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ST's Meta-Paracompleteness

Abstract. The logic **ST** [10] has been alternatively identified with classical logic due to its set of valid inferences, and with the logic **LP**, as demonstrated by Barrio et al. [7], through the ability to translate every valid metainference of **ST** into a valid inference of **LP**. Furthermore, [18] have argued that **ST** and **LP** are the same logic given their membership in the same equivalence class according to Blok-Jónsson's theory of consequence. These two results connecting **ST** with **LP** have led to the perception of **ST** as a form of *paraconsistent* logic, at least metainferentially. In this work, a different characterization of the metainferences of **ST** is proposed through the inferences of the paracomplete logic **K3**, by arguing that this presentation is as legitimate as the other perspectives on this logic available in the literature. Utilizing new insights into the nature of the duality between the logics **LP** and **K3** [as featured in 9], a translation from every valid metainference of **ST** into a valid inference of the logic **K3** is offered. Additionally, a possible understanding of the notion of *external logic* is considered, allowing **K3** to be viewed as the external logic of **ST**. Finally, an interpretation of the notion of consequence is presented that provides a natural understanding of the metainferences of **ST** as metainferential counterparts of the inferences of **K3**.

Keywords: strict-tolerant logic; metainferences; duality; paraconsistency; paracompleteness

1. Introduction

The past few years have seen significant developments in the study of the metainferential level of a logic, beginning with the application by some authors of several *substructural* logics to the solution of different seman-

tic paradoxes.¹ Roughly speaking, metainferences are inferences between collections of inferences, and thus *substructural logics* can be regarded as those logics which have fewer valid metainferences than classical logic, the usual reference point for logicians and philosophers.² In particular, the Strict-Tolerant logic, which is the focus of this work, was introduced by Cobreros et al. [10] as an approach based on a non-transitive notion of consequence that is suitable for addressing semantic paradoxes while maintaining a transparent truth predicate alongside classical logic. One of the main advantages of this logic is that it has exactly the same set of valid inferences as classical logic, and this particular feature has led the proponents of this logic to depict it as *a version* of **CL** itself.³ However, it is well-known that [7] demonstrated a significant connection between **ST** and the paraconsistent logic **LP** (the Logic of Paradox, which was introduced by Priest [26] and Asenjo [1]). The aim of this article is to discuss the extent to which the Strict-Tolerant logic, **ST**, is related to classical logic or to the Logic of Paradox, **LP**. We will offer a characterization of the metainferences of **ST** through the inferences of the paracomplete logic **K3**, and we will argue that this presentation is as legitimate as the other perspectives on this logic available in the literature.

The authors in [7] demonstrated that, although similar to **CL** at the level of inferences, **ST** has a set of metainferences that can be translated into valid inferences of the logic **LP**. Due to this fact, it is argued that the logic of the metainferences of **ST** *is*, in fact, **LP**. They also explain that, if the distinction between *external* and *internal* logic⁴ is considered, **LP** can be understood as the external logic of a given calculus for **ST**. Furthermore, authors such as [18] claim that **ST** and **LP** can be regarded as the same logic because they belong in the same equivalence class according to Blok-Jónsson's theory of consequence [see 8]. This view of the logic **ST** as similar in a special sense to the paraconsistent **LP** suggests that logics with failures of the principle of Transitivity can be regarded as metainferential versions of paraconsistent logics. In this

¹ A non-contractive approach can be found in [36]; a non-transitive solution, in [11], and there are non-reflexive and non-monotonic strategies as well [see 16, 19, 20].

² This definition of *substructural* follows the approach proposed by Barrio and Egré [3].

³ **ST** proponents defend, when arguing in this way, a conception of logic linked to the set of valid inferences. This approach is not the only one available in the literature, and we can find various criticisms, for example in [4].

⁴ The precise definitions of the terms introduced here can be found in Section 2.

paper, we present a different characterization of the metainferences of **ST**, one that allows us to see this logic also close to a paracomplete logic: **K3**. There is a certain consensus regarding what it means for a logic to be paraconsistent at the level of inferences [see 35], but when we consider metainferential levels, it is no longer so clear what exactly paraconsistency is. However, some authors have taken the trouble to investigate this question and offer possible answers [17]. Based on these conceptual clarifications, it is possible to draw a relationship between the Cut rule⁵ at the meta-level and the Explosion rule at the inferential level, and to say that Cut is a kind of *metainferential Explosion*. In particular, when metainferences are translated into inferences in the manner proposed by Barrio et al. [7], the Cut rule becomes an instance of Explosion, and conversely, it can be shown that Explosion is translatable via translation functions such as those used by Barrio et al. [7] or Dicher and Paoli [18], in a certain version of Cut. Thus, the failure of Cut in the logic **ST** can be seen as analogous to the failure of Explosion in the logic **LP**. We show that these translations, which allow an understanding of the logic **ST** as metainferentially paraconsistent, arise from a habit of reading the notion of consequence *disjunctively*. If a *negated conjunctive* reading of the relation of implication is adopted, then the metainferences of **ST** can be perceived in a very different manner. Crucially, the interpretation of the rule of Cut goes from a *metainferential version* of the principle of Explosion in a disjunctive reading to a *metainferential version* of the principle of Excluded Middle in a negated conjunctive reading. In this way, a failure of Cut can be viewed as a failure of *metainferential Excluded Middle*. We will take some space to think about the (relatively unexplored) notion of *metainferential paracompleteness* and argue that this view of Cut makes it plausible to interpret **ST** as metainferentially paracomplete.

The structure of the paper is as follows. After a brief technical introduction, in Section 2, to the main concepts and tools we will be working with, in Section 3 we review the proof given by Barrio et al. [7] regarding the possibility of translating every valid metainference of **ST** into a valid inference of **LP**. In Section 4, we prove that there is also a translation from every valid metainference of **ST** into a valid inference of the logic **K3**. Additionally, in Section 4.2, we provide an approach to the

⁵ In particular, a certain version of Cut without contexts, which Přenosil [28] calls *explosive Cut*.

notion of *external* logic that allows for the interpretation of **K3** as the external logic of the logic **ST**. Later, in Section 5, we address a possible objection to our proposal: we consider the fact that some may view our results as merely technical and argue that there is an intuitive reading of the metainferences of **ST** that naturally connects them with inferences of **LP**. We answer this philosophical challenge with an argument that pays attention to the concepts of *metainferential paraconsistency* and *metainferential paracompleteness*. Finally, in Section 6 we clarify that our intention is not to begin viewing **ST** as more related to **K3** than **LP**, or **CL**. Instead, we propose considering the unique character of this non-transitive logic, which is similar to classical logic if we consider the set of valid inferences, and on the level of metainferences it can alternatively, depending on the reading of the notion of consequence that is adopted, be characterized as similar to **LP** (and therefore metainferentially paraconsistent) or to **K3** (and hence metainferentially paracomplete).

2. ST and other three-valued Strong Kleene logics

This work concerns some of the logics usually known as *Strong Kleene Logics*. These logics are defined by means of consequence relations obtained from three-valued Strong Kleene algebras ($\mathbb{K}$). In this section, we present the technical concepts and definitions used throughout the paper, beginning with the algebraic setting and valuations before introducing validity for inferences and metainferential structures.

2.1. Strong Kleene logics

We are going to work mainly with the logics **CL**, the *Strict-Tolerant* logic **ST** [see 10], the *Strong Kleene* logic **K3** [see 23], and the *Logic of Paradox*, **LP** [see 26]. We will also be mentioning the *Tolerant-Strict* logic **TS** [see 20]. It is possible to characterize these five different consequence relations using the Strong Kleene algebra, $\mathbb{K}$. Let $\mathcal{L}$ be a propositional language, and $\text{FOR}(\mathcal{L})$ the algebra of well-formed formulas defined in the usual way. We use lowercase Greek letters $\phi, \psi, \dots$ for schematic formulas, and uppercase Greek letters $\Gamma, \Delta, \Sigma, \Theta, \dots$ for sets of formulas.

DEFINITION 2.1 (Strong Kleene Algebra). The Strong Kleene algebra $\mathbb{K}$ is defined over the set of truth-values $V_{\text{SK}} = \{1, 1/2, 0\}$ with the standard

order $1 > 1/2 > 0$, together with the operations $\neg, \wedge, \vee$ of arities 1, 2, and 2, respectively: $\neg x = 1 - x$, $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$.

A K -valuation v is a homomorphism $v: \text{FOR}(\mathcal{L}) \rightarrow \mathbb{K}$. We let Val be the set of all K -valuations over $\mathcal{L}$. For any set of formulas $\Gamma \subseteq \text{FOR}(\mathcal{L})$, we denote its image under v by $v[\Gamma] = \{v(\gamma) : \gamma \in \Gamma\}$. Following the framework of standards introduced by Cobreros et al. [10], a *standard* is a non-empty subset of V_{SK} . In particular, two standards are central to our discussion: the strict standard $s = \{1\}$ and the tolerant standard $t = \{1/2, 1\}$. These standards give rise to two fundamental notions of formula satisfaction. We say that a formula ϕ is *strictly satisfied* by a valuation v , written $v \Vdash_s \phi$, when $v(\phi) \in s$, and *tolerantly satisfied*, written $v \Vdash_t \phi$, when $v(\phi) \in t$. Note that these two notions are dual in the sense that $v \Vdash_s \neg\phi$ iff $v \not\Vdash_t \phi$, and $v \Vdash_t \neg\phi$ iff $v \not\Vdash_s \phi$. From these two notions of satisfaction, it is possible to draw four different notions of *inference validity*⁶ by requiring preservation of satisfaction from premises to conclusions, either homogeneously (ss or tt) or heterogeneously (st or ts).

DEFINITION 2.2 (Standards and Consequence Relations). Let $v[\Gamma] \subseteq x$ mean that $v(\gamma) \in x$ for all $\gamma \in \Gamma$, and $v[\Delta] \cap y \neq \emptyset$ mean that $v(\delta) \in y$ for some $\delta \in \Delta$. A valuation v xy -satisfies an inference $\Gamma \Rightarrow \Delta$ (denoted $v \Vdash_{xy} \Gamma \Rightarrow \Delta$) iff whenever $v[\Gamma] \subseteq x$, then $v[\Delta] \cap y \neq \emptyset$. An inference is xy -valid ($\models_{xy}^{V_{SK}} \Gamma \Rightarrow \Delta$) iff every $v \in Val$ xy -satisfies it.

Accordingly, we define the following systems:

- Logic **LP** is associated with the relation $\models_{LP} = \models_{tt}^{V_{SK}}$
- Logic **K3** is associated with the relation $\models_{K3} = \models_{ss}^{V_{SK}}$
- Logic **ST** is associated with the relation $\models_{ST} = \models_{st}^{V_{SK}}$
- Logic **TS** is associated with the relation $\models_{TS} = \models_{ts}^{V_{SK}}$

If we restrict valuations to $V_{CL} = \{v \in Val : v[\mathcal{L}] \subseteq \{0, 1\}\}$, Classical Logic (CL) corresponds to $\models_{CL} = \models_{tt}^{V_{CL}} = \models_{ss}^{V_{CL}} = \models_{st}^{V_{CL}} = \models_{ts}^{V_{CL}}$.

To briefly characterize these logics, we can say that **K3** is *paracomplete* in the sense that $\not\models_{K3} \emptyset \Rightarrow \phi \vee \neg\phi$, and **LP** is *paraconsistent* in the sense that $\phi \wedge \neg\phi \Rightarrow \emptyset \not\models_{LP}$. If we define *negation duality* between logics using the previously shown duality of the notions of satisfaction, then we can say that, being $\models_x$ a given notion of logical consequence, its

⁶ If we were to consider meta-inferential validity, we would have to discuss 16 different concepts; I would like to thank an anonymous reviewer for this clarification.

dual is the consequence $\models_y$ such that $\models_x \Gamma \Rightarrow \Delta$ iff $\models_y \neg(\Delta) \Rightarrow \neg(\Gamma)$ (where $\neg(\Delta) = \{\neg\delta \mid \delta \in \Delta\}$). Thus, as proven in [10], **LP** and **K3** are *negation-dual*, which means that $\models_{\mathbf{LP}} \phi_1, \dots, \phi_n \Rightarrow \psi_1, \dots, \psi_m$ just in case $\models_{\mathbf{K3}} \neg\psi_1, \dots, \neg\psi_m \Rightarrow \neg\phi_1, \dots, \neg\phi_n$, and vice versa. It is well-known that, if we assume that language does not contain constant proposition symbols such as $\top$ or $\perp$, **K3** is “classical on the left” but empty on the right in the sense that for any sets of formulas Γ and Δ :

$$\models_{\mathbf{K3}} \Gamma \Rightarrow \emptyset \text{ iff both } \models_{\mathbf{CL}} \Gamma \Rightarrow \emptyset \text{ and } \not\models_{\mathbf{K3}} \emptyset \Rightarrow \Delta.$$

Therefore, Γ is **K3**-unsatisfiable just in case it is classically unsatisfiable and no set Δ is **K3**-valid. Similarly, **LP** is empty on the left but “classical on the right”:

$$\models_{\mathbf{CL}} \emptyset \Rightarrow \Delta \text{ iff both } \not\models_{\mathbf{LP}} \Gamma \Rightarrow \emptyset \text{ and } \models_{\mathbf{LP}} \emptyset \Rightarrow \Delta.$$

No set Γ is **LP**-unsatisfiable but, for any Δ , it is **LP**-valid just in case it is classically valid. Focusing now on the mixed notions of validity presented above, **ST**, and **TS**, we can say that **ST** is non-transitive in the sense that, despite having the same validities as **CL** [see, e.g., 10] it locally invalidates many classically valid metainferences, such as Cut:

$$\frac{\Gamma, \phi \Rightarrow \Delta \quad \Gamma \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \Delta} \text{ Cut}$$

Finally, **TS** is non-reflexive: the axiom of Identity, $\phi \Rightarrow \phi$, is not a valid inference of **TS**. In fact, in a language without the means for expressing semantic values, **TS** has no validities at all, though it validates many classically valid metainferences.

2.2. Formal preliminaries: Inferences and Metainferences

We now formally define inferences and metainferences. To keep the technical framework lightweight and directly tied to our semantic results, we treat inferences and metainferences directly as schematic structures under uniform substitution.

DEFINITION 2.3 (Inferences and Metainferences). An *inference* of a language $\mathcal{L}$ is a pair (Γ, Δ) , written $\Gamma \Rightarrow \Delta$, where $\Gamma, \Delta \subseteq \text{FOR}(\mathcal{L})$.

A *metainference* is a pair (Γ, Δ) , written Γ/Δ , where Γ and Δ are sets of inferences. A metainference is *structural* if it contains no object-language connectives or constants.

To evaluate whether a metainference is valid in a given logic $\mathbf{L}$, different notions of validity have been proposed in the literature:

1. *Local Validity*: A metainference $\Gamma \Rightarrow \Delta / \Sigma \Rightarrow \Pi$ is *locally valid* in $\mathbf{L}$ iff for every valuation v , if v satisfies every premise inference $\Gamma \Rightarrow \Delta$ according to $\mathbf{L}$, then v satisfies the conclusion inference $\Sigma \Rightarrow \Pi$ according to $\mathbf{L}$.
2. *Simple Global Validity*: A metainference is *simply globally valid* in $\mathbf{L}$ iff whenever the premise inferences are valid in $\mathbf{L}$ ($\models_{\mathbf{L}} \Gamma \Rightarrow \Delta$), the conclusion inference is also valid in $\mathbf{L}$ ($\models_{\mathbf{L}} \Sigma \Rightarrow \Pi$). This notion is implicitly present in [11] when assessing the impact of Cut-elimination and rule admissibility.
3. *Global Substitutional Validity*: A metainference is *global-substitutionally valid* in $\mathbf{L}$ iff for every uniform substitution σ of propositional variables, if every substituted premise $\models_{\mathbf{L}} \sigma(\Gamma \Rightarrow \Delta)$ is valid in $\mathbf{L}$, then the substituted conclusion $\models_{\mathbf{L}} \sigma(\Sigma \Rightarrow \Pi)$ is also valid in $\mathbf{L}$. This reading has been explicitly defended and developed by Teijeiro [34] and Kortenbach [24], and is the framework underlying the meta-results in [7].

In this paper, we focus primarily on *local validity*, as it provides a uniform semantic treatment across inferential levels without collapsing the distinct behavior of non-transitive logics like **ST** [6].

2.3. Internal and External Validity

In order to clarify the discussion that will take place in Section 4.2, we give here a formal account of the notions of *external* and *internal* validity. We assume that the reader is already somewhat familiar with the basic features of a sequent calculus, as explained, for example, in [25]. A sequent on $\mathcal{L}$ is an ordered pair (Γ, Δ) of finite, possibly empty subsets of $\text{FOR}(\mathcal{L})$, written as $\Gamma \triangleright \Delta$ for ease of notation. $\text{SEQ}(\mathcal{L})$ is the set of all sequents on $\mathcal{L}$, and $\text{SEQ}^*(\mathcal{L})$ the set of all sequents $\Gamma \triangleright \Delta$ on $\mathcal{L}$ such that $\Gamma \cup \Delta \neq \emptyset$. We borrow the definitions of external and internal validity provided in [13], which are presented using multiple conclusions:

DEFINITION 2.4 (Internal validity). An inference from premises Γ to conclusions Δ is valid in the internal logic of a sequent system $\mathcal{C}$ iff the sequent $\Gamma \triangleright \Delta$ is derivable in that system.

DEFINITION 2.5 (External validity). An inference from premises Γ to conclusions Δ is valid in the external logic of a sequent system $\mathcal{C}$ iff the

sequent $\triangleright \Delta$ is derivable in the new system arrived at by adding initial sequents $\triangleright \gamma$ for every $\gamma \in \Gamma$ to the original system.

This is the most common way of conceiving the difference between what we call *external logic* and *internal logic*. That is, these are notions fundamentally linked to certain calculi. As noted in [13], the distinction between internal and external is used to distinguish the external logic of a sequent system from its internal logic, and it appears in these terms in [2], where they are referred to as “two methods that are traditionally used to associate a consequence relation with a Gentzen-type formalism.” Therefore, it is somewhat forced to speak of the external logic of a given logic, and claiming that the external logic of **ST** is **LP** could be problematic for the reasons stated in [13]. These authors propose approaching the problem in terms of a *desideratum* that could be held with respect to a given notion of consequence, i.e., something that could be asked of a logic is that it should have *some* particular calculus, for which a coincidence between its internal and external validity could be established. In this work, we propose looking at the issue from a semantic point of view and considering a semantic definition of external logic. We can then stay on familiar ground with respect to the logics we know and not rely on results that are relative to the calculi that can be given for those logics. In this way, in Section 4.2, an argument is presented around the notion of external and internal logic based on the following semantic definition of external validity:

DEFINITION 2.6 (Semantic external validity). An inference from premises Γ to conclusions Δ is valid in the external logic ($\mathbf{L}_e$) of a logic $\mathbf{L}$ iff, for every valuation v , if v satisfies in $\mathbf{L}$ the inferences $\Rightarrow \gamma$ for every $\gamma \in \Gamma$, then the valuation v also satisfies some inference $\Rightarrow \delta$ in $\mathbf{L}$ for some $\delta \in \Delta$ in $\mathbf{L}$.

This semantic definition is also presented in terms of multiple conclusions, since it is of vital importance for the similarity we will show between a given notion of semantic external logic for **ST** and the logic **K3**. Now, to conclude the formal preliminaries, in the next section we will present and briefly describe the logics we will focus on throughout this paper.

3. ST as LP

The non-transitive logic **ST** was originally proposed by Cobreros et al. [10] [see also 11, 32] as a formal framework for dealing with semantic paradoxes, and one of the main advantages of this logic is that, as [12] show, it has exactly the same valid inferences as **CL**. This fact has led **ST** advocates to argue for a possible identification between **ST** and **CL**. The authors defend a criterion of identity for logics based on the set of valid inferences. However, this has not been without discussion; in particular, [7] have proven that every **ST**-valid metainference can be translated into an **LP**-valid inference and vice versa, and while doing so, these authors have argued that the set of valid inferences is not enough for characterizing a logic. For this specific case, if one pays attention to the level of metainferences, it is possible to consider a very interesting similarity between the paraconsistent logic **LP** and **ST**. They also consider the point from a proof-theoretical perspective, and using a three-sided sequent calculus show that it is possible to consider the logic **LP** as the external logic for **ST**. Later, they use this fact as an additional argument for their proposal of an identification between these two logics, arguing in this way against a conception of **ST** as a non-transitive version of classical logic.

Alternatively, [18] have advocated that **ST** is actually **LP**, as they belong in the same equivalence class according to Blok-Jónsson's theory of consequence [see 8]. The basic idea of this theory is that one and the same logic can be presented as a consequence relation over different sets of formal objects (e.g. formulas, sequents or equations), and that whenever we intend to single out a logic we should abstract away from the specifics of these presentations. We usually think of consequence relations as relations between sets of formulas obeying the principles of Reflexivity, Monotonicity, and Transitivity. Despite this, Blok and Jónsson observe that the same three principles govern many other contexts where we are reasoning from given premises to a certain conclusion, regardless of whether we are manipulating formulas or other more complicated objects in the course of our deductions. Thus, they propose replacing, in the usual (Tarskian) definition of logical consequence, the set of formulas for any arbitrary set, and define what they call an *abstract consequence relation* (*ACR*). An *ACR* over the set A is a relation $\vdash \subseteq \wp(A) \times A$ obeying the following conditions for all $a \in A$ and for all $X, Y \subseteq A$:

1. $X \vdash a$ if $a \in X$ (*reflexivity*);
2. If $X \vdash a$ and $X \subseteq Y$, then $Y \vdash a$ (*monotonicity*);
3. If $Y \vdash a$ and $X \vdash b$ for every $b \in Y$, then $X \vdash a$ (*transitivity*).

This notion of *ACR* has a syntactic counterpart which are the so-called *abstract proof systems (APS)*. An abstract proof system is basically an axiom system where elements of an arbitrary set act as stand-ins for formulas in their various capacities as axioms, members of derivations, etc.

DEFINITION 3.1. An *abstract proof system (APS)* C is an ordered pair $\langle A, R \rangle$ consisting of a set A and a set R of rules for A .

Any propositional Hilbert-style calculus over the language $\mathcal{L}$ can be viewed as an example of an *APS* $\langle A, R \rangle$, with $A = \text{FOR}(\mathcal{L})$. In the same way, any propositional sequent calculus over the language $\mathcal{L}$ can be regarded as an instance of an *APS* $\langle A, R \rangle$, with $A = \text{FOR}(\mathcal{L})$. Dicher and Paoli connect the two concepts of *ACR* and *APS* in a precise way. First, they define a relation of derivability in *APS*s that can be a primary example of an abstract consequence relation:

DEFINITION 3.2. Let $C = \langle A, R \rangle$ be an abstract proof system. A *C-derivation* d of $a \in A$ from $X \in A$ is a finite tree labeled by members of A such that:

1. a labels the root;
2. For each node labeled a_0 , either $a_0 \in X$ or its child nodes are labeled $a_1, \dots, a_n$, and $(\{a_1, \dots, a_n\}, a_0)$ is an instance of a rule of C .

If there is a *C-derivation* d of a from X , then a is said to be *C-derivable* from X , written as $d; X \vdash_C a$ or simply $X \vdash_C a$. a is a theorem of C iff $\emptyset \vdash_C a$.

And then they prove the following theorem:

THEOREM 3.1. Let $C = \langle A, R \rangle$ be an *APS*. Then $\vdash_C$ is an *ACR* over A .

LK, Gentzen's calculus for classical propositional logic [see 21], is an instance of an *APS*, and therefore its derivability relation $\vdash_{\text{LK}}$ on *sequents* is an instance of an *ACR*. Dicher and Paoli show that the sequent calculus LK_{INV}^- [see 29] (which is the calculus LK without the rule of Cut, plus the inverted rules for the connectives) is sound and complete with respect to locally **ST**-valid metainferences. And the abstract

consequence relation $(ACR) \vdash_{\text{LK}_{\text{INV}}^-}$, defined over the calculus LK_{INV}^- , is equivalent to the $ACR \models_{\text{LP}}$ via the mutually inverse transformers:

$$\begin{aligned}\tau(\alpha) &= \{\Rightarrow \alpha\}, \\ \rho(\gamma_1, \dots, \gamma_n \Rightarrow \delta_1, \dots, \delta_m) &= \text{ft}\{\gamma_1, \dots, \gamma_n \Rightarrow \delta_1, \dots, \delta_m\},\end{aligned}$$

where τ is a transformer from formulas to sequents, namely $\tau : \gamma \rightsquigarrow \Rightarrow \gamma$, that preserves and reflects derivability. And ρ works in the opposite direction, from sequents to formulas, also preserving and reflecting derivability.⁷ ft is a formula translation defined as follows:

DEFINITION 3.3. If $\Gamma \Rightarrow \Delta \in \text{SEQ}^*(\mathcal{L})$, its formula translation $\text{ft}(\Gamma \Rightarrow \Delta)$ is

$$\text{ft}(\Gamma \Rightarrow \Delta) = \begin{cases} \neg\gamma_1 \vee \dots \vee \neg\gamma_n \vee \delta_1 \vee \dots \vee \delta_m & \text{if } \Gamma = \{\gamma_1, \dots, \gamma_n\} \text{ and } \\ & \Delta = \{\delta_1, \dots, \delta_m\}, \\ \neg\gamma_1 \vee \dots \vee \neg\gamma_n & \text{if } \Gamma = \{\gamma_1, \dots, \gamma_n\} \text{ and } \\ & \Delta = \emptyset, \\ \delta_1 \vee \dots \vee \delta_m & \text{if } \Gamma = \emptyset, \Delta = \{\delta_1, \dots, \delta_m\}. \end{cases}$$

In virtue of this Dicher and Paoli conclude that the logic **ST** is in a very precise way equivalent to the logic **LP**. Although this approach centered on Blok and Jónsson's theory has an inescapable element of proof theory (by virtue of the relationship it establishes between *ACR* and *APS*), our intention in this paper is not to discuss it in depth in these terms. As we have previously pointed out, the scope of our proposal in this paper is only semantic. However, we will offer a conceptual critique of the approach of these authors given the alternative reading of the notion of consequence that we propose in what follows and that justifies, from a philosophical point of view, the technical result of translation that this work explores.⁸

4. ST as K3

In this article, we propose questioning this series of identifications between **ST** and **LP** by showing that, in the same way that metainferential

⁷ In some other works about the relation of equivalence between formulas and sequents, such as [30], these transformers are defined in reverse, i.e., *rho* goes from formulas to sequents and *tau* from sequents to formulas.

⁸ We leave for future work the *syntactic* discussion as to whether LK_{INV}^- can also be considered, under a given notion of equivalence, a calculus for **K3**.

ST can be seen as inferential **LP**, there is a possible interpretation that allows us to understand the logic of the metainferences of **ST** as the paracomplete logic **K3**. In this section, we provide two possible characterizations of the metalogic of **ST** as the logic **K3**, and argue that this picture of **ST** is as legitimate as the ones presented by Barrio et al. [7] and [18]. Let us begin by recalling that we know two important things: 1. There is a translation from the valid metainferences of the non-transitive logic **ST** into valid inferences of the paraconsistent logic **LP** (according to [7]), and 2. **LP** is *negation dual* to a paracomplete logic: **K3** [see 12]. Could it also be possible to translate every **ST**-valid metainference into a **K3**-valid inference? The answer involves a certain complexity, because the truth is that the duality via negation that exists between **LP** and **K3** is not enough to prove a result equivalent to that of [7]. It is necessary to understand that the relation between these two logics, one paraconsistent and the other paracomplete, is somewhat deeper than what is usually considered. Therefore, the answer is yes: it is possible to translate every **ST**-valid metainference into a **K3**-valid inference, but only if new results regarding the nature of the duality between **LP** and **K3** are taken into account. In what follows, we will provide a proof of the translation of the metainferences of **ST** into inferences of **K3**, basing our strategy partially on the proof presented by Barrio et al. [7], and also using more recent results on the duality between **LP** and **K3** provided by Blomet and Égré [9].

4.1. The translation t' : Metainferences of **ST** into inferences of **K3**

First, let us consider the proof, provided by Barrio et al. [7], of the possibility of translating every valid metainference of the logic **ST** into a valid inference of the logic **LP** and vice versa. From that result, we will show that it is also possible to prove a similar claim with respect to the logic **K3**. The core of the proof will be in Theorem 7, in which we propose the equivalence between metainferences of the logic **ST** and its correspondent translation into inferences of **K3**, via the composed function t' . [7] show that every **ST** metainference token can be translated into an **LP** inference token using a conditional, $\rightarrow$,⁹ instead of a consequence symbol. More precisely, they define a translation t , such that:

⁹ This is none other than the conditional that arises in Strong Kleene from the negation and disjunction defined above.

DEFINITION 4.1. If $\Gamma \Rightarrow \Delta$ is an **ST**-valid inference token, its **LP** translation $t(\Gamma \Rightarrow \Delta)$ is defined as follows:

- $t(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \rightarrow \bigvee \Delta$, and
- $t(\Rightarrow \Delta) = \bigvee \Delta$,

where $\bigwedge \Gamma$ is the conjunction of all the formulas in Γ and $\bigvee \Delta$ is the disjunction of all the formulas in Δ .

And, based on this translation, they prove the following Theorem:¹⁰

THEOREM 4.1 (**ST** and **LP** [7]). *The following two sentences are equivalent:*

- $$(a) \models_{\mathbf{ST}} \frac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_n \Rightarrow \Delta_n}{\Sigma \Rightarrow \Theta}$$
- $$(b) \models_{\mathbf{LP}} t(\Gamma_1 \Rightarrow \Delta_1), \dots, t(\Gamma_n \Rightarrow \Delta_n) \Rightarrow t(\Sigma \Rightarrow \Theta)$$

Our aim is to prove an equivalent result, but between the metainferences of **ST** and the inferences of **K3**.¹¹ For this, we need to define another translation and prove a new Theorem based on that translation. In order to build our translation, which we will call t' , we are going to use a stronger version of duality than the one usually considered between **LP** and **K3**. As we mentioned earlier, these two logics are dual via negation, i.e., $\models_{\mathbf{LP/K3}} \Gamma \Rightarrow \Delta$ iff $\models_{\mathbf{LP/K3}} \neg \Delta \Rightarrow \neg \Gamma$. Nevertheless, there is a stronger sense in which these logics are dual. In a recent paper, [9] define what they call *operational duality*. This notion of duality can be found in Kleene (1952) (and goes back to Schroder [1877]), where it is proven that for any two formulas ϕ, ψ , if the inference $\phi \Rightarrow \psi$ is classically valid, then so is the inference $\psi' \Rightarrow \phi'$, with ϕ' and ψ' obtained from ϕ and ψ by interchanging $\wedge$ with $\vee$. In their case, they define a mapping $\sim$ that will substitute $\wedge$ and $\vee$ for each other in all formulas. If we consider a language $\mathcal{L}$, and let Var be the set of all propositional variables, $INF(\mathcal{L})$ be the set of all the inferences belonging to $\mathcal{L}$, and L^+ be the set of valid inferences of $\mathcal{L}$, then the mapping $\sim$ is defined as follows:

DEFINITION 4.2. Let $\sim : \mathcal{L} \rightarrow \mathcal{L}$ be a mapping such that:

¹⁰ Theorem 3 in the original paper. As an anonymous referee rightly pointed out, it is worth emphasizing that the meta-results in [7] rely on a global substitutional account of metainferential validity. In contrast, under the local validity account adopted in this paper, these structural properties exhibit the distinct metaparacomplete behavior explored below.

¹¹ Note that in this extended proof we use multiple conclusions, in order to facilitate the translation we are looking for.

- $\sim p = p$, if $p \in Var$,
- $\sim \neg \phi = \neg \sim \phi$,
- $\sim(\phi \wedge \psi) = \sim \phi \vee \sim \psi$,
- $\sim(\phi \vee \psi) = \sim \psi \wedge \sim \phi$.

The function $\sim$ is extended to all $\Gamma \subseteq \mathcal{L}$, by $\sim(\Gamma) := \{\sim \gamma : \gamma \in \Gamma\}$, and to all $X \subseteq \text{INF}(\mathcal{L})$, by $\sim(X) := \{\sim(\Gamma) \Rightarrow \sim(\Delta) : \Gamma \Rightarrow \Delta \in X\}$. For instance, this function applied to the formula $p \wedge q$ will produce the formula $p \vee q$, and applied to the formula $p \wedge (q \vee \neg q)$ it will produce the formula $p \vee (q \wedge \neg q)$. Blomet and Égré use the mapping $\sim$ for the following revised notion of operational duality:

DEFINITION 4.3 (Operational duality for formulas and inferences). For any two formulas ϕ and ψ , ψ is called the (revised) operational dual of ϕ when $\sim \phi = \psi$. For any two inferences $\Gamma \Rightarrow \Delta$ and $\Delta' \Rightarrow \Gamma'$, $\Delta' \Rightarrow \Gamma'$ is called the operational dual of $\Gamma \Rightarrow \Delta$ when $\sim \Gamma = \Gamma'$ and $\sim \Delta = \Delta'$.

The next two lemmas express useful properties of $\sim$, namely that it is involutive and commutes with the relational inverse function (x^{-1}). The proofs for lemmas and propositions below can be found in [9].

LEMMA 4.1 ([9]). *For any $\phi \in \mathcal{L}$, $\sim \sim \phi = \phi$. For any $X \subseteq \text{INF}(\mathcal{L})$, $(\sim X)^{-1} = \sim(X^{-1})$.*

The next lemma describes the operational duality between two formulas in terms of dual valuations. Two $\mathbb{K}$ valuations v and v' are said to be dual when for all $p \in Var$, $v'(p) = 1 - v(p)$. The dual of a valuation v will be noted v^* .

LEMMA 4.2 ([9]). *For any valuation v and any $\phi \in \mathcal{L}$, $v(\phi) = 0$ if and only if $v^*(\sim \phi) = 1$ and $v(\phi) = 1$ if and only if $v^*(\sim \phi) = 0$.*

DEFINITION 4.4. (Operational duality for logics). For any two logics $\mathbf{L}_1$ and $\mathbf{L}_2$, we will say that $\mathbf{L}_1$ is operationally dual to $\mathbf{L}_2$ when $\models_{\mathbf{L}_1} \Gamma \Rightarrow \Delta$ if and only if $\models_{\mathbf{L}_2} \sim(\Delta) \Rightarrow \sim(\Gamma)$ for all $\Gamma, \Delta \subseteq \mathcal{L}_{1/2}$.

This allows [9] to prove that **K3** and **LP** are operationally dual:

PROPOSITION 4.1 ([9]). $\models_{\mathbf{K3}} \Gamma \Rightarrow \Delta$ if and only if $\models_{\mathbf{LP}} \sim(\Delta) \Rightarrow \sim(\Gamma)$ and $\models_{\mathbf{LP}} \Gamma \Rightarrow \Delta$ if and only if $\models_{\mathbf{K3}} \sim(\Delta) \Rightarrow \sim(\Gamma)$.

With this result from [9] that proves the operational duality of **LP** and **K3**, we are now able to define another translation t' , which is an *operational dualization* of the translation t :

DEFINITION 4.5. If $\Gamma \Rightarrow \Delta$ is an **ST**-valid inference schema, its **K3** translation $t'(\Gamma \Rightarrow \Delta)$ is defined as follows:

- $t'(\Gamma \Rightarrow \Delta) = \sim(t(\Gamma \Rightarrow \Delta))$, and
- $t'(\Rightarrow \Delta) = \sim(t(\Rightarrow \Delta))$.

Thanks to this translation, we are now in a position to formulate a Theorem with the main result we are after. We present Theorem 7, in which we propose the equivalence between metainferences of the logic **ST** and its corresponding translation into inferences of **K3**, via the composed function t' .

THEOREM 4.2 (**ST** as **K3**). *The following two sentences are equivalent:*

$$\begin{aligned} \text{(a)} \models_{\mathbf{ST}} & \frac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_n \Rightarrow \Delta_n}{\Sigma \Rightarrow \Theta} \\ \text{(b)} \models_{\mathbf{K3}} & t'(\Sigma \Rightarrow \Theta) \Rightarrow t'(\Gamma_1 \Rightarrow \Delta_1), \dots, t'(\Gamma_n \Rightarrow \Delta_n). \end{aligned}$$

We already have everything we need to prove the first direction of this theorem. However, for the second direction, we will need to take into account one more result from [9]. It is easy to see that, if we were only considering the usual negation duality of **LP** and **K3**, that would be insufficient to prove that we can express every **K3**-valid inference by means of a valid inference of **LP**: this can be noticed by letting $n: \wp(\text{INF}(\mathcal{L})) \rightarrow \wp(\text{INF}(\mathcal{L}))$ be a mapping such that $n(x) = \{\neg(\Delta \Rightarrow \neg(\Gamma) : \Gamma \Rightarrow \Delta \in \text{LP})\}$. Clearly, $\mathbf{K3} \neq n(\mathbf{LP})$, since not all **K3**-valid inferences are of the form $\neg(\Delta \Rightarrow \neg(\Gamma))$. Nevertheless, [9] prove that their notion of operational duality allows two logics to be extensionally interdefinable. The following lemma allows us to show that, for every valid inference in logic **K3**, there is a corresponding valid inference in **LP**.

LEMMA 4.3 ([9]). *Given two logics $\mathbf{L}_1$ and $\mathbf{L}_2$, $\mathbf{L}_1$ is operationally dual to $\mathbf{L}_2$ if and only if $\mathbf{L}_1 = \sim \mathbf{L}_2^{-1}$. (Where $\mathbf{L}_1$ is the set of valid inferences of a logic $\mathbf{L}_1$.)*

Now we have all the necessary elements for the test we want to provide. The proof of this lemma can also be found in [9].

PROOF. The first direction, from metainferences of **ST** to inferences of **K3**, is proven by the fact that t' is a dualization, in the very precise terms of the definition of operational duality presented by Blomet and Égré [9]. By virtue of the first direction of Theorem 2, we know that for every locally valid metainference of **ST** there is a corresponding valid

inference in **LP**. And we also know that for every **LP**-valid inference there is an inference valid in **K3** that is the result of applying t' , which gives the corresponding dual of t . Therefore, for every metainference that is valid in **ST** there is a translation t' which gives us a valid inference in **K3**.

We prove the second direction, from inferences of **K3** to metainferences of **ST**, using Lemma 8, the fact that **K3** and **LP** are extensionally interdefinable. By virtue of that, we can express every valid inference of **K3** with a corresponding valid inference of **LP**. And, given the result of [7] proving that we can go from valid inferences of **LP** to valid metainferences of **ST**, we can now prove that for every valid inference of **K3** there is a corresponding valid metainference of **ST**. $\dashv$

We have shown that the usually assumed relation between the metainferences of **ST** and the inferences of **LP** can be emulated using the logic **K3** instead of the logic **LP**. But there is another possible approach to the similarities between logics that we could explore to see what that perspective brings to us in terms of a deeper understanding of the nature of the relations between some of these Strong Kleene logics.

4.2. External and internal logic

In this section, we offer a different approach for the identification of the logic **ST** with the logic **K3**. We will focus on the distinction between *external* and *internal* logic,¹² and we will prove that, under an alternative definition of *external validity*, the external logic of **ST** can also be identified with the internal logic of **K3**. In what follows, it can be useful to keep in mind the definitions of Theorem and Antitheorem: *Theorems* of a logic **L** are sets of formulas Δ such that $\emptyset \vdash_{\mathbf{L}} \Delta$. *Antitheorems* of **L** are sets of formulas Γ such that $\Gamma \vdash_{\mathbf{L}} \emptyset$. In this work, we are keeping the argumentation focused solely on a semantic approach, so we will consider the equivalent semantic notions of theorem and antitheorem: *tautologies* and *contradictions*. *Tautologies* of a logic **L** are sets of formulas Δ such that $\emptyset \models_{\mathbf{L}} \Delta$, and *contradictions* of a logic **L** are sets of formulas Δ such that $\Gamma \models_{\mathbf{L}} \emptyset$. Using a proof-theoretical perspective in their paper, [7] offer a second collapse result connecting the logics **ST** and **LP**. They introduce the distinction between the notions of (syntactic) *external* and *internal* validity for a given calculus and show that for the three-sided

¹² The formal definitions of these concepts can be found in Section 2.1.

sequent calculus that they consider in their paper,¹³ $\mathcal{S}$, they argue in favor of a picture of the external logic of **ST** that coincides with the internal logic of **LP**. Hereafter, we will refer to their definition of external validity (the one provided in Section 2.1) as *External validity 1* (E_1), in order to distinguish it from a second option of external validity we will discuss later. [7] prove that, when using this conception of external validity, there is an additional collapse between the logics **ST** and **LP**:

THEOREM 4.3 (Second Collapse Result¹⁴). $\vdash_{\mathbf{ST}_{E_1}} \Gamma \Rightarrow \Delta$ iff $\vdash_{\mathbf{LP}} \Gamma \Rightarrow \Delta$.

We want to question the priority given in the literature to this conception of external validity. The main idea we propose considering is that this definition is somehow unbalanced and gives at least arguable superiority to the right side of the turnstile over the left. [13] offer an interesting conceptual reflection on external validity that is a good starting point for our argument. They note that the internal logic of a system considers the premises of an inference as located on the left side of a sequent. By contrast, the external logic of a system considers premises and conclusions alike as inhabiting the right side, with the difference between premises and conclusions being that the premises live at the top end of a sequent derivation. In this way, roughly, internal logic is about *following from*, whereas external logic is about *preservation of theoremhood/tautologicity*. Because of this, the internal logic of a system is often useful for connecting the system to natural deduction proof systems, and the external logic for connecting to axiomatic proof systems. But we wonder: Why is the right side of the turnstile more important in characterizing an *external* notion of validity? We can think of the dual movement of passing premises and conclusions to the left side as an equally legitimate way of *externalizing* a consequence relation. It is certainly possible to question the place given to the notion of *tautologicity* (or *theoremhood*) in the definition of external validity. Moreover, when it comes to connecting a notion of validity with an axiomatic presentation of a logic, we could certainly have a set of anti-axioms that may work as a suitable characterization of the same notion of consequence that we are thinking of. The usual notion of external validity emphasizes the relationship between the logical implication and the conditional, but

¹³ Which can provide a proof system for all Strong Kleene logics (by changing the definition of proof-theoretical validity).

¹⁴ This name is the one used in [7].

given that a conditional claim can be contraposed, negation is also a key feature in the definition of logical consequence. Our point is that external logic can be understood in an equally legitimate way if we consider the contraposed reading of the conditional claim as characteristic of logical implication. This conditional expressed in an inference or metainference can be read in two directions:¹⁵ The *forward reading* tells us that if such and such inferences are valid, then so are some of these inferences, and the *backward reading* tells us that if such and such inferences are not valid, then some of these are not valid either. Therefore, we can think of another alternative definition of external validity with a focus on this *contrapositive* notion of validity, thereby understanding external validity as preservation of *contradictoriness* (or *anti-theoremhood*) instead.

DEFINITION 4.6 (External validity 2; E_2). An inference from premises Γ to conclusions Δ is valid in the external logic $\mathbf{L}_E$ of a logic $\mathbf{L}$ iff, for every valuation v , if $v \Vdash_{\mathbf{L}} \delta \Rightarrow$ for every $\delta \in \Delta$, then $v \Vdash_{\mathbf{L}} \gamma \Rightarrow$ for some $\gamma \in \Gamma$.

This second definition understands external validity as preservation of contradictoriness. It tells us that if such and such inferences are contradictions, then some of these other inferences are also contradictions. And what can we conclude from this? Cobreros et al. [13] show that if one contraposes the metainferential validity of **ST**, there is a correspondence between the meta-(anti)-inferences of the logic **ST** and the false inferences of the logic **K3**. Here, we were able to prove something more *direct* in a way: that there is a correspondence between valid *metainferences* of **ST** and valid *inferences* of **K3**. And, by virtue of this contraposed notion of external validity, we are also able to show the following alternative, semantic, collapse:

THEOREM 4.4 (Contraposed Collapse Result).

$$\models_{\mathbf{ST}_{E_2}} \Gamma \Rightarrow \Delta \text{ iff } \models_{\mathbf{K3}} \Gamma \Rightarrow \Delta.$$

PROOF. The first direction can be proven by reductio ad absurdum as follows: suppose that $\models_{\mathbf{ST}_{E_2}} \Gamma \Rightarrow \Delta$ but $\not\models_{\mathbf{K3}} \Gamma \Rightarrow \Delta$. Then there is a valuation v such that v satisfies every $\gamma \in \Gamma$ and does not satisfy some $\delta \in \Delta$. So $v(\gamma) = 1$ for every $\gamma \in \Gamma$ and $v(\delta) = 0$ for every $\delta \in \Delta$ (by the definition of validity in **K3**). But we have supposed that $\models_{\mathbf{ST}_{E_2}} \Gamma \Rightarrow \Delta$, so if v does not satisfy every $\delta \in \Delta$, then v does not satisfy some $\gamma \in \Gamma$.

¹⁵ We follow here, also, the thoughts of [13].

Now, we have that $v(\delta)$ has to be 0 for every $\delta \in \Delta$, so the conclusion is not satisfied in **ST**, and therefore there has to be a $\gamma \in \Gamma$ such that $v(\gamma) = 0$ or $v(\gamma) = 1/2$ (by the definition of *external validity 2* in **ST**).

Similarly, for the second direction, suppose that: $\models_{\mathbf{K3}} \Gamma \Rightarrow \Delta$ but $\not\models_{ST_{e2}} \Gamma \Rightarrow \Delta$. Then there is a valuation v such that $v(\delta) = 0$ for every $\delta \in \Delta$ and $v(\gamma) = 1$ for every $\gamma \in \Gamma$ (by the definition of *external validity 2* in **ST**). But, by the definition of validity in **K3**, if $v(\gamma) = 1$ for every $\gamma \in \Gamma$ then $v(\delta)$ should be 1 for at least some $\delta \in \Delta$. $\dashv$

This makes sense if we simply think about the fact that **ST** has the same tautologies (or theorems) as **LP** and the same contradictions (or anti-theorems) as **K3**. For this reason, a definition of external validity as preservation of tautologicity will directly relate **ST** with **LP**, and a definition of external validity as preservation of contradictoriness will do so for **ST** with **K3**. Here we have two structures that mirror each other, and the two are candidates for characterizing the external logic of **ST**. Why should we prioritize one over the other? It could be argued that the tautologicity (or theoremhood) approach is more suitable by virtue of the translation defined in Barrio et al. [2014] from metainferences of **ST** into inferences of **LP**, *especially* if we were only able to translate meta-(anti)-inferences of **ST** into inferences of **K3**, as [13] do. But we have shown that, by virtue of the (*operational*) duality between **LP** and **K3**, the correspondence between the metainferential level of **ST** and the inferential level of **K3** is stronger than what we usually think. Therefore, we maintain that the two possible definitions of external validity are equally legitimate and have the same value for characterizing **ST**.

5. Two readings of the sequent

The results we have offered so far are mainly technical: we have proven that there is a translation from every valid metainference of **ST** into a valid inference of **K3**, and also that the internal logic of **K3** can be seen as a possible *externalization* of the logic **ST**. Or, in other words, under an alternative (contraposed) conception of *external logic*, as the external logic of **ST**. However, one possible objection to our proposal that we would like to address is of a conceptual nature, and it is related to the interpretation we can give to certain metainferences. It could be argued that if we look at the metainferences of **ST**, there is a very natural interpretation which links them directly with inferences of the logic **LP**.

In particular, the (meta-)rules of Cut and Identity can be understood as the meta-versions of some well-known principles. An advocate of the similarities between **ST** and **LP** may see one of the most simple (in a logical sense) instances of Cut,¹⁶

$$\frac{\Rightarrow \phi \quad \phi \Rightarrow}{\Rightarrow}$$

as a metainferential form of the principle of Explosion ($\models \phi, \neg\phi \Rightarrow \emptyset$). And, because of this, they may value a perspective according to which **ST** can be seen as *metainferentially paraconsistent*. As we pointed out in the Introduction to this paper, there is still debate about what we mean by the properties of paraconsistency or paracompleteness when we focus on the metainferential level of a logic. With regard to paraconsistency, there is at least one paper in the literature exclusively devoted to this topic, which can be taken as a reference point for considering this question. In [17], the authors reflect on what a logic is and what we understand by paraconsistency in general, and then provide a definition of metainferential paraconsistency that is achieved by identifying the metainferential equivalent of the Explosion principle. Given that paraconsistency is usually understood as the failure of at least some form of the Explosion principle, it is necessary to establish what we understand by Explosion at the meta-level, and this will depend, crucially, on the idea of metainferential negation that is considered. A previous attempt to define meta-Explosion can be found in [5], who consider the following principle to be the expression of Explosion at the metainferential level:

$$\frac{\Rightarrow \phi \quad \Rightarrow \neg\phi}{\Rightarrow \psi}$$

[17] criticize this option given that the authors in [5] consider that, in order to formalize the contradiction between two inferences, we need to use the negation of the object language. And, therefore, paraconsistency could be seen as a property that depends on how the negation of the object language behaves. If the inferential negation is paraconsistent, the meta-logic of that logic will also become meta-paraconsistent. This is why they offer as an alternative one way of expressing metainferential negation that they use in another work ([14]) to characterize the duality between logics. They divide inferences and metainferences into negative and positive, they argue that the negative ones are the negation of the

¹⁶ This is what Přenosil [28] calls *explosive Cut*.

positive ones and vice versa, and they denote regular inferences as $\Gamma \Rightarrow^+ \Delta$ and express their negation as $\Gamma \Rightarrow^- \Delta$. The satisfaction conditions for negative inferences in a logic $\mathbf{L}$ are established as follows:

DEFINITION 5.1 ([14]). A valuation v l -satisfies a negative inference or metainference $\Gamma \Rightarrow^- \Delta$ if and only if v l -satisfies $\Rightarrow^+ \gamma$, for all $\gamma \in \Gamma$, and v l -satisfies $\delta \Rightarrow^+$ for all $\delta \in \Delta$ (written as $v \models_{\mathbf{L}} \Gamma \Rightarrow^- \Delta$). A negative inference is valid in $\mathbf{L}$ if and only if for every valuation v , $v \models \Gamma \Rightarrow^- \Delta$.

Just as the negation of formulas contrasts truth and falsehood in different ways, depending on the logic in question, the (meta-)negation of [14] operates at the meta-level, affecting the conditions of satisfaction. And negative and positive inferences form contradictory pairs, no longer in terms of truth and falsehood, but in terms of satisfaction and unsatisfaction. In classical logic, for every pair of inferences $\Gamma \Rightarrow^+ \Delta$ and $\Gamma \Rightarrow^- \Delta$, every classical valuation v satisfies at least one of the two (this is what these authors call *exhaustion*), and no classical valuation v satisfies both inferences at the same time (*exclusion*). However, in non-classical logics such as **ST**, it may happen that there exists a valuation v that **ST**-satisfies both $\Gamma \Rightarrow^+ \Delta$ and $\Gamma \Rightarrow^- \Delta$ (in particular, when everything in Γ is $\frac{1}{2}$ and everything in Δ is also $\frac{1}{2}$). In this way, [17] enable an interpretation of **ST** as metaparaconsistent using the following definition of metaparaconsistency:

DEFINITION 5.2. ([17]). A logic is paraconsistent if and only if either Explosion is invalid or the following rule (*metaExplosion*):

$$\frac{\Gamma \Rightarrow^+ \Delta \quad \Gamma \Rightarrow^- \Delta}{\Sigma \Rightarrow^{+/-} \Pi}$$

is locally invalid.

They say that a logic is metainferentially paraconsistent if metaExplosion is locally invalid in it. That is the case with **ST**, so in this approach this logic can be considered metainferentially paraconsistent.

How, then, according to a conception of metainferential negation such as the one we have just reconstructed, can we understand the property of *paracompleteness* as applied to metainferences? It is not as easy to define a metainferential equivalent of the Law of Excluded Middle with this negation of [14]. In their book on metainferential logics, [15] state that since [14] adopt a set-set framework, metapara-completeness could be defined as follows:

DEFINITION 5.3. A logic is paracomplete if and only if either LEM is invalid or the following rule (*metaLEM*):

$$\frac{}{\Gamma \Rightarrow^+ \Delta \quad \Gamma \Rightarrow^- \Delta}$$

is locally invalid.

If this were the appropriate version of the Law of Excluded Middle for the metainferential level, then a logic such as **TS** (see Section 2.2 for details) could be considered *metainferentially paracomplete* by virtue of the fact that there are **TS**-valuations that do not satisfy any of the conclusions of what we have just called *metaLEM*. However, in our view, formulating this principle in this way unnecessarily complicates matters. Although we also adopt a set-set framework in this paper, there are problems in applying a principle like this when we move, for example, to sequent calculus, because it is not so easy to conceive of sequent calculi in which the rules have more than one conclusion. And, as we will explain below, it is not essential to define the principles of Explosion and LEM for the metalanguage, nor to take a detour by distinguishing between negative and positive inferences in order to account for the properties of metaparainconsistency and metaparacompleteness, nor to appreciate the phenomenon of duality between logics at the metainferential level. For these reasons, we believe that the version of metainferential negation provided by Da Ré et al. [14] and used by Da Ré et al. [17] to characterize metaparainconsistency is not entirely adequate. In this paper, we will not delve into this particular discussion¹⁷ but we nevertheless propose approaching this issue from a different angle and thinking about these properties of metaparacompleteness and metaparainconsistency in a considerably different way.

Let us assume that we largely agree that a negation, at whatever level, is a generator of contradictions [see 22, 27], and that it is the element of the system that expresses how truth values (or satisfaction conditions in the case of metainferences) relate to each other, that is, whether they (the values/satisfaction conditions) are mutually exclusive and/or jointly exhaustive. Perhaps it is not necessary to define a negation for the meta-level; we believe that the function of a negation operator at the meta-level of a logic is fulfilled through the rules of Id

¹⁷ We leave for future work the exhaustive development of the notions of both metainferential negation and *metaparacompleteness* which we believe should be treated following a similar approach to that in [17].

and Cut.¹⁸ Cut and Identity are the rules that express, at the meta-level, the way in which the truth values of a system are linked, whether they are mutually exclusive or jointly exhaustive. In a multivalued context, depending on the reading of the notion of consequence that is adopted, they alternatively express exhaustiveness and exclusion. But then, which principles of the object language express the same thing via inferential negation? Explosion and LEM, which is why Id and Cut can alternatively be identified with one or the other. In what follows, we will explain this point in detail and further clarify everything we have just mentioned.

There is a reason why Cut is the rule that is usually considered similar to the Explosion principle. We consider this to be due to the automatic adoption of a particular reading of the notion of consequence. The main core of the literature about this particular subject uses by default what can be called a *disjunctive* reading of the turnstile. We will explain this point in detail because it is crucial for understanding the reason why the usual interpretation of the metainferences of **ST** is not particularly wrong; it is, however, partial. From a conceptual point of view, it all begins with a revision of the impact that the two possible readings of the turnstile have on the metainferences of a given system once we no longer deal with only two values. If we adopt a multivalued setting, as we are doing here by considering Strong Kleene logics, the two different readings of the notion of consequence induce crucial changes in the metainferences of the systems. As [32] shows, depending on the reading of the consequence relation, Cut and Identity are able to tell us different things with respect to the values or standards. When an inference, or a metainference, $\Gamma \Rightarrow \Delta$ is valid in a logic **L**, there are two possible interpretations of that fact that produce a significant change in the conceptual role of the metainferences of Cut and Identity:

The disjunctive reading: On the one hand, it might tell us that, for every valuation v , either v does not satisfy in **L** some $\gamma \in \Gamma$ or v satisfies in **L** some $\delta \in \Delta$. It tells us that every valuation meets a particular disjunctive (and existential) condition.

The negated conjunctive reading: On the other hand, it could tell us something different: that, for every valuation v , it is not the case that v satisfies in **L** every $\gamma \in \Gamma$ and v does not satisfy in **L** every $\delta \in \Delta$.

¹⁸ We will expand on this point in much greater detail in another paper.

It tells us that every valuation meets a particular negated conjunctive (and universal) condition.

The usual reading is the disjunctive one, and many of the results we have presented about the identification of certain logics with others depend crucially on this fact. But the truth is that we cannot see metainferences such as Identity or Cut in the same way once we consider alternative readings of $\Rightarrow$. Identity in the disjunctive (usual) reading expresses an *exhaustivity condition* on values (standards): consider an inference of the form $\phi \Rightarrow \phi$, and read it disjunctively. It tells us that every valuation either satisfies ϕ or does not satisfy ϕ : an *exhaustion constraint*. On the other hand, Cut in the disjunctive (usual) reading is an *exclusivity condition* on values (standards).

$$\frac{\Gamma, \phi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta} \text{ Cut}$$

The disjunctive reading tells us that ϕ cannot be satisfied and not satisfied at the same time. This is an *exclusion constraint*. If we change the reading of the arrow, all this is affected. In the negated conjunctive reading, that $\phi \Rightarrow \phi$ is valid guarantees that there is no valuation v that both satisfies and does not satisfy ϕ . Now, in this negated conjunctive reading, Identity is the metainference that expresses an *exclusivity condition* on the values or standards. And, alternatively, in this reading Cut tells us that so long as we have a model of $\Gamma \Rightarrow \Delta$, ϕ has to be able either to be satisfied or not satisfied. Cut, in the negated conjunctive reading, is an *exhaustivity condition* on values or standards. Given that the usual reading of the notion of consequence is the disjunctive one, failures of Cut have been seen as failures of some meta-version of the principle of Explosion, because it has been interpreted as a weakening of the exclusivity constraint over the standards of satisfaction. For instance, in **ST**, the metainference of Cut is not valid because both inferences-premises can be satisfied by virtue of the mixing of standards of premises and conclusions. In **ST**, an inference $\phi \Rightarrow$ is satisfied whenever $v(\phi) = 1/2$ (on the left $1/2$ counts as a *bad* value, because it does not belong to the standard S), and, under the same valuation, the inference $\Rightarrow \phi$ also gets satisfied (on the right $1/2$ is a *good* value: it belongs to T). This has been largely interpreted as a failure of an exclusivity constraint, *since $\phi \Rightarrow$ and $\Rightarrow \phi$ can be satisfied at the same time, and therefore ϕ cannot be cut*. In this extended view, Cut can be seen as meta-Explosion. But

the interesting thing is that this is not the only way in which we can understand a failure of Cut. A given formula ϕ can be *cut* if being on the left and on the right of the sequent are mutually exclusive and jointly exhaustive. If one of these two conditions fails, we are no longer able to cut. In a *negated conjunctive reading of the consequence relation*, it is the exhaustivity condition that gets weakened. We can say that Cut fails because $\phi \Rightarrow$ and $\Rightarrow \phi$ are not exhaustive options, in a bilateralist reading [see, e.g., 31, 33], and we could say that *asserting* and *denying* ϕ are not the only possible attitudes. In this reading, Cut can be assimilated to a metainferential version of the Law of Excluded Middle, which expresses exhaustiveness among the values of the system. And, for this reason, a failure of Cut can be interpreted as a failure of meta-LEM, and a logic in which this occurs can be considered *metainferentially paracomplete*. We argue that this is a better way of understanding metainferential negation, and that the properties of metaparacompleteness and metaparaconsistency are thus better characterized. Id and Cut are, alternatively, Explosion and LEM, and a logic where Cut fails, such as **ST**, is alternatively metaparaconsistent (as in the approach of [7] or [18]), or metaparacomplete, as we have attempted to characterize it in this paper.

Finally, another way in which someone could understand **ST** as a sort of *glutty* or paraconsistent logic is by paying attention to the relation between the sets of designated and undesignated values. Both in **LP** and in **K3**, the sets of designated and undesignated values are jointly exhaustive and mutually exclusive. For instance, in **LP** the set of designated values $\{1, 1/2\}$ is exclusive and exhaustive with respect to the set $\{0\}$ of undesignated values. However, in **ST**, given the notion of consequence, those sets can be seen as overlapping, if we consider what follows from what in that logic. A conclusion Δ follows from premises Γ if and only if either all the premises have a value of 0 or $1/2$, or something in the conclusion has a value of 1 or $1/2$. The intermediate value appears as an option in the valuation for both premises and conclusions of a valid inference. Nevertheless, note that the reading here is, again, a disjunctive one. If we consider, alternatively, the notion of counterexample,¹⁹ what we have instead is a *gappy* picture of the relation between the sets of designated and undesignated values in **ST**. An inference is not

¹⁹ We thank Pablo Cobreros for the addition of this argument, which comes from a conversation with this author about Tableaux for **ST**.

valid in **ST** whenever all premises have the value 1 and all conclusions have the value 0. Here, the intermediate value appears neither in the premises nor in the conclusions, and therefore the relation between the sets of designated and undesignated values is no longer non-exclusive, but rather non-exhaustive. We can easily relate the notion of counterexample and the negated conjunctive reading of the consequence relation. The crucial point here is: *we could interpret failures of Cut as failures of a meta-version of the Law of Excluded Middle*. And, by virtue of this, we could start seeing non-transitive logics, in particular **ST**, not only as metaparainconsistent but, alternatively, as metaparacomplete logics.

6. What is **ST**?

After all this discussion, what is **ST** after all? If we were inclined to assert an identification between this logic and **LP**, we could not help accepting that **ST** can be understood as well as **K3** in an equally legitimate way. Therefore, we argue that **ST** is *neither* **LP** nor **K3**. It is a very interesting and unique combination of **LP**, **K3**, and **CL** as well. A more precise way of picturing what we are defending comes from considering how [18] propose characterizing consequence relations. As we mentioned earlier, these authors introduce a criterion for identifying abstract consequence relations based on Blok and Jónsson's theory of consequence. They maintain that we can identify two logics if they belong in the same equivalence class according to Blok and Jónsson's theory. Nevertheless, they themselves recognize that their criterion can be seen as somewhat partial, because the translators τ and ρ (see Section 3 and Definition 14) they consider in order to define equivalence between **ST** and **LP** are unbalanced if one were to adopt a bilateralist reading of the sequent (which an advocate of **TS** may very well do). As these authors note, such a mapping is supposed to be saying which sequents translate formulas of the given language. However, in their own words:

It does so by completely ignoring the left-hand side of such sequents. The partisan of **ST** could argue that τ is a *bad* transformer in that it is unbalanced on the side of assertions. A *good* transformer should respect the fundamental symmetry between assertions and denials, arising from the fact that neither is logically prior to the other. [18]

It could be argued that equivalence à la Blok and Jónsson is a necessary but not sufficient condition for identifying *ACRs*, and that further con-

straints, designed to single out acceptable transformers, are needed. If *ACRs* draw unnecessary distinctions, perhaps their equivalence classes have the opposite drawback of bundling up logics that must be kept apart.

What we propose is that an understanding of external validity in terms of preservation of theoremhood has the same problem, and it is part of this view of **ST** as **LP**, which at the end of the day is revealed as partial. It gives us necessary but not sufficient conditions for identifying abstract consequence relations, or logics. Our own proposal based on a conception of external logic as preservation of anti-theoremhood is also biased, only in the other direction: it gives us sufficient but not necessary conditions. We therefore think that both views are equally unable to fully characterize a logic such as **ST**. We can think about all this in the following way: classical logic can be related to **K3** (through its anti-theorems) and to **LP** (through its theorems), but when we climb up to the meta-level by means of what we can call a *Tarskian jump*, i.e., going up to the metalanguage through a Tarskian consequence relation, we obtain classical logic again. If we take a different route, and consider which metainferences we can get by means of a *non-Tarskian jump*, we find ourselves in a very different environment — one that can be alternatively seen as metaparainconsistent or metaparacomplete depending on how we decide to characterize it. Either perspective is equally possible and equally partial; the complete understanding of the unique properties of non-Tarskian logics comes from this acceptance of their dual nature at the metainferential level. **ST** is, arguably, doubly non-classical at the meta-level, but by virtue of this it allows us to deal with paradoxes by keeping a considerable space for classical reasoning.

7. Final remarks

In this work, we have shown that there is a translation of every valid metainference of the logic **ST** into a valid inference of the logic **K3**. This is a departure from a generally accepted line of interpretation according to which there is a strong connection between **ST** and **LP**. In the same way that one can accept the usual understanding of the paraconsistent character of the metainferences of **ST**, we are obliged to acknowledge another equally legitimate similarity between **ST** and **K3**, and therefore a picture of **ST** as metainferentially paracomplete. What does this tell

us about the nature of the logics involved in this debate? Is **ST** classical logic, as its advocates have argued? Is it **LP**, as [7] or [18] so interestingly defended? Is it all of them? We think that the problem lies in the identity criteria for logics that we have available at present. There is not a sufficiently refined criterion that could allow us to understand the particular nature of some non-Tarskian logics, in the sense that once we drop one of the core principles of the Tarskian notion of consequence, such as Transitivity or Reflexivity, we break the uniformity through the different inferential levels of a logic and, therefore, that logic can no longer be plausibly identified with its set of valid inferences or metainferences. And not even, as proposed by Barrio et al. [6], with the set of all the inferences and metainferences of the given logic, because, as we have shown here, there are different possible interpretations of the metainferences of non-Tarskian logics. The main contribution of this work is, we believe, that it sheds some light on the complexities of any attempt to characterize the nature of a logic such as **ST**. It is not as easy as to say that **ST** is **CL**, or **LP**, or **K3**: it is a highly unique formal framework, and we still have to develop the conceptual tools in order to fully characterize it.

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