ((\sim\sim p \to \sim p) \to \sim\circ\sim p))$	2, 3, (MP)
5.	$\circ\sim\sim p \wedge \circ\sim p \to \circ(\sim\sim p \to \sim p)$	(ca)
6.	$\circ\sim\sim p \wedge \circ\sim p \to ((\sim\sim p \to \sim p) \to \sim\circ\sim p)$	(HS), 4, 5, (MP)
7.	$\circ\sim\sim p \to (\circ\sim p \to ((\sim\sim p \to \sim p) \to \sim\circ\sim p))$	(LE1), 6, (MP)
8.	$\circ\sim\sim p \to ((\sim\sim p \to \sim p) \to (\circ\sim p \to \sim\circ\sim p))$	(LC*), 7, (MP)
9.	$\circ\sim\sim p \wedge (\sim\sim p \to \sim p) \to (\circ\sim p \to \sim\circ\sim p)$	(LE2), 8, (MP)
10.	$(\circ\sim p \to \sim\circ\sim p) \to \sim\circ\sim p$	(CM2)
11.	$\circ\sim\sim p \wedge (\sim\sim p \to \sim p) \to \sim\circ\sim p$	(HS), 9, 10, (MP)

12. $\sim\circ\sim p \to (\sim p \wedge \sim\sim p)$ (ci)
13. $\circ\sim\sim p \wedge (\sim\sim p \to \sim p) \to (\sim p \wedge \sim\sim p)$ (HS), 11, 12, (MP)
14. $(\sim p \wedge \sim\sim p) \to \sim\sim p$ (A4)
15. $\circ\sim\sim p \wedge (\sim\sim p \to \sim p) \to \sim\sim p$ (HS), 13, 14, (MP)
16. $\circ\sim\sim p \to ((\sim\sim p \to \sim p) \to \sim\sim p)$ (LE1), 15, (MP)
17. $((\sim\sim p \to \sim p) \to \sim\sim p) \to \sim\sim p$ (PL)
18. $\circ\sim\sim p \to \sim\sim p$ (HS), 16, 17, (MP)
19. $\circ\sim\sim p \to (\sim\sim p \to (\sim\sim\sim p \to \sim\circ\sim\sim p)$ (bcl)
20. $\circ\sim\sim p \to (\circ\sim\sim p \to (\sim\sim\sim p \to \sim\circ\sim\sim p)$ (LC), (HS), 18, 19, (MP)
21. $\circ\sim\sim p \to (\sim\sim\sim p \to \sim\circ\sim\sim p)$ (C), 20, (MP)
22. $\sim\sim\sim p \to (\circ\sim\sim p \to \sim\circ\sim\sim p)$ (LC), 21, (MP)
23. $(\circ\sim\sim p \to \sim\circ\sim\sim p) \to \sim\circ\sim\sim p$ (CM2)
24. $\sim\sim\sim p \to \sim\circ\sim\sim p$ (HS), 22, 23, (MP)
25. $\sim\circ\sim\sim p \to (\sim\sim p \wedge \sim\sim\sim p)$ (ci)
26. $\sim\sim\sim p \to (\sim\sim p \wedge \sim\sim\sim p)$ (HS), 24, 25, (MP)
27. $(\sim\sim p \wedge \sim\sim\sim p) \to \sim\sim p$ (A3)
28. $\sim\sim\sim p \to \sim\sim p$ (HS), 26, 27, (MP)
29. $(\sim\sim\sim p \to \sim\sim p) \to \sim\sim p$ (CM1)
30. $\sim\sim p$ 28, 29, (MP)
31. p (NN1), 30, (MP).

For ACi+(ca), when the schema (AT*) is used in the proof:

1. $\circ(p \to \sim p) \to ((p \to \sim p) \to (\sim(p \to \sim p) \to \sim\circ\sim p))$ (bcl)
2. $\circ p \wedge \circ\sim p \to \circ(p \to \sim p)$ (ca)
3. $\circ p \wedge \circ\sim p \to ((p \to \sim p) \to (\sim(p \to \sim p) \to \sim\circ\sim p))$ (HS), 1, 2, (MP)
4. $\circ p \wedge \circ\sim p \to ((p \to \sim p) \to \sim\circ\sim p)$ (LC*), (LC), 3, (AT*), (MP)
5. $\circ p \to (\circ\sim p \to ((p \to \sim p) \to \sim\circ\sim p))$ (LE1), 4, (MP)
6. $\circ p \to ((p \to \sim p) \to (\circ\sim p \to \sim\circ\sim p))$ (LC*), 5, (MP)
7. $\circ p \wedge (p \to \sim p) \to (\circ\sim p \to \sim\circ\sim p)$ (LE2), 6, (MP)
8. $(\circ\sim p \to \sim\circ\sim p) \to \sim\circ\sim p$ (CM2)
9. $\circ p \wedge (p \to \sim p) \to \sim\circ\sim p$ (HS), 7, 8, (MP)
10. $\sim\circ\sim p \to (\sim p \wedge \sim\sim p)$ (ci)
11. $\circ p \wedge (p \to \sim p) \to (\sim p \wedge \sim\sim p)$ (HS), 9, 10, (MP)
12. $\circ p \wedge (p \to \sim p) \to \sim\sim p$ (A4), (HS), 11, (MP)
13. $\circ p \wedge (p \to \sim p) \to p$ (NN1), (HS), 12, (MP)
14. $(p \to \sim p) \to (\circ p \to p)$ (LE2), (LC), 13, (MP)
15. $\sim p \to (\circ p \to p)$ (A1), (HS), 14, (MP)
16. $\circ p \to (\sim p \to p)$ (LC), 15, (MP)
17. $\circ p \to p$ (CM1), (HS), 16, (MP)
18. $p \to (\circ p \to (\sim p \to \sim\circ p)$ (bcl), (LC)
19. $\circ p \to (\circ p \to (\sim p \to \sim\circ p)$ (HS), 18, 17, (MP)
20. $\circ p \to (\sim p \to \sim\circ p)$ (C), 19, (MP)
21. $\sim p \to (\circ p \to \sim\circ p)$ (LC), 20, (MP)

22. $\sim p \to \sim \circ p$ (CM2), (HS), 21, (MP)
23. $\sim p \to (p \wedge \sim p)$ (ci), (HS), 22, (MP)
24. $\sim p \to p$ (A3), (HS), 23, (MP)
25. p (CM1), 24, (MP).

ACi+(co): Note first that $(\alpha \wedge \beta) \to (\alpha \vee \beta)$ is a thesis of C_ω. Then, from $(\circ\,\alpha \wedge \circ\,\beta) \to (\circ\,\alpha \vee \circ\,\beta)$, (co) and (HS), we derive (ca) by (MP). The remaining part of the proof proceeds as in the case of *ACi*+(ca). $\dashv$

Proposition 3.2 implies that no non-trivial calculus can be defined by simultaneously extending *Ci* with (ca) (or (co)) and at least one of Aristotle's theses. Similar results can be observed for the theses of Boethius; more precisely, we have:

PROPOSITION 3.3. *The extension of Ci with both* (ca) (*or* (co)) *and either* (BT) *or* (BT*) *trivializes Ci.*

PROOF. *Ci*+(ca)+(BT): By (BT), we have $(\sim\sim p \to p) \to \sim(\sim\sim p \to \sim p)$. From (NN1), we derive $\sim(\sim\sim p \to \sim p)$ using (MP). Next, we proceed as outlined in Proposition 3.2.

Ci+(ca)+(BT*): From an instance of (BT*), we have $(\sim p \to \sim p) \to \sim(\sim p \to p)$. By (IL) and (MP), we obtain $\sim(\sim p \to p)$. The remaining part of the proof is identical to that of Proposition 3.2. $\dashv$

Alternatively, one may observe that there is no non-trivial extension of *Cia* (or *Cio*) obtained by adding Aristotle's or Boethius' theses. In the next section, we address this limitation by introducing a weakened variant of *Ci* whose negation-free fragment corresponds to the positive fragment of intuitionistic propositional calculus.

4. A weakening of *Ci*, its partially connexive extensions and Peirce's law

We begin by establishing several results for *Ci*. It is worth noting that most of these results coincide with those provided for the calculus *Cil*, which is an extension of *Ci* by the schema (cl) (see [8, §3] and [4, §3.8] for details on *Cil*).

PROPOSITION 4.1. *In Ci, axiom* (bcl) *can be replaced by the pair*:

$$\circ\,\alpha \to (\sim\alpha \to (\sim\sim\alpha \to \beta)) \tag{bcl$^\sim$}$$

$$\circ\,\alpha \wedge \alpha \to \sim\sim\alpha \tag{cn}$$

PROOF. (bcl) $\Rightarrow$ (bcl$^\sim$): Apply (DT), (NN1), (bcl) and (MP).

(bcl) $\Rightarrow$ (cn): Apply (DT), (bcl), (LE2), (CM2) and (MP).

(bcl$^\sim$)+(cn) $\Rightarrow$ (bcl): By (DT), we have $\circ\,\alpha$, α, and $\sim\alpha$. From (A5) and (MP), we derive $\circ\,\alpha \wedge \alpha$, and consequently, $\sim\sim\alpha$, by (cn). By applying (bcl$^\sim$) and (MP), we obtain β, and, finally, $\circ\,\alpha \to (\alpha \to (\sim\alpha \to \beta))$, by (DT). $\dashv$

LEMMA 4.1. *The axioms of C_ω plus* (bcl$^\sim$), (cn), (ci), *and (MP) axiomatize Ci.*

PROPOSITION 4.2 (8, Remark 9). (cn) *is independent of the other axioms of Ci.*

PROOF. Consider $\mathfrak{M}_3 = \langle\{1,2,0\},\{1,2\},\sim,\circ,\to,\vee,\wedge\rangle$, where $\sim$, $\wedge$, and $\vee$ are interpreted as in $\mathfrak{M}_1$, while $\circ$ and $\to$ are defined as follows:

	$\circ$
1	1
2	2
0	1

$\to$	1	2	0
1	1	2	0
2	1	2	0
0	1	1	1

Formula (cn) is invalid in $\mathfrak{M}_3$ when α is assigned the value 2. $\dashv$

Let *iCi* denote the calculus obtained from *Ci* by removing the formula (cn) from its axioms.

PROPOSITION 4.3 (8, Remark 10). (PL), (bcl), *and* $\alpha \vee (\alpha \to \beta)$ *are not provable in iCi.*

PROOF. Consider the matrix $\mathfrak{M}_4 = \langle\{1,2,0\},\{1\},\sim,\circ,\to,\vee,\wedge\rangle$, where $\sim$, $\wedge$, and $\vee$ are interpreted as in $\mathfrak{M}_1$, while $\circ$ and $\to$ are defined by the following truth-tables:

	$\circ$
1	1
2	1
0	1

$\to$	1	2	0
1	1	2	0
2	1	1	0
0	1	1	1

The assignment $p/2$ and $q/0$ in $\mathfrak{M}_4$ falsifies (PL), (bcl), and $\alpha \vee (\alpha \to \beta)$ for $\alpha = p$ and $\beta = q$. $\dashv$

Although the negation-free fragment of *iCi* coincides with intuitionistic propositional calculus, a variant of Peirce's law remains provable in *iCi*. Since (bcl$^\sim$) is an axiom schema of *iCi*, proceeding as in the proof of (PL) (see Section 2), we obtain:

PROPOSITION 4.4. *$((\sim\alpha \to \beta) \to \sim\alpha) \to \sim\alpha$ is a thesis of iCi.*

It is noteworthy that an alternative weakening of *Ci* has already been examined in the literature (see the conclusions of [8]). This alternative axiomatization, which employs (MP) as its sole rule of inference, includes the axioms of C_ω, (ci), and the following pair of axioms:

$$\circ\alpha \to (\sim\circ\alpha \to \beta) \tag{DS$^\circ$}$$

$$(\sim\alpha \wedge \sim\sim\alpha) \to \sim\circ\alpha \tag{ic$^\sim$}$$

It is straightforward to verify that (bcl$^\sim$) is a thesis of this calculus. Conversely, by means of the matrix $\mathfrak{M}_3$, one can show that (DS$^\circ$) is not provable in *iCi*. Hence, *iCi* is a proper subsystem of the axiomatization proposed in [8]. Since *iCi* is itself a proper subsystem of *Ci*, the set of axioms of *iCi* may be safely expanded by adding (cl), (ATa), and (ATb) without inducing triviality. This leads to the following question: Can the axioms of *iCi* be further extended by including (ca) (or (co)) together with Aristotle's theses (equivalently, (ATa) and (ATb)) while still avoiding triviality? We denote these extensions by *AiCia* (respectively, *AiCio*).

PROPOSITION 4.5. *Enriching iCi with* (ATa), (ATb), *and* (ca) (*or* (co)) *does not trivialize iCi.*

PROOF. Let $\mathfrak{M}_5 = \langle\{1,2,3,0\},\{1,2\},\sim,\circ,\to,\vee,\wedge\rangle$, where the connectives are defined by the following truth-tables:

$\to$	1	2	3	0
1	2	2	3	0
2	2	2	3	0
3	2	2	2	0
0	2	2	2	2

$\vee$	1	2	3	0
1	1	1	1	1
2	1	2	2	2
3	1	2	3	3
0	1	2	3	0

$\wedge$	1	2	3	0
1	1	2	3	0
2	2	2	3	0
3	3	3	3	0
0	0	0	0	0

	$\sim$	$\circ$
1	0	1
2	1	1
3	1	1
0	1	1

The resulting calculi, *AiCia* and *AiCio*, are sound with respect to the matrix $\mathfrak{M}_5$. The assignment $p/0$ and $q/3$ in $\mathfrak{M}_4$ falsifies $(p \to q) \to (q \to p)$. The assignment $p/2$ and $q/0$ in $\mathfrak{M}_4$ falsifies (bcl) for $\alpha = p$ and $\beta = q$. The assignment $p/3$ and $q/0$ in $\mathfrak{M}_4$ falsifies (PL) for $\alpha = p$ and $\beta = q$. ⊣

Since $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$ and $(\alpha \wedge \beta) \to (\alpha \vee \beta)$ are theses of the negation-free fragment of intuitionistic propositional calculus, it follows immediately that

$$\circ\,\alpha \wedge \circ\,\beta \to \circ(\alpha \sharp \beta) \in \mathrm{Th}(AiCio), \text{ for } \sharp \in \{\wedge, \vee, \to\}.$$

Consequently, *AiCia* is a subsystem of *AiCio*. Moreover, not only *AiCia* and *AiCio*, but also their extensions by $\sim(\alpha \wedge \sim\alpha) \to \circ\,\alpha$, are sound with respect to $\mathfrak{M}_5$, which shows that their axiom sets can be safely expanded by adding (cl) without leading to triviality. This raises the question of whether a similar outcome can be obtained when extending *AiCio* (and, trivially, *AiCia* as well) by Peirce's law (PL).

PROPOSITION 4.6. *The extension of AiCio (resp. AiCia) with Peirce's law — denoted ACio (resp. ACia) — does not lead to the trivialization of AiCio (resp. ACia).*

PROOF. Let $\mathfrak{M}_6 = \langle \{1, 2, 0\}, \{1, 2\}, \sim, \circ, \to, \vee, \wedge \rangle$, where $\sim$, $\wedge$, $\to$, and $\vee$ are interpreted as in $\mathfrak{M}_1$, while $\circ$ is defined as follows:

	$\circ$
1	1
2	1
0	1

ACio (and *ACia*) is sound with respect to the matrix $\mathfrak{M}_6$. However, neither $(\alpha \to \beta) \to (\beta \to \alpha)$ nor Boethius' theses are valid in $\mathfrak{M}_6$. $\dashv$

Every *ACio*-bivaluation satisfies the following conditions for all α, $\beta \in \mathcal{F}$:

(v1) $v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$,
(v2) $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$,
(v3) $v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$,
(v4) if $v(\sim\alpha) = 0$, then $v(\alpha) = 1$,
(v5) if $v(\sim\sim\alpha) = 1$, then $v(\alpha) = 1$,
(v6*) if $v(\circ\,\alpha) = 1$, then $v(\sim\alpha) = 0$ or $v(\sim\sim\alpha) = 0$,
(v7) if $v(\sim\circ\,\alpha) = 1$, then $v(\alpha) = 1 = v(\sim\alpha)$,
(v8) if $v(\alpha) = 1$, then $v(\sim(\sim\alpha \to \alpha)) = 1$,
(v9) if $v(\sim\alpha) = 1$, then $v(\sim(\alpha \to \sim\alpha)) = 1$,
(v10) if $v(\circ\,\alpha) = 1$ or $v(\circ\,\beta) = 1$, then $v(\circ(\alpha \sharp \beta)) = 1$, for $\sharp \in \{\wedge, \vee, \to\}$.

THEOREM 4.1. *For all $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$, $\Gamma \vdash_{ACio} \alpha$ iff $\Gamma \vDash_{ACio} \alpha$.*

Note that the definition of *ACia*-bivaluations is obtained by replacing clause ($v10$) of the *ACio*-bivaluation with the following one:

(v10*) if $v(\circ\,\alpha) = 1 = v(\circ\,\beta)$, then $v(\circ(\alpha \,\sharp\, \beta)) = 1$, for $\sharp \in \{\wedge, \vee, \rightarrow\}$.

THEOREM 4.2. *For all $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$, $\Gamma \vdash_{ACia} \alpha$ iff $\Gamma \vDash_{ACia} \alpha$.*

Remark 4.1. *ACia* is a proper subsystem of *ACio*.

Similarly to *AiCia* and *AiCio*, the calculi obtained by adding the formula $\sim(\alpha \wedge \sim\alpha) \rightarrow \circ\,\alpha$ to *ACia* and *ACio* yield their non-trivial extensions, denoted by *ACila* and *ACilo*, respectively. Every *ACila*-bivaluation satisfies all the conditions imposed on *ACia*-bivaluations for all $\alpha, \beta \in \mathcal{F}$, and, in addition, must satisfy the following clause:

(v11) if $v(\sim(\alpha \wedge \sim\alpha)) = 1$, then $v(\circ\,\alpha) = 1$.

THEOREM 4.3. *For all $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$, $\Gamma \vdash_{ACila} \alpha$ iff $\Gamma \vDash_{ACila} \alpha$.*

Likewise, every *ACilo*-bivaluation satisfies all the conditions imposed on *ACio*-bivaluations for all $\alpha, \beta \in \mathcal{F}$, together with the following clause:

(v11) if $v(\sim(\alpha \wedge \sim\alpha)) = 1$, then $v(\circ\,\alpha) = 1$.

THEOREM 4.4. *For all $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$, $\Gamma \vdash_{ACilo} \alpha$ iff $\Gamma \vDash_{ACilo} \alpha$.*

ACilo is the top partially connexive, strongly paraconsistent extension proposed in Sections 2–4. The calculi developed therein constitute the structure illustrated in Figure 1.

5. Maximal *LFI*s and Aristotle's theses

In this section, we present several extensions of *iCi* obtained by adding axioms known from the maximal *LFI*s. Let $\mathcal{L}^\circ$ be the standard propositional language extended by the consistency operator symbol, and let $\mathcal{CPL}^\circ$ be the calculus defined over $\mathcal{L}^\circ$, obtained by adding the axiom $\circ\,\alpha$ to the postulates of classical propositional calculus. We say that a calculus $\mathcal{C}$ is maximal with respect to $\mathcal{CPL}^\circ$ iff $\mathcal{C}$ is a proper subsystem of $\mathcal{CPL}^\circ$, and for any formula α such that $\alpha \in \mathrm{Th}(\mathcal{CPL}^\circ)$ and $\alpha \notin \mathrm{Th}(\mathcal{C})$, we have $\mathcal{C} \cup \{\alpha\} = \mathcal{CPL}^\circ$ [see 3, §5.3]. The well-known examples of maximal *LFI*s are *LFI1*, *LFI2*, and P_1'. The former is an extension of *Ci* by the following postulates [cf. 4, Theorem 3.69]:

$$\alpha \rightarrow \sim\sim\alpha \tag{NN2}$$

$$\sim\circ(\alpha \wedge \beta) \leftrightarrow (\sim\circ\,\alpha \wedge \beta) \vee (\sim\circ\,\beta \wedge \alpha) \tag{cj1}$$

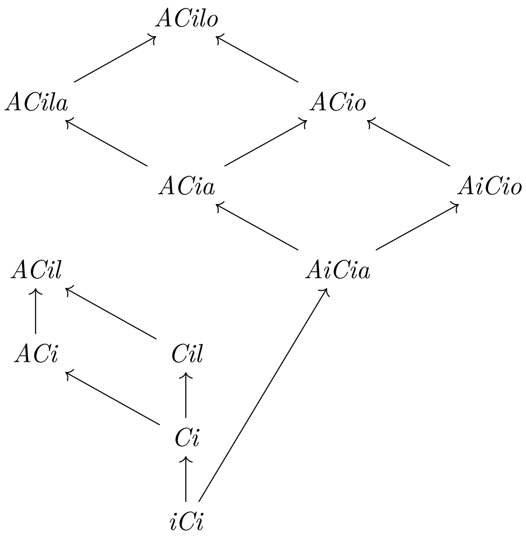

Figure 1. Partially connexive non-trivial extensions of *iCi*

$$\sim\circ(\alpha \vee \beta) \leftrightarrow (\sim\circ\alpha \wedge \sim\beta) \vee (\sim\circ\beta \wedge \sim\alpha) \tag{cj2}$$
$$\sim\circ(\alpha \rightarrow \beta) \leftrightarrow (\alpha \wedge \sim\circ\beta) \tag{cj3}$$

Proposition 5.1. *The extension of LFI1 with any thesis of Aristotle or Boethius trivializes LFI1.*

Proof. For *LFI1*+(AT):

1. $\circ(\sim\sim p \rightarrow \sim p) \rightarrow ((\sim\sim p \rightarrow \sim p) \rightarrow$
   $(\sim(\sim\sim p \rightarrow \sim p) \rightarrow \sim\circ(\sim\sim p \rightarrow \sim p)))$ (bcl)
2. $(\sim\sim p \rightarrow \sim p) \rightarrow (\circ(\sim\sim p \rightarrow \sim p) \rightarrow \sim\circ(\sim\sim p \rightarrow \sim p))$ (LC*), (LC), (AT), 1, (MP)
3. $(\circ(\sim\sim p \rightarrow \sim p) \rightarrow \sim\circ(\sim\sim p \rightarrow \sim p)) \rightarrow \sim\circ(\sim\sim p \rightarrow \sim p)$ (CM2)
4. $(\sim\sim p \rightarrow \sim p) \rightarrow \sim\circ(\sim\sim p \rightarrow \sim p)$ (HS), 2, 3, (MP)
5. $\sim p \rightarrow \sim\circ(\sim\sim p \rightarrow \sim p)$ (A1), (HS), 4, (MP)
6. $\sim p \rightarrow (\sim\sim p \wedge \sim\circ\sim p)$ (cj3), (HS), 5, (MP)
7. $\sim p \rightarrow \sim\sim p$ (A3), (HS), 6, (MP)
8. $\sim\sim p$ (CM2), (HS), 7, (MP)
9. p (NN1), 8, (MP).

For *LFI1*+(AT*):

1. $\circ(p \rightarrow \sim p) \rightarrow ((p \rightarrow \sim p) \rightarrow (\sim(p \rightarrow \sim p) \rightarrow \sim\circ(p \rightarrow \sim p)))$ (bcl)

2. $(p \to \sim p) \to (\circ(p \to \sim p) \to \sim\circ(p \to \sim p)))$ (LC⋆), (LC), (AT⋆), 1, (MP)
3. $(p \to \sim p) \to \sim\circ(p \to \sim p)$ (CM2), (HS), 2, (MP)
4. $(p \to \sim p) \to (p \wedge \sim\circ\sim p)$ (cj3), (HS), 3, (MP)
5. $(p \to \sim p) \to p$ (A3), (HS), 4, (MP)
6. p (PL), (HS), 5, (MP)

For *LFI1*+(BT) and *LFI1*+(BT⋆): Proceed similarly to the proof of Proposition 3.3. ⊣

The strategy of replacing axiom (bcl) with a weaker form, namely $\circ\alpha \to (\sim\alpha \to (\sim\sim\alpha \to \beta))$, fails in *LFI1*. This is not surprising, given that *LFI1* includes the schema (NN2) among its axioms. To address this issue, we introduce a weakening of *LFI1* by replacing not only (bcl) but, respectively, also (NN2) with (bcl~) and

$$\sim\alpha \to \sim\sim\sim\alpha \tag{NN2$^\sim$}$$

We shall call this weakened system *iLFI1*. By applying the matrix $\mathfrak{M}_4$ (see the proof of Proposition 4.3), one verifies that neither Peirce's law (PL), nor (bcl) and (NN2), are provable in *iLFI1*. Moreover, proceeding as in the proof of Proposition 4.2, we obtain the following result:

PROPOSITION 5.2. *The extension of iLFI1 by* (PL) *does not yield LFI1.*

We may now consider a partially connexive variant of *LFI1*, *ALFI1* for brevity, defined by all axioms of da Costa's calculus C_ω, the detachment rule (MP), Peirce's law, (bcl~), (NN2~), (ci), (ATa), (ATb), and (cj1) through (cj3). Every *ALFI1*-bivaluation satisfies the following conditions for all $\alpha, \beta \in \mathcal{F}$:

(v1) $v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$,
(v2) $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$,
(v3) $v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$,
(v4) if $v(\sim\alpha) = 0$, then $v(\alpha) = 1$,
(v5) if $v(\sim\sim\alpha) = 1$, then $v(\alpha) = 1$,
(v6⋆) if $v(\circ\alpha) = 1$, then $v(\sim\alpha) = 0$ or $v(\sim\sim\alpha) = 0$,
(v7) if $v(\sim\circ\alpha) = 1$, then $v(\alpha) = 1 = v(\sim\alpha)$,
(v8) if $v(\alpha) = 1$, then $v(\sim(\sim\alpha \to \alpha)) = 1$,
(v9⋆) if $v(\sim\alpha) = 1$, then $v(\sim(\alpha \to \sim\alpha)) = 1$ and $v(\sim\sim\sim\alpha) = 1$,
(v10) $v(\sim\circ(\alpha \wedge \beta)) = 1$ iff $v(\sim\circ\alpha \wedge \beta)= 1$ or $v(\sim\circ\beta \wedge \alpha) = 1$,
(v11) $v(\sim\circ(\alpha \wedge \beta)) = 1$ iff $v(\sim\circ\alpha \wedge \sim\beta) = 1$ or $v(\sim\circ\beta \wedge \sim\alpha) = 1$,
(v12) $v(\sim\circ(\alpha \wedge \beta)) = 1$ iff $v(\alpha \wedge \sim\circ\beta) = 1$.

THEOREM 5.1. *For all* $\Gamma \subseteq \mathcal{F}$ *and* $\alpha \in \mathcal{F}$, $\Gamma \vdash_{ALFI1} \alpha$ *iff* $\Gamma \vDash_{ALFI1} \alpha$.

Another maximal *LFI* is *LFI2*, which is obtained by extending the calculus *Ci* with the following axiom schemata [see 7, §7]:[1] (NN2) and

$$(\sim\circ\alpha\wedge\sim\circ\beta)\leftrightarrow\sim\circ(\alpha\sharp\beta),\quad \text{for } \sharp\in\{\wedge,\vee,\to\} \qquad (\mathrm{cr}_\sharp)$$

PROPOSITION 5.3. *The extension of LFI2 with any thesis of Aristotle or Boethius trivializes LFI2.*

PROOF. For *LFI2*+(AT): After steps 1–5 in the proof of *LFI1*+(AT)

6.	$\sim p\to(\sim\circ\sim\sim p\wedge\sim\circ\sim p)$	($\mathrm{cr}_\sharp$), (HS), 5, (MP)
7.	$\sim p\to\sim\circ\sim p$	(A4), (HS), 6, (MP)
8.	$\sim p\to(\sim p\wedge\sim\sim p)$	(ci), (HS), 7, (MP)
9.	$\sim p\to\sim\sim p$	(A4), (HS), 8, (MP)
10.	$\sim\sim p$	(CM2), (HS), 9, (MP)
11.	p	(NN1), 10, (MP).

For *LFI2*+(AT*): After steps 1–3 in the proof of *LFI1*+(AT*)

4.	$(p\to\sim p)\to(\sim\circ p\wedge\sim\circ\sim p)$	($\mathrm{cr}_\sharp$), (HS), 3, (MP)
5.	$(p\to\sim p)\to\sim\circ p$	(A3), (HS), 4, (MP)
6.	$(p\to\sim p)\to(p\wedge\sim p)$	(ci), (HS), 5, (MP)
7.	$(p\to\sim p)\to p$	(A4), (HS), 6, (MP)
8.	p	(PL), (HS), 7, (MP)

For *LFI2*+(BT) and *LFI2*+(BT*): Proceed similarly to the proof of Proposition 3.3. ⊣

A strategy similar to that used for *LFI1* is essential for defining a partially connexive variant of *LFI2*. By analogy with *LFI1*, we first introduce a weakening of *LFI2*, denoted *iLFI2*. The axioms of *iLFI2* consist of those from C_ω, along with (bcl$^\sim$), (NN2$^\sim$), (ci), ($\mathrm{cr}_\sharp$), and the only rule (MP). Proceeding as in the proof of Proposition 4.2, we get:

PROPOSITION 5.4. *The extension of iLFI2 by (PL) does not yield LFI2.*

A partially connexive variant of *LFI2*, abbreviated as *ALFI2*, is obtained by adding Peirce's law and (ATa) and (ATb) to these of *iLFI2*. Every *ALFI2*-bivaluation meets the following conditions for all $\alpha,\beta\in\mathcal{F}$:

(v1) $v(\alpha\wedge\beta)=1$ iff $v(\alpha)=1$ and $v(\beta)=1$,
(v2) $v(\alpha\vee\beta)=1$ iff $v(\alpha)=1$ or $v(\beta)=1$,
(v3) $v(\alpha\to\beta)=1$ iff $v(\alpha)=0$ or $v(\beta)=1$,

[1] The language of *LFI2* originally includes the symbol '•' to denote the inconsistency operator, which is represented in this paper as '$\sim\circ$'. The logic *LFI2* is also known as *Ciore* (see [4, Theorem 3.69] and [2, Theorem 4.4.29]).

(v4) if $v({\sim}\alpha) = 0$, then $v(\alpha) = 1$,
(v5) if $v({\sim}{\sim}\alpha) = 1$, then $v(\alpha) = 1$,
(v6*) if $v(\circ\alpha) = 1$, then $v({\sim}\alpha) = 0$ or $v({\sim}{\sim}\alpha) = 0$,
(v7) if $v({\sim}\circ\alpha) = 1$, then $v(\alpha) = 1 = v({\sim}\alpha)$,
(v8) if $v(\alpha) = 1$, then $v({\sim}({\sim}\alpha \to \alpha)) = 1$,
(v9*) if $v({\sim}\alpha) = 1$, then $v({\sim}(\alpha \to {\sim}\alpha)) = 1$ and $v({\sim}{\sim}{\sim}\alpha) = 1$,
(v10) $v({\sim}\circ\alpha) = 1 = v({\sim}\circ\beta)$ iff $v({\sim}\circ(\alpha\sharp\beta)) = 1$, where $\sharp \in \{\wedge, \vee, \to\}$.

THEOREM 5.2. *For all $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$, $\Gamma \vdash_{ALFI2} \alpha$ iff $\Gamma \vDash_{ALFI2} \alpha$.*

There is another maximal *LFI* discussed in the literature. This calculus, denoted P_1', is an extension of *Ci* by two axiom schemata for $\sharp \in \{\wedge, \vee, \to\}$ (see [4, Theorem 3.69] and [2, Definition 4.4.19]):

$$\circ{\sim}\alpha \tag{cw}$$

$$\circ(\alpha \,\sharp\, \beta) \tag{cv$_\sharp$}$$

Note that any extension of P_1' with Aristotle's theses results in trivial logic. Indeed, by applying the detachment rule (MP) to (cv$_\sharp$), (AT*) (for (AT), proceed similarly), (LC), and the following instance of (bcl): $\circ(p \to {\sim}p) \to ((p \to {\sim}p) \to ({\sim}(p \to {\sim}p) \to p))$, we obtain $(p \to {\sim}p) \to p$, and consequently, p by Peirce's law. A better approach is to take *iCi* as a basic calculus and then extend it by adding the axioms (cw) and (cv$_\sharp$). We will refer to the resulting calculus as iP_1'. Proceeding as in the proof of Proposition 4.6, we obtain:

PROPOSITION 5.5. *The extension of iP_1' with Peirce's law and the schemata* (ATa) *and* (ATb)*, AP_1' for brevity, does not trivialize iP_1'.*

Unlike in *ALFI1* and *ALFI2*, the formulas ${\sim}{\sim}\alpha \to ({\sim}{\sim}{\sim}\alpha \to \beta)$ and ${\sim}(\alpha \to \beta) \to ({\sim}{\sim}(\alpha \to \beta) \to \gamma)$ are provable in AP_1'. On the other hand, $\circ{\sim}\alpha$ is not a thesis of either *ALFI1* or *ALFI2*. Moreover, since both ${\sim}(\alpha \to {\sim}\alpha) \in \mathrm{Th}(AP_1')$ and ${\sim}({\sim}\alpha \to \alpha) \in \mathrm{Th}(AP_1')$, it follows for all $\alpha, \beta \in \mathcal{F}$ that

$$\begin{gathered}{\sim}{\sim}({\sim}\alpha \to \alpha) \vdash_{AP_1'} \beta,\\ {\sim}{\sim}(\alpha \to {\sim}\alpha) \vdash_{AP_1'} \beta.\end{gathered}$$

Every AP_1'-bivaluation fulfills the following conditions for all $\alpha, \beta \in \mathcal{F}$:

(v1) $v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$,
(v2) $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$,

(v3) $v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$,
(v4) if $v({\sim}\,\alpha) = 0$, then $v(\alpha) = 1$,
(v5) if $v({\sim}\,{\sim}\,\alpha) = 1$, then $v(\alpha) = 1$,
(v6*) if $v(\circ\,\alpha) = 1$, then $v({\sim}\,\alpha) = 0$ or $v({\sim}\,{\sim}\,\alpha) = 0$,
(v7) if $v({\sim}\circ\alpha) = 1$, then $v(\alpha) = 1 = v({\sim}\,\alpha)$,
(v8) if $v(\alpha) = 1$, then $v({\sim}({\sim}\,\alpha \to \alpha)) = 1$,
(v9) if $v({\sim}\,\alpha) = 1$, then $v({\sim}(\alpha \to {\sim}\,\alpha)) = 1$,
(v10) $v(\circ\,\alpha) = 1$ for $\alpha \notin Var$.

THEOREM 5.3. *For all $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$, $\Gamma \vdash_{AP_1'} \alpha$ iff $\Gamma \vDash_{AP_1'} \alpha$.*

The calculi discussed in this section form the structure shown in Figure 2.

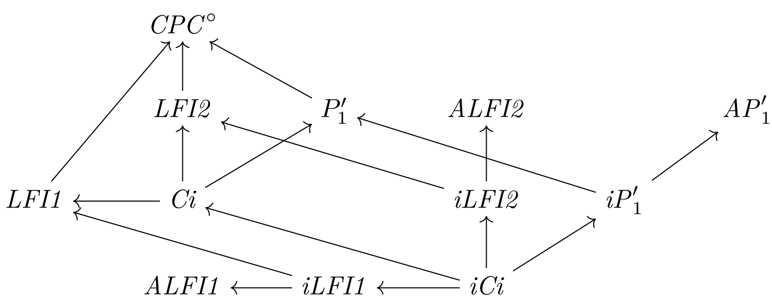

Figure 2. Non-trivial extensions of *iCi*

6. From *LFI*s to hierarchies of *LFqI*s

The crucial requirement for a calculus $\mathcal{C}$ to qualify as an *LFI* is that $\{\circ\,\alpha, \alpha, {\sim}\,\alpha\} \vdash_{\mathcal{C}} \beta$ holds for all $\alpha, \beta \in \mathcal{F}$. More precisely, we say that a calculus $\mathcal{C}$ is an *LFI* if and only if the following conditions are satisfied for $\mathcal{C}$ (Definition 23 in [3]):

(1) $\mathcal{C}$ is paraconsistent, i.e., $\{\alpha, {\sim}\,\alpha\} \nvdash_{\mathcal{C}} \beta$ for some $\alpha, \beta \in \mathcal{F}$.
(2) For some $\alpha, \beta \in \mathcal{F}$, we have:
 (a) $\{\circ\,\alpha, \alpha\} \nvdash_{\mathcal{C}} \beta$,
 (b) $\{\circ\,\alpha, {\sim}\,\alpha\} \nvdash_{\mathcal{C}} \beta$.
(3) $\{\circ\,\alpha, \alpha, {\sim}\,\alpha\} \vdash_{\mathcal{C}} \beta$ for every $\alpha, \beta \in \mathcal{F}$.

Most calculi discussed in this paper, particularly those that are extensions of *iCi*, do not meet the third criterion. To address this issue, we first need to revise the definition of *LFI*s. We say that $\mathcal{C}$ is a *Logic of Formal quasi-Inconsistency*, *LFqI* for brevity, iff the following holds:

(1) $\{\sim^{k-1}\alpha, \sim^{k}\alpha\} \nvdash_{\mathcal{C}} \beta$ for some $\alpha, \beta \in \mathcal{F}$ and every $k \geqslant 1$.

(2) For some $\alpha, \beta \in \mathcal{F}$ and $k \geqslant 1$, we have:

(k-1) $\{\circ\,\alpha, \sim^{k-1}\alpha\} \nvdash_{\mathcal{C}} \beta$,

(k) $\{\circ\,\alpha, \sim^{k}\alpha\} \nvdash_{\mathcal{C}} \beta$

(3) $\{\circ\,\alpha, \sim^{k-1}\alpha, \sim^{k}\alpha\} \vdash_{\mathcal{C}} \beta$, for every $\alpha, \beta \in \mathcal{F}$ and $k \geqslant 1$.

The definition is more general than that given for *LFI*s: Logics of Formal Inconsistency are special cases of *LFqI*. Based on this definition, we can extend the results presented in the previous sections to certain hierarchies of *LFqI*s. We begin by introducing a hierarchy of ACi_n-calculi, where $n \geqslant 1$. From a technical perspective, this hierarchy is obtained by replacing axiom (bcl) of *ACi* with the following schema:

$$\circ\,\alpha \to (\sim^{k-1}\alpha \to (\sim^{k}\alpha \to \beta)), \tag{bcl^k}$$

where $\sim^{k}\alpha$ is an abbreviation for $\overbrace{\sim\sim\ldots\sim}^{\text{k-times}}\alpha$, with $k \geqslant 1$ (and $n = k-1$). Of course, if $k = 1$, then (bcl^1) is simply (bcl) and ACi_0 is *ACi*, that is, a partially connexive *LFI* introduced in Section 2.

The axiom set of ACi_n-calculi involves all instances of the axiom schemata for da Costa's C_ω, along with the following additional axioms: (ci), (ATa), (ATb), (bcl^k), where $k \geqslant 1$ and $n = k-1$. The sole inference rule is (MP).

Every ACi_n-bivaluation satisfies the following conditions for all α, $\beta \in \mathcal{F}$ and $k, n \in \mathbb{N}$ such that $k \geqslant 1$ and $n = k-1$:

(v1) $v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$,
(v2) $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$,
(v3) $v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$,
(v4) if $v(\sim\alpha) = 0$, then $v(\alpha) = 1$,
(v5) if $v(\sim\sim\alpha) = 1$, then $v(\alpha) = 1$,
(v6^k) if $v(\circ\,\alpha) = 1$, then $v(\sim^{k-1}\alpha) = 0$ or $v(\sim^{k}\alpha) = 0$,
(v7) if $v(\sim\circ\,\alpha) = 1$, then $v(\alpha) = 1 = v(\sim\alpha)$,
(v8) if $v(\alpha) = 1$, then $v(\sim(\sim\alpha \to \alpha)) = 1$,
(v9) if $v(\sim\alpha) = 1$, then $v(\sim(\alpha \to \sim\alpha)) = 1$.

Theorem 6.1. *For all $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$, $\Gamma \vdash_{ACi_n} \alpha$ iff $\Gamma \vDash_{ACi_n} \alpha$.*

Each calculus ACi_i, where $i > 1$, is strictly weaker than all its predecessors in the hierarchy. Furthermore, all the calculi except for ACi_0

are *LFqI*s, but not *LFI*s. A similar result can be obtained by extending ACi_n-calculi with the axiom $\sim(\alpha \wedge \sim \alpha) \to \circ\, \alpha$. Every $ACil_n$-bivaluation satisfies all the conditions of an ACi_n-bivaluation, and the following clause:

if $v(\sim(\alpha \wedge \sim \alpha)) = 1$, then $v(\circ\, \alpha) = 1$.

Notice that $ACil_0$, the strongest calculus in the hierarchy, is an *LFI*; however, starting from $ACil_1$ they are all solely *LFqI*s. This paradigm changes for the hierarchies based on the calculus *iCi*, specifically for the following calculi:[2]

$ACia_n = ACi_n + \circ\, \alpha \wedge \circ\, \beta \to \circ(\alpha \sharp \beta)$ for $\sharp \in \{\wedge, \vee, \to\}$,
$ACio_n = ACi_n + \circ\, \alpha \vee \circ\, \beta \to \circ(\alpha \sharp \beta)$ for $\sharp \in \{\wedge, \vee, \to\}$,
$ACila_n = ACia_n + \sim(\alpha \wedge \sim \alpha) \to \circ\, \alpha$,
$ACilo_n = ACio_n + \sim(\alpha \wedge \sim \alpha) \to \circ\, \alpha$.

Starting from X_n (where X is either *ACia*, *ACio*, *ACila*, or *ACilo*), all are *LFqI*s but not *LFI*s. This follows from the assumption that $\circ\, \alpha \to (\sim^{k-1} \alpha \to (\sim^k \alpha \to \beta))$ must be an axiom of X_n if and only if $k > 1$; otherwise, the logic becomes trivial. Every $ACia_n$ (resp. $ACio_n$)-bivaluation satisfies all the conditions of an ACi_n-bivaluation with the restriction that $k > 1$ in clause (v6^k), in addition to the following:

(v10) if $v(\circ\, \alpha) = 1 = v(\circ\, \beta)$, then $v(\circ(\alpha \sharp \beta)) = 1$, where $\sharp \in \{\wedge, \vee, \to\}$, for $ACia_n$-bivaluation,

(v10) if $v(\circ\, \alpha) = 1$ or $v(\circ\, \beta) = 1$, then $v(\circ(\alpha \sharp \beta)) = 1$, where $\sharp \in \{\wedge, \vee, \to\}$, for $ACio_n$-bivaluation.

Every $ACila_n$ (resp. $ACilo_n$)-bivaluation satisfies all the conditions of an $ACia_n$ (resp. $ACio_n$)-bivaluation, along with the additional clause:

(v11) if $v(\sim(\alpha \wedge \sim \alpha)) = 1$, then $v(\circ\, \alpha) = 1$.

THEOREM 6.2. *For all $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$: $\Gamma \vdash_{X_n} \alpha$ iff $\Gamma \vDash_{X_n} \alpha$, where X_n is either $ACia_n$, $ACio_n$, $ACila_n$, or $ACilo_n$, and $n = k-1$ with $k > 1$.*

7. Conclusions

The paper presents initial results on the connexive variants of *LFI*s. We primarily focused on the extensions of *LFI*s, which were obtained by adding the alternatives of Aristotle's theses into the postulates of *LFI*s.

[2] The abbreviation '$X = Y + c$' means 'the postulates of Y and c axiomatize X'.

This approach led to the definition of several non-trivial calculi, including *ACi*, *ACil*, *ACia*, *ACio*, *ACila*, *ACilo*, *ALFI1*, *ALFI2*, and AP'_1. Some of them do not admit the principle of gentle explosion and therefore can no longer be considered original *LFI*s. To address this issue, we proposed an alternative and more general definition of *LFI*s by introducing the Logics of Formal quasi-Inconsistency (*LFqI*s). In contrast to *LFI*s, the calculi defined over *LFqI*s are all non-trivial, partially connexive, strongly paraconsistent logics.

As mentioned, this paper provides an initial discussion on the partially connexive variants of *LFqI*s. An open problem is to develop their fully connexive variants, i.e., to introduce calculi based on *LFqI*s that incorporate not only Aristotle's but also Boethius' theses among their axiom schemata. Moreover, the paper employs a non-deterministic bivaluational semantics originating from the work of da Costa and Alves. This type of semantics is not fully compositional. On the other hand, since some *LFI*s, for instance, *LFI1* and *LFI2*, have fully compositional trivalent semantics, a subsequent challenge is to propose a fully compositional many-valued semantics for the *LFqI*s and their connexive extensions.

Appendix

Listed below are the calculi discussed in this paper. The only rule is (MP). '$X = a{+}b{+}c$' (resp. '$X = Y{+}c$') means 'the postulates: a, b and c axiomatize X' (resp. 'the postulates of Y and c axiomatize X').

$C_\omega = \text{(A1)}{+}\cdots{+}\text{(A8)}{+}\text{(NN1)}{+}\text{(ExM)}$
$Ci = C_\omega{+}\text{(bcl)}{+}\text{(ci)} = C_\omega{+}(\text{bcl}^{\sim}){+}\text{(cn)}{+}\text{(ci)}$
$Cil = Ci{+}\text{(cl)}$
$Cila = Cil{+}\text{(ca)}$
$Cilo = Cil{+}\text{(co)}$
$ACi = Ci{+}\text{(ATa)}{+}\text{(ATb)} = Ci{+}\text{(AT)}{+}(\text{AT}^\star)$
$ACil = ACi{+}\text{(cl)}$
$iCi = C_\omega{+}(\text{bcl}^{\sim}){+}\text{(ci)}$
$AiCia = iCi{+}\text{(ATa)}{+}\text{(ATb)}{+}\text{(ca)}$
$AiCio = iCi{+}\text{(ATa)}{+}\text{(ATb)}{+}\text{(co)}$
$ACia = AiCia{+}\text{(PL)}$
$ACio = AiCio{+}\text{(PL)}$
$ACila = ACia{+}\text{(cl)}$

$ACilo = ACio$+(cl)
$LFI1 = Ci$+(NN2)+(cj1)+(cj2)+(cj3)
$iLFI1 = iCi$+(NN2$^\sim$)+(cj1)+(cj2)+(cj3)
$ALFI1 = iLFI1$+(PL)+(ATa)+(ATb)
$LFI2 = Ci$+(NN2)+(cr$_\sharp$)
$iLFI2 = iCi$+(NN2$^\sim$)+(cr$_\sharp$)
$ALFI2 = iLFI2$+(PL)+(ATa)+(ATb)
$P'_1 = Ci$+(cw)+(cv$_\sharp$)
$iP'_1 = iCi$+(cw)+(cv$_\sharp$)
$AP'_1 = iP'_1$+(PL)+(ATa)+(ATb)
ACi+(ca) = ACi+(co) = $ALFI1$+(NN2) = $ALFI2$+(NN2) =
$TRIV$ (trivial logic)

Acknowledgement. I am grateful to Prof. Andrzej Pietruszczak for helpful comments on an earlier draft.

References

[1] Avron, A., B. Konikowska, and A. Zamansky, "Systematic construction of analytic calculi for logics of formal inconsistency", in J.-Y. Béziau and M. E. Coniglio (eds.), *Logic Without Frontiers. Festschrift for Walter Alexandre Carnielli on the Occasion of His 60th Birthday*, College Publications, 2011.

[2] Carnielli, W., and M. E. Coniglio, *Paraconsistent Logic: Consistency, Contradiction and Negation, Logic, Epistemology, and the Unity of Science*, Springer, Berlin-Heidelberg, 2016.

[3] Carnielli, W. A., M. E. Coniglio and J. Marcos, "Logics of formal inconsistency", pages 1–93 in D. M. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*, Vol. 14, 2nd edn., Springer, 2007. DOI: 10.1007/978-1-4020-6324-4_1

[4] Carnielli, W. A., and J. Marcos, "A taxonomy of C-systems", pages 1–94 in W. A. Carnielli, M. E. Coniglio and I. M. L. D'Ottaviano (eds.), *Paraconsistency. The Logical Way to the Inconsistent*, Routledge, 2002.

[5] Carnielli, W., and A. Rodrigues, "An epistemic approach to paraconsistency: a logic of evidence and truth", *Synthese*, 196: 3789–3813, 2019. DOI: 10.1007/s11229-017-1621-7

[6] Carnielli, W., J. Bueno-Soler, "Where the truth lies: A paraconsistent approach to Bayesian epistemology", *Studia Logica*, 113(2): 397–418, 2025. DOI: 10.1007/s11225-024-10144-y

[7] Carnielli, W., J. Marcos and S. de Amo, "Formal inconsistency and evolutionary databases", *Logic and Logical Philosophy*, 8: 115–152, 2000. DOI: 10.12775/LLP.2000.008

[8] Ciuciura, J., "Intuitionistic implication and logics of formal inconsistency", *Axioms*, 13: 738, 2024. DOI: 10.3390/axioms13110738

[9] Ciuciura, J., "Variations on the calculi C_n of da Costa", *Studia Logica*, 113(4): 1101–1137, 2025. DOI: 10.1007/s11225-025-10170-4

[10] Ciuciura, J., "Gently paraconsistent extensions of C_1, intuitionistic implication, De Morgan laws, and the law of non-contradiction", *Studia Logica*, 2025. DOI: 10.1007/s11225-025-10171-3

[11] Ciuciura, J., "C-systems of da Costa and Aristotle's theses", *Journal of Logic and Computation*, 35(4): 1–19, 2025. DOI: 10.1093/logcom/exaf026

[12] Ciuciura, J., "Partial connexivity and the logic CG'_3", *Logic Journal of the IGPL*, 34(3): 1–20, 2026. DOI: 10.1093/jigpal/jzaf094

[13] da Costa, N. C. A., "On the theory of inconsistent formal systems", *Notre Dame Journal of Formal Logic*, 15(4): 497–510, 1974. DOI: 10.1305/ndjfl/1093891487

[14] da Costa, N. C. A., and E. Alves, "A semantical analysis of the calculi C_n", *Notre Dame Journal of Formal Logic*, 18: 621–630, 1977. DOI: 10.1305/ndjfl/1093888132

[15] Estrada-González, L., and E. Ramírez-Cámara, "A comparison of connexive logics", *The IfCoLog Journal of Logics and Their Applications*, 3(3): 341–356, 2016.

[16] Lenzen, W., "Rewriting the history of connexive logic", *Journal of Philosophical Logic*, 51: 525–553, 2022. DOI: 10.1007/s10992-021-09640-6

[17] Mortensen, C., "Aristotle's thesis in consistent and inconsistent logics", *Studia Logica*, 43: 107–116, 1984. DOI: 10.1007/BF00935744

[18] Omori H., "A simple connexive extension of the basic relevant logic BD", *IfCoLog Journal of Logics and their Applications*, 3: 467–478, 2016.

[19] Pogorzelski, W. A., and P. Wojtylak, *Completeness Theory for Propositional Logics*, Studies in Universal Logic, Birkhäuser Basel, 2008.

[20] Routley, R., and R. K. Meyer, "Dialectical logic, classical logic, and the consistency of the world", *Studies in Soviet Thought*, 16(1–2): 1–25, 1976. DOI: 10.1007/BF00832085

[21] Routley, R., and H. Montgomery, "On systems containing Aristotle's thesis", *The Journal of Symbolic Logic*, 33(1): 82–96, 1968. DOI: 10.2307/2270055

[22] Sylvan, R., "Variations on da Costa C systems and dual-intuitionistic logics I. Analyses of C_ω and CC_ω", *Studia Logica*, 49: 47–65, 1990. DOI: 10.1007/BF00401553

[23] Urbas, I., "Paraconsistency and the C-systems of da Costa", *Notre Dame Journal of Formal Logic*, 30(4): 583–597, 1989.

[24] Wansing, H., "Connexive logic", in E. N. Zalta and U. Nodelman (eds.), *The Stanford Encyclopedia of Philosophy*, Summer 2023 Edition. https://plato.stanford.edu/archives/sum2023/entries/logic-connexive

[25] Wansing, H., and H. Omori, "Connexive logic, connexivity, and connexivism: Remarks on terminology", *Studia Logica*, 112(1–2): 1–35, 2024. DOI: 10.1007/s11225-023-10082-1

JANUSZ CIUCIURA
Department of Logic and Methodology of Science
Institute of Philosophy
University of Łódź, Poland
janusz.ciuciura@uni.lodz.pl
https://orcid.org/0000-0001-9965-9822