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## Stoic Syllogistic

**Abstract.** This paper reconstructs the Stoic syllogistic as equivalent to classical propositional calculus. The method that has been used has three distinctive features: (i) the Stoics' proof routine is considered a source on a par with explicit declarations, (ii) all atomic formulae are consequently considered schemata, i.e. true or false relative to a value assignment, (iii) the view is held that only some of the permissible proof steps are classified as based upon syllogisms. The Stoic syllogistic turns out to be a complete axiomatic system of derivation rules and consists of three parts: (a) a decision procedure for formulae, which is a part of an axiomatic system rather than a formal semantics, (b) an axiomatic system of designated derivation rules, called indemonstrable syllogisms, (c) a method to effectively generate any valid derivation rule by means of the true formulae and indemonstrable syllogisms. A completeness result for the Stoic syllogistic is provided. Arguably, this result is highly congruent with the account known to have existed in the lost sources of antiquity.

**Keywords:** Stoic syllogistic; syllogism; proof; completeness

## Introduction

Teachings of the Stoa, with Zeno of Citium as the founder and Chrysippus of Soli as the principal systematiser, include a system of logic, known as syllogistic. Essential questions regarding the system remain unanswered, because authoritative sources are missing. No primary sources survive. Sextus Empiricus and Diogenes Laertius, to be considered secondary sources, know works by prominent Stoics, including Chrysippus and Diogenes of Babylon, and quote from them. Some scarce and obscure ternary or quaternary sources also survive. A majority of the

extant source material comes from authors averse either to the Stoicism or to logic.

The contemporary studies of the logical legacy of the Stoa begins with Jan Łukasiewicz's claim that the Stoic syllogistic is an inchoate stage of classical propositional calculus (Łukasiewicz, 1927, 1935). However, the case Łukasiewicz makes for the claim leaves a lot to be desired. In particular, despite the fact that Chrysippus insists on the completeness of the syllogistic, the proof of the result is considered missing. Attempted reconstructions (starting with Becker, 1957, followed by Mueller, 1979) are not altogether satisfactory, because they are fragmentary and, what is worse, make assumptions foreign to or even directly denied by the sources. At the opposite extreme there are views that Stoic logicians deliberately reject some rules of classical propositional calculus to create a deviant logic, typically associated with the notion of relevance. Such accounts originate in works by Martha Kneele née Hurst (Hurst, 1935; Kneale and Kneale, 1962) and are developed by Susanne Bobzien (1996), Alan Hájek (2009), Miguel López-Astorga (2015), Piotr Łukowski (2025), Graham Priest (2007), Richard Sylvan (2000) and others. These views seem to have prevailed since the turn of the 20th century, becoming received wisdom (see e.g. Bonevac and Dever, 2012), even though it has been demonstrated, both formally and textually, that the Stoics' connectives are truth-functional and that the relevance criteria are irreconcilable with the Stoic logic (Barnes, 2005; Tkaczyk, 2024a,b).

In this study I reiterate, actually, even strengthen, Łukasiewicz's original thesis to the effect that the Stoic syllogistic simply is an axiomatisation of classical propositional calculus. The axiomatisation, I contend, is correct, though formative. The set of axioms contains

- all true formulae, effectively generated by means of the truth tables,
- selected derivation rules, demonstrated from another axiomatic system,

and proves all the remaining derivation rules of classical propositional calculus. I deliver the completeness result, making no assumption unless it is grounded in the extant sources. However, I consider Stoics' proof routine a fully-fledged source on a par with explicit declarations. I suggest, the result I offer and the lost one, Chrysippus is known to have asserted, are essentially the same.

## 1. The object language

The Stoic syllogistic, as depicted by Diogenes Laertius (1964), Sextus Empiricus (1842; 1935) and Aulus Gellius (1544), is propositional in the sense that in the object language, firstly, the alphabet consists of atomic formulae (i.e. atoms) and connectives, secondly, all formulae are sentences and divide into atomic and molecular, the latter compounded of atomic formulae by means of connectives.

Atomic formulae are schematic sentences, i.e. they are sentences with the content artificially excised. By default, the ordinal numerals serve as atoms: the first one (το πρῶτον), the second one (το δεύτερον), the third one (το τρίτον), ... Thuswise, whether intentionally or not, the number of atomic formulae is infinite but countable, just like in contemporary propositional calculi. I call atoms of that group ordinary. Occasionally, vernacular clauses serve as atomic formulae too, e.g.: it is day, it is light, Dion is walking, I am conversing. It is not at all clear whether in the lost primary sources they actually qualify as atoms or merely as didactic application examples. Therefore, I call those atoms extraordinary. It is to be presumed that allowing extraordinary atoms does not impact the total number of atomic formulae. Additionally, the alphabet contains the connectives of negation (οὐ, οὐχ, οὐχί, οὐκ), implication (εἰ, εἴπερ), conjunction (καί), and disjunction (ἢ, ἢτοι). I use the symbols: ‘ $\neg$ ’, ‘ $\rightarrow$ ’, ‘ $\wedge$ ’, ‘ $\vee$ ’, respectively, instead. I also use parentheses.

**DEFINITION 1.** The set of formulae is the smallest collection including an infinite but countable set of atomic formulae, and containing

$$\neg\varphi, \quad (\varphi \rightarrow \psi), \quad (\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n) \quad \text{and} \quad (\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n),$$

for any positive integer  $n$  and for any formulae  $\varphi_1, \varphi_2, \dots, \varphi_n, \varphi, \psi$ .

A formula  $\neg\varphi$  is called the negation of  $\varphi$ , to be read: it is not the case that  $\varphi$ . A formula  $(\varphi \rightarrow \psi)$  is called the implication with the antecedent  $\varphi$  and the consequent  $\psi$ , to be read: if  $\varphi$ , (then)  $\psi$ . A formula  $(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n)$  is called the conjunction with the conjuncts:  $\varphi_1, \varphi_2, \dots, \varphi_n$  in sequence, to be read:  $\varphi_1$  and  $\varphi_2$  ... and  $\varphi_n$ . A formula  $(\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n)$  is called the disjunction with the disjuncts:  $\varphi_1, \varphi_2, \dots, \varphi_n$  in sequence, to be read:  $\varphi_1$  or  $\varphi_2$  ... or  $\varphi_n$ .

Definition 1, on the one hand, seems to be a sufficiently accurate approximation of the Stoics’ primaeval regimented Greek, appearing in

the extant sources, on the other, it enables a straightforward comparison with contemporary propositional calculi. Two reservations, however, seem required. Firstly, in definition 1, I allow  $n = 1$ , assuming that any formula is a single-conjunct conjunction as well as a single-disjunct disjunction. The assumption is merely to shorten proofs to follow and of no impact on the provable theorems. Secondly, when translated into Greek, two conjunctions of my object language are indistinguishable if they are equal in everything but the distribution of parentheses. Therefore, the following associative rule for conjunction applies, even though the sources are silent about it.

OBSERVATION 2. The formulae:  $((\varphi \wedge \chi) \wedge \psi)$ ,  $(\varphi \wedge (\chi \wedge \psi))$  and  $(\varphi \wedge \chi \wedge \psi)$  can be used interchangeably for any formulae  $\varphi, \chi, \psi$ .  $\square$

Observation 2 becomes immediate once it is noted that in the primaeval Greek of the Stoic syllogistic a conjunction  $(\varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n)$  takes either of the following equally correct and freely interchangeable shapes:

$$\begin{array}{cccc} \kappa\alpha\iota \varphi_1, & \kappa\alpha\iota \varphi_2, & \dots, & \kappa\alpha\iota \varphi_n, \\ \varphi_1, & \kappa\alpha\iota \varphi_2, & \dots, & \kappa\alpha\iota \varphi_n, \end{array}$$

i.e., either each conjunct is prefixed with an instance of the respective connective, or each but the initial one. It is essential to understand that, unlike the Polish notation, the Stoics' object language does not involve systematic nested iterations of the occurrences of the connective in question. Recollect also that the sources are silent about the use of parentheses or similar punctuation signs in the Stoa.

The object language concept of negation ( $\alpha\pi\omicron\varphi\alpha\tau\iota\chi\acute{o}\nu$ ) and the metalinguistic concept of being contradictory ( $\acute{\alpha}\nu\tau\iota\kappa\epsilon\acute{\iota}\mu\epsilon\nu\omicron\nu$ ) are to be distinguished carefully. By definition, any formula and its negation are contradictories of each other. In the metalanguage I use the superscript character of star with a variable to signify that whenever two variables are the same except one of them is unstarred and the other is starred, e.g.,

$$\varphi, \varphi^*,$$

one of the formulae spoken of is the negation of the other. It is not assumed, however, that the starred formula in the negation of the unstarred one. Hence, both of the formulae  $\varphi$  and  $\neg\neg\varphi$  are contradictories of the formula  $\neg\varphi$ .

Hereafter,  $\tau$ ,  $\varphi$ ,  $\chi$ ,  $\psi$ , whether with or without subscripts or superscripts, are any formulae, whether atomic or molecular, with the reservation that the superscript star always indicates a pair of contradictories. Moreover,  $\Phi$  and  $\Psi$  are any finite (possibly empty) sequences of formulae.

## 2. The objective of the Stoic syllogistic

The objective of a logical theory is to discriminate between valid and invalid steps in a proof. The Stoic syllogistic is no exception, it is a theory of provability (proof theory in short), i.e. a system of rules of derivation or similar creations. However, the logicians of the Stoa seem to consequently distinguish between two technical terms:

$$\acute{\alpha}\pi\acute{o}\delta\epsilon\iota\zeta\iota\varsigma, \quad (1)$$

$$\acute{\alpha}\nu\acute{\alpha}\lambda\upsilon\sigma\iota\varsigma, \quad (2)$$

both of which are to be translated as proof or demonstration. The proof in sense (1) is the subject of the Stoic syllogistic. On the one hand, at the initial stage, it is the vernacular word for proof and means a proof in the generic sense. On the other, at the final stage, it is a technical term referring to the formalised proofs spoken of in the Stoic syllogistic. Term (2), in contrast, always means a formal proof within a particular, well-defined, effectively generated axiomatic theory, attributed to Chrysippus and known as a system of indemonstrable syllogisms, to be discussed in Section 4. I contend, the system of indemonstrable syllogisms, employing the proofs in sense (2), is merely a part, albeit a very important one, of the Stoics' theory of provability in sense (1).

The available sources contain no comprehensive description of the theory under consideration. Instead, they quote proofs accompanied by explanatory notes, enabling one to reconstruct the theory. Sextus (1935, B 281–282, cf. B 466–467) quotes Stoics' curious theorem:

$$\text{a sign exists,} \quad (3)$$

with the following proof.

$$\text{if a sign does not exist, then a sign exists} \quad (3a)$$

$$\text{if a sign exists, then a sign exists} \quad (3b)$$

$$\text{either a sign exists or does not exist} \quad (3c)$$

$$\text{a sign exists} \quad (3d)$$

Line (3a) seems to have been established within a theory of language, more or less, as follows. Assume that a sign does not exist. It follows, the sentence “a sign does not exist” is true. It follows, the sentence exists. But every sentence is a sign, therefore, a sign exists. Lines (3b) and (3c) are considered permissible, as Sextus clearly mentions, *because they are bound* (ὁφείλει) *to be true*. That means, Sextus explains, they instantiate formulae, which are instantiated only by true sentences. Finally, line (3d) is permissible, because it is derived in a permissible way from permissible lines (3a), (3b) and (3c), i.e. by means of a *sylogism*.

The Stoic syllogistic deals with the axiomatic method. Line (3a), as well as Theorem (3) itself, belong to a specific theory. And the Stoics make it clear that such statements remain outside the scope of syllogistic. Lines of kind (3b) and (3c), in contrast, are permissible in any proof within any branch of knowledge. Therefore, I contend, they fall within the subject of syllogistic, even though they do not count as syllogisms.

As can be seen, the derivation rules the Stoic syllogistic deals with fall into two categories: rules that unconditionally authorise a derivation of specific sentences, e.g.,

$$\begin{aligned} &\text{derive } (\varphi \rightarrow \varphi), \\ &\text{derive } (\varphi \mid \neg\varphi), \end{aligned}$$

applied in lines (3b) or (3c), respectively, and rules that authorise a derivation of some specific sentences on condition that other specific sentences have been derived, e.g.,

$$\text{derive } \varphi \text{ on condition that } (\neg\varphi \rightarrow \varphi), (\varphi \rightarrow \varphi), (\varphi \mid \neg\varphi) \text{ are derived,}$$

applied in line (3d). Generally speaking, the two categories of derivation rules to deal with are:

$$\text{derive } \varphi, \tag{4}$$

$$\text{derive } \varphi \text{ on condition that } \varphi_1, \varphi_2, \dots, \varphi_n \text{ are derived.} \tag{5}$$

Observe that in the rules of category (5) the integer  $n$  is always greater than zero. It is critical to understand that in antiquity no logical theory is ripe to treat the rules of category (4) as a subcategory of the rules of category (5), viz. with  $n = 0$ . Any ancient concept of syllogism, whether Stoic or Aristotelian, is confined to rules of category (5). It is designed to reduce all valid proof steps of category (5) to an effectively generated

collection of evidently valid proof steps. Failure to take into account idiosyncrasies of the rules of category (4) is the key error, I believe, in many attempts to reconstruct the Stoic syllogistic and to deliver for its completeness.

It is worth noticing that the Stoics are not the first to recognise the rules of category (4) and their role in the axiomatic method. In *Posterior Analytics* Aristotle speaks of formulae, which are unconditionally permissible proof steps within any theory, and mentions the law of excluded middle as an example. And in Euclid's *Elements* the term “ἀξίωμα” (proper axiom or postulate) is established for the axioms which are characteristic of a theory, and the term “κοινὴ ἔννοια” (common axiom) for those suitable for use in many different theories, i.e. for the logical ones. As the scanty surviving source material suggests, Chrysippus, the keynote logician of the Stoa, makes use of the latter term (Arnim, 1964, II 473, III 65, cf. II 83, II 403).

If a syllogistic is to be understood as a proof theory or a consequence operation and compared with contemporary systems of logic, it should be treated as including the rules of category (4) as well as those of category (5). Therefore, I contend, if a formula is *bound to be true*, it is to be regarded as a common or logical axiom, and thereby as derivable in the Stoic syllogistic. A novelty and a major development introduced in the Stoa is the use of the truth tables to identify the formulae in question. If I am correct, the truth tables present a part of the Stoics' axiomatic system rather than a formal semantics. The Stoics aim to demonstrate rules of derivation by means of the formulae that are bound to be true and of indemonstrable syllogisms combined. And the truth tables present a means of effectively generating the true formulae, while indemonstrable syllogisms are organised into a separate axiomatic system. Consequently, I distinguish the Stoics' system of indemonstrable syllogisms and their proof theory, the latter being an extension of the former with the algorithmically generated true formulae. Whenever I speak of *the Stoic syllogistic*, I mean the latter, stronger theory, while to the former one I refer as *the system of indemonstrable syllogisms*. In Section 3, I deal with the formulae bound to be true, i.e. the rules of category (4), while in Section 4, with the system of indemonstrable syllogisms, i.e. designated rules of category (5). In Section 5, the results of sections 3 and 4 serve to deliver a completeness result of the Stoic syllogistic, i.e. the Stoics proof theory in its entirety.

### 3. The logical axioms, i.e. the (unreservedly) true formulae

The principle of bivalence asserts that, on the one hand, every sentence without exception is either true or false, on the other, no sentence whatsoever is both true and false (traditionally, I call truth and falsity truth values, values in short). It is an essential doctrine of the Stoa (Diogenes, 1964, Z 73; Sextus, 1935; B 85), and not by any means considered a linguistic convention at choice, but rather a kind of law of nature. The Stoics are zealous defenders of bivalence in an unending controversy with the Epicureans and the Aristotelians over determinism. As followers of Zeno and Chrysippus anxiously renounce contingency, it is an indispensable article of their faith that

nothing is possible unless it is, sooner or later, the case. (6)

Under belief (6) the modal distinctions collapse, unless they reduce to purely linguistic phenomena, as in David Hume or The Early Wittgenstein. With the disappearance of extralinguistic modal distinctions, the contingent states of affairs also disappear. As in antiquity non-Sceptics agree that no sentence is exempt from the principle of bivalence unless the sentence reports on a future contingent state of affairs, determinists endorse the principle, while indeterminists either restrict it, like the Epicureans, or at least show hesitation regarding it, like the Aristotelians. For the Stoics, the principle of bivalence is an equivalent of their firm belief that every state of affairs is either necessary or impossible (cf. Long and Sedley, 1987).

Moving from actual (unabridged) sentences to schemata, Stoics improvise truth values by means of value assignments. The atomic formulae are assigned truth values arbitrarily, while the molecular formulae have their truth values determined by the values assigned to atoms.

Sources (including Diogenes, 1964 and Sextus, 1842, 1935) know two ways of specifying assignments of truth values to atomic formulae. When ordinary atoms come into play, truth values are assigned directly, just as in the contemporary matrices, e.g. the implication “if the first one, then the second one” is false if the first one is true and the second one is false. Extraordinary atoms are presumed to be variable in time in terms of truth value, and assignments are specified by postulating atoms or their negations at a time, e.g. the implication “if it is day, then I am conversing” is false whenever it is day but I am not conversing. Despite using the word “whenever” specified times are pure constructions rather than

descriptions of existing states of affairs. In either case, the Stoics know how to algorithmically exhaust the relevant assignments to construct a truth table. My hypothesis is that these two procedures are essentially identical. The practice of assigning truth values to propositions with respect to time is not, I believe, an expression of a specific philosophy of language, but rather an alternative, more vivid way of assigning truth values to propositional schemata.

For the sake of succinctness, I represent falsity by 0 and truth by 1. Hence, any function from the set of atomic formulae to the pair  $\{0, 1\}$  counts as a value assignment, and for any formula  $\varphi$  and any value assignment  $v$ ,

$$\|\varphi\|_v = 0 \quad \text{or} \quad \|\varphi\|_v = 1,$$

mean, respectively, that the formula  $\varphi$  is false or true relative to the value assignment  $v$ . I also have “ $\prod$ ” and “ $\sum$ ” express the arithmetic product or sum, respectively. Truth values of atomic formulae are decided directly by an assignment, hence, for any atomic formula  $\varphi$  and value assignment  $v$ ,

$$\|\varphi\|_v = v(\varphi).$$

According to the Stoics, the connectives of the object language are to be understood truth-functionally. Hence, the negation of a false formula is true and conversely (Alexander, 1883, p. 402; Diogenes, 1964, Z 69; Sextus, 1935, B 103). In other words, for any formula  $\varphi$  and any value assignment  $v$ ,

$$\|\neg\varphi\|_v = \begin{cases} 1, & \text{if } \|\varphi\|_v = 0, \\ 0, & \text{otherwise.} \end{cases} \quad (7)$$

An implication is true if it has either a false antecedent or a true consequent. It is false if it has both a true antecedent and a false consequent (Diogenes, 1964, Z 81, cf. Z 71; Sextus, 1842, B 104–106; 1935, B 245–248, B 332). In other words, for any formulae  $\varphi, \psi$  and value assignment  $v$ ,

$$\|(\varphi \rightarrow \psi)\|_v = \begin{cases} 1, & \text{if } \|\varphi\|_v \leq \|\psi\|_v, \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

A conjunction is true if it has only true conjuncts, and is false otherwise (Gellius, 1544, XVI 8; Sextus, 1935, B 125). More precisely, for any value assignment  $v$ , any integer  $n$ , and any formulae  $\varphi_1, \varphi_2, \dots, \varphi_n$ ,

$$\|(\varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n)\|_v = \begin{cases} 1, & \text{if } \prod_{i=1}^n \|\varphi_i\|_v = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (9)$$

A disjunction is true if it has exactly one true disjunct, and is false otherwise (Diogenes, 1964, Z 72; Gellius, 1544, XVI 8; Sextus, 1935, B 282, B 467.) More precisely, for any integer  $n$ , any formulae  $\varphi_1, \varphi_2, \dots, \varphi_n$ , and any value assignment  $v$ ,

$$\|(\varphi_1 \mid \varphi_2 \mid \dots \mid \varphi_n)\|_v = \begin{cases} 1, & \text{if } \sum_{i=1}^n \|\varphi_i\|_v = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (10)$$

Notice that the Stoics' connective of disjunction is no straightforward generalisation of any familiar dyadic connective. It could be referred to as the connective of *partitive disjunction* or as the *Stoic stroke*.

Rules (7) and (8) are equal to the respective rules of the bivalent Boolean matrix. Rules (9) and (10) in general are not truth functions in the contemporary sense, since, strictly speaking, a function has a fixed number of variables (arity). However, for any fixed integer  $n$ , these rules clearly constitute truth functions. Therefore, they may be considered infinite collections of truth functions. Based on those observations, the connectives of the Stoics' object language are definable in the bivalent Boolean matrix and vice versa.

The Stoic understanding of negation and conjunction is entirely uncontroversial. Admittedly, the Stoic interpretation of implication and disjunction has been a subject of lively debate for decades. The justification of the approach proposed above is deferred to Section 7. As mentioned, a key observation is that formulae of the object language, whether they consist of ordinary or extraordinary atoms, are schemata and, therefore, are true or false relative to a value assignment. The loci seemingly suggesting a non truth functional interpretation of the connectives in question refer to formulae true or false relative to every value assignment.

It should be clear that, instead of saying that a formula is true relative to every, to some, or to no value assignment, the extant sources tend to speak in a temporal or in a modal manner. They say, respectively, that the formula in question

- is always ( $\acute{\alpha}\epsilon\acute{\iota}$ ) true or is bound ( $\acute{\omicron}\varphi\epsilon\acute{\iota}\lambda\epsilon\acute{\iota}$ ) to be true,
- is sometimes ( $\pi\omicron\tau\acute{\epsilon}$ ) true or is capable ( $\delta\acute{\upsilon}\nu\alpha\tau\alpha\iota$ ) of being true,
- is never ( $\mu\eta\delta\acute{\epsilon}\pi\omicron\tau\epsilon$ ) true or is incapable ( $\acute{\alpha}\delta\acute{\upsilon}\nu\alpha\tau\alpha\iota$ ) of being true,

and analogically with falsity. Both ways of speaking, utilising temporal or modal vocabulary, are interchangeable and refer to nothing else than quantities of value assignments, in accordance with article (6) of the

Stoics' creed. A formula true relative to every value assignment is also said to be

$$\text{unreservedly true or true without reserve } (\xi\upsilon\theta\dot{\iota}\varsigma \ \acute{\alpha}\lambda\eta\theta\acute{\epsilon}\varsigma). \quad (11)$$

To speak of those formulae I introduce the following abbreviation:

$$\|\varphi\| = \begin{cases} 1, & \text{if for any value assignment } v, \|\varphi\|_v = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (12)$$

In this context, I claim that in the extant sources the formulae *true without reserve* are often called briefly just *true*. For — being schemata rather than actual sentences — they cannot be just true or false in the proper sense. It is an important point in the case for rules (8) and (10), delivered in Section 7.

The above synopsis leaves no room for doubt that the Stoics are familiar with the truth table method and apply it to formulae. There is no trace in the extant sources of using that method for inference schemata or rules. Whether the method has been invented in the Stoa or taken over from the Megarians remains unknown. Anyway, the Stoics are associated with the method — and often ridiculed for it — throughout history. What is critical, is that there is absolutely nothing in the sources to indicate that the truth tables are meant to present a formal semantics in a contemporary sense. Rather, I hypothesise, they are a part of the axiomatic system of the Stoic syllogistic and serve to produce the formulae *bound to be true*, i.e. the derivation rules of category (4) for a theory of proof in sense (1). The method opens up for the Stoic logicians a virtually unlimited source of such rules, while the Aristotelians remain confined to the conceptual analysis in the sense of Plato's διαίρεσις (consult especially Plato's, *Sophist* and *Phaedrus*).

#### 4. The system of indemonstrable syllogisms

The Stoics' system of indemonstrable syllogisms is the best documented part of the Stoic syllogistic (cf. [Mates, 1953](#)). It is an effectively generated axiomatic theory, which means that the theorems are formally derived from a set of axioms and transformation rules. I write

$$\Phi \Vdash \varphi$$

to assert that a rule, which authorises the derivation of the formula  $\varphi$  on condition that all members of the sequence  $\Phi$  are derived, is a theorem of the Stoics' system of indemonstrable syllogisms.

Observe that the theorems of the system of indemonstrable syllogisms are themselves derivation rules, authorising proof steps in sense (1), whereas the transformation rules of the system authorise proof steps in sense (2), namely take and return rules of derivation in sense (1). It is common to refer to the rules which take and return rules with the misleading term “meta-rules”. To minimise the risk of confusion, I speak of the

- first-degree rules, taking and returning formulae of the object language,
- second-degree rules, taking and returning first-degree rules.

The theorems of the system of indemonstrable syllogisms are first-degree rules, while the transformation rules of that system are second-degree. Speaking of rules of derivation, whenever I use the term “rule” without further specification, I mean a first-degree rule.

Canonically, by Chrysippus' authority, the system of syllogisms contains five historical axioms (i.a. [Diogenes, 1964](#), Z 80–81; [Sextus, 1842](#), B 156–157; [Sextus, 1935](#), B 223–226). In the object language I consider, those historical axioms convert into eight. Chrysippus' first axiom consists of two premises and a conclusion, the first premise being an implication with an antecedent  $\varphi$  and a consequent  $\psi$ , the second premise being the antecedent  $\varphi$ , the conclusion being the consequent  $\psi$ :

$$(\varphi \rightarrow \psi), \varphi \Vdash \psi. \quad (13)$$

Chrysippus' second axiom consists of two premises and a conclusion, the first premise being an implication with an antecedent  $\varphi$  and a consequent  $\psi$ , the second premise being a contradictory  $\psi^*$  of the consequent, the conclusion being any contradictory  $\varphi^*$  of the antecedent:

$$(\varphi \rightarrow \psi), \psi^* \Vdash \varphi^*. \quad (14)$$

Chrysippus' third axiom consists of two premises and a conclusion, the first premise being the negation of a conjunction with two conjuncts  $\varphi$  and  $\psi$ , the second premise being any of the conjuncts,  $\varphi$  or  $\psi$ , the conclusion being any contradictory of the other conjunct,  $\psi^*$  or  $\varphi^*$ , respectively:

$$\neg(\varphi \wedge \psi), \varphi \Vdash \psi^*, \quad (15)$$

$$\neg(\varphi \wedge \psi), \psi \Vdash \varphi^*. \quad (16)$$

Chrysippus' fourth axiom consists of two premises and a conclusion, the first premise being a disjunction with two disjuncts  $\varphi$  and  $\psi$ , the second premise being any contradictory,  $\varphi^*$  or  $\psi^*$ , of any of the disjuncts, the conclusion being the other disjunct  $\psi$  or  $\varphi$ , respectively:

$$(\varphi \mid \psi), \varphi^* \Vdash \psi, \quad (17)$$

$$(\varphi \mid \psi), \psi^* \Vdash \varphi. \quad (18)$$

Chrysippus' fifth axiom consists of two premises and a conclusion, the first premise being a disjunction with two disjuncts  $\varphi$  and  $\psi$ , the second premise being any of the disjuncts,  $\varphi$  or  $\psi$ , the conclusion being any contradictory,  $\psi^*$  or  $\varphi^*$ , of the other disjunct, respectively:

$$(\varphi \mid \psi), \varphi \Vdash \psi^*, \quad (19)$$

$$(\varphi \mid \psi), \psi \Vdash \varphi^*. \quad (20)$$

As mentioned, axioms (13) to (20) are first-degree rules of derivation. Reportedly, in the system there are four second-degree rules, of which only two survive. Second-degree rule 1 is a version of transposition ([Apuleius, 1991](#), p. 209). It authorises, in any two-premise theorem, to simultaneously replace the conclusion with a contradictory of a premise, and the premise with a contradictory of the conclusion:

$$\frac{\varphi^*, \psi \Vdash \chi,}{\chi^*, \psi \Vdash \varphi,} \quad \text{and} \quad \frac{\varphi, \psi^* \Vdash \chi,}{\varphi, \chi^* \Vdash \psi.} \quad (21)$$

Second-degree rule 3 is a limited version of cut ([Alexander, 1883](#), p. 274; [Simplicius, 1894](#), p. 237). If a formula appears as a premise in a two-premise theorem and as a conclusion in any other one, the rule authorises, in the former theorem, to replace the formula in question with all the premises of the other theorem: for any positive integer  $n$ ,

$$\frac{\varphi_1, \varphi_2, \dots, \varphi_n \Vdash \varphi, \quad \varphi, \psi \Vdash \chi,}{\varphi_1, \varphi_2, \dots, \varphi_n, \psi \Vdash \chi,} \quad \text{and} \quad \frac{\psi_1, \psi_2, \dots, \psi_n \Vdash \psi, \quad \varphi, \psi \Vdash \chi,}{\varphi, \psi_1, \psi_2, \dots, \psi_n \Vdash \chi.} \quad (22)$$

The set of theorems of the Stoics' system of indemonstrable syllogisms is the smallest collection containing axioms (13)–(20), being first-degree rules of derivation, and closed under second-degree rules (21), (22), probably together with two additional missing ones. The derivations from the listed axioms and second-degree rules are the proofs in sense (2).

Since Łukasiewicz’s seminal works the research on the Stoics’ system of indemonstrable syllogisms has been intense. However, a key detail has been typically overlooked, namely that the ancient concept of syllogism is narrower than the contemporary concept of consequence. In particular, as mentioned, the syllogisms are confined to the rules of category (5). Therefore, to reconstruct and examine the Stoic syllogistic as a system of logic, the truth tables and the system of indemonstrable syllogisms are to be combined, rather than one treated as a formal semantics for the other. Particularly, to compare the logic of the Stoa with contemporary logical calculi, one should seek to establish the completeness result for the Stoic syllogistic in its entirety rather than for the system of indemonstrable syllogisms alone.

## 5. The Stoics’ proof theory in its entirety

I contend that the Stoics’ counterpart of a contemporary consequence operation is generated by the truth tables applied to formulae and by the system of indemonstrable syllogisms combined. In other words, I allow myself to formulate the following hypothesis: in the lost logical works by Chrysippus and other logicians of the Stoa a formal proof is understood (and, perhaps, defined) as a finite sequence comprising only of:

- proper axioms (characteristic of a given theory),
- logical axioms (instances of unreservedly true formulae),
- results of applications of syllogisms to preceding terms of the sequence,

and ending with the sentence to be established. My hypothesis is based upon the logical routine of the Stoics—as well as other logicians of antiquity—as described in Section 2. Evidently, when exercising the axiomatic method they deliberately apply instances of the unreservedly true formulae, i.e. rules of category (4), on a par with syllogisms that belong to the rules of category (5). As a result, the proof theory of the Stoa turns out to be equivalent with classical propositional calculus treated as a consequence operation.

Let, hence, the symbol “ $\vdash$ ” stand for the Stoic syllogistic treated as a consequence operation in its entirety, in accordance with the above hypothesis, viz.

$$\Phi \vdash \varphi,$$

means that, in accordance with the logical teaching and practice of the Stoa, the line  $\varphi$  is permissible in any proof, providing all members of  $\Phi$ , in any order, are among preceding lines of that proof. By “any proof” I mean any proof within any theory, to the effect that the lines that are permissible within one theory but not within another, e.g. line (3a), are excluded. Based on sections 2, 3 and 4, the Stoic syllogistic may be summarised in two statements:

$$\text{if } \langle \varphi_i \rangle_{i=1}^n \Vdash \varphi, \text{ then } \langle \varphi_i \rangle_{i=1}^n \vdash \varphi, \quad (23)$$

$$\text{if both } \tau, \Phi \vdash \varphi \text{ and } \|\tau\| = 1, \text{ then } \Phi \vdash \varphi, \quad (24)$$

stating: claim (23) that any proof step is admissible, if it results from an application of an indemonstrable syllogism; claim (24) that any proof step is admissible, if it instantiates an unreservedly true formula. The symbol “ $\vdash$ ”, I hypothesise, is an articulation of concept (1) of proof, while the symbol “ $\Vdash$ ” is an articulation of the other one, viz. (2).

The system generated by claims (23) and (24) is equal with classical propositional calculus in the sense of the following concept of classical validity, applied to the object language I employ.

DEFINITION 3.  $\Phi \models \psi$  if and only if for any value assignment  $v$ , either  $\|\varphi\|_v \neq 1$  for some  $\varphi$  in  $\Phi$ , or  $\|\psi\|_v = 1$ .

My claim that the Stoic syllogistic is complete or equal with classical propositional calculus means specifically that

$$\Phi \vdash \varphi \quad \text{if and only if} \quad \Phi \models \varphi, \quad (25)$$

in other words, the operation of derivability the Stoics endorse and that of classical propositional calculus are coextensive. That is to be demonstrated.

LEMMA 4. If  $\langle \varphi_i \rangle_{i=1}^n \Vdash \varphi$ , then  $\langle \varphi_i \rangle_{i=1}^n \models \varphi$ .

PROOF. The proof proceeds by induction on the length of the derivation: clearly, axioms (13)–(20) are valid and rules (21)–(22) preserve validity.  $\square$

The above result is quite straightforward with the reservation that, as mentioned, two of four second-degree rules are missing. Ignorance of those rules is the single most painful obstacle to the reconstruction of the Stoic syllogistic. In an unlikely, but not absolutely impossible, event

of finding that validity is not preserved by some of the missing rules, Lemma 4 would turn out to be questionable and the entire picture of the Stoic syllogistic could change dramatically.

**THEOREM 5.** If  $\Phi \vdash \varphi$ , then  $\Phi \models \varphi$ .

**PROOF.** Proof by induction on the number of applications of claim (24). In the base case claim (23) is exploited and the thesis follows directly from Lemma 4. That assumed, preserving the thesis under each application of claim (24) is secured by well known properties of the Boolean matrices.  $\square$

**LEMMA 6.**  $\neg(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \wedge \varphi), \langle \varphi_i \rangle_{i=1}^n \Vdash \varphi^*$ , for any  $n \geq 1$ .

**PROOF.** Proof by induction on  $n$ . If  $n = 1$ , the thesis is an instance of axiom (15). Assume for the induction step that the thesis holds for some  $n$ , therefore:

$$\neg(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \wedge (\varphi_{n+1} \wedge \varphi)), \langle \varphi_i \rangle_{i=1}^n \Vdash \neg(\varphi_{n+1} \wedge \varphi). \quad (\dagger)$$

Notice that the following syllogism is an instance of axiom (15):

$$\neg(\varphi_{n+1} \wedge \varphi), \varphi_{n+1} \Vdash \varphi^*. \quad (\ddagger)$$

Form the above syllogisms  $(\dagger)$  and  $(\ddagger)$  by second-degree rule (22),

$$\neg(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \wedge (\varphi_{n+1} \wedge \varphi)), \langle \varphi_i \rangle_{i=1}^{n+1} \Vdash \varphi^*.$$

From this, however, by observation 2, it follows:

$$\neg(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \wedge \varphi_{n+1} \wedge \varphi), \langle \varphi_i \rangle_{i=1}^{n+1} \Vdash \varphi^*.$$

Thus the induction step and the entire proof are complete.  $\square$

**THEOREM 7.** If  $\Phi \models \varphi$ , then  $\Phi \vdash \varphi$ .

**PROOF.** Assume  $\Phi \models \varphi$ . Either the sequence  $\Phi$  is empty or  $\Phi = \langle \varphi_i \rangle_{i=1}^n$  for some positive integer  $n$ . In the first case, assume that  $\Phi$  is empty. By definition 3, it follows that

$$\|\varphi\| = 1.$$

Notice that  $\|\neg(\varphi \wedge \neg\varphi)\| = 1$  also, and consider axiom (15):

$$\neg(\varphi \wedge \neg\varphi), \varphi \Vdash \varphi.$$

By statement (23),

$$\neg(\varphi \wedge \neg\varphi), \varphi \vdash \varphi.$$

Complete the first part of the proof, applying statement (24) twice to infer  $\vdash \varphi$ . In the other case, assume for some positive integer  $n$ , that  $\Phi = \langle \varphi_i \rangle_{i=1}^n$ . By definition 3, since  $\Phi \models \varphi$  has been already assumed, it follows that

$$\|\neg(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \wedge \neg\varphi)\| = 1. \quad (\dagger)$$

By Lemma 6,

$$\neg(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \wedge \neg\varphi), \langle \varphi_i \rangle_{i=1}^n \Vdash \varphi$$

and, therefore, by statement (23),

$$\neg(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \wedge \neg\varphi), \langle \varphi_i \rangle_{i=1}^n \vdash \varphi. \quad (\ddagger)$$

Apply statement (24) to the above lines ( $\ddagger$ ) and ( $\dagger$ ), deducing that

$$\langle \varphi_i \rangle_{i=1}^n \vdash \varphi.$$

Thus both parts of the proof are complete.  $\square$

Theorems 5 and 7 put together result in claim (25) that the Stoic syllogistic is equivalent with classical propositional calculus. It should be understood in such a way that the steps in a proof the Stoic syllogistic and classical propositional calculus authorise are the same.

## 6. The absence of independence

What immediately catches the eye, is that only one of the eight axioms of the system of indemonstrable syllogisms plays a role in the proof of Theorem 7, viz. axiom (15). Of course, axioms (15) and (16) historically constitute one. However, axioms: (13), (14), and (17) to (20) seem redundant when it comes to the completeness result. Similarly, of the second-degree rules, rule (22) is the only one cited in the proof in question.

Based on the surviving material, with two second-degree rules missing, no alternative result seems available. However, the following lemma may be demonstrated, citing axiom (16) and second-degree rules (21) and (22).

LEMMA 8.  $\langle \varphi_i \rangle_{i=1}^n \Vdash (\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n)$ , for any  $n > 1$ .

PROOF. Proof by induction on  $n$ . If  $n = 2$ , the theorem is derivable from axiom (16) by a single application of rule (21). Assume for the induction step that the thesis holds for some  $n > 1$ . Observe the following instance of axiom (16):

$$\neg(\varphi_1 \wedge (\varphi_2 \wedge \varphi_3 \wedge \cdots \wedge \varphi_{n+1})), (\varphi_2 \wedge \varphi_3 \wedge \cdots \wedge \varphi_{n+1}) \Vdash \neg\varphi_1$$

From that, by rule (21), derive

$$\varphi_1, (\varphi_2 \wedge \varphi_3 \wedge \cdots \wedge \varphi_{n+1}) \Vdash (\varphi_1 \wedge (\varphi_2 \wedge \varphi_3 \wedge \cdots \wedge \varphi_{n+1})). \quad (\dagger)$$

By the inductive assumption, it is the case that

$$\langle \varphi_i \rangle_{i=2}^{n+1} \Vdash (\varphi_2 \wedge \varphi_3 \wedge \cdots \wedge \varphi_{n+1}). \quad (\ddagger)$$

From the above lines,  $(\dagger)$  and  $(\ddagger)$ , by rule (22), it follows:

$$\langle \varphi_i \rangle_{i=1}^{n+1} \Vdash (\varphi_1 \wedge (\varphi_2 \wedge \varphi_3 \wedge \cdots \wedge \varphi_{n+1})).$$

And, therefore, by observation (2), it follows:

$$\langle \varphi_i \rangle_{i=1}^{n+1} \Vdash (\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \cdots \wedge \varphi_{n+1}).$$

Thus the proof is complete.  $\square$

Based on Lemma 8 and on either of axioms (13) and (14), the following lemmata may be straightforwardly demonstrated, analogous to Lemma 6.

LEMMA 9.  $((\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n) \rightarrow \varphi), \langle \varphi_i \rangle_{i=1}^n \Vdash \varphi$ , for any  $n \geq 1$ .

LEMMA 10.  $(\neg\varphi \rightarrow \neg(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n)), \langle \varphi_i \rangle_{i=1}^n \Vdash \varphi$ , for any  $n \geq 1$ .

Instead of Lemma 6, either of lemmata 9, 10 may serve to obtain the completeness result. As, through Lemma 8, lemmata 9 and 10 are based on axiom (16), neither of axioms (13), (14) suffices to deliver for completeness. However, in a possible case that Lemma 8 is a tacit assumption of ancient syllogistics, each of the axioms: (13), (14), (15) and (16) allows for the completeness result to be demonstrated. It cannot be ruled out that the missing second-degree rules of the system of indemonstrable syllogisms allow for similar results with axioms (17)–(20) being cited.

The lack of independence in the system of indemonstrable syllogisms is by no means an oversight on the part of the Stoics. As deriving axiom (14) from axiom (13) — and conversely — only requires a single application of second-degree rule (21), it has certainly not been missed by ancient logicians. They have other goals in mind, I conclude, besides formal economy — symmetry in the treatment of connectives, methodological, linguistic, psychological, content related goals or others.

## 7. A controversy over implication and disjunction

As mentioned in Section 3, it is an uncontested view that the Stoics understand the connectives of negation and of conjunction in accordance with rules (7) and (9). However, the way they understand the connectives of implication and of disjunction has long been a matter of doubt and, occasionally, a subject of lively discussion. Since the turn of the 21st century it has become a received wisdom, though not of a unanimous consensus (cf. Barnes, 2005), that the connectives of implication and of disjunction employed in the Stoic syllogistic are somehow deviant. Consequently, rules (8) and (10), I attribute to the Stoa, are controversial. I maintain, however, it is next to certain that the Stoics do understand the connectives in question in accordance with the aforementioned rules (8) and (10). I refer to the way to understand a connective in accordance with rules (7) to (10) or analogical as truth functional.

Below, I make my case for the truth functional nature of the Stoics' connectives of implication and of disjunction. I take it as established that no relevantistic interpretation of the connectives is reconcilable with the Stoic syllogistic (cf. Tkaczyk, 2024a,b). Consequently, the issue is narrowed down to the respective arguments supporting modal and the truth functional interpretations. A critical premise of mine is that all atoms, whether ordinary or extraordinary, are schemata — true or false relative to a value assignment — and that the modal concepts reduce to the quantifying of such value assignments, as discussed in detail in Section 3.

First, contrary to a widely held belief, the truth functional understanding of the connectives of implication and of disjunction are multitudinously attested to and never clearly denied in the extant sources. Second, the loci allegedly suggesting an alternative understanding of the

connectives can easily be explained, providing one discriminates carefully between actual sentences and schemata.

With respect to the connective of implication, its truth functional understanding in the Stoa is multitudinously unequivocally attested to both by Sextus (1842, B 104–106; 1935, B 245–248, B 332) and by Diogenes (1964, Z 81, cf. Z 71). It includes three extensive quotes of the entire truth table, row by row, one in Sextus’ *Outlines of Pyrrhonism*, another in his *Against Logicians*, and an independent one in Diogenes. The cited loci are no marginal notes, rather they intentionally account for the Stoics’ understanding of the connective of implication.

The evidence for rule (10) is equally clear. Sextus describes the truth table in question twice (Sextus, 1935, B 282, B 467). Diogenes’ account of disjunction is far from precise, but at least it indicates a truth functional understanding of the connective (1964, Z 72). There is also a particularly clear quote of rule (10) in Aulus Gellius, to the effect that for a disjunction to be true one disjunct is to be true and all other disjuncts to be false: “ex omnibus, quae disiunguntur, unum esse verum debet, falsa cetera” (Gellius, 1544, XVI 8, p. 438).

However, according to a widely held belief, made popular by Mates (1953), there are two loci, one in Sextus (1935, B 415–426) and the other in Diogenes (1964, Z 73), to suggest a non truth functional understanding of the connective of implication, and a locus in Gellius to suggest a non truth functional understanding of the connective of disjunction in the Stoa. Those very loci are thought to suggest that for an implication and for a disjunction to be true a necessary connection of sort is required between the component sentences, i.e. a modal understanding of the connectives in question. Clearly, such a requirement, if existed, would be incompatible with rules (8) and (10). I believe, however, that the aforementioned difficulties are spurious.

Before all else, notice that advancing a study of modal concepts in the Stoa is a priori extremely unlikely. Firstly, the thinkers of the Stoa seriously stand up for tenet (6) and earnestly reject modal distinctions. Secondly, compared to those confirming the truth functional understanding of the connectives in the Stoa, the loci allegedly suggesting a modal interpretation of the connectives in question are of low quality. Not one of them intends to explicate truth conditions of a connective, they only mention the conditions in passing. Thirdly, and more importantly, the problematic loci can be easily explained in full accordance with rules (8) and (10). To achieve that, I believe, it is sufficient to discriminate

between those loci which speak of a formula true relative to a value assignment, on the one hand, and those which speak of a formula true without reserve, on the other.

Sextus' problematic locus is a long passage containing a criterion for a valid inference (Sextus, 1935, B 415–426):

an inference is valid (ὕγιέος) if and only if the implication is true (ἀληθές)  
having a conjunction of the premises as the antecedent and the conclu-  
sion as the consequent. (Sextus, 1935, B 426)

Otherwise the inference is invalid. Scholars tend to think that a necessary connection between the premises and the conclusion of a valid inference remains beyond dispute and, therefore, it is an attribute of a true implication. They overlook, however, that Sextus speaks of schemata rather than of actual sentences and is more than likely to mean an implication true without reserve in sense (11). For observe that validity is generally not thought of as depending on the value assignment. And neither should be the other side of the above quoted equivalence. In the Stoic syllogistic the only concept of truth that is independent on the value assignment is the concept of truth without reserve, referred to by expression (11). The following analysis of the entire passage in question (Sextus, 1935, B 415–426) strengthens the case for such an interpretation of Sextus even further.

Again and again, over the course of several pages, Sextus switches from speaking of truth relative to a value assignment to speaking of truth without reserve and conversely, making extensive use of extraordinary atoms. And he does it carelessly. When applying the criterion of validity Sextus speaks in a slightly confusing manner, and he does it three times (Sextus, 1935, B 416, B 419, B 421), because he gives one example each of an inference which is sound (i.e. both is valid and has only true premises), an inference which is valid but not sound, and an invalid inference. Twice he says that an inference under consideration is valid, because the implication with a conjunction of the premises in the antecedent and the conclusion in the consequent is true, and the implication is true, because it never (μῆδέποτε) has both a true antecedent and a false consequent (Sextus, 1935, B 416, B 419). It is the very adverb “never” that bothers scholars and suggests a modal concept of implication rather than truth functional. The difficulty vanishes, however, once it has become clear that “never” means “under no value assignment” and, hence, “true” means here “unreservedly true” (consult Section 3

for details). It becomes evident when the problematic loci are compared with the third application of the validity criterion within the same passage (Sextus, 1935, B 421). Here Sextus is more careful in choosing words. He says, the inference under consideration is invalid because the implication with a conjunction of the premises in the antecedent and the conclusion in the consequent

is capable of having both a true antecedent and a false consequent and, therefore, is capable of being false (δύναται γὰρ τὸ συνημμένον τοῦτο νυκτὸς οὐσης ἀπὸ ἀληθοῦς ἄρχεσθαι τῆς συμπλοκῆς, λήγειν ἐπὶ ψεῦδος τὸ ἡμέρα ἔστιν, καὶ διὰ τοῦτο εἶναι ψεῦδος).

Clearly, if Sextus meant anything but the truth functional understanding of implication, he would say “it is false” (ψεῦδος ἐστίν) not “it is capable of being false” (δύναται εἶναι ψεῦδος). On the contrary, since the word “δύναται” serves as the verbum regens for all infinitives: “ἄρχεσθαι”, “λήγειν” and “εἶναι”, it follows that to be false is meant to mean the same as to have both a true antecedent and a false consequent. Hence, the other two loci within the same context should be understood as stating in a careless manner that the inference in question is valid because the implication in question is bound (ὀφείλει) to be true, i.e. is valid. After a thorough examination, the quoted loci seem to further confirm the truth functional understanding of implication in the Stoa rather than to undermine it.

To historians Diogenes is very well known for his negligence of details. Regarding the Stoics’ understanding of implication, he gives two explanations: in one he quotes the entire truth table (1964, Z 81), in the other he obscurely says: “an implication is true if and only if a contradictory of its consequent disagrees (μάχομαι) with the antecedent”, with no in depth explanation of the concept of disagreement (1964, Z 73). It seems to me that there is only one consistent reading of those two loci combined. What Diogenes means is either one or both of the following statements:

$$\begin{aligned} \|( \varphi \rightarrow \psi ) \|_v &= 1 && \text{if and only if} && \| \neg( \varphi \wedge \psi^* ) \|_v &= 1, \\ \|( \varphi \rightarrow \psi ) \| &= 1 && \text{if and only if} && \| \neg( \varphi \wedge \psi^* ) \| &= 1, \end{aligned}$$

i.e. the disagreement of formulae is that the conjunction of the formulae is false either relative to a given value assignment or relative to every value assignment. Such a construction is not unfamiliar to the Stoic syllogistic, as it appears in axioms (15) and (16). Observe also that although scholars translate Diogenes’ expression “μάχομαι” as “to be incompatible”, in

Greek the expression is not modal, viz. it means “to not agree” rather than “to cannot agree”. The modal tone may be introduced to it only by quantifying over value assignments, especially in Stoics, who by means of claim (6) reject genuine modal distinctions.

Gellius’ problematic locus is quite similar. He expresses dissatisfaction with the disjunction “the commands of a father are honourable or base”, and proposes, instead, the following one “the commands of a father are honourable or base or are neither honourable nor base” (quoted in [Mates, 1953](#), p. 52). However, notice that Gellius does not claim that the dubious disjunction is false. Rather he claims that it is not valid ( $\psi\gamma\iota\acute{\epsilon}\varsigma$ ) and not lawlike ( $\nu\acute{o}\mu\iota\mu\omicron\nu$ ). And, as mentioned above, in the same work he quotes an accurate description of rule (10). Hence, it is more than likely that he rejects the sentence in question not as false relative to a mysterious value assignment but as an instance of an invalid schema. Thus the locus is in perfect agreement with the truth functional understanding of the connective of disjunction.

In my view, the evidence for attributing rules (7)–(10) to the Stoa is strong. All surviving loci support the attribution either evidently or arguably. Actually, the evidence for the truth functional understanding of implication is even stronger than in the cases of negation and conjunction. One should only keep in mind that formulae of the object language are schemata rather actual sentences, to solve the alleged enigmas in the sources. However, even if I were to turn out to be wrong on that point, Theorem 7 would still hold. For observe that — due to the above mentioned lack of independence — Theorem 7 employs the connectives of negation and of conjunction exclusively, that definitely are understood in the Stoa in accordance with rules (7) and (9). Furthermore, the Stoics are well known to define connectives using one another ([Mates, 1953](#)), and the connectives of negation and of conjunction define every truth functional connective. Therefore, even if in the Stoa the connectives of implication and of disjunction were not truth functional, the Stoic syllogistic would be an extension of classical propositional calculus, certainly not a deviant logic.

## 8. The Chrysippean completeness

Stoics, and Chrysippus in particular, insist that their syllogistic is complete. The claim is unequivocally attested to by Sextus ([1842](#), B 156–

157, 166–167, 194) and by Diogenes (1964, Z 79). However, the available sources do not specify either a proof of completeness or even any clues that would allow its straightforward reconstruction, as stated by Mates:

Evidently, some such thesis [i.e. one of completeness] must have been a part of the Stoic introductions with which Sextus and Diogenes were familiar. Whether Stoic logic was in fact complete cannot be decided until we know all four of the meta-rules for analyzing arguments.

(Mates, 1953, p. 82, cf. p. 4, 64)

And yet, I find Mates’s statement a bit too pessimistic. With some acceptable degree of speculation, Chrysippus’ completeness theorem can be trustworthily identified. The Stoic syllogistic, as I understand it, consists of three separate, although interrelated, parts:

- (a) a theory of the rules of category (4), i.e. the unreservedly true formulae, in the form of a decision procedure by means of the truth tables, as described in Section 3,
- (b) a theory of designated rules of category (5), i.e. indemonstrable syllogisms, in the form of an axiomatic system based on the concept of formal proof in sense (2), as described in Section 4,
- (c) a theory of the remaining rules, based on the concept of formal proof in sense (1), i.e. a complete proof theory, employing parts (a) and (b) as primitives in the way similar to the derivative rules of the contemporary logic.

Part (a) presents an infinite, but effectively generated, collection of axioms rather than a formal semantics, and, hence, Chrysippus’ completeness claim concerns the syllogistic in its entirety, i.e. parts from (a) to (c) combined rather than part (b) alone. When insisting on completeness of the syllogistic, Chrysippus means, I hypothesise, that every valid rule of derivation within the propositional logic reduces to the effectively generated rules of category (4), i.e. the unreservedly true formulae, and a collection of designated rules of category (5), i.e. indemonstrable syllogisms, as described in Section 2.

The argument here is that the completeness result, asserted by the Stoics, is essentially equivalent with the result I presented in Section 5. Firstly, the result has been achieved exclusively by means of assumptions that are attested to on the extant source material. Secondly, unlike implausibly advanced or convoluted interpretations of the logical legacy of the Stoa, the argument presented herein — namely, the generation of a complete system of logical rules via the decision procedure for classi-

cal propositional calculus and the syllogistic system — remains entirely within the reach of Stoic logicians during the nascent, formative developmental stage of formal logic. And thirdly, although Chrysippus's completeness proof is lost, a noteworthy vestige of it is preserved in the sources. It is the statement contemporary scholars identify as the Stoics' criterion of a valid inference, quoted in (Sextus, 1935, B 426, see above, p. 21), to the following effect:

an inference with the premises:  $\varphi_1, \varphi_2, \dots, \varphi_n$  and the conclusion  $\varphi$   
is valid if and only if  $\|((\varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n) \rightarrow \varphi)\| = 1$ . (26)

According to Mates (1953) criterion (26) is universally accepted in the Stoa but never applied in a proof. I hypothesise, it is because for Chrysippus claim (26) is a result, not a definition. Observe that claim (26) corresponds with my Lemma 9. The former is likely to be originally a derivative of the latter. As mentioned, although the Stoics do use the truth table method quite efficiently, they seem to apply it solely to formulae, never to inference schemata or rules. Therefore, the reduction of the validity of the inference schemata and rules to the validity of formulae may be a key result of the Stoic syllogistic. Although in the 21st century the result is well known to every freshman, it has to have been achieved at some point, being at that time quite a discovery. Arguably, the Stoics are to be credited for it.

Instead of the complex procedure described above, it is part of the mindset of a contemporary logician to immediately extend the system of indemonstrable syllogisms, described in Section 4, by the following second-degree rule:

$$\frac{\tau, \Phi \Vdash \varphi,}{\Phi \Vdash \varphi,} \quad \text{on condition that } \|\tau\| = 1. \quad (27)$$

Clearly, the system of indemonstrable syllogisms supplemented by rule (27) is equivalent with classical propositional calculus and, as the set of valid formulae is decidable, from the contemporary point of view it is a proper axiomatic theory. Hence, it may be tempting to think of rule (27) as one of the Stoics' missing second-degree rules. It does not seem, however, that ancient logicians, whether Stoic or Aristotelian, are ready to take that step. Instead, as mentioned in Section 2, they consider the rules of categories (4) and (5) separately, to combine them into a complete proof theory. Otherwise it is difficult to explain why they distinguish

terms (1) and (2), and refer as indemonstrable to all theorems — and not only the axioms — of the system of indemonstrable syllogisms. The fact that the term “indemonstrable” (ἀναπόδεικτος) derives from term (1) suggests that the indemonstrable syllogisms are not demonstrable in sense (1) on the ground of part (c) of syllogistic without committing a circularity, even though they may be demonstrated in sense (2) on the ground of part (b). A question naturally arises, why ancient logicians seem to have failed to qualify the rules of category (4) as a special case of the rules of category (5) with  $n = 0$ , and to endorse rule (27) within their system of indemonstrable syllogisms. It is so, perhaps, because in antiquity 0 is not considered a number?

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