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A Completeness Theorem for a Functionally Complete Łukasiewicz Logic

Abstract. We present a new axiomatization for a truth-functionally complete version of $\mathbf{L3}$, a three-valued propositional logic, using two propositional constants and the Łukasiewicz implication as primitive symbols. We develop a corresponding proof system that incorporates Łukasiewicz’s axioms, a variant of Śłupecki’s postulates for a 0-ary connective, and some properties of the Baaz *delta operation*. Using a weakened Deduction Theorem and adapting Henkin’s method to this three-valued setting, we establish a Completeness Theorem for the system.

Keywords: many-valued logics; three-valued logics; propositional logic; non-classical logics

1. Introduction

In (1974), Goldberg et al. proved that Wajsberg’s axiomatization of $\mathbf{L3}$ (a three-valued propositional calculus) is *strongly complete*, and therefore *weakly complete*. Although the result was not entirely original (1974, p. 325), they argued that the “simplicity and directness” of their proof made it “worthy of reporting”. In this paper, we provide a new axiomatization of a truth-complete version of $\mathbf{L3}$ and prove its strong completeness as well.

As it is well known, $\mathbf{L3}$ is not a *functionally complete* logic: some truth-functions cannot be defined in terms of the familiar \neg , \rightarrow , \wedge , \vee , \equiv (see, e.g. Malinowski, 2007, p. 37). To solve this problem, Śłupecki (1946, 1967) first added a new 1-ary connective assigning the intermediate truth-value, 0.5, to every proposition; then, he extended Wajsberg’s

axioms for **L3** with two new principles governing that connective. Goldberg et al. observed that their proof could be easily generalized to this system, but did not provide the details. We have replaced Słupecki's unary connective by a 0-ary one (whose interpretation is 0.5), but, since our set of primitive symbols still allows us to define the original operator, the *expressive power* is kept.

The strategy to obtain the Completeness Theorem is the following. Firstly, we present the language and semantics of **LS**, a truth-functionally complete Łukasiewicz logic. Then, we offer a proof system consisting of Łukasiewicz's axioms, some properties of the Baaz (1996) *delta operation*, one of the axioms formulated by Avron (1991, p. 289) (which plays an essential role in the completeness proof) and a variant of Słupecki's two postulates for the new 0-ary connective, \top . We prove a weaker version of the Deduction Theorem, taken from Novak's work (2005, p. 261) and involving the Baaz delta operation. Finally, in the last section, we adapt the usual definitions of *consistency*, *maximality* and *completeness* (of a set of formulas) to our three-valued framework and get a Completeness Theorem following Henkin's procedure.

2. Preliminaries

In this section, we first introduce the language of the truth-functionally complete **L3**, but replacing Słupecki's unary connective by a 0-ary one. We call this logic **LS**.

The logic **LS** has propositional letters p_1, p_2, \dots , two 0-ary connectives, \perp and \top , and implication \rightarrow . Formulas are defined as follows:

$$p \mid \perp \mid \top \mid \varphi \rightarrow \psi$$

Now, we define the semantics of our system (*valuation*, *tautology*, *model* and *consequence*).

DEFINITION 2.1 (Valuation). A *valuation* is a function \mathbf{v} assigning to each propositional variable p its truth-value $\mathbf{v}(p) \in \{0, 0.5, 1\}$. This is extended to all formulas as follows:

1. $\mathbf{v}(\perp) = 0$,
2. $\mathbf{v}(\top) = 0.5$,
3. $\mathbf{v}(\varphi \rightarrow \psi) = \min\{1, (1 - \mathbf{v}(\varphi)) + \mathbf{v}(\psi)\}$.

Clearly, \rightarrow is a Łukasiewicz implication.

We fix 1 as the only *designated* truth-value. Therefore: $\models \varphi$ if $\mathbf{v}(\varphi) = 1$ for every valuation \mathbf{v} .

DEFINITION 2.2 (Model). A *model* of a set of formulas Γ is a valuation \mathbf{v} such that $\mathbf{v}(\gamma) = 1$ for all $\gamma \in \Gamma$.

DEFINITION 2.3 (Consequence). $\Gamma \models \varphi$ if, for every valuation \mathbf{v} , if \mathbf{v} is a model of Γ , then $\mathbf{v}(\varphi) = 1$.

The *Lukasiewicz negation* and *strong conjunction* are definable in terms of the set of primitive symbols $\{\perp, \top, \rightarrow\}$ as follows:

$$\begin{aligned}\neg\varphi &:= \varphi \rightarrow \perp \\ \varphi \&\psi &:= \neg(\varphi \rightarrow \neg\psi)\end{aligned}$$

We have $\mathbf{v}(\neg\varphi) = 1 - \mathbf{v}(\varphi)$ and $\mathbf{v}(\varphi \&\psi) = \max\{0, (\mathbf{v}(\varphi) + \mathbf{v}(\psi)) - 1\}$. Moreover, new connectives, \wedge , \vee , and \equiv , the Baaz operator, Δ , and the original Słupecki's T operator are introduced:

$$\begin{aligned}\varphi \wedge \psi &:= \varphi \&(\varphi \rightarrow \psi) \\ \varphi \vee \psi &:= (\varphi \rightarrow \psi) \rightarrow \psi \\ \varphi \equiv \psi &:= (\varphi \rightarrow \psi) \&(\psi \rightarrow \varphi) \\ \Delta\varphi &:= \varphi \&\varphi \\ T\varphi &:= \top \rightarrow (\perp \&\varphi)\end{aligned}$$

THEOREM 2.1. *The set $\{\perp, \top, \rightarrow\}$ is truth-functionally complete.*

PROOF. In (1946), Jerzy Słupecki has showed that $\{T, \neg, \rightarrow\}$ is truth-functionally complete (see also an English translation Słupecki (1967) of Słupecki's article (1946)). Since $\{\perp, \top, \rightarrow\}$ can be used to define $\{\neg, T\}$, it is truth-functionally complete. \square

3. The system \mathbf{LS}

Axiom schemas:

1. *Lukasiewicz's axioms:*

- (L1) $\varphi \rightarrow (\psi \rightarrow \varphi)$,
- (L2) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$,
- (L3) $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$,
- (L4) $((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)$.

2. *Axioms of Δ .*

- ($\Delta 1$) $\Delta \varphi \rightarrow \varphi$,
 ($\Delta 2$) $(\Delta \varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\Delta \varphi \rightarrow \psi) \rightarrow (\Delta \varphi \rightarrow \chi))$,
 ($\Delta 3$) $\Delta(\varphi \rightarrow \psi) \rightarrow (\Delta \varphi \rightarrow \Delta \psi)$,
 ($\Delta 4$) $\Delta \varphi \rightarrow \Delta \Delta \varphi$,
 ($\Delta 5$) $\Delta(\varphi \rightarrow \psi) \vee \Delta(\psi \rightarrow \varphi)$,
 ($\Delta 6$) $\neg \Delta(\neg \Delta \varphi) \rightarrow \Delta \varphi$.

3. *The Axiom of Avron:*

$$(Av) \quad (\varphi \vee (\varphi \rightarrow \psi)) \vee (\psi \rightarrow \chi).$$

4. *New axioms of Stupecki:*

- (S11) $\top \rightarrow \neg \top$,
 (S12) $\neg \top \rightarrow \top$.

Rules of inference:

- (MP) From φ and $\varphi \rightarrow \psi$ infer ψ .
 (N) From φ infer $\Delta \varphi$.

A *proof* in a set of formulas Γ is a sequence $\varphi_1, \dots, \varphi_n$ of formulas such that each φ_i :

1. is an axiom of **LS**, or
2. is in Γ , or
3. follows from some preceding φ_j, φ_k by (MP), or
4. follows from some preceding φ by (N).

- φ is *provable from Γ in LS* (in symbols, $\Gamma \vdash \varphi$) if φ is the last member of a proof in Γ .
- φ is *provable in LS* (in symbols, $\vdash \varphi$) if φ is the last member of a proof from \emptyset .

The formulas below follow from Łukasiewicz's axioms (see, e.g., [Hájek, 1998](#), p. 66):

$$\varphi \rightarrow ((\varphi \rightarrow \psi) \rightarrow \psi) \tag{1}$$

$$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi)), \tag{2}$$

$$\varphi \rightarrow \varphi \tag{3}$$

$$\perp \rightarrow \varphi \tag{4}$$

$$\neg\neg\varphi \rightarrow \varphi, \quad (5)$$

$$(\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi) \quad (6)$$

$$\varphi \rightarrow \neg\neg\varphi \quad (7)$$

Moreover, the following axioms and properties of Basic Fuzzy Logic (BL) follow from Łukasiewicz's axioms (see, e.g., [Hájek, 1998](#), Lemma 3.1.9, Lemma 2.2.8 and 2.2.10(16)):

$$(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi \rightarrow \chi)) \quad (8)$$

$$(\varphi \rightarrow (\psi \rightarrow \chi) \rightarrow ((\varphi \& \psi) \rightarrow \chi), \quad (9)$$

$$((\varphi \& \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi)), \quad (10)$$

$$(\varphi \& \psi) \rightarrow \varphi \quad (11)$$

$$(\varphi \& \psi) \rightarrow (\psi \& \varphi) \quad (12)$$

$$\varphi \rightarrow (\psi \rightarrow \varphi \& \psi) \quad (13)$$

$$(\varphi \& \psi) \& \chi \rightarrow \varphi \& (\psi \& \chi) \quad (14)$$

$$\varphi \& (\psi \& \chi) \rightarrow (\varphi \& \psi) \& \chi \quad (15)$$

Hence we obtain:

PROPOSITION 3.1. *The formulas (1)–(15) are provable in **LS**.*

THEOREM 3.1 (Deduction theorem). *For all $\Gamma \subseteq \text{Fm}$ and $\varphi, \psi \in \text{Fm}$,*

$$\Gamma \cup \{\varphi\} \vdash \psi \text{ iff } \Gamma \vdash \Delta\varphi \rightarrow \psi.$$

PROOF. Let $\Gamma \cup \{\varphi\} \vdash \psi$. The proof is by induction on the number of formulas, k , in the sequence ψ_1, \dots, ψ_n forming the derivation of ψ from $\Gamma \cup \{\varphi\}$.

Base step: $k = 1$.

1. ψ_1 is an Axiom. Then:

a. $\Gamma \vdash \psi_1$

ψ_1 is an Axiom

b. $\Gamma \vdash \Delta\varphi \rightarrow \psi_1$

(Ł1), 1, (MP)

2. $\psi_1 \in \Gamma$. Then, $\Gamma \vdash \psi_1$, so the same argument works here.

3. $\psi_1 = \varphi$. Since $\Delta\varphi \rightarrow \varphi$ is an instance of Axiom ($\Delta 1$), $\Gamma \vdash \Delta\varphi \rightarrow \varphi$.

Inductive step: suppose that the derivation finishes with the step φ_k and the theorem holds for all the previous ones.

1. ψ_k have been obtained using (MP). Thus, $\psi_j \rightarrow \psi_k$ and ψ_j appear earlier in the derivation, so $\Gamma \cup \{\varphi\} \vdash \psi_j \rightarrow \psi_k$ and $\Gamma \cup \{\varphi\} \vdash \psi_j$. Hence:

1. $\Gamma \vdash \Delta \varphi \rightarrow (\psi_j \rightarrow \psi_k)$ IH
2. $\Gamma \vdash \Delta \varphi \rightarrow \psi_j$ IH
3. $\Gamma \vdash (\Delta \varphi \rightarrow \psi_j) \rightarrow (\Delta \varphi \rightarrow \psi_k)$ ($\Delta 2$), 1, (MP)
4. $\Gamma \vdash \Delta \varphi \rightarrow \psi_k$ 3, 2, (MP)

2. ψ_k have been obtained using Rule (N). It follows that ψ_k is of the form $\Delta \theta$, so θ appears earlier in the derivation. In consequence, we have $\Gamma \cup \{\varphi\} \vdash \theta$ and hence:

1. $\Gamma \vdash \Delta \varphi \rightarrow \theta$ IH
2. $\Gamma \vdash \Delta(\Delta \varphi \rightarrow \theta)$ 1, Rule (N)
3. $\Gamma \vdash \Delta \Delta \varphi \rightarrow \Delta \theta$ ($\Delta 3$), 2, (MP)
4. $\Gamma \vdash \Delta \varphi \rightarrow \Delta \theta$ 3, ($\Delta 4$), ($\text{L}2$), (MP)

Conversely, let $\Gamma \vdash \Delta \varphi \rightarrow \psi$. Then:

1. $\Gamma \cup \{\varphi\} \vdash \Delta \varphi \rightarrow \psi$ hypothesis
2. $\Gamma \cup \{\varphi\} \vdash \varphi$ $\varphi \in \Gamma \cup \{\varphi\}$
3. $\Gamma \cup \{\varphi\} \vdash \Delta \varphi$ 2, Rule (N)
4. $\Gamma \cup \{\varphi\} \vdash \psi$ 1, 3, (MP) \square

THEOREM 3.2 (Soundness). *If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.*

PROOF. Straightforward. All the axioms are tautologies and both (MP) and (N) preserve validity:

1. $\Gamma \vdash \varphi \rightarrow \psi$ hypothesis
2. $\Gamma \vdash \varphi$ hypothesis
3. $\Gamma \models \varphi \rightarrow \psi$ IH
4. $\Gamma \models \varphi$ IH
 - a. Γ does not have any model
 - 5a. $\Gamma \models \psi$ Definition 2.3
 - b. Every model \mathbf{v} of Γ is a model of φ and $\varphi \rightarrow \psi$
 - 5b. $\mathbf{v}(\gamma) = 1$, for all $\gamma \in \Gamma$ Definition 2.2
 - 6b. $\mathbf{v}(\varphi) = 1$
 - 7b. $\mathbf{v}(\varphi \rightarrow \psi) = 1$
 - 8b. $\mathbf{v}(\psi) = 1$ 6b, 7b, Definition 2.1
 - 9b. $\Gamma \models \psi$ 5b, 8b, Definition 2.3
10. $\Gamma \models \psi$ 5a, 9b

The proof for (N) is analogous. \square

4. Completeness

A set of formulas Γ is *inconsistent* iff $\Gamma \vdash \perp$. Otherwise it is *consistent*. A set of formulas Γ is *maximal* iff for every formula φ such that $\varphi \notin \Gamma$, $\Gamma \cup \{\varphi\}$ is inconsistent. A set of formulas Γ is *complete* iff for every two formulas φ and ψ , $\Gamma \vdash \varphi \rightarrow \psi$ or $\Gamma \vdash \psi \rightarrow \varphi$.

PROPOSITION 4.1. *If Γ is a consistent set of formulas, then $\top, \neg\top \notin \Gamma$.*

PROOF. By contrapositive. Suppose that $\top \in \Gamma$. Then:

- | | |
|--|-------------------|
| 1. $\Gamma \vdash \top$ | $\top \in \Gamma$ |
| 2. $\Gamma \vdash \top \rightarrow (\top \rightarrow \perp)$ | (SŁ1) |
| 3. $\Gamma \vdash \top \rightarrow \perp$ | 2, 1, (MP) |
| 4. $\Gamma \vdash \perp$ | 3, 1, (MP) |

Suppose that $\neg\top \in \Gamma$. Hence, $\Gamma \vdash \neg\top$, so we can use Axiom (S12) and (MP) to get $\Gamma \vdash \top$, making Γ inconsistent (by the previous argument). \square

PROPOSITION 4.2. *For any maximal consistent and complete set Γ of formulas,*

1. *If $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$.*
2. *$\varphi \ \& \ \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$.*
3. *If $\varphi \vee \psi \in \Gamma$, then $\varphi \in \Gamma$ or $\psi \in \Gamma$.*

PROOF. 1. Suppose that $\Gamma \vdash \varphi$, but let us assume, for the sake of contradiction, that $\varphi \notin \Gamma$. Since Γ is maximal, it follows that $\Gamma \cup \{\varphi\} \vdash \perp$. Hence:

- | | |
|--|--|
| 1. $\Gamma \vdash \varphi$ | hypothesis |
| 2. $\Gamma \cup \{\varphi\} \vdash \perp$ | hypothesis |
| 3. $\Gamma \vdash \Delta(\varphi) \rightarrow \perp$ | 2, DT |
| 4. $\Gamma \vdash \Delta \varphi$ | 3, Rule (N) |
| 5. $\Gamma \vdash \perp$ | 3, 4, (MP) contradicting the consistency of Γ . |

2, Let $\varphi \ \& \ \psi \in \Gamma$. Suppose, for the sake of contradiction, that $\varphi \notin \Gamma$ or $\psi \notin \Gamma$, say φ . Since Γ is maximal, it follows that $\Gamma \cup \{\varphi\} \vdash \perp$. Thus:

- | | |
|--|------------|
| 1. $\Gamma \vdash \varphi \ \& \ \psi$ | hypothesis |
| 2. $\Gamma \cup \{\varphi\} \vdash \perp$ | hypothesis |
| 3. $\Gamma \vdash \Delta(\varphi) \rightarrow \perp$ | 2, DT |

4. $\Gamma \vdash \varphi$ (BL4), 1, (MP)
5. $\Gamma \vdash \Delta \varphi$ 4, Rule (N)
6. $\Gamma \vdash \perp$ 3, 5, (MP), contradicting the consistency of Γ .

In case $\psi \notin \Gamma$, the proof still works, as (12) guarantees that $\&$ is commutative.

Conversely, let $\varphi \in \Gamma$ and $\psi \in \Gamma$, but assume that $\varphi \& \psi \notin \Gamma$. Because Γ is maximal, $\Gamma \cup \{\varphi \& \psi\} \vdash \perp$ and hence we have:

1. $\Gamma \vdash \varphi$ hypothesis
2. $\Gamma \vdash \psi$ hypothesis
3. $\Gamma \cup \{\varphi \& \psi\} \vdash \perp$ hypothesis
4. $\Gamma \vdash \Delta(\varphi \& \psi) \rightarrow \perp$ 3, DT
5. $\Gamma \vdash \psi \rightarrow (\varphi \& \psi)$ (13), 1, (MP)
6. $\Gamma \vdash \varphi \& \psi$ 5, 2, (MP)
7. $\Gamma \vdash \Delta(\varphi \& \psi)$ 6, Rule (N)
8. $\Gamma \vdash \perp$ 4, 7, (MP), contradicting the consistency of Γ .

3. Let $\varphi \vee \psi \in \Gamma$. Assume, for the sake of contradiction, that $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. Since Γ is maximal, $\Gamma \cup \{\varphi\} \vdash \perp$ and $\Gamma \cup \{\psi\} \vdash \perp$. Therefore:

1. $\Gamma \vdash \varphi \vee \psi$ hypothesis
2. $\Gamma \cup \{\varphi\} \vdash \perp$ hypothesis
3. $\Gamma \cup \{\psi\} \vdash \perp$ hypothesis
4. $\Gamma \vdash \Delta \varphi \rightarrow \perp$ 3, DT
5. $\Gamma \vdash \Delta \psi \rightarrow \perp$ 4, DT
6. $\Gamma \vdash (\varphi \rightarrow \psi) \rightarrow \psi$ 1
7. $\Gamma \vdash (\psi \rightarrow \varphi) \rightarrow \varphi$ (L4), 6, (MP)
8. $\Gamma \vdash \Delta((\varphi \rightarrow \psi) \rightarrow \psi)$ 6, Rule (N)
9. $\Gamma \vdash \Delta((\psi \rightarrow \varphi) \rightarrow \varphi)$ 7, Rule (N)
10. $\Gamma \vdash \Delta(\varphi \rightarrow \psi) \rightarrow \Delta \psi$ (Δ 3), 8, (MP)
11. $\Gamma \vdash \Delta(\psi \rightarrow \varphi) \rightarrow \Delta \varphi$ (Δ 3), 9, (MP)
12. $\Gamma \vdash \Delta(\varphi \rightarrow \psi) \rightarrow \perp$ (L2), 10, 5, (MP)
13. $\Gamma \vdash \Delta(\psi \rightarrow \varphi) \rightarrow \perp$ (L2), 11, 4, (MP)

However, since Γ is complete, we know that either $\Gamma \vdash \varphi \rightarrow \psi$ or $\Gamma \vdash \psi \rightarrow \varphi$. In any case, we can use Rule (N), 12 (or 13) and (MP) to get $\Gamma \vdash \perp$, which is a contradiction. \square

PROPOSITION 4.3. *For any maximal consistent and complete set Γ of formulas,*

1. If $\neg\psi \in \Gamma$, then $\psi \rightarrow \chi \in \Gamma$.
2. If $\chi \in \Gamma$, then $\psi \rightarrow \chi \in \Gamma$.
3. If $\psi \notin \Gamma$ and $\neg\chi \notin \Gamma$, then $\psi \rightarrow \chi \in \Gamma$.

PROOF. 1 Let $\neg\psi \in \Gamma$. Then:

1. $\Gamma \vdash \neg\psi$ $\neg\psi \in \Gamma$
2. $\Gamma \vdash \psi \rightarrow \perp$ 1,
3. $\Gamma \vdash \perp \rightarrow \chi$ (4)
4. $\Gamma \vdash \psi \rightarrow \chi$ 2, 3, (L2), (MP)
5. $\psi \rightarrow \chi \in \Gamma$ (4), Proposition 4.2(1)

2. Let $\chi \in \Gamma$. Then:

1. $\Gamma \vdash \chi$ $\chi \in \Gamma$
2. $\Gamma \vdash \chi \rightarrow (\psi \rightarrow \chi)$ (L1)
3. $\Gamma \vdash \psi \rightarrow \chi$ 2, 1, (MP)
4. $\psi \rightarrow \chi \in \Gamma$ 3, Proposition 4.2(1)

3. Let $\psi \notin \Gamma$ and $\neg\chi \notin \Gamma$. We have that:

1. $\Gamma \vdash (\psi \vee (\psi \rightarrow \chi)) \vee (\chi \rightarrow \perp)$ (Av)
2. $\Gamma \vdash (\psi \vee (\psi \rightarrow \chi)) \vee \neg\chi$ 1
3. $(\psi \vee (\psi \rightarrow \chi)) \vee \neg\chi \in \Gamma$ 2, Proposition 4.2(1)
4. $(\psi \vee (\psi \rightarrow \chi)) \in \Gamma$ or $\neg\chi \in \Gamma$ 3, Proposition 4.2(3)
5. $(\psi \vee (\psi \rightarrow \chi)) \in \Gamma$ 4, $\neg\chi \notin \Gamma$
6. $\psi \in \Gamma$ or $\psi \rightarrow \chi \in \Gamma$ 5, Proposition 4.2(3)
7. $\psi \rightarrow \chi \in \Gamma$ 6, $\psi \notin \Gamma$ \square

In the standard way we get:

LEMMA 4.1 (Lindenbaum). *Every consistent set of formulas Γ can be extended to a maximal consistent set Γ^* .*

So, we obtain:

PROPOSITION 4.4. *Γ^* is complete.*

PROOF. Suppose that there are formulas φ and ψ such that $\Gamma^* \not\vdash \varphi \rightarrow \psi$ and $\Gamma^* \not\vdash \psi \rightarrow \varphi$. Since Γ^* is maximal, $\Gamma^* \cup \{\varphi \rightarrow \psi\} \vdash \perp$ and $\Gamma^* \cup \{\psi \rightarrow \varphi\} \vdash \perp$. Therefore:

1. $\Gamma^* \vdash \Delta(\varphi \rightarrow \psi) \rightarrow \perp$ DT
2. $\Gamma^* \vdash \Delta(\psi \rightarrow \varphi) \rightarrow \perp$ DT

3. $\Gamma^* \vdash (\Delta(\psi \rightarrow \varphi) \rightarrow \perp) \rightarrow ((\Delta(\varphi \rightarrow \psi) \vee \Delta(\psi \rightarrow \varphi)) \rightarrow \perp)$ (8), 1, (MP)
4. $\Gamma^* \vdash (\Delta(\varphi \rightarrow \psi) \vee \Delta(\psi \rightarrow \varphi)) \rightarrow \perp$ 3, 2, (MP)
5. $\Gamma^* \vdash \perp$ 4, ($\Delta 5$), (MP), contradicting the consistency of Γ^* \square

DEFINITION 4.1. For any maximal set Γ^* of formulas and propositional letter p , let

$$\mathbf{v}(p) := \begin{cases} 1 & \text{if } p \in \Gamma^*; \\ 0 & \text{if } \neg p \in \Gamma^*; \\ 0.5 & \text{otherwise.} \end{cases}$$

LEMMA 4.2 (Truth lemma). *For every $\varphi \in \text{Fm}$: $\mathbf{v}(\varphi) = 1$ iff $\varphi \in \Gamma^*$.*

PROOF. We prove both directions simultaneously, by induction on φ .

Base step: $\varphi := p$. Immediate by Definition 4.1 of a valuation \mathbf{v} .
 $\varphi := \top$. $[[\top]]^{\mathcal{M}^{\Gamma^*}, g} = 0.5$ by the definition of satisfaction. On the other hand, $\top, \neg\top \notin \Gamma^*$, since Γ^* is consistent (Proposition 4.1). $\varphi := \perp$. $[[\perp]]^{\mathcal{M}^{\Gamma^*}, g} = 0$ by the definition of satisfaction. Because Γ^* is consistent, we also know that $\perp \notin \Gamma^*$, as required.

Inductive step: $\varphi := \psi \rightarrow \chi$. By the definition of satisfaction, if $[[\psi \rightarrow \chi]]^{\mathcal{M}^{\Gamma^*}, g} = 1$, we shall distinguish three cases: $\mathbf{v}(\psi) = 0$ iff $\neg\psi \in \Gamma^*$, (by IH) iff $\psi \rightarrow \chi \in \Gamma^*$ (by Proposition 4.3(1)).

$\mathbf{v}(\chi) = 1$ iff $\chi \in \Gamma^*$ (by IH) iff $\psi \rightarrow \chi \in \Gamma^*$ (by Proposition 4.3(2)).

$\mathbf{v}(\psi) = 0.5$ and $\mathbf{v}(\chi) = 0.5$ iff $\psi, \neg\psi \notin \Gamma^*$ and $\chi, \neg\chi \notin \Gamma^*$ (by IH) iff $\psi \rightarrow \chi \in \Gamma^*$ (by Proposition 4.3(3)). \square

THEOREM 4.1 (Completeness). *If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.*

PROOF. Assume that $\Gamma \not\vdash \varphi$. Obviously, Γ has to be consistent (otherwise $\Gamma \vdash \perp$ and, hence, $\Gamma \vdash \varphi$ by Proposition 3.1(4) and (MP)). We shall prove that $\Gamma \cup \{\neg\Delta\varphi\}$ is consistent:

1. $\Gamma \cup \{\neg\Delta\varphi\} \vdash \perp$ hypothesis
2. $\Gamma \vdash \Delta(\neg\Delta\varphi) \rightarrow \perp$ 1, DT
3. $\Gamma \vdash \neg\Delta(\neg\Delta\varphi)$ 2
4. $\Gamma \vdash \Delta\varphi$ 3, ($\Delta 6$), (MP)
5. $\Gamma \vdash \varphi$ 4, ($\Delta 1$), (MP), contradicting the starting assumption

Thus, by Lemma 4.1, there is a Γ^* such that $\Gamma \subseteq \Gamma^*$, Γ^* is maximal consistent and contains $\Gamma \cup \{\neg\Delta\varphi\}$. Take the valuation \mathbf{v} for Γ^* as it is defined in Definition 4.1. It follows (by Lemma 4.2) that $\mathbf{v}(\varphi) = 1$

for all $\gamma \in \Gamma$ and $\mathfrak{v}(\neg \Delta \varphi) = 1$. Hence, $\mathfrak{v}(\Delta \varphi) = 0$, so $\mathfrak{v}(\varphi) \neq 1$ by Definition 2.1. In consequence, by Definition 2.3, $\Gamma \not\models \varphi$. \square

5. Conclusion

In the present paper, we have addressed the issue of the completeness of a particular non-classical formalism: the truth-functionally complete **L3**. To do so, we have replaced Ślupecki's 1-ary connective by a 0-ary one, included the Baaz delta operation and established a (weaker) version of the Deduction Theorem. Henkin's method has been successfully adapted to this three-valued setting.

The extension of this logic to a first-order setting is a matter of future work. We think that the intermediate truth value, 0.5, can be attributed to those statements about non-existing objects, like 'Pegasus has a white hind leg'. The idea is to develop a formal system for a *neutral free logic*, based upon our truth-functionally complete **L3**.

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References

- Avron, A., 1991, “Natural 3-valued logics: Characterization and proof theory”, *The Journal of Symbolic Logic*, 56(1): 276–294. DOI: [10.2307/2274919](https://doi.org/10.2307/2274919)
- Baaz, M., 1996, “Infinite-valued Gödel logics with 0-1-projections and relativizations”, pages 23–33 in P. Hájek (ed.), *GÖDEL’96 – Logical Foundations of Mathematics, Computer Science and Physics*, Lecture Notes in Logic, 6, Springer Verlag.
- Goldberg, H., H. LeBlanc and G. Weaver, 1974, “A strong completeness theorem for 3-valued logic”, *Notre Dame Journal of Formal Logic*, 15(2): 325–330. DOI: [10.1305/ndjfl/1093891310](https://doi.org/10.1305/ndjfl/1093891310)
- Hájek, P., 1998, *Metamathematics of Fuzzy Logic*, Springer Science & Business Media. DOI: [10.1007/978-94-011-5300-3](https://doi.org/10.1007/978-94-011-5300-3)
- Malinowski, G., 2007, “Many-valued logic and its philosophy”, pages 13–94 in D. M. Gabbay and J. Woods (eds.) *Handbook of the History of Logic*, Vol. 8, North-Holland, Amsterdam. DOI: [10.1016/S1874-5857\(07\)80004-5](https://doi.org/10.1016/S1874-5857(07)80004-5)
- Novak, V., 2005, “On fuzzy type theory”, *Fuzzy Sets and Systems*, 149(2): 235–273. DOI: [10.1016/j.fss.2004.03.027](https://doi.org/10.1016/j.fss.2004.03.027)
- Ślupecki, J., 1946, “Pełny trójwartościowy rachunek zdań” (The complete three-valued propositional calculus), *Annales Universitatis Mariae Curie-Skłodowska*, 1: 193–209.
- Ślupecki, J., 1967, “The full three-valued propositional calculus”, pages 335–337 in S. McCall (ed.), *Polish Logic 1920–1939*, Clarendon Press, Oxford.

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