CC BY ND

Logic and **L**ogical **P**hilosophy
Online First Articles (2026)
DOI: 10.12775/LLP.2026.002

Bartłomiej Uzar

Two Semantics for Zalta's Object Theory

Abstract. This paper investigates the relationship between two semantics for Edward Zalta's Elementary Object Theory (OT): one proposed by Dana Scott and the other by Peter Aczel. We present some philosophical motivations underlying OT, characterize its second-order monadic fragment (MOT), and prove some of its theses. We define Scott and Aczel structures and establish a soundness theorem for MOT with respect to the latter. We indicate a class of Aczel structures in which a given formula is true iff it is true in all Scott structures. We also investigate two formulas: one concerning the extensionality of the identity of properties and another related to the overloading of extensions containing abstracta, meaning that if one abstract object exemplifies a property, then all abstract objects do.

Keywords: object theory; abstract objects; formalized ontology; Edward Zalta

Introduction

The aim of this paper is to investigate a relationship between two semantics for a non-modal fragment of Edward Zalta's Object Theory (OT). This formalized theory is intended to describe various types of abstract objects and analyze philosophical concepts that assume their existence. The theory is based on classical logic and is formulated as a Hilbert-style deductive system. Zalta provided an axiomatization of the non-modal version of his theory and its extensions in (Zalta, 1983), where he also characterized the semantics suggested by Dana Scott. This semantics was presented in two variants: for the monadic Elementary OT (Zalta, 1983, pp. 160–162) and for its modal extension (Zalta, 1983, pp. 164–166). The book includes sketches of soundness theorems for both. Another semantics, proposed by Peter Aczel, was introduced in

Formal Methods in Philosophy. Guest Editors: K. Świętorzecka and K. Zdanowski
Received July 19, 2025. *Accepted* January 19, 2026. *Published online* February 17, 2026
© The Author(s), 2026. Published by Nicolaus Copernicus University in Toruń

(Zalta, 1997, pp. 270–275) for the modal OT with polyadic predicates. Further descriptions of Aczel's semantics appear in (Zalta, 1999, pp. 626–627) and (Nodelman and Zalta, 2024, pp. 1192–1194). Aczel models for modal OT with polyadic predicates were constructed by Kirchner in HOL (Kirchner, 2022, Section 4). Here we construct Aczel models for OT directly using set-theoretic tools and without the use of higher-order logic. The comparison of Scott's and Aczel's models, which interests us, is, as far as we know, an open question. A separate issue, which we do not address here, is whether each of the models discussed in this paper can be extended to a Kirchner model.[1]

In (Kirchner et al., 2020, footnote 2) it is noted that models of Scott's semantics are special cases of models of Aczel's semantics with a single urelement. Here we clarify this remark and provide its proof. We focus on the monadic fragment of Elementary OT and demonstrate precisely which classes of Scott and Aczel models are equivalent. These results are of particular interest because Scott's semantics validates certain formulas that extensionalize some concepts which are intended to be intensional, while Aczel's semantics falsify these formulas.

In Section 1, we sketch the philosophical background of Zalta's theory. Section 2 contains the axiomatics of the second-order moandic fragment of Elementary OT (MOT), along with several derivable theses. In Section 3, we define both Scott and Aczel structures and provide a soundness proof with respect to the latter. Finally, in Section 4, we compare the two interpretations.

1. On the philosophical background of OT

Before presenting the formal theory, let us outline some of its pretheoretical principles. Zalta discusses these principles in several works for various extensions of OT (see Zalta, 1983, 1988, 2025a). The basic and original distinction present in all these formalisms is that between two kinds of predication: *exemplification* and *encoding*. Informally speaking, properties can be *exemplified* by individuals and on the other hand they are *encoded* by individuals. Whereas only *abstract objects* can encode properties, both abstract and *concrete objects* exemplify properties. The two different types of predication are notated by a different juxtaposition

[1] And this is not obvious in view of the fact that the theory we are considering is weaker than the theory investigated by Kirchner.

of terms for individuals and properties, respectively. The expression Fx is read as x *exemplifies* F, and xF as x *encodes* F.

Here, we restrict ourselves to Elementary OT[2] from (Zalta, 1983, pp. 15–39). This theory includes a primitive term for the property of *being concrete* and the property of *being abstract* is defined (see definition D1 below).[3]

Let us anticipate the axioms and definitions assumed by Zalta with a few introductory remarks. As we have already said, concreta and abstracta differ concerning ways of property attribution. Concreta exemplify properties (cf. axiom A2 below), while abstracta can also encode them. For example, Charles III exemplifies the property of *being a king*, while Sherlock Holmes encodes the property of *being a detective*. Holmes does not exemplify this property, as he cannot literally perform detective duties (Zalta, 1988, p. 17). Exemplified properties are those that an object really possesses, while encoded properties serve to identify abstracta.

Criteria for identity differ between concreta and abstracta. Two concrete objects are identical when they exemplify the same properties (see definition D2 below), while two abstract objects are identical when they encode the same properties (see axiom A1 below). The *identity of the properties* is also defined in terms of encoding (see definition D3 below). Two properties are identical when they are encoded by the same objects. We distinguish property identity from *property equivalence* (see definition D4 below). Only expressions denoting identical properties can be substituted *salva veritate* (see axiom A3 below).

Zalta introduces two principles regarding the existence of properties and abstract objects. The first one has a logical origin: it is a comprehension scheme but limited to formulas without encoding contexts. According to it, for any formula φ without encoding contexts, there exists a property exemplified by all and only those objects that satisfy φ (see (Comp) below). This scheme, characteristic of second-order logic

[2] Elementary OT is not elementary in the sense that it is formalized in a first-order language, but elementary in the sense that it is formulated in a language without modalities.

[3] In the language of the modal extension of OT, which uses the modal connectives for necessity and possibility, a distinction is made between abstract objects, for which it is *not possible* to exemplify the property of *being concrete*, and three kinds of the so-called *ordinary objects*: contingently concrete, contingently non-concrete, and necessarily concrete (Zalta, 2025a, p. 378).

allows one to define various new properties (see, e.g., Manzano, 1996, pp. 56–57). The second principle specifies that for any formula φ where x is not free, there exists an abstract object x encoding property F if and only if F satisfies φ (A4). Intuitively, one can say that this principle ensures that for every describable set of properties there exists an abstract object that encodes precisely those properties. These principles together highlight the force of Zalta's theory and express a rich universe of properties and abstract objects, including e.g., many possibly different properties that are not exemplified and many abstacta that encode them. As shown in the literature, OT and its extensions can be used to model a wide range of interesting philosophical concepts. Among others, these include Plato's Forms, Leibniz's Monads, Frege's Senses, Bolzano's Substances and Adherences (Zalta, 1983, 2025a; Świętorzecka, 2019).

2. Monadic second-order fragment of OT: MOT

We focus on second-order monadic OT without modalities. We take a language with variables of two kinds. For $i \geq 1$, indexed letters x_i are *object variables*, and indexed letters F_i are *property variables*. Var_1 is the set of all object variables, and Var_2 is the set of all property variables. We will omit subscripts for variables and use symbols x, y, z, ..., and F, G, H, ..., respectively. Object variables are object terms. The alphabet also includes two primitive constants: a property constant $\mathsf{E!}$, denoting the property of *being concrete*, and a binary constant $=_{\mathsf{E}}$, denoting the *identity between concreta*. We use logical symbols: $\neg$, $\rightarrow$, $\forall$, λ, and parentheses. Next, we define restricted formulas, property terms, and formulas of the language.

Definition 2.1 (Restricted formula). The set of restricted formulas For^* is the smallest set such that:

- if $F \in \mathsf{Var}_2$ and $x \in \mathsf{Var}_1$, then $Fx \in \mathsf{For}^*$;
- if $\varphi^*, \psi^* \in \mathsf{For}^*$, then $(\neg\varphi^*) \in \mathsf{For}^*$ and $(\varphi^* \rightarrow \psi^*) \in \mathsf{For}^*$.

Definition 2.2 (Property term). The set of properties Π is the smallest set such that:

- if $F \in \mathsf{Var}_2$, then $F \in \Pi$;
- $\mathsf{E!} \in \Pi$;
- if $x \in \mathsf{Var}_1$ and $\varphi^* \in \mathsf{For}^*$, then $[\lambda x\, \varphi^*] \in \Pi$.

DEFINITION 2.3 (Formula). The set of formulas For is the smallest set such that:

- if $\pi \in \Pi$ and $x \in \mathsf{Var}_1$, then $\pi x \in \mathsf{For}$;
- if $x \in \mathsf{Var}_1$ and $\pi \in \Pi$, then $x\pi \in \mathsf{For}$;
- if $x, y \in \mathsf{Var}_1$, then $(x =_{\mathsf{E}} y) \in \mathsf{For}$;
- if $\varphi, \psi \in \mathsf{For}$, then $(\neg\varphi) \in \mathsf{For}$ and $(\varphi \to \psi) \in \mathsf{For}$;
- if $\varphi \in \mathsf{For}$, $x \in \mathsf{Var}_1$, $F \in \mathsf{Var}_2$, then $(\forall x\varphi) \in \mathsf{For}$ and $(\forall F\varphi) \in \mathsf{For}$.

We omit parentheses in the usual way. Metavariables $\varphi^*, \psi^*, \chi^*$ range over For^*, π over Π, and φ, ψ, χ over For. We will use metavariables α, β to represent variables of any kind. The metavariable t will represent terms of any kind.

Zalta does not impose as strong constraints on the expressions occurring within property terms as above. He assumes that $[\lambda x\, \varphi^\star]$ is a term, if $\varphi^\star$ is a formula that contains neither encoding subformulas nor subformulas with quantifiers binding property variables (Zalta, 1983, pp. 17–18). The first of these restrictions helps to avoid the inconsistency known as *Clark's paradox* (Zalta, 1983, pp. 158–159).[4] In this paper we impose stricter constraints on λ-terms: only atomic formulas of the form Fx, closed under $\neg$ and $\to$, may occur in them.[5]

We will also use the defined symbols: $\mathsf{A!}$, $=^1$, $=^2$, and $\equiv$.

The constant $\mathsf{A!}$ denotes the property of *being abstract*, and is understood as follows:

$$\mathsf{A!}x := \neg\mathsf{E!}x. \tag{D1}$$

Intuitively, we can say that the set of all objects is divided into two disjoint kinds: concrete objects and abstract objects.

The predicates $=^1$ and $=^2$ denote, respectively: *identity between objects* and *identity between properties*, and are understood so that:

$$x =^1 y := x =_{\mathsf{E}} y \vee (\mathsf{A!}x \wedge \mathsf{A!}y \wedge \forall F(xF \leftrightarrow yF)), \tag{D2}$$

$$F =^2 G := \forall x(xF \leftrightarrow xG). \tag{D3}$$

[4] Compare the derivation of the paradox in (Świętorzecka, 2019, p. 276, footnote 13).

[5] The constraint on constructing λ-terms so that they do not include other nested λ-operators or quantifiers for object variables simplifies the nuances of the semantic description that are not relevant to the proof of the key result (see Theorem 4.8). Nevertheless, some philosophical motivations might induce one to take these additional nuances into account.

Where it will not lead to confusion, we will omit the superscripts in the symbols $=^1$ and $=^2$.

Identities defined above are not understood in the standard sense. They differ from the definitions usually adopted in second-order logic, where Leibniz indiscernibility principle and extensionality principle, respectively, are adopted (see, e.g., Manzano, 1996, 54–55). We will discuss the issue of identity in more detail in the following.

We define *equivalence of properties* and $\exists!$ as follows:

$$F \equiv G := \forall x(Fx \leftrightarrow Gx), \tag{D4}$$

$$\exists! x\varphi := \exists x\varphi \wedge \forall y(\varphi(x/y) \rightarrow y = x), \tag{D5}$$

where $\varphi(x/y)$ is the result of substituting x in place of all occurrences of the y in the formula φ.

Our fragment of Zalta's theory, which we call MOT here, is the smallest set of formulas containing:

- classical propositional logic (PC);
- formulas of the form:

$$\forall\alpha\varphi \rightarrow \varphi(t/\alpha), \tag{$\forall$1}$$

where α and t are object variables, or α is a property variable and t is a property term;

$$\forall\alpha(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall\alpha\psi), \quad \alpha \text{ is not free in } \varphi; \tag{$\forall$2}$$

$$\forall x([\lambda y\, \varphi^*]x \leftrightarrow \varphi^*(x/y)), \tag{β-Conv}$$

$$[\lambda x\, Fx] = F, \quad \text{for any } x \in \mathsf{Var}_1, \tag{η-Conv}$$

$$\exists F\forall x(Fx \leftrightarrow \varphi^*), \quad \text{provided } F \text{ is not free in } \varphi^*; \tag{Comp}$$

- specific axioms:

$$x =_{\mathsf{E}} y \leftrightarrow \mathsf{E}!x \wedge \mathsf{E}!y \wedge \forall F(Fx \leftrightarrow Fy), \tag{A1}$$

$$\mathsf{E}!x \rightarrow \neg\exists FxF, \tag{A2}$$

$$\alpha = \beta \rightarrow (\varphi(\alpha) \leftrightarrow \varphi(\beta//\alpha)), \tag{A3}$$

where α and β are either both object variables or both property variables, and '=' is either $=^1$ or $=^2$ depending on whether α, β are object or property variables, and $\varphi(\beta//\alpha)$ is the result of substituting β in place of all or some occurrences of α in φ;

$$\exists x(\mathsf{A}!x \wedge \forall F(xF \leftrightarrow \varphi)), \quad \text{where } x \text{ is not free in } \varphi; \tag{A4}$$

- closed under *modus ponens* and the rule of generalization:
  (MP) $\varphi \to \psi, \varphi \in \mathsf{MOT} \Rightarrow \psi \in \mathsf{MOT}$,
  (G) $\varphi \in \mathsf{MOT} \Rightarrow \forall\alpha\varphi \in \mathsf{MOT}$.

Note that, as in (Zalta, 1983, pp. 28–30), we omit the α-conversion principle $[\lambda x\, \varphi] = [\lambda y\, \varphi(x/y)]$.

For MOT the deduction theorem holds, so we will also use natural deduction proofs.

Let us briefly return to the question of identity. In classical logic, the identity of objects is defined in higher-order logic within standard models, where quantifiers range over all subsets of the domain (Manzano and Moreno, 2017). Objects are identified by Leibniz's principle of indiscernibility, while properties are defined by the principle of extensionality and the comprehension axiom, which allows to define various new properties (Manzano, 1996, p. 56). In OT and MOT, identity definitions differ due to the distinction between encoding and exemplifying, as well as between abstracta and concreta. Abstracta are identified by encoded properties, while concreta are identified by exemplified properties. In Zalta's axiomatization, the identity of concreta, $=_{\mathsf{E}}$, is taken as primitive (cf. A1).[6] As we shall see, from the perspective of Scott and Aczel semantics, where concreta are urelements and abstracta are sets of properties, the definability of identity between abstracta is straightforward (see Section 4).

The identity between properties is more intriguing, as it reveals the intensional character of Zalta's theory.[7] The distinction between encoding and exemplifying formulas allows us to distinguish two counterparts of the extensionality principle: (D3) and (D4). Note that only identity $=^2$ in the sense of (D3) gives contexts in which variables for properties can be substituted *salva veritate* (cf. A3). The aim is to ensure an intensional approach to properties, which avoids the well-known examples of co-extensive but non-identical properties like being 'cordate' and being 'renate' (Zalta, 1983, p. 32) and (Zalta, 1988, p. 29), and to ensure that intensions determine extensions (cf. T4), rather than vice versa (cf. Section 4).

---

[6] In (Zalta, 1983) $=_{\mathsf{E}}$ is treated as primitive because constraints on λ-terms prevented using the right-hand side of (A1) within a λ-terms. In recent versions of modal OT, $=_{\mathsf{E}}$ is definable because impredicative λ-terms are allowed (see Zalta, 2025b, 15). Thanks to the reviewer for pointing this out.

[7] In the modal extension of OT, the identity conditions for properties exhibit the theory's *hyperintensional* character, since there may be distinct properties that are necessarily equivalent (Zalta, 2025a, pp. 221–223).

Considering the semantic interpretations of MOT that will be discussed later in Section 3, the following derivable formulas are noteworthy. Directly from (A4) we get that *there is an abstract object*:

$$\exists x \mathsf{A}!x.$$

However, $\exists x \mathsf{E}!x$ is not derivable (see Section 4). Now, we prove the foolowing stronger version of (A4), where x is not free in φ:

$$\exists! x(\mathsf{A}!x \wedge \forall F(xF \leftrightarrow \varphi)). \tag{T1}$$

PROOF. Assume for reductio that 1. $\mathsf{A}!x \wedge \forall F(xF \leftrightarrow \varphi)$; 2. $\mathsf{A}!y \wedge \forall F(yF \leftrightarrow \varphi)$ and 3. $x \neq y$. Then: 4. $\neg\mathsf{A}!x \vee \neg\mathsf{A}!y \vee \neg\forall F(xF \leftrightarrow yF)$ (3, D2); 5. $\neg\forall F(xF \leftrightarrow yF)$ (1, 2, 4); 6. $\exists F\neg(xF \leftrightarrow yF)$ (5); 7. $\neg(xF \leftrightarrow yF)$ (6); 8. $xF \leftrightarrow \varphi$ (1); 9. $yF \leftrightarrow \varphi$ (2); 10. $xF \leftrightarrow yF$ (8, 9). Contradiction: 7, 10. ⊣

Regarding identity, the following theses are derivable. Directly from (D3) and (D4) we have the reflexivity of the properties of identity and equivalence:

$$F = F, \tag{T2}$$

$$F \equiv F. \tag{T3}$$

Using (T3) and (A3), we get that Identical properties are equivalent:

$$F = G \rightarrow F \equiv G. \tag{T4}$$

The inverse implication is not derivable (see Section 4).

Now, we obtain that the identity of objects is reflexive:

$$x = x.$$

PROOF. We know that 1. $\mathsf{E}!x \vee \mathsf{A}!x$ (D1).

Assume 1.1. $\mathsf{E}!x$. Thus: 1.2. $\mathsf{E}!x \wedge \forall F(Fx \leftrightarrow Fx)$ (1.1); 1.3. $x =_{\mathsf{E}} x$ (1.2, A1); 1.4. $x =_{\mathsf{E}} x \vee (\mathsf{A}!x \wedge \mathsf{A}!x \wedge \forall F(xF \leftrightarrow xF))$ (1.3).

Now assume 2.1. $\mathsf{A}!x$. Thus: 2.2. $\mathsf{A}!x \wedge \forall F(xF \leftrightarrow xF)$ (2.1); 2.3. $x =_{\mathsf{E}} x \vee (\mathsf{A}!x \wedge \mathsf{A}!x \wedge \forall F(xF \leftrightarrow xF))$ (2.2).

Hence 2. $x =_{\mathsf{E}} x \vee (\mathsf{A}!x \wedge \mathsf{A}!x \wedge \forall F(xF \leftrightarrow xF))$ (1.1 ⊢ 1.4, 2.1 ⊢ 2.3). So, 3. $x = x$ (2, D2). ⊣

Moreover, there is exactly one object encoding no property and there is exactly one object encoding all properties:

$$\exists! x(\mathsf{A}!x \wedge \forall F \neg xF), \tag{T5}$$

$$\exists! x(\mathsf{A}!x \wedge \forall F xF). \tag{T6}$$

PROOF. *For* (T5) 1. $\exists!x(\mathsf{A}!x \wedge \forall F(xF \leftrightarrow F \neq F))$ (T1); 2. $\forall F(xF \leftrightarrow F \neq F) \leftrightarrow \forall F\neg xF$ (T2, PC). Hence: 3. $\exists!x(\mathsf{A}!x \wedge \forall F\neg xF)$ (1, 2).

For (T6) 1. $\exists!x(\mathsf{A}!x \wedge \forall F(xF \leftrightarrow F = F))$ (T1); 2. $\forall F(xF \leftrightarrow F = F) \leftrightarrow \forall FxF$ (T2, PC). Hence: 3. $\exists!x(\mathsf{A}!x \wedge \forall FxF)$ (1, 2). ⊣

Using (T5) and (T6) we introduce the definitions of the abstract object that does not encode any property and the abstract object that encodes all properties (see Świętorzecka, 2019, p. 272). They are called by Zalta (2025a, p. 404) '*the Null object*' and '*the Universal object*', respectively:

$$x = \bigcirc := \mathsf{A}!x \wedge \forall F\neg xF, \tag{D6}$$

$$x = \bullet := \mathsf{A}!x \wedge \forall FxF. \tag{D7}$$

We observe that the universal object both encodes and exemplifies the property of being abstract:

$$\bullet\mathsf{A}! \quad \text{and} \quad \mathsf{A}!\bullet.$$

So, there is at least one object that is subject to both types of predication:

$$\exists F\exists x(Fx \wedge xF). \tag{T7}$$

Following (Świętorzecka, 2017, p. 419) we introduce the predicate '*con*', used to talk about *characteristic encoders* of various properties. We say that x *is a characteristic encoder of a property* F if every property it encodes is identical to F, i.e.:

$$x\ con\ F := \forall G(xG \leftrightarrow G = F). \tag{D8}$$

In our theory it is provable that each property has the only one characteristic encoder, i.e. (the proof as in (Świętorzecka, 2017, p. 419)):

$$\forall F\exists!x\ x\ con\ F.$$

Based on the above fact, we introduce two constants, respectively, for the abstract object that is a *characteristic encoder of property* E! and for the abstract object that is a *characteristic encoder of property* A!:

$$x = \mathsf{e} := x\ con\ \mathsf{E}!, \tag{D9}$$

$$x = \mathsf{a} := x\ con\ \mathsf{A}!. \tag{D10}$$

Using the above definitions one can show that the Null object, the Universal object, the characteristic encoder of E! and the characteristic encoder of $\mathsf{A!}$ are distinct:

$$\bigcirc \neq \bullet \neq \mathsf{e} \neq \mathsf{a}.$$

From the above fact, it directly follows that in the universe there are at least four abstract objects:

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 (\mathsf{A!}x_1 \wedge \mathsf{A!}x_2 \wedge \mathsf{A!}x_3 \wedge \mathsf{A!}x_4 \wedge \textstyle\bigwedge_{1 \leqslant i,j \leqslant 4} x_i \neq x_j).$$

3. Scott's semantics and Aczel's semantics

Now we will characterize Scott structures and Aczel structures, interpret the language of MOT in them, and show that our fragment of Zalta's theory is satisfied in all Aczel models. As we will show in Section 4, from this fact the analogous theorem for Scott models can be easily derived. Our presentation of both semantics is based on (Zalta, 1983, pp. 160–162) and (Zalta, 1997, 1999; Świętorzecka, 2019; Nodelman and Zalta, 2014, 2024).

3.1. Scott's semantics

We define the Scott structure:

Definition 3.1 (Scott structure). An Scott structure (for short: $\mathcal{S}$-structure) is 5-tuple $\mathcal{S} = (\mathcal{E}_s, \mathcal{P}r_s, exp_s, enc_s, op_s)$, where:

- $\mathcal{E}_s$ is a set of concrete objects (possibly empty);
- $\mathcal{P}r_s = \{+, -\} \times 2^{\mathcal{E}_s}$ is a set of properties, where each property is a pair in which the first member is either $+$ or $-$, and the second member is a set of concrete objects;
- $2^{\mathcal{P}r_s}$ is a set of abstract objects.

Let o be any object in $\mathcal{E}_s \cup 2^{\mathcal{P}r_s}$ and r any property in $\mathcal{P}r_s$. Then:

- $exp_s \colon \mathcal{P}r_s \to 2^{\mathcal{E}_s \cup 2^{\mathcal{P}r_s}}$ is an exemplification extension, such that:
  $o \in exp_s(r)$ iff either (i) $\exists X \in 2^{\mathcal{E}_s}(o \in X$ and $(r = \langle +, X\rangle$ or $r = \langle -, X\rangle))$, or (ii) $\exists X \in 2^{\mathcal{E}_s}(o \in 2^{\mathcal{P}r_s}$ and $r = \langle +, X\rangle)$;
- $enc_s \colon \mathcal{P}r_s \to 2^{\mathcal{E}_s \cup 2^{\mathcal{P}r_s}}$ is an encoding extension, such that:
  $o \in enc_s(r)$ iff $o \in 2^{\mathcal{P}r_s}$ and $r \in o$;
- op_s is a set of operations $neg_s \colon \mathcal{P}r_s \to \mathcal{P}r_s$ and $cond_s \colon \mathcal{P}r_s \times \mathcal{P}r_s \to \mathcal{P}r_s$, such that:
  $neg_s(\langle +, X\rangle) = \langle -, \mathcal{E}_s/X\rangle$ and $neg_s(\langle -, X\rangle) = \langle +, \mathcal{E}_s/X\rangle$;
  $cond_s(\langle a, X\rangle, \langle b, Y\rangle) = \langle a \rightsquigarrow b, X' \cup Y\rangle$,

$$\text{where for } a, b \in \{+,-\}\text{: } a \rightsquigarrow b = \begin{cases} + & \text{iff } a = - \text{ or } b = + \\ - & \text{iff } a = + \text{ and } b = - \end{cases}$$

To simplify notation, we will omit brackets when writing elements of the set $\mathcal{P}r_s$. Note that according to our definition of exp_s, abstract objects belong to the exemplification extensions only of *positive properties*, i.e. properties of the form $+X$, which are exemplified by all abstract objects (from (ii)) and possibly some concrete objects (from (i)). According to our definition, $neg_a(+X)$ is exemplified by no abstract objects (again from (ii)) and all concrete objects that are not elements of X. So the intuition of negation is preserved here. Analogously for $neg_a(-X)$.

Example 3.1. Consider a structure $\mathcal{S}^1 = (\{d\}, \{+\{d\}, -\{d\}, +\emptyset, -\emptyset\}, exp_s, enc_s, op_s)$. We have one concrete object, four properties, and sixteen abstracta. For $+\{d\}$ and $-\{d\}$ we have: $exp_s(+\{d\}) = \{d\} \cup 2^{\mathcal{P}r_s}$, $exp_s(-\{d\}) = \{d\}$, and to $enc_s(+\{d\})$ belong:

1. $\{+\{d\}\}$,
2. $\{+\{d\}, -\{d\}\}$,
3. $\{+\{d\}, +\emptyset\}$,
4. $\{+\{d\}, -\emptyset\}$,
5. $\{+\{d\}, -\{d\}, +\emptyset\}$,
6. $\{+\{d\}, -\{d\}, -\emptyset\}$,
7. $\{+\{d\}, +\emptyset, -\emptyset\}$,
8. $\{+\{d\}, -\{d\}, +\emptyset, -\emptyset\}$.

Let us note that for every $r \in \mathcal{P}r_s$, we have that $enc_s(r)$ is the principal ultrafilter on $\mathcal{P}r_s$ generated by r. It is straightforward to verify the following fact:

Fact 3.1. *The function exp_s satisfies the following conditions*:

- $exp_s(neg_s(r)) = \{o : o \notin exp_s(r)\}$;
- $exp_s(cond_s(r_1, r_2)) = \{o : o \notin exp_s(r_1) \text{ or } o \in exp_s(r_2)\}$.

It is also straightforward to see, that for $z = \text{card}(\mathcal{E}_s)$ we have $2^{2^{z+1}}$ abstract objects (it depends on $\text{card}(\mathcal{P}r_s)$). So for $\text{card}(\mathcal{E}_s) = 2$ we have $\text{card}(2^{\mathcal{P}r_s}) = 256$, for $\text{card}(\mathcal{E}_s) = 3$ we have $\text{card}(2^{\mathcal{P}r_s}) = 65536$ etc.

Definition 3.2 (Scott model). An Scott model ($\mathcal{S}$-model) is a pair $(\mathcal{S}, v_s)$, where $\mathcal{S}$ is Scott structure and v_s is valuation function, which assigns denotations to each term:

$v_s(x) \in \mathcal{E}_s \cup 2^{\mathcal{P}r_s}$, for any $x \in \mathsf{Var}_1$;
$v_s(F) \in \mathcal{P}r_s$, for any $F \in \mathsf{Var}_2$;
$v_s(\mathsf{E!}) = -\mathcal{E}_s$;
$v_s([\lambda x\, Fx]) = v_s(F)$;

$v_s([\lambda x\, \neg\psi^*]) = neg_s(v_s([\lambda x\, \psi^*]))$;
$v_s([\lambda x\, (\psi^* \to \chi^*)]) = cond_s(v_s([\lambda x\, \psi^*]), v_s([\lambda x\, \chi^*]))$.

We define satisfaction in $\mathcal{S}$-model and truth in $\mathcal{S}$-structure.

Definition 3.3 (Satisfaction in $\mathcal{S}$-model). For any model $(\mathcal{S}, v_s)$ we define $\vDash$ as follows. For any $x, y \in \mathsf{Var}_1$:

$\mathcal{S}, v_s \vDash \pi x$ iff $v_s(x) \in exp_s(v_s(\pi))$;
$\mathcal{S}, v_s \vDash x\pi$ iff $v_s(x) \in enc_s(v_s(\pi))$;
$\mathcal{S}, v_s \vDash x =_{\mathsf{E}} y$ iff $v_s(x) = v_s(y)$ and $v_s(x) \in \mathcal{E}_s$;
$\mathcal{S}, v_s \vDash \neg(\varphi)$ iff $\mathcal{S}, v_s \nvDash \varphi$;
$\mathcal{S}, v_s \vDash (\varphi \to \psi)$ iff $\mathcal{S}, v_s \nvDash \varphi$ or $\mathcal{S}, v_s \vDash \psi$;
$\mathcal{S}, v_s \vDash \forall\alpha(\varphi)$ iff for all $v'_s : v'_s \sim_\alpha v_s$ it holds that $\mathcal{S}, v'_s \vDash \varphi$, where for any v_s and v'_s: $v'_s \sim_\alpha v_s$ iff v'_s differs from v_s at most in assigning a different value to α.

Definition 3.4 (Truth in $\mathcal{S}$-structure). Formula φ is true in a structure $\mathcal{S} = (\mathcal{E}_s, \mathcal{P}r_s, exp_s, enc_s, op_s)$ iff for every valuation v_s: $\mathcal{S}, v_s \vDash \varphi$.

Having defined models, we can note that in any model based on the structure $\mathcal{S}^1$ from Example 3.1 we have: $v_s(\bigcirc) = \emptyset$, $v_s(\bullet) = \{+\{d\}, -\{d\}, +\emptyset, -\emptyset\}$, $v_s(\mathsf{e}) = \{-\mathcal{E}_s\}$, $v_s(\mathsf{a}) = \{+\emptyset\}$.

For this semantics, the soundness theorem holds. We will show this in section 4 using the soundness theorem for Aczel's semantics.

3.2. Aczel's semantics

Now we move on to the second semantics. We define the Aczel structure:

Definition 3.5 (Aczel structure). An Aczel structure (for short: $\mathcal{A}$-structure) is a 8-tuple:

$$\mathcal{A} = (\mathcal{E}_a, \mathcal{P}s, \mathcal{P}r_a, \mathfrak{e}, |\cdot|, exp_a, enc_a, op_a),$$

where:

- $\mathcal{E}_a$ is a set of concrete objects (possibly empty);
- $\mathcal{P}s$ is a non-empty and disjoint with $\mathcal{E}_a$ set of *proxies* of abstract objects;
- $\mathcal{P}r_a$ is a set of properties such that the distinguished element $\mathfrak{e}$ belongs to $\mathcal{P}r_a$ and $\text{card}(\mathcal{P}r_a) \geqslant \text{card}(2^{\mathcal{E}_a \cup \mathcal{P}s})$;
- $2^{\mathcal{P}r_a}$ is a set of abstract objects.

Let o be any object in $\mathcal{E}_a \cup \mathcal{P}s \cup 2^{\mathcal{P}r_a}$ and r any property in $\mathcal{P}r_a$. Then:

- $|\cdot|\colon \mathcal{E}_a \cup 2^{\mathcal{P}r_a} \to \mathcal{E}_a \cup \mathcal{P}s$ is an *imitation function* such that:
  (i) if $o \in \mathcal{E}_a$, then $|o| = o$;
  (ii) if $o \in 2^{\mathcal{P}r_a}$, then $\exists o'(o' \in \mathcal{P}s$ and $|o| = o')$;
- $exp_a\colon \mathcal{P}r_a \to 2^{\mathcal{E}_a \cup \mathcal{P}s}$ is an exemplification extension such that $exp_a(\mathfrak{e}) = \mathcal{E}_a$;
- $enc_a\colon \mathcal{P}r_a \to 2^{\mathcal{E}_a \cup 2^{\mathcal{P}r_a}}$ is an encoding extension such that: $o \in enc_a(r)$ iff $o \in 2^{\mathcal{P}r_a}$ and $r \in o$.
- op_a is a set of operations $neg_a\colon \mathcal{P}r_a \to \mathcal{P}r_a$ and $cond_a\colon \mathcal{P}r_a \times \mathcal{P}r_a \to \mathcal{P}r_a$, such that the function exp_a satisfies two conditions:
  $exp_a(neg_a(r)) = \{o : o \notin exp_a(r)\}$;
  $exp_a(cond_a(r_1, r_2)) = \{o : o \notin exp_a(r_1) \text{ or } o \in exp_a(r_2)\}$.

Observe that in Scott semantics, Fact 3.1 concerning exp_s and operations from op_s follows directly from the definition of the structure, whereas in Aczel semantics, the analogous fact about exp_a and operations from op_a must be assumed as a condition in the definition of the structure. This is because no special conditions are imposed on $\mathcal{P}r_a$.

Example 3.2. Consider a structure $\mathcal{A}^1 = (\{d\}, \{\mathfrak{s}\}, \{\mathfrak{e}, r_1, r_2, r_3\}, \mathfrak{e}, |\cdot|, exp_a, enc_a, op_a)$. We have one concrete object d, one proxy $\mathfrak{s}$, four properties, and sixteen abstracta (members of $2^{\{\mathfrak{e}, r_1, r_2, r_3\}}$). We have $|d| = d$ and $|a| = \mathfrak{s}$, for any $a \in 2^{\mathcal{P}r_a}$. For property $\mathfrak{e}$ we have: $exp_a(\mathfrak{e}) = \{d\}$. For r_1 we take: $exp_a(r_1) = \{d, \mathfrak{s}\}$. Following the definition of enc_a we see that elements of $enc_a(r_1)$ are:

1. $\{r_1\}$,
2. $\{r_1, r_2\}$,
3. $\{r_1, r_3\}$,
4. $\{r_1, \mathfrak{e}\}$,
5. $\{r_1, r_2, r_3\}$,
6. $\{r_1, r_2, \mathfrak{e}\}$,
7. $\{r_1, r_3, \mathfrak{e}\}$,
8. $\{r_1, r_2, r_3, \mathfrak{e}\}$.

$\mathcal{A}^1$ is a case of a structure in which a single proxy represents all abstract objects. We will return to the features of such structures in Section 4.1. However, no specific conditions are imposed on the set $\mathcal{P}s$. There are $\mathcal{S}$-structures in which there are many proxies imitating different abstract objects, as well as structures in which there are many proxies but only one of them imitates all abstract objects.

It is also straightforward to see that for $z = \text{card}(\mathcal{E}_a \cup \mathcal{P}s)$ we have at least 2^{2^z} abstract objects (it depends on $\text{card}(\mathcal{P}r_a)$). So for structures with $\text{card}(\mathcal{E}_a) = 2$ we have at least $\text{card}(2^{\mathcal{P}r_a}) = 256$, with $\text{card}(\mathcal{E}_a) = 3$ we have at least $\text{card}(2^{\mathcal{P}r_a}) = 65536$, etc.

Definition 3.6 (Aczel model). An Aczel model ($\mathcal{A}$-model) is a pair $(\mathcal{A}, v_a)$, where $\mathcal{A}$ is Aczel structure and v_a is valuation function, which assigns denotations to each term:

$v_a(x) \in \mathcal{E}_a \cup 2^{\mathcal{P}r_a}$, for any $x \in \mathsf{Var}_1$;
$v_a(F) \in \mathcal{P}r_a$, for any $F \in \mathsf{Var}_2$;
$v_a(\mathsf{E!}) = \mathfrak{e}$;
$v_a([\lambda x\, Fx]) = v_a(F)$;
$v_a([\lambda x\, \neg\psi^*]) = \mathit{neg}_a(v_a([\lambda x\, \psi^*]))$;
$v_a([\lambda x\, (\psi^* \to \chi^*)]) = \mathit{cond}_a(v_a([\lambda x\, \psi^*]), v_a([\lambda x\, \chi^*]))$.

Note that in this semantics object variables do not range over proxies. We define satisfaction in $\mathcal{A}$-model and truth in the $\mathcal{A}$-structure.

Definition 3.7 (Satisfaction in $\mathcal{A}$-model). For any model $(\mathcal{A}, v_a)$ we define $\vDash$ as follows. For any $x, y \in \mathsf{Var}_1$:

$\mathcal{A}, v_a \vDash \pi x$ iff $|v_a(x)| \in \mathit{exp}_a(v_a(\pi))$;
$\mathcal{A}, v_a \vDash x\pi$ iff $v_a(x) \in \mathit{enc}_a(v_a(\pi))$;
$\mathcal{A}, v_a \vDash x =_{\mathsf{E}} y$ iff $v_a(x) = v_a(y)$ and $v_a(x) \in \mathcal{E}_a$;
$\mathcal{A}, v_a \vDash \neg(\varphi)$ iff $\mathcal{A}, v_a \nvDash \varphi$;
$\mathcal{A}, v_a \vDash (\varphi \to \psi)$ iff $\mathcal{A}, v_a \nvDash \varphi$ or $\mathcal{A}, v_a \vDash \psi$;
$\mathcal{A}, v_a \vDash \forall\alpha(\varphi)$ iff for all v'_a: $v'_a \sim_\alpha v_a$ it holds that $\mathcal{A}, v'_s \vDash \varphi$, where for any v_s and v'_s: $v'_s \sim_\alpha v_s$ iff v'_s differs from v_s at most in assigning a different value to α.

Definition 3.8 (Truth in $\mathcal{A}$-structure). The formula φ is true in a structure $\mathcal{A} = (\mathcal{E}_a, \mathcal{P}s, \mathcal{P}r_a, \mathfrak{e}, |\cdot|, \mathit{exp}_a, \mathit{enc}_a, \mathit{op}_a)$ iff for every valuation v_a: $\mathcal{A}, v_a \vDash \varphi$.

Note that in any model based on the structure $\mathcal{A}^1$ from the example 3.2 we have: $v_a(\bigcirc) = \emptyset$, $v_a(\bullet) = \{\mathfrak{e}, r_1, r_2, r_3\}$, $v_a(\mathsf{e}) = \{\mathfrak{e}\}$. Note that in the $\mathcal{A}$-models where $\mathcal{E}_a \neq \emptyset$ we need not identify $v_a(\mathsf{a})$ (we already know $v_a(\mathsf{a}) = \{\mathit{neg}_a(\mathfrak{e})\}$).

Let us note an interesting point regarding Aczel's semantics. Consider the smallest structure of MOT: $\mathcal{A}^2 = (\emptyset, \{\mathfrak{s}\}, \{\mathfrak{e}, r_1\}, \mathfrak{e}, |\cdot|, \mathit{exp}_a, \mathit{enc}_a, \mathit{op}_a)$. We have no concrete objects, one proxy, two properties, and four abstracta. In this structure it is true that: $\exists x \exists y\, (\mathsf{A!}x \wedge \mathsf{A!}y \wedge x \neq y \wedge \forall F\, (Fx \leftrightarrow Fy))$. This claim is proved by Kirchner (2022, pp. 43–45). Since our restrictions on λ-terms are stricter than Zalta's, the argument cannot be reconstructed in the fragment of the theory examined in (cf. Zalta, 2025a, pp. 407, 1190).

3.3. Soundness theorem for Aczel's semantics

Now we move on to the proof of the soundness theorem.

Theorem 3.1. MOT *is satisfied in the class of all $\mathcal{A}$-models.*

Proof. Proofs for PC, (∀1), (∀2), (MP) and (G) are standard. It remains to show that (β-Conv), (η-Conv), (Comp) and each of the specific axioms is true in any $\mathcal{A}$-model.

(β-Conv) Proof by induction on the structure of φ^*.

($\varphi^* = Fy$) From the definition of valuation we know that $v_a([\lambda y\, Fy]) = v_a(F)$. Therefore, for any o: $o \in exp_a(v_a([\lambda y\, Fy]))$ iff $o \in exp_a(v_a(F))$. From the definition of satisfaction it follows that for every $\mathcal{A}$-model: $\mathcal{A}, v_a \vDash \forall x([\lambda y\, Fy]x \leftrightarrow Fx)$, and, therefore, $\mathcal{A}, v_a \vDash \forall x([\lambda y\, Fy]x \leftrightarrow Fy(x/y))$.

($\varphi^* = \neg(\psi^*)$) Assume that $\mathcal{A}, v_a \vDash [\lambda y\, \psi^*]x \leftrightarrow \psi^*(x/y)$. From this, using the definition of satisfaction we get: $|v_a(x)| \in exp_a(v_a([\lambda y\, \psi^*]))$ iff $\mathcal{A}, v_a \vDash \psi^*(x/y)$. Thus, $|v_a(x)| \notin exp_a(v_a([\lambda y\, \psi^*]))$ iff $\mathcal{A}, v_a \nvDash \psi^*(x/y)$. The following equivalence holds:

$$\begin{aligned}
\mathcal{A}, v_a \vDash [\lambda y\, \neg\psi^*]x &\Leftrightarrow |v_a(x)| \in exp_a(v_a([\lambda y\, \neg\psi^*])) \\
&\Leftrightarrow |v_a(x)| \in exp_a(neg_a(v_a([\lambda y\, \psi^*]))) \\
&\Leftrightarrow |v_a(x)| \notin exp_a(v_a([\lambda y\, \psi^*])) \\
&\Leftrightarrow \mathcal{A}, v_a \nvDash \psi^*(x/y) \\
&\Leftrightarrow \mathcal{A}, v_a \vDash \neg\psi^*(x/y).
\end{aligned}$$

Therefore, for every $\mathcal{A}$-model: $\mathcal{A}, v_a \vDash \forall x([\lambda y\, \neg\psi^*]x \leftrightarrow \neg\psi^*(x/y))$.

($\varphi^* = (\psi^* \to \chi^*)$) Assume that $\mathcal{A}, v_a \vDash [\lambda y\, \psi^*]x$ iff $\psi^*(x/y)$ and $\mathcal{A}, v_a \vDash [\lambda z\, \chi^*]x \leftrightarrow \chi^*(x/z)$. Using the definition of satisfaction, we get: $|v_a(x)| \in exp_a(v_a([\lambda y\, \psi^*]))$ iff $\mathcal{A}, v_a \vDash \psi^*(x/y)$ and $|v_a(x)| \in exp_a(v_a([\lambda z\, \chi^*]))$ iff $\mathcal{A}, v_a \vDash \chi^*(x/z)$. From this, it follows that $|v_a(x)| \notin exp_a(v_a([\lambda y\, \psi^*]))$ or $|v_a(x)| \in exp_a(v_a([\lambda y\, \chi^*]))$ iff $\mathcal{A}, v_a \nvDash \psi^*(x/y)$ or $\mathcal{A}, v_a \vDash \chi^*(x/y)$. The following equivalence holds:

$$\begin{aligned}
\mathcal{A}, v_a \vDash [\lambda y\, (\psi^* \to \chi^*)]x &\Leftrightarrow |v_a(x)| \in exp_a(v_a([\lambda y\, (\psi^* \to \chi^*)])) \\
&\Leftrightarrow |v_a(x)| \in exp_a(cond_a(v_a([\lambda y\, \psi^*]), v_a([\lambda y\, \chi^*]))) \\
&\Leftrightarrow |v_a(x)| \notin exp_a(v_a([\lambda y\, \psi^*])) \\
&\qquad \text{or } |v_a(x)| \in exp_a(v_a([\lambda y\, \chi^*])) \\
&\Leftrightarrow \mathcal{A}, v_a \nvDash \psi^*(x/y) \text{ or } \mathcal{A}, v_a \vDash \chi^*(x/y) \\
&\Leftrightarrow \mathcal{A}, v_a \vDash (\psi^* \to \chi^*)(x/y).
\end{aligned}$$

Therefore, for every $\mathcal{A}$-model we have: $\mathcal{A}, v_a \vDash \forall x([\lambda y\,(\psi^* \to \chi^*)]x \leftrightarrow (\psi^* \to \chi^*)(x/y))$.

(η-Conv) From the definition of v_a, we know that $v_a([\lambda y\, Fy]) = v_a(F)$. Therefore, for any o: $o \in enc_a(v_a([\lambda y\, Fy]))$ iff $o \in enc_a(v_a(F))$. From the definition of satisfaction, it directly follows that for every $\mathcal{A}$-model: $\mathcal{A}, v_a \vDash \forall x(x[\lambda y\, Fy] \leftrightarrow xF)$. Hence, for every $\mathcal{A}$-model: $\mathcal{A}, v_a \vDash [\lambda y\, Fy] = F$.

(Comp) Since $\mathrm{card}(\mathcal{P}r_a) \geq \mathrm{card}(2^{\mathcal{E}_a \cup \mathcal{P}s})$, operations neg_a and $cond_a$ are defined on $\mathcal{P}r_a$ with appropriate conditions about exemplification extension, and φ^* is restricted formula so that only atomic exemplification formulas closed under $\neg$ and $\to$ may occur in it, then for any condition expressed by φ^* we have a corresponding property in $\mathcal{P}r_a$. So there exists a property exemplified by an object o if and only if o satisfies the condition expressed by φ^*. Therefore, for every $\mathcal{A}$-model: $\mathcal{A}, v_a \vDash \exists F \forall x(Fx \leftrightarrow \varphi^*)$, where F is not free in φ^*.

(A1) Assume for reductio that there exists an $\mathcal{A}$-model such that $\mathcal{A}, v_a \nvDash x =_{\mathsf{E}} y \leftrightarrow \mathsf{E}!x \wedge \mathsf{E}!y \wedge \forall F(Fx \leftrightarrow Fy)$. Thus, 2. ($\mathcal{A}, v_a \vDash x =_{\mathsf{E}} y$ and $\mathcal{A}, v_a \nvDash \mathsf{E}!x \wedge \mathsf{E}!y \wedge \forall F(Fx \leftrightarrow Fy)$) or ($\mathcal{A}, v_a \nvDash x =_{\mathsf{E}} y$ and $\mathcal{A}, v_a \vDash \mathsf{E}!x \wedge \mathsf{E}!y \wedge \forall F(Fx \leftrightarrow Fy)$). Consider both cases from 2.

Assume 1.1. $\mathcal{A}, v_a \vDash x =_{\mathsf{E}} y$ and 1.2. $\mathcal{A}, v_a \nvDash \mathsf{E}!x \wedge \mathsf{E}!y \wedge \forall F(Fx \leftrightarrow Fy)$. From 1.1 and the definition of satisfaction we have that 1.3. $v_a(x) = v_a(y)$ and 1.4. $v_a(x) \in \mathcal{E}_a$, thus 1.5. $v_a(y) \in \mathcal{E}_a$. Therefore, we have that 1.6. $v_a(x)$ and $v_a(y)$ belong to $exp_a(v_a(\mathsf{E}!))$, thus 1.7. $\mathcal{A}, v_a \vDash \mathsf{E}!x$ and 1.8. $\mathcal{A}, v_a \vDash \mathsf{E}!y$. From 1.2 we have that 1.9. $\mathcal{A}, v_a \nvDash \mathsf{E}!x$ or $\mathcal{A}, v_a \nvDash \mathsf{E}!y$ or $\mathcal{A}, v_a \nvDash \forall F(Fx \leftrightarrow Fy)$. From disjunction elimination, 1.7, 1.8, and 1.9 we get 1.10. $\mathcal{A}, v_a \nvDash \forall F(Fx \leftrightarrow Fy)$. Thus, 1.11. for some $v'_a : v'_a \sim_F v_a$ we have $\mathcal{A}, v'_a \nvDash Fx \leftrightarrow Fy$. Therefore, 1.12. it is not true that ($v'_a(x) \in exp_a(v'_a(F))$ iff $v'_a(y) \in exp_a(v'_a(F))$). Since v_a and v'_a differ at most in the value for F, from 1.12 we have 1.13. $v_a(x) \neq v_a(y)$. Contradiction: 1.3, 1.13.

Assume 2.1. $\mathcal{A}, v_a \nvDash x =_{\mathsf{E}} y$ and 2.2. $\mathcal{A}, v_a \vDash \mathsf{E}!x \wedge \mathsf{E}!y \wedge \forall F(Fx \leftrightarrow Fy)$. From 2.2 we have that 2.3. $\mathcal{A}, v_a \vDash \mathsf{E}!x$ and 2.4. $\mathcal{A}, v_a \vDash \mathsf{E}!y$ and 2.5. $\mathcal{A}, v_a \vDash \forall F(Fx \leftrightarrow Fy)$. Thus, 2.6. $|v_a(x)| \in exp_a(\mathfrak{e})$ and 2.7. $|v_a(y)| \in exp_a(\mathfrak{e})$, hence 2.8. $|v_a(x)| \in \mathcal{E}_a$ and 2.9. $|v_a(y)| \in \mathcal{E}_a$, and thus from the definition of valuation 2.10. $|v_a(x)| = v_a(x)$ and 2.11. $|v_a(y)| = v_a(y)$. From 2.5 we have that 2.12. for every $v''_a : v''_a \sim_F v_a$ it holds that $\mathcal{A}, v''_a \vDash Fx \leftrightarrow Fy$. Therefore, 2.13. $v''_a(x) \in exp_a(v''_a(F))$ iff $v''_a(y) \in exp_a(v''_a(F))$. Since v_a and v''_a differ at most in the value for F, from 2.13 we have 2.14. $v_a(x) = v_a(y)$. From 2.1 we have that

2.15. $v_a(x) \neq v_a(y)$ or $v_a(x) \notin \mathcal{E}_a$. From disjunction elimination, 2.14 and 2.15 we have 2.16. $v_a(x) \notin \mathcal{E}_a$. But from 2.8 and 2.10 we have that 2.17. $v_a(x) \in \mathcal{E}_a$. Contradiction: 2.16, 2.17.

(A2) Assume for reductio that 1. there exists an $\mathcal{A}$-model such that $\mathcal{A}, v_a \nvDash \mathsf{E}!x \rightarrow \neg\exists F\ xF$. Thus, 2. $\mathcal{A}, v_a \vDash \mathsf{E}!x$ and 3. $\mathcal{A}, v_a \nvDash \neg\exists F\ xF$. From 2 we have that 4. $|v_a(x)| \in exp_a(v_a(\mathsf{E}!))$, hence 5. $|v_a(x)| \in exp_a(\mathfrak{e})$, hence 6. $|v_a(x)| \in \mathcal{E}_a$, and thus 7. $|v_a(x)| = v_a(x)$, which means that 8. $v_a(x) \in \mathcal{E}_a$. From 3 we have that 9. for some $v'_a : v'_a \sim_F v_a$ it holds that $\mathcal{A}, v'_a \vDash xF$, and thus 10. $v'_a(x) \in enc_a(v'_a(F))$. Therefore, 11. $v'_a(x) \in 2^{\mathcal{P}r_a}$. Since v_a and v'_a differ at most in the value for F, thus 12. $v_a(x) \in 2^{\mathcal{P}r_a}$. We know that 13. $\mathcal{E}_a \cap 2^{\mathcal{P}r_a} = \emptyset$, so we get contradiction from 8, 12, 13.

(A3) *For object variables.* Assume for reductio that 1. there exists an $\mathcal{A}$-model such that $\mathcal{A}, v_a \nvDash x = y \rightarrow (\varphi(x) \leftrightarrow \varphi(x//y))$. Thus, 2. $\mathcal{A}, v_a \vDash x = y$ and 3. $\mathcal{A}, v_a \nvDash \varphi(x) \leftrightarrow \varphi(x//y)$. From 3 we have that 4. ($\mathcal{A}, v_a \vDash \varphi(x)$ and $\mathcal{A}, v_a \nvDash \varphi(x//y)$) or ($\mathcal{A}, v_a \nvDash \varphi(x)$ and $\mathcal{A}, v_a \vDash \varphi(x//y)$). From 2 and (D2) we get 5. $\mathcal{A}, v_a \vDash x =_{\mathsf{E}} y \vee (\mathsf{A}!x \wedge \mathsf{A}!y \wedge \forall F(xF \leftrightarrow yF))$, so 6. $\mathcal{A}, v_a \vDash x =_{\mathsf{E}} y$ or $\mathcal{A}, v_a \vDash \mathsf{A}!x \wedge \mathsf{A}!y \wedge \forall F(xF \leftrightarrow yF)$. Let us consider both cases from 6.

Assume 1.1. $\mathcal{A}, v_a \vDash x =_{\mathsf{E}} y$. Then 1.2. $|v_a(x)| = |v_a(y)|$ and 1.3. $v_a(x) \in \mathcal{E}_a$, then from 1.3 we have 1.4. $|v_a(x)| = v_a(x)$. From 1.2 and 1.4 we have 1.5. $|v_a(y)| = v_a(x)$, thus 1.6. $|v_a(y)| \in \mathcal{E}_a$, and so 1.7. $|v_a(y)| = v_a(y)$. From 1.5 and 1.7 we have 1.8. $v_a(x) = v_a(y)$. Let us consider both cases from 4.

Assume 1.1.1. $\mathcal{A}, v_a \vDash \varphi(x)$ and 1.1.2. $\mathcal{A}, v_a \nvDash \varphi(x//y)$. From 1.8 and 1.1.2 we have 1.1.3. $\mathcal{A}, v_a \nvDash \varphi(x//x)$. Contradiction: 1.1.1, 1.1.3.

Assume 1.2.1. $\mathcal{A}, v_a \nvDash \varphi(x)$ and 1.2.2. $\mathcal{A}, v_a \vDash \varphi(x//y)$. From 1.8 and 1.2.2 we have 1.2.3. $\mathcal{A}, v_a \vDash \varphi(x//x)$. Contradiction: 1.2.1, 1.2.3.

We come back to the second case from 6. Assume 2.1. $\mathcal{A}, v_a \vDash \mathsf{A}!x \wedge \mathsf{A}!y \wedge \forall F(xF \leftrightarrow yF)$. Then 2.2. $\mathcal{A}, v_a \vDash \mathsf{A}!x$ and 2.3. $\mathcal{A}, v_a \vDash \mathsf{A}!y$ and 2.4. $\mathcal{A}, v_a \vDash \forall F(xF \leftrightarrow yF)$. From 2.2, 2.3 and (D1) we get 2.5. $\mathcal{A}, v_a \nvDash \mathsf{E}!x$ and 2.6. $\mathcal{A}, v_a \nvDash \mathsf{E}!y$, so 2.7. $v_a(x) \notin \mathcal{E}_a$ and 2.8. $v_a(y) \notin \mathcal{E}_a$, hence 2.9. $v_a(x) \in 2^{\mathcal{P}r_a}$ and 2.10. $v_a(y) \in 2^{\mathcal{P}r_a}$. From 2.4 we have that 2.11. for every $v''_a : v''_a \sim_F v_a$ we have $\mathcal{A}, v''_a \vDash xF \leftrightarrow yF$, i.e., 2.12. $v''_a(x) \in enc_a(v_a(F))$ iff $v''_a(y) \in enc_a(v_a(F))$, and thus 2.13. $v''_a(x) \in 2^{\mathcal{P}r_a}$ and $v''_a(F) \in v''_a(x)$ iff $v''_a(y) \in 2^{\mathcal{P}r_a}$ and $v''_a(F) \in v''_a(y)$. Thus from PC: 2.14. if $v''_a(x) \in 2^{\mathcal{P}r_a}$ and $v''_a(y) \in 2^{\mathcal{P}r_a}$, then ($v''_a(F) \in v''_a(x)$ iff $v''_a(F) \in v''_a(y)$). From 2.9, 2.10 and 2.14 we have 2.15. $v'_a(F) \in v'_a(x)$ iff $v'_a(F) \in v'_a(y)$. From extensionality we have 2.16. $v'_a(x) = v'_a(y)$, and since v_a

and v'_a differ at most in the value for F, we have 2.17. $v_a(x) = v_a(y)$. Let us consider both cases from 4.

In each case, we get a contradiction: the same as in 1.1.1–1.1.3 and 1.2.1–1.2.3.

For property variables. Assume for reductio that 1. there exists an $\mathcal{A}$-model such that $\mathcal{A}, v_a \nvDash F = G \to (\varphi(F) \leftrightarrow \varphi(F//G))$. Thus 2. $\mathcal{A}, v_a \vDash F = G$ and 3. $\mathcal{A}, v_a \nvDash \varphi(F) \leftrightarrow \varphi(F//G)$. From 3 we have 4. ($\mathcal{A}, v_a \vDash \varphi(F)$ and $\mathcal{A}, v_a \nvDash \varphi(F//G)$) or ($\mathcal{A}, v_a \nvDash \varphi(F)$ and $\mathcal{A}, v_a \vDash \varphi(F//G)$). From 2 and (D3) we have that $\mathcal{A}, v_a \vDash \forall x(xF \leftrightarrow xG)$, thus 5. for every $v'_a \sim_x v_a$ we have $v'_a(x) \in enc_a(v'_a(F))$ iff $v'_a(x) \in enc_a(v'_a(G))$, which means that 6. $v'_a(x) \in 2^{\mathcal{P}r_a}$ and $v'_a(F) \in v'_a(x)$ iff $v'_a(x) \in 2^{\mathcal{P}r_a}$ and $v'_a(G) \in v'_a(x)$. Thus from PC: 7. if $v'_a(x) \in 2^{\mathcal{P}r_a}$, then $v'_a(F) \in v'_a(x)$ iff $v'_a(G) \in v'_a(x)$. We also have the implication 8. $v'_a(x) \notin 2^{\mathcal{P}r_a}$, then $v'_a(F) \in v'_a(x)$ iff $v_a(G) \in v'_a(x)$, since in such a case $v'_a(x) \in \mathcal{E}_a$, and then both components of the equivalence are false, hence the equivalence is true, and thus the entire implication is true. From 7, 8 and PC we have 9. $v'_a(F) \in v'_a(x)$ iff $v'_a(G) \in v'_a(x)$. Then we have that 10. $v'_a(F) = v'_a(G)$, and since $v'_a \sim_x v_a$, then 11. $v_a(F) = v_a(G)$. Consider cases from 4.

Assume 1.1. $\mathcal{A}, v_a \vDash \varphi(F)$ and 1.2. $\mathcal{A}, v_a \nvDash \varphi(F//G)$. From 11 and 1.2 we have 1.3. $\mathcal{A}, v_a \nvDash \varphi(F)$. Contradiction: 1.1, 1.3.

Assume 2.1. $\mathcal{A}, v_a \nvDash \varphi(F)$ and 2.2. $\mathcal{A}, v_a \vDash \varphi(F//G)$. From 11 and 2.2 we have 2.3. $\mathcal{A}, v_a \vDash \varphi(F)$. Contradiction: 2.1, 2.3.

(A4) The proof proceeds in the same way as for Scott models (Zalta, 1983, p. 163). We need to show that in every $\mathcal{A}$-model there exists an $o \in 2^{\mathcal{P}r_a}$ such that for every property r, it holds that $o \in enc_a(r)$ if and only if $\mathcal{A}, v_a \vDash \varphi$. Let us take any condition ϕ expressed by formula φ. Since it is true that $\exists o \forall r(r \in o \text{ iff } r \in \mathcal{P}r_a \wedge \phi)$, there exists an $o \subseteq \mathcal{P}r_a$ to which exactly those properties belong that satisfy ϕ.

Therefore, for every $\mathcal{A}$-model: $\mathcal{A}, v_a \vDash \exists x(\mathsf{A}!x \wedge \forall F(xF \leftrightarrow \varphi))$, where x is not free in φ. ⊣

From Theorem 3.1 it directly follows that:

Corollary 3.1. MOT *is consistent.*

4. Comparison of Scott and Aczel models

Scott and Aczel models differ in key aspects. Scott models distinguish between positive and negative properties, with abstracta exemplifying only positive ones. Aczel models introduce an imitation function and a separate set of urelements as proxies for abstracta, where these proxies, not the abstracta themselves, belong to exemplification extensions. Notably, neither Scott nor Aczel models are standard second-order models; rather, they are Henkin models, where the domain of relations is not the full power set of all objects (see Nodelman and Zalta, 2024, pp. 1165, 1193, footnote 10)). Aczel's approach, however, is philosophically more attractive in the sense that it avoids the *overloading of extensions containing abstracta* and the *extensionality of identity of properties*, captured by the following formulas:

$$\exists x(\mathsf{A}!x \wedge Fx) \rightarrow \forall x(\mathsf{A}!x \rightarrow Fx), \qquad (\exists\mathsf{A}!/\forall\mathsf{A}!)$$

$$F \equiv G \rightarrow F = G. \qquad (\equiv/=)$$

Formula ($\exists\mathsf{A}!/\forall\mathsf{A}!$) states that if some abstract object exemplifies a property, then every abstract object exemplifies that property. As a result, abstract objects cannot differ with respect to exemplification. This is not in line with basic intuitions underlying OT. For example, the fact that the abstract object Sherlock Holmes is a character created by Conan Doyle does not require that every abstract object be a character created by Conan Doyle. In turn, formula ($\equiv/=$) states that equivalent properties are identical. This claim excludes distinctions between properties that the theory is intended to allow for: it rules out cases in which two properties have the same exemplification extension but differ in content. For example, the properties *being-Sherlock-Holmes-or-not-being-Sherlock-Holmes* and *being-Watson-or-not-being-Watson* are both tautological, so are exemplified by all objects, yet they are distinct properties with different contents (cf. Zalta, 1983, pp. 72–73). Another known example, often invoked by Zalta, involves the properties *being a rational animal* and *being a featherless biped* (Zalta, 1983, p. 32): even if they are exemplified by exactly the same objects, it does not thereby follow that they are identical. As mentioned earlier (Section 2), the notion of identity of properties in OT is supposed to be intensional.

What is crucial for us, however, is that formulas ($\exists\mathsf{A}!/\forall\mathsf{A}!$) and ($\equiv/=$) are verified by Sotta models. Directly from condition (ii) in exp_s, we get:

Theorem 4.1. *Formula* ($\exists\mathsf{A}!/\forall\mathsf{A}!$) *is true in all* $\mathcal{S}$*-structures.*

However:

Theorem 4.2. *There is an* $\mathcal{A}$*-structure in which* ($\exists\mathsf{A}!/\forall\mathsf{A}!$) *is not true.*

PROOF. Consider the model $\mathcal{A}^3$, where $\mathcal{E}_a = \emptyset$, $\mathcal{P}s = \{\mathfrak{s}_1, \mathfrak{s}_2\}$, $\mathcal{P}r_a = \{\mathfrak{e}, r_1, r_2, r_3\}$, $o_1, ..., o_{16} \in 2^{\mathcal{P}r_a}$, $|o_1| = \mathfrak{s}_1$, $|o_i| = \mathfrak{s}_2$, for $i \in \{2, \dots, 16\}$, $exp_a(\mathfrak{e}) = \emptyset$ and $exp_a(r_1) = \{\mathfrak{s}_1\}$. Let $v_a(x) = o_1$ and $v_a(F) = r_1$. Then the antecedent of the implication is satisfied: we have an abstract object o_1, whose proxy is $\mathfrak{s}_1$, located in the extension of property r_1. Let us take $v'_a : v'_a \sim_x v_a$ so that $v'_a(x) = o_2$. Since the proxy for o_2, that is, $\mathfrak{s}_2$, is not in the extension of r_1, the consequent is false. We have a countermodel. ⊣

Regarding the second formula, it holds that:

THEOREM 4.3. *Formula* ($\equiv/=$) *is true in all* $\mathcal{S}$*-structures.*

PROOF. We will show that in every $\mathcal{S}$-structure the formulas $\forall x(Fx \leftrightarrow Gx) \to \forall x(xF \to xG)$ and $\forall x(Fx \leftrightarrow Gx) \to \forall x(xG \to xF)$ are true.

($\Rightarrow$) Let us take any $\mathcal{S}$-model. Assume that 1. $\mathcal{S}, v_s \vDash \forall x(Fx \leftrightarrow Gx)$ and 2. $\mathcal{S}, v_s \vDash xF$. From 2 we have that 3. $v_s(x) \in enc_s(v_s(F))$, hence 4. $v_s(x) \in 2^{\mathcal{P}r_s}$ and 5. $v_s(F) \in v_s(x)$. From 1 we have that 6. for every $v'_s : v'_s \sim_x v_s$ it is such that $\mathcal{S}, v'_s \vDash Fx \leftrightarrow Gx$, hence 8. ($\mathcal{S}, v'_s \vDash Fx$ and $\mathcal{S}, v'_s \vDash Gx$) or ($\mathcal{S}, v'_s \nvDash Fx$ and $\mathcal{S}, v'_s \nvDash Gx$). Let us consider both cases from 8.

Assume that 1.1. $\mathcal{S}, v'_s \vDash Fx$ and 1.2. $\mathcal{S}, v'_s \vDash Gx$. Hence we have that 1.3. $v'_s(x) \in exp_s(v'_s(F))$ and 1.4. $v'_s(x) \in exp_s(v'_s(G))$. Since $v_s(x) \in 2^{\mathcal{P}r_s}$, then 1.5. $v'_s(F)$ and $v'_s(G)$ are properties of the form $+X$. $v'_s(x)$ is arbitrary, hence from the subset axiom 1.6. there exists such $a = v'_s(x)$ that $v'_s(F) \in a$ and $v'_s(G) \in a$. Hence 1.7. $v'_s(x) \in enc_s(v'_s(F))$ and 1.8. $v'_s(x) \in enc_s(v'_s(G))$, hence 1.9. $\mathcal{S}, v'_s \vDash Fx \leftrightarrow Gx$, hence 1.10. $\mathcal{S}, v'_s \vDash \forall x(xF \leftrightarrow xG)$, which means that 1.11. $\mathcal{S}, v'_s \vDash F = G$. Since v_s and v'_s are identical, 1.12. $\mathcal{S}, v_s \vDash F = G$.

Assume that 2.1. $\mathcal{S}, v'_s \nvDash Fx$ and 2.2. $\mathcal{S}, v'_s \nvDash Gx$. Hence 2.3. $v'_s(x) \notin exp_s(v'_s(F))$ and 2.4. $v'_s(x) \notin exp_s(v'_s(G))$. From 4 we have that 2.5. $v'_s(x) \in 2^{\mathcal{P}r_s}$ and $v'_s(F)$ and $v'_s(G)$ are properties of the form $-X$. $v'_s(x)$ is arbitrary, hence from the axiom of subsets we have that 2.6. there exists such $a = v'_s(x)$ that $v'_s(F) \in a$ and $v'_s(G) \in a$. Further the same as in 1.7–1.12.

The proof ($\Leftarrow$) is analogous. From the satisfaction conditions we will get that formula $\forall x(Fx \leftrightarrow Gx) \to \forall x(xF \leftrightarrow xG)$ is true in all $\mathcal{S}$-models. Hence from (D3) and (D4) we have that the formula $F \equiv G \to F = G$ is true in every $\mathcal{S}$-model. ⊣

THEOREM 4.4. *There is an* $\mathcal{A}$*-structure in which* ($\equiv/=$) *is not true.*

PROOF. Consider the previous model $\mathcal{A}^3$. Let us also add that $exp_a(r_2) = \{\mathfrak{s}_1\}$, $enc_a(r_1) = \{o_1\}$ and $enc_a(r_2) = \{o_2\}$. Let $v_a(x) = o_1$, $v_a(F) = r_1$ and $v_a(G) = r_2$. Then the antecedent is satisfied, but consequent is not. We have a countermodel. ⊣

From Theorems 3.1, 4.2, and 4.4, we know that (∃A!/∀A!) and (≡/=) are not theses of MOT.

4.1. Translation of Scott structures into Aczel structures

Now our goal is to reconstruct Scott structures in Aczel structures. We define the *unitary Aczel structure* and the translation function #.

DEFINITION 4.1 (unitary Aczel structure). An Aczel structure $\mathcal{A} = (\mathcal{E}_a, \mathcal{P}s, \mathcal{P}r_a, \mathfrak{e}, |\cdot|, exp_a, enc_a, op_a)$ is a unitary Aczel structure iff:

1. $\mathcal{P}s$ is a singleton: $\mathcal{P}s = \{\mathfrak{s}\}$;
2. $\mathrm{card}(\mathcal{P}r_a) = \mathrm{card}(2^{\mathcal{E}_a \cup \mathcal{P}s})$;
3. exp_a is surjective: $exp_a[\mathcal{P}r_a] = 2^{\mathcal{E}_a \cup \mathcal{P}s}$.

For example, the structure $\mathcal{A}^1 = (\{d\}, \{\mathfrak{s}\}, \{\mathfrak{e}, r_1, r_2, r_3\}, |\cdot|, exp_a, enc_a)$ in Example 3.2 is a unitary Aczel structure.

DEFINITION 4.2 (Translation #). For a given Scott structure $\mathcal{S} = (\mathcal{E}, \mathcal{P}r_s, exp_s, enc_s, op_s)$ and unitary Aczel structure $\mathcal{A} = (\mathcal{E}, \mathcal{P}s, \mathcal{P}r_a, \mathfrak{e}, |\cdot|, exp_a, enc_a, op_a)$, a translation of $\mathcal{S}$ into $\mathcal{A}$ is a function $\#\colon \mathcal{E} \cup \mathcal{P}r_s \cup 2^{\mathcal{P}r_s} \to \mathcal{E} \cup \mathcal{P}r_a \cup 2^{\mathcal{P}r_a}$ such that:

- (i) # is a bijection;
- (ii) $o \in \mathcal{E} \Rightarrow \#(o) = o$;
- (iii) $\forall r_s\ \#(r_s) \in \mathcal{P}r_a$, where:
  - (a) $\forall o \in \mathcal{E}(o \in exp_s(r_s)$ iff $o \in exp_a(\#(r_s)))$ and
  - (b) $\exists o \in 2^{\mathcal{P}r_s}(o \in exp_s(r_s))$ iff $\mathfrak{s} \in exp_a(\#(r_s))$;
- (iv) $o \in enc_s(r_s)$ iff $\#(o) \in enc_a(\#(r_s))$.

We now turn to the three key theorems.

THEOREM 4.5. *For any Scott structure $\mathcal{S} = (\mathcal{E}, \mathcal{P}r_s, exp_s, enc_s, op_s)$, unitary Aczel structure $\mathcal{A} = (\mathcal{E}, \mathcal{P}s, \mathcal{P}r_a, \mathfrak{e}, |\cdot|, exp_a, enc_a, op_a)$ and any valuation v_s:*

$$\mathcal{S}, v_s \vDash \varphi \quad \textit{iff} \quad \mathcal{A}, v_a \vDash \varphi,$$

where v_a is a composition of v_s and #, i.e. $v_a(x) = (\# \circ v_s)(x) = \#(v_s(x))$.

Proof. We prove the theorem by induction on the complexity of φ.

($\varphi = \pi x$) ($\Rightarrow$) Assume that $\mathcal{S}, v_s \models \pi x$. Hence we have $v_s(x) \in exp_s(v_s(\pi))$. Given that $v_s(x) \in \mathcal{E} \cup 2^{\mathcal{P}r_s}$, then either $v_s(x) \in \mathcal{E}$ or $v_s(x) \in 2^{\mathcal{P}r_s}$.

We consider two cases. Assume that $v_s \in \mathcal{E}$. Then $\#(v_s(x)) \in exp_a(\#(v_s(\pi)))$. So. $(\# \circ v_s)(x) \in exp_a((\# \circ v_s)(\pi))$. Since $o \in \mathcal{E} \Rightarrow |o| = o$, $|(\# \circ v_s)(x)| \in exp_a((\# \circ v_s)(\pi))$. So, $\mathcal{A}, \# \circ v_s \models \pi x$. Thus, $\mathcal{A}, v_a \models \pi x$.

Assume that $v_s(x) \in 2^{\mathcal{P}r_s}$. Then $\mathfrak{s} \in exp_a(\#(v_s(\pi)))$. Since $\mathfrak{s}$ is the only element of $\mathcal{Ps}$, $\mathfrak{s} = |\#(v_s(x))|$. Thus, we get $|\#(v_s(x))| \in exp_a(\#(v_s(\pi)))$. Hence, $|(\# \circ v_s)(x)| \in exp_a((\# \circ v_s)(\pi))$. So, $\mathcal{A}, \# \circ v_s \models \pi x$. Thus. $\mathcal{A}, v_a \models \pi x$.

($\Leftarrow$) Assume that $\mathcal{A}, v_a \models \pi x$, i.e. $\mathcal{A}, \# \circ v_s \models \pi x$. Then $|(\# \circ v_s)(x)| \in exp_a((\# \circ v_s)(\pi))$. So, $|\#(v_s(x))| \in exp_a(\#(v_s(\pi)))$. We know that either $|\#(v_s(x))| \in \mathcal{E}$ or $|\#(v_s(x))| = \mathfrak{s}$. We consider two cases.

Assume that $|\#(v_s(x))| \in \mathcal{E}$. From the fact that $o \in \mathcal{E} \Rightarrow |o| = o$, we get that $|\#(v_s(x))| = v_s(x)$. Hence, we get $v_s(x) \in exp_a(\#(v_s(\pi)))$, and $v_s(x) \in exp_s(v_s(\pi))$. So, $\mathcal{S}, v_s \models \pi x$.

Assume that $|\#(v_s(x))| = \mathfrak{s}$. Then $\mathfrak{s} \in exp_a(\#(v_s(\pi)))$. Since $\mathfrak{s}$ is a proxy of all abstract objects, $v_s(x) \in exp_s(v_s(\pi))$. So, $\mathcal{S}, v_s \models x$.

($\varphi = x\pi$) ($\Rightarrow$) Assume that $\mathcal{S}, v_s \models x\pi$. Then $v_s(x) \in enc_s(v_s(\pi))$. From the conditions for enc_s, we get that $v_s(x) \in 2^{\mathcal{P}r_s}$ and $v_s(\pi) \in v_s(x)$. Hence, we get that $\#(v_s(x)) \in enc_a(\#(v_s(\pi)))$. So, $(\# \circ v_s)(x) \in enc_a((\# \circ v_s)(\pi))$. Thus, $\mathcal{A}, \# \circ v_s \models x\pi$. Therefore, $\mathcal{A}, v_a \models x\pi$.

($\Leftarrow$) Assume that $\mathcal{A}, v_a \models x\pi$, i.e. $\mathcal{A}, \# \circ v_s \models x\pi$. Then $(\# \circ v_s)(x) \in enc_a((\# \circ v_s)(\pi))$, which means that $\#(v_s(x)) \in enc_a(\#(v_s(\pi)))$. From the conditions for enc_a, we get that $\#(v_s(x)) \in 2^{\mathcal{P}r_s}$ and $\#(v_s(\pi)) \in \#(v_s(x))$. Thus, $v_s(x) \in enc_s(v_s(\pi))$. So, $\mathcal{S}, v_s \models x\pi$.

($\varphi = (x =_{\mathsf{E}} y)$) ($\Rightarrow$) Assume that $\mathcal{S}, v_s \models x =_{\mathsf{E}} y$. Then $v_s(x) = v_s(y)$ and $v_s(x) \in exp_s(v_s(\mathsf{E!}))$. Therefore, $v_s(x) \in exp_s(-\mathcal{E})$. So, from the condition for exp_s, we have that $v_s(x) \in \mathcal{E}$. Hence, $v_s(x) = \#(v_s(x))$. So, $\#(v_s(x)) \in \mathcal{E}$, $\#(v_s(x)) = v_s(y)$, and $v_s(y) \in \mathcal{E}$. So, $\#(v_s(y)) = v_s(y)$. Hence $\#(v_s(x)) = \#(v_s(y))$. Moreover, we get $(\# \circ v_s)(x) = (\# \circ v_s)(y)$ and $(\# \circ v_s)(x) \in \mathcal{E}$. So, $\mathcal{A}, \# \circ v_s \models x =_{\mathsf{E}} y$. Thus, $\mathcal{A}, v_a \models x =_{\mathsf{E}} y$.

($\Leftarrow$) Assume that $\mathcal{A}, v_a \models x =_{\mathsf{E}} y$, i.e. $\mathcal{A}, \# \circ v_s \models x =_{\mathsf{E}} y$. Then $(\# \circ v_s)(x) = (\# \circ v_s)(y)$ and $(\# \circ v_s)(x) \in \mathcal{E}$. Hence, $\#(v_s(x)) = \#(v_s(y))$ and $\#(v_s(x)) \in \mathcal{E}$. So, we get $\#(v_s(x)) = v_s(x)$ and $\#(v_s(y)) \in \mathcal{E}$. Moreover, we get $\#(v_s(y)) = v_s(y)$. Therefore, $v_s(x) = v_s(y)$. Thus, $v_s(x) \in \mathcal{E}$, and $\mathcal{S}, v_s \models x =_{\mathsf{E}} y$.

($\varphi = \neg(\psi)$) From the inductive hypothesis we have that $\mathcal{S}, v_s \vDash \psi$ iff $\mathcal{A}, v_a \vDash \psi$. Hence $\mathcal{S}, v_s \nvDash \psi$ iff $\mathcal{A}, v_a \nvDash \psi$. From the conditions for negation we have that $\mathcal{S}, v_s \vDash \neg(\psi)$ iff $\mathcal{A}, v_a \vDash \neg(\psi)$.

($\varphi = (\psi \rightarrow \chi)$) From the inductive hypothesis we have that $\mathcal{S}, v_s \vDash \psi$ iff $\mathcal{A}, v_a \vDash \psi$ and $\mathcal{S}, v_s \vDash \chi$ iff $\mathcal{A}, v_a \vDash \chi$. Hence, $\mathcal{S}, v_s \nvDash \psi$ iff $\mathcal{A}, v_a \nvDash \psi$. Moreover, $\mathcal{S}, v_s \nvDash \psi$ or $\mathcal{S}, v_s \vDash \chi$ iff $\mathcal{A}, v_a \nvDash \psi$ or $\mathcal{A}, v_a \vDash \chi$, which means that $\mathcal{S}, v_s \vDash (\psi \rightarrow \chi)$ iff $\mathcal{A}, v_a \vDash (\psi \rightarrow \chi)$.

($\varphi = \forall\alpha(\psi)$) From the inductive hypothesis we have that $\mathcal{S}, v_s \vDash \psi$ iff $\mathcal{A}, v_a \vDash \psi$. Since $v_a = (\# \circ v_s)$ and $v_s, \#$ are arbitrary, for any variable α and for any functions v_s' and $(\# \circ v_s)'$, which differ from v_s and $(\# \circ v_s)$ at most in the value for α, it is such that $\mathcal{S}, v_s' \vDash \psi$ iff $\mathcal{A}, (\# \circ v_s)' \vDash \psi$. Therefore, from the conditions for the universal quantifier we have that $\mathcal{S}, v_s \vDash \forall\alpha(\psi)$ iff $\mathcal{A}, \# \circ v_s \vDash \forall\alpha(\psi)$. Thus, $\mathcal{S}, v_s \vDash \forall\alpha(\psi)$ iff $\mathcal{A}, v_a \vDash \forall\alpha(\psi)$. $\dashv$

THEOREM 4.6. *For any Scott structure* $\mathcal{S} = (\mathcal{E}, \mathcal{P}r_s, exp_s, enc_s, op_s)$, *unitary Aczel structure* $\mathcal{A} = (\mathcal{E}, \mathcal{P}s, \mathcal{P}r_a, \mathfrak{e}, |\cdot|, exp_a, enc_a, op_a)$ *and any valuation* v_a:

$$\mathcal{A}, v_a \vDash \varphi \quad \textit{iff} \quad \mathcal{S}, v_s \vDash \varphi,$$

where v_s *is a composition of* v_a *and* $\#^{-1}$, *i.e.* $v_s(x) = (\#^{-1} \circ v_a)(x) = \#^{-1}(v_a(x))$.

PROOF. Since any composition of functions is associative and composition of any bijection with its inverse is an identity function, then we get $(\# \circ v_s) = v_a$ iff $(\#^{-1} \circ v_a) = v_s$. Thus, by Theorem 4.5, we get $\mathcal{A}, v_a \vDash \varphi$ iff $\mathcal{S}, \#^{-1} \circ v_a \vDash \varphi$. Thus, $\mathcal{A}, v_a \vDash \varphi$ iff $\mathcal{S}, v_s \vDash \varphi$. $\dashv$

THEOREM 4.7. *For a given Scott structure* $\mathcal{S} = (\mathcal{E}, \mathcal{P}r_s, exp_s, enc_s, op_s)$, *there exists exactly one unitary Aczel structure* $\mathcal{A} = (\mathcal{E}, \mathcal{P}s, \mathcal{P}r_a, \mathfrak{e}, |\cdot|, exp_a, enc_a, op_a)$ *for which there is a translation* $\#$ *of* $\mathcal{S}$ *into* $\mathcal{A}$.

PROOF. Assume for reductio that there exist an $\mathcal{S}$-structure and two distinct structures $\mathcal{A}$ and $\mathcal{A}'$ such that there are translations $\#$ from $\mathcal{S}$ into $\mathcal{A}$ and $\mathcal{A}'$. From Definition 4.1 and (i) of Definition 4.2, $\mathcal{A}$ and $\mathcal{A}'$ have the same domain of concrete objects $\mathcal{E}$. From (1), $\mathcal{A}$ and $\mathcal{A}'$ have the same domain of proxies, that is, $\mathcal{P}s = \{\mathfrak{s}\}$. From (2) and (3), $\mathcal{A}$ and $\mathcal{A}'$ have the same domain of properties $\mathcal{P}r_a$. Therefore, $\mathcal{A}$ and $\mathcal{A}'$ do not differ in the domain of abstract objects $2^{\mathcal{P}r_a}$. $\mathcal{A}$ and $\mathcal{A}'$ also do not differ in $|\cdot|$, since for every $o \in \mathcal{E} : |o| = o$ and for every $o \in 2^{\mathcal{P}r_a} : |o| = \mathfrak{s}$. From (iii) and (iv) of Definition 4.2, $\mathcal{A}$ and $\mathcal{A}'$ do not differ in exp_a

and enc_a, respectively; therefore, they do not differ with respect to op_a either. Hence, contrary to the assumption, $\mathcal{A}$ and $\mathcal{A}'$ are not distinct. ⊣

These theorems imply that every formula true in a $\mathcal{S}$-structure is also true in a unitary $\mathcal{A}$-structure with the same domain of concreta, and conversely, every formula true in a unitary $\mathcal{A}$-structure is also true in a $\mathcal{S}$-structure with the same domain of concreta. Thus, it follows from Theorems 4.5, 4.6 and 4.7 that:

Theorem 4.8. *The formula φ is true in the class of all $\mathcal{S}$-structures iff φ is true in the class of all unitary $\mathcal{A}$-structures.*

The above theorems imply a result already known from elsewhere (Zalta, 1983, pp. 160–164):

Corollary 4.1. MOT *is satisfied in the class of all $\mathcal{S}$-models.*

Finally, let us note the following corollaries:

Corollary 4.2. *Formulas* ($\exists$A!/$\forall$A!) *and* ($\equiv$/$=$) *are true in all unitary $\mathcal{A}$-structures.*

5. Conclusion

The two semantic interpretations differ in modeling the ontology of objects and the modes of attribution. Neither approach perfectly mirrors the ontology proposed by Zalta. Scott's semantics can be challenged with justifying the distinction between positive and negative properties, while Aczel's semantics can be criticized for introduction of proxies for abstracta, which can be seen as 'ontological danglers'. However, Aczel's semantics has the advantage of falsifying the philosophically problematic formulas ($\exists$A!/$\forall$A!) and ($\equiv$/$=$). Keeping in mind the philosophical motivations of Zalta's theory discussed in Section 1, the features of Scott structures and unitary Aczel structures are undesirable.

However, according to Zalta, it is crucial not to confuse the model-theoretic interpretation of OT with its ontological interpretation (Zalta, 1988, pp. 34–35). Specifically, abstract objects cannot be identified with sets of properties because sets are, ontologically speaking, a special kind of abstracta (Zalta, 1997, p. 271). He argues that model theory is a pragmatic tool for proving meta-theoretical properties such as consistency. The goal, Zalta asserts, is not to build a model but to construct a formal theory that mirrors the world's structure. Another reason is connected with (T7), which asserts that there is an object x that both encodes and exemplifies some property F. If abstract objects were identified with sets of properties, this would yield $x \in F$ and $F \in x$, thereby violating the Axiom of Foundation (Zalta, 1983, p. 36) and (Zalta, 1997, p. 270, footnote 18). From this perspective, the significance of Aczel's semantics becomes apparent, as it enables the proof of consistency without the extensionalization of certain concepts in OT that are intended to be intensional.

Acknowledgments. I am grateful to Edward N. Zalta for granting access to the original drafts of the Scott and Aczel semantics and for numerous valuable suggestions that have significantly improved this paper.

References

Kirchner, D., 2022, "Computer-verified foundations of metaphysics and an ontology of natural numbers in isabelle/hol", PhD thesis, Freie Universität Berlin. DOI: 10.17169/refubium-35141

Kirchner, D. , C. Benzmüller, and E. N. Zalta, 2020, "Mechanizing principia logico-metaphysica in functional type theory", *Review of Symbolic Logic*, 13 (1): 206–218, DOI: 10.1017/s1755020319000297

Manzano, M., 1996, *Extensions of First Order Logic*, Cambridge University Press, New York.

Manzano, M., and M. C. Moreno, 2017, "Identity, equality, nameability and completeness", *Bulletin of the Section of Logic*, 46(3/4): 169–195. DOI: 10.18778/0138-0680.46.3.4.02

Nodelman, U., and E. N. Zalta, 2014, "Foundations for mathematical structuralism", *Mind*, 123(489): 39–78. DOI: 10.1093/mind/fzu003

Nodelman, U., and E. N. Zalta, 2024, "Number theory and infinity without mathematics", *Journal of Philosophical Logic*, 53(5): 1161–1197. DOI: 10.1007/s10992-024-09762-7

Świętorzecka, K., 2017, "Bolzano's argument for the existence of substances: a formalization with two types of predication", *Acta Analytica*, 32: 411–426. DOI: 10.1007/s12136-017-0317-4

Świętorzecka, K., 2019, "Two formal interpretations of Bolzano's theory of substances and adherences", *Axiomathes*, 29(3): 265–284. DOI: 10.1007/s10516-018-9390-6

Zalta, E. N., 1983, *Abstract Objects: An Introduction to Axiomatic Metaphysics*, Dordrecht: D. Kluwer Academic Publishers.

Zalta, E. N., 1988, *Intensional Logic and the Metaphysics of Intentionality*, Cambridge: MIT Press.

Zalta, E. N., 1997, "The modal object calculus and its interpretation", pages 249–279 in M. de Rijke (ed.), *Advances in Intensional Logic*, Dordrecht: Kluwer Academic Publishers.

Zalta, E. N., 1999, "Natural numbers and natural cardinals as abstract objects: A partial reconstruction of Frege's grundgesetze in object theory", *Journal of Philosophical Logic*, 28: 617–658. DOI: 10.1023/A:1004330128910

Zalta, E. N., 2025a, "Principia logico-metaphysica". https://mally.stanford.edu/principia.pdf

Zalta, E. N., 2025b, "The power of predication and quantification", *Open Philosophy*, 8(1): 20240059. DOI: 10.1515/opphil-2024-0059

Bartłomiej Uzar
Department of Philosophy
Cardinal Stefan Wyszyński University in Warsaw, Poland
b.uzar@doktorant.uksw.edu.pl
https://orcid.org/0000-0002-2240-1310