Logic and Logical Philosophy
Online First Articles (2025)
DOI: 10.12775/LLP.2025.014

Víctor Aranda

A Completeness Theorem for a Functionally Complete Łukasiewicz Logic

Abstract. Radzki has recently claimed the incompleteness of the axioms given by Słupecki for the functionally complete Ł3: some of its tautologies are not provable. In this paper, we provide a new axiom system for this logic (choosing a variant with two propositional constants and the Łukasiewicz implication as primitive symbols) and prove a Completeness Theorem.

Keywords: many-valued logics; three-valued logics; propositional logic; non-classical logics; fuzzy logic.

1. Introduction

In 2017, Radzki claimed that, although Jerzy Słupecki (1967, p. 336) had postulated the semantic completeness of his axiom system for the functionally complete Ł3, there are tautologies of this logic which are not provable. For this reason, he finished his paper saying that his final result was "to reveal the need for constructing a new axiom system for the functionally complete three-valued propositional logic" (Radzki, 2017, p. 413). We will show how such a system can be constructed.

As it is well known, Ł3 is not a *functionally complete* logic: some truth-functions cannot be defined in terms of the familiar $\neg$, $\to$, $\land$, $\lor$, $\equiv$ (Malinowski, 2007, p. 37). To solve this problem, Słupecki first added a new 1-ary connective assigning the intermediate truth-value, 0.5, to every proposition; then, he extended Wajsberg's axioms for Ł3 with two new principles governing that connective. In the present paper, we have replaced Słupecki's unary connective by a 0-ary one (whose interpretation is 0.5), thus avoiding the difficulties pointed out by Radzki

Formal Methods in Philosophy. Guest Editors: K. Świętorzecka and K. Zdanowski
Received March 1, 2025. *Accepted* July 2, 2025. *Published online* August 28, 2025

(2017, pp. 410–411) leading to incompleteness. Our set of primitive symbols still allows us to define the original operator, so the *expressive power* is kept.

The strategy to obtain the Completeness Theorem is the following. Firstly, we present the language and semantics of ŁS, a truth-functionally complete Łukasiewicz logic. Then, we offer a proof system consisting of Łukasiewicz's axioms, some properties of the Baaz (1996) *delta operation*, one of the axioms formulated by Avron (1991, p. 289) (which plays an essential role in the completeness proof) and a variant of Słupecki's two postulates for the new 0-ary connective, **T**. We prove a weaker version of the Deduction Theorem, taken from Novak's work (2005, p. 261) and involving the Baaz delta operation. Finally, in the last section, we adapt the usual definitions of *consistency*, *maximality* and *completeness* (of a set of formulas) to our three-valued framework and get a Completeness Theorem following Henkin's procedure.

2. Preliminaries

In this section, we first introduce the language of the truth-functionally complete Ł3, but replacing Słupecki's unary connective by a 0-ary one. We call this logic ŁS.

The logic ŁS has propositional letters $p_1, p_2, \ldots$, two 0-ary connectives, $\bot$ and $\mathbf{T}$, and implication $\to$. Formulas are defined as follows:

$$p \mid \bot \mid \mathbf{T} \mid \varphi \to \psi$$

Now, we define the semantics of our system (*valuation*, *tautology*, *model* and *consequence*).

DEFINITION 2.1 (Valuation). A *valuation* is a function $\mathfrak{v}$ assigning to each propositional variable p its truth-value $\mathfrak{v}(p) \in \{0, 0.5, 1\}$. This is extended to all formulas as follows:

1. $\mathfrak{v}(\bot) = 0$,
2. $\mathfrak{v}(\mathbf{T}) = 0.5$,
3. $\mathfrak{v}(\varphi \to \psi) = \min\{1, (1 - \mathfrak{v}(\varphi)) + \mathfrak{v}(\psi)\}$.

Clearly, $\to$ is a Łukasiewicz implication.

We fix 1 as the only *designated* truth-value. Therefore: $\models \varphi$ if $\mathfrak{v}(\varphi) = 1$ for every valuation $\mathfrak{v}$.

Definition 2.2 (Model). A *model* of a set of formulas Γ is a valuation $\mathfrak{v}$ such that $\mathfrak{v}(\gamma) = 1$ for all $\gamma \in \Gamma$.

Definition 2.3 (Consequence). $\Gamma \models \varphi$ if, for every valuation $\mathfrak{v}$, if $\mathfrak{v}$ is a model of Γ, then $\mathfrak{v}(\varphi) = 1$.

The Łukasiewicz negation is definable in terms of the set of primitive symbols $\{\bot, \mathbf{T}, \to\}$:

Definition 2.4. Negation is defined as follows:

$$\neg\varphi := \varphi \to \bot$$

By Definitions 2.4 and 2.1, we can check that $\mathfrak{v}(\neg\varphi) = 1 - \mathfrak{v}(\varphi)$. Hence, $\neg$ is a Łukasiewicz negation.

Definition 2.5. Strong conjunction is defined as follows:

$$\varphi \,\&\, \psi := \neg(\varphi \to \neg\psi)$$

By Definitions 2.5 and 2.1, we have $\mathfrak{v}(\varphi \,\&\, \psi) = \max\{0, (\mathfrak{v}(\varphi) + \mathfrak{v}(\psi)) - 1\}$.

Definition 2.6. New connectives, $\wedge$, $\vee$, and $\equiv$, the Baaz operator, $\mathbf{\Delta}$, and the original Słupecki's T operator are introduced:

1. $\varphi \wedge \psi := \varphi \,\&(\varphi \to \psi)$.
2. $\varphi \vee \psi := (\varphi \to \psi) \to \psi$.
3. $\varphi \equiv \psi := (\varphi \to \psi) \,\&(\psi \to \varphi)$.
4. $\mathbf{\Delta}\varphi := \varphi \,\&\, \varphi$.
5. $T\varphi := \mathbf{T} \to (\bot \,\&\, \varphi)$.

Theorem 2.1. *The set* $\{\bot, \mathbf{T}, \to\}$ *is truth-functionally complete.*

Proof. Słupecki (1946) has showed that $\{T, \neg, \to\}$ is truth-functionally complete. Since $\{\bot, \mathbf{T}, \to\}$ can be used to define $\{\neg, T\}$ (Definitions 2.4, 2.5 and 2.6), so is $\{\bot, \mathbf{T}, \to\}$. □

3. The system ŁS

Axiom schemas:

1. *Łukasiewicz's axioms*:
   (Ł1) $\varphi \to (\psi \to \varphi)$,
   (Ł2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,

(Ł3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$,
(Ł4) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$.

2. *Axioms of* $\mathbf{\Delta}$.
   ($\mathbf{\Delta}$1) $\mathbf{\Delta}\varphi \to \varphi$,
   ($\mathbf{\Delta}$2) $(\mathbf{\Delta}\varphi \to (\psi \to \chi)) \to ((\mathbf{\Delta}\varphi \to \psi) \to (\mathbf{\Delta}\varphi \to \chi))$,
   ($\mathbf{\Delta}$3) $\mathbf{\Delta}(\varphi \to \psi) \to (\mathbf{\Delta}\varphi \to \mathbf{\Delta}\psi)$,
   ($\mathbf{\Delta}$4) $\mathbf{\Delta}\varphi \to \mathbf{\Delta}\mathbf{\Delta}\varphi$,
   ($\mathbf{\Delta}$5) $\mathbf{\Delta}(\varphi \to \psi) \vee \mathbf{\Delta}(\psi \to \varphi)$,
   ($\mathbf{\Delta}$6) $\neg\mathbf{\Delta}(\neg\mathbf{\Delta}\varphi) \to \mathbf{\Delta}\varphi$.
3. *The Axiom of Avron*:
   (Av) $(\varphi \vee (\varphi \to \psi)) \vee (\psi \to \chi)$.
4. *New axioms of Słupecki*:
   (Sł1) $\mathbf{T} \to \neg\mathbf{T}$,
   (Sł2) $\neg\mathbf{T} \to \mathbf{T}$.

Rules of inference:
(MP) From φ and $\varphi \to \psi$ infer ψ.
(N) From φ infer $\mathbf{\Delta}\varphi$.

A *proof* in a set of formulas Γ is a sequence $\varphi_1, \ldots, \varphi_n$ of formulas such that each φ_i:

1. is an axiom of ŁS, or
2. is in Γ, or
3. follows from some preceding φ_j, φ_k by MP, or
4. follows from some preceding φ by N.

- φ is *provable in* Γ (in symbols, $\Gamma \vdash \varphi$) if φ is the last member of a proof in Γ.
- φ is *provable in* ŁS (in symbols, $\vdash \varphi$) if φ is the last member of a proof in $\varnothing$.

The formulas below follow from Łukasiewicz's axioms (see Hájek, 1998, p. 66):

$$\varphi \to ((\varphi \to \psi) \to \psi) \tag{1}$$

$$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)), \tag{2}$$

$$\varphi \to \varphi \tag{3}$$

$$\bot \to \varphi \tag{4}$$

$$\neg\neg\varphi \to \varphi, \tag{5}$$

$$(\varphi \to \neg\psi) \to (\psi \to \neg\varphi) \tag{6}$$

$$\varphi \to \neg\neg\varphi \tag{7}$$

Moreover, the following axioms and properties of Basic Fuzzy Logic (BL) follow from Łukasiewicz's axioms (see Hájek, 1998, Lemma 3.1.9):

$$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi)) \tag{8}$$
$$(\varphi \to (\psi \to \chi) \to ((\varphi \,\&\, psi) \to \chi), \tag{9}$$
$$((\varphi \,\&\, \psi) \to \chi) \to (\varphi \to (\psi \to \chi), \tag{10}$$
$$(\varphi \,\&\, \psi) \to \varphi \tag{11}$$
$$(\varphi \,\&\, \psi) \to (\psi \,\&\, \varphi) \tag{12}$$
$$\varphi \to (\psi \to \varphi \,\&\, \psi) \tag{13}$$
$$(\varphi \,\&\, \psi) \,\&\, \chi \to \varphi \,\&(\psi \,\&\, \chi) \tag{14}$$
$$\varphi \,\&(\psi \,\&\, \chi) \to (\varphi \,\&\, \psi) \,\&\, \chi \tag{15}$$

Hence we obtain:

Proposition 3.1. *The formulas* (1)–(15) *are provable in* ŁS.

Theorem 3.1 (Deduction theorem). *For all* $\Gamma \subseteq \mathrm{Fm}$ *and* $\varphi, \psi \in \mathrm{Fm}$,

$$\Gamma \cup \{\varphi\} \vdash \psi \text{ iff } \Gamma \vdash \mathbf{\Delta}\varphi \to \psi.$$

Proof. Let $\Gamma \cup \{\varphi\} \vdash \psi$. The proof is by induction on the number of formulas, k, in the sequence $\psi_1, \dots, \psi_n$ forming the derivation of ψ from $\Gamma \cup \{\varphi\}$.

Base step: $k = 1$.

1. ψ_1 is an Axiom. Then:
   a. $\Gamma \vdash \psi_1$ — ψ_1 is an Axiom
   b. $\Gamma \vdash \mathbf{\Delta}\varphi \to \psi_1$ — (Ł1), 1, MP
2. $\psi_1 \in \Gamma$. Then, $\Gamma \vdash \psi_1$, so the same argument works here.
3. $\psi_1 = \varphi$. Since $\mathbf{\Delta}\varphi \to \varphi$ is an instance of Axiom ($\mathbf{\Delta}$1), $\Gamma \vdash \mathbf{\Delta}\varphi \to \varphi$.

Inductive step: suppose that the derivation finishes with the step φ_k and the theorem holds for all the previous ones.

1. ψ_k have been obtained using MP. Thus, $\psi_j \to \psi_k$ and ψ_j appear earlier in the derivation, so $\Gamma \cup \{\varphi\} \vdash \psi_j \to \psi_k$ and $\Gamma \cup \{\varphi\} \vdash \psi_j$. Hence:

1. $\Gamma \vdash \mathbf{\Delta}\varphi \to (\psi_j \to \psi_k)$ — IH
2. $\Gamma \vdash \mathbf{\Delta}\varphi \to \psi_j$ — IH
3. $\Gamma \vdash (\mathbf{\Delta}\varphi \to \psi_j) \to (\mathbf{\Delta}\varphi \to \psi_k)$ — ($\mathbf{\Delta}$2), 1, MP
4. $\Gamma \vdash \mathbf{\Delta}\varphi \to \psi_k$ — 3, 2, MP

2. ψ_k have been obtained using Rule (N). It follows that ψ_k is of the form $\mathbf{\Delta}\theta$, so θ appears earlier in the derivation. In consequence, we have

$\Gamma \cup \{\varphi\} \vdash \theta$ and hence:

1.	$\Gamma \vdash \mathbf{\Delta}\varphi \to \theta$	IH
2.	$\Gamma \vdash \mathbf{\Delta}(\mathbf{\Delta}\varphi \to \theta)$	1, Rule (N)
3.	$\Gamma \vdash \mathbf{\Delta}\mathbf{\Delta}\varphi \to \mathbf{\Delta}\theta$	($\mathbf{\Delta}$3), 2, MP
4.	$\Gamma \vdash \mathbf{\Delta}\varphi \to \mathbf{\Delta}\theta$	3, ($\mathbf{\Delta}$4), (Ł2), MP

Conversely, let $\Gamma \vdash \mathbf{\Delta}\varphi \to \psi$. Then:

1.	$\Gamma \cup \{\varphi\} \vdash \mathbf{\Delta}\varphi \to \psi$	hypothesis
2.	$\Gamma \cup \{\varphi\} \vdash \varphi$	$\varphi \in \Gamma \cup \{\varphi\}$
3.	$\Gamma \cup \{\varphi\} \vdash \mathbf{\Delta}\varphi$	2, Rule (N)
4.	$\Gamma \cup \{\varphi\} \vdash \psi$	1, 3, MP

□

THEOREM 3.2 (Soundness). *If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.*

PROOF. Straightforward. All the axioms are tautologies and both MP and N preserve validity:

1.	$\Gamma \vdash \varphi \to \psi$	hypothesis
2.	$\Gamma \vdash \varphi$	hypothesis
3.	$\Gamma \models \varphi \to \psi$	IH
4.	$\Gamma \models \varphi$	IH
a.	Γ does not have any model	
5a.	$\Gamma \models \psi$	Definition 2.3
b.	Every model $\mathfrak{v}$ of Γ is a model of φ and $\varphi \to \psi$	
5b.	$\mathfrak{v}(\gamma) = 1$, for all $\gamma \in \Gamma$	Definition 2.2
6b.	$\mathfrak{v}(\varphi) = 1$	
7b.	$\mathfrak{v}(\varphi \to \psi) = 1$	
8b.	$\mathfrak{v}(\psi) = 1$	6b, 7b, Definition 2.1
9b.	$\Gamma \models \psi$	5b, 8b, Definition 2.3
10.	$\Gamma \models \psi$	5a, 9b

The proof for (N) is analogous. □

4. Completeness

A set of formulas Γ is *inconsistent* iff $\Gamma \vdash \bot$. Otherwise it is *consistent*. A set of formulas Γ is *maximal* iff for every formula φ such that $\varphi \notin \Gamma$, $\Gamma \cup \{\varphi\}$ is inconsistent. A set of formulas Γ is *complete* iff for every two formulas φ and ψ, $\Gamma \vdash \varphi \to \psi$ or $\Gamma \vdash \psi \to \varphi$.

PROPOSITION 4.1. *If Γ is a consistent set of formulas, then $\mathbf{T}, \neg\mathbf{T} \notin \Gamma$.*

Proof. By contrapositive. Suppose that $\mathbf{T} \in \Gamma$. Then:

1.	$\Gamma \vdash \mathbf{T}$	$\mathbf{T} \in \Gamma$
2.	$\Gamma \vdash \mathbf{T} \to (\mathbf{T} \to \bot)$	(Sł1), Definition 2.4
3.	$\Gamma \vdash \mathbf{T} \to \bot$	2, 1, MP
4.	$\Gamma \vdash \bot$	3, 1, MP

Suppose that $\neg\mathbf{T} \in \Gamma$. Hence, $\Gamma \vdash \neg\mathbf{T}$, so we can use Axiom (Sł2) and MP to get $\Gamma \vdash \mathbf{T}$, making Γ inconsistent (by the previous argument). □

Proposition 4.2. *For any maximal consistent and complete set Γ of formulas,*

1. *If $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$.*
2. *$\varphi \,\&\, \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$.*
3. *If $\varphi \vee \psi \in \Gamma$, then $\varphi \in \Gamma$ or $\psi \in \Gamma$.*

Proof. 1. Suppose that $\Gamma \vdash \varphi$, but let us assume, for the sake of contradiction, that $\varphi \notin \Gamma$. Since Γ is maximal, it follows that $\Gamma \cup \{\varphi\} \vdash \bot$. Hence:

1.	$\Gamma \vdash \varphi$	hypothesis
2.	$\Gamma \cup \{\varphi\} \vdash \bot$	hypothesis
3.	$\Gamma \vdash \mathbf{\Delta}(\varphi) \to \bot$	2, DT
4.	$\Gamma \vdash \mathbf{\Delta}\varphi$	3, Rule (N)
5.	$\Gamma \vdash \bot$	3, 4, MP contradicting the consistency of Γ.

2, Let $\varphi \,\&\, \psi \in \Gamma$. Suppose, for the sake of contradiction, that $\varphi \notin \Gamma$ or $\psi \notin \Gamma$, say φ. Since Γ is maximal, it follows that $\Gamma \cup \{\varphi\} \vdash \bot$. Thus:

1.	$\Gamma \vdash \varphi \,\&\, \psi$	hypothesis
2.	$\Gamma \cup \{\varphi\} \vdash \bot$	hypothesis
3.	$\Gamma \vdash \mathbf{\Delta}(\varphi) \to \bot$	2, DT
4.	$\Gamma \vdash \varphi$	(BL4), 1, MP
5.	$\Gamma \vdash \mathbf{\Delta}\varphi$	4, Rule (N)
6.	$\Gamma \vdash \bot$	3, 5, MP, contradicting the consistency of Γ.

In case $\psi \notin \Gamma$, the proof still works, as (12) guarantees that & is commutative.

Conversely, let $\varphi \in \Gamma$ and $\psi \in \Gamma$, but assume that $\varphi \,\&\, \psi \notin \Gamma$. Because Γ is maximal, $\Gamma \cup \{\varphi \,\&\, \psi\} \vdash \bot$ and hence we have:

1.	$\Gamma \vdash \varphi$	hypothesis
2.	$\Gamma \vdash \psi$	hypothesis
3.	$\Gamma \cup \{\varphi \,\&\, \psi\} \vdash \bot$	hypothesis

4. $\Gamma \vdash \mathbf{\Delta}(\varphi \,\&\, \psi) \to \bot$ 3, DT
5. $\Gamma \vdash \psi \to (\varphi \,\&\, \psi)$ (13), 1, MP
6. $\Gamma \vdash \varphi \,\&\, \psi$ 5, 2, MP
7. $\Gamma \vdash \mathbf{\Delta}(\varphi \,\&\, \psi)$ 6, Rule (N)
8. $\Gamma \vdash \bot$ 4, 7, MP, contradicting the consistency of Γ.

3. Let $\varphi \vee \psi \in \Gamma$. Assume, for the sake of contradiction, that $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. Since Γ is maximal, $\Gamma \cup \{\varphi\} \vdash \bot$ and $\Gamma \cup \{\psi\} \vdash \bot$. Therefore:

1. $\Gamma \vdash \varphi \vee \psi$ hypothesis
2. $\Gamma \cup \{\varphi\} \vdash \bot$ hypothesis
3. $\Gamma \cup \{\psi\} \vdash \bot$ hypothesis
4. $\Gamma \vdash \mathbf{\Delta}\varphi \to \bot$ 3, DT
5. $\Gamma \vdash \mathbf{\Delta}\psi \to \bot$ 4, DT
6. $\Gamma \vdash (\varphi \to \psi) \to \psi$ 1, Definition 2.6(2)
7. $\Gamma \vdash (\psi \to \varphi) \to \varphi$ (Ł4), 6, MP
8. $\Gamma \vdash \mathbf{\Delta}((\varphi \to \psi) \to \psi)$ 6, Rule (N)
9. $\Gamma \vdash \mathbf{\Delta}((\psi \to \varphi) \to \varphi)$ 7, Rule (N)
10. $\Gamma \vdash \mathbf{\Delta}(\varphi \to \psi) \to \mathbf{\Delta}\psi$ ($\mathbf{\Delta}$3), 8, MP
11. $\Gamma \vdash \mathbf{\Delta}(\psi \to \varphi) \to \mathbf{\Delta}\varphi$ ($\mathbf{\Delta}$3), 9, MP
12. $\Gamma \vdash \mathbf{\Delta}(\varphi \to \psi) \to \bot$ (Ł2), 10, 5, MP
13. $\Gamma \vdash \mathbf{\Delta}(\psi \to \varphi) \to \bot$ (Ł2), 11, 4, MP

However, since Γ is complete, we know that either $\Gamma \vdash \varphi \to \psi$ or $\Gamma \vdash \psi \to \varphi$. In any case, we can use Rule (N), 12 (or 13) and MP to get $\Gamma \vdash \bot$, which is a contradiction. □

PROPOSITION 4.3. *For any maximal consistent and complete set Γ of formulas,*

1. *If $\neg\psi \in \Gamma$, then $\psi \to \chi \in \Gamma$.*
2. *If $\chi \in \Gamma$, then $\psi \to \chi \in \Gamma$.*
3. *If $\psi \notin \Gamma$ and $\neg\chi \notin \Gamma$, then $\psi \to \chi \in \Gamma$.*

PROOF. 1 Let $\neg\psi \in \Gamma$. Then:

1. $\Gamma \vdash \neg\psi$ $\neg\psi \in \Gamma$
2. $\Gamma \vdash \psi \to \bot$ 1, Definition 2.4
3. $\Gamma \vdash \bot \to \chi$ (4)
4. $\Gamma \vdash \psi \to \chi$ 2, 3, (Ł2), MP
5. $\psi \to \chi \in \Gamma$ (4), Proposition 4.2(1)

2. Let $\chi \in \Gamma$. Then:

1. $\Gamma \vdash \chi$ — $\chi \in \Gamma$
2. $\Gamma \vdash \chi \to (\psi \to \chi)$ — (Ł1)
3. $\Gamma \vdash \psi \to \chi$ — 2, 1, MP
4. $\psi \to \chi \in \Gamma$ — 3, Proposition 4.2(1)

3. Let $\psi \notin \Gamma$ and $\neg\chi \notin \Gamma$. We have that:

1. $\Gamma \vdash (\psi \vee (\psi \to \chi)) \vee (\chi \to \bot)$ — (Av)
2. $\Gamma \vdash (\psi \vee (\psi \to \chi)) \vee \neg\chi$ — 1, Definition 2.4
3. $(\psi \vee (\psi \to \chi)) \vee \neg\chi \in \Gamma$ — 2, Proposition 4.2(1)
4. $(\psi \vee (\psi \to \chi)) \in \Gamma$ or $\neg\chi \in \Gamma$ — 3, Proposition 4.2(3)
5. $(\psi \vee (\psi \to \chi)) \in \Gamma$ — 4, $\neg\chi \notin \Gamma$
6. $\psi \in \Gamma$ or $\psi \to \chi \in \Gamma$ — 5, Proposition 4.2(3)
7. $\psi \to \chi \in \Gamma$ — 6, $\psi \notin \Gamma$ □

In the standard way we get:

LEMMA 4.1 (Lindenbaum). *Every consistent set of formulas Γ can be extended to a maximal consistent set Γ^*.*

So, we obtain:

PROPOSITION 4.4. *Γ^* is complete.*

PROOF. Suppose that there are formulas φ and ψ such that $\Gamma^* \nvdash \varphi \to \psi$ and $\Gamma^* \nvdash \psi \to \varphi$. Since Γ^* is maximal, $\Gamma^* \cup \{\varphi \to \psi\} \vdash \bot$ and $\Gamma^* \cup \{\psi \to \varphi\} \vdash \bot$. Therefore:

1. $\Gamma^* \vdash \mathbf{\Delta}(\varphi \to \psi) \to \bot$ — DT
2. $\Gamma^* \vdash \mathbf{\Delta}(\psi \to \varphi) \to \bot$ — DT
3. $\Gamma^* \vdash (\mathbf{\Delta}(\psi \to \varphi) \to \bot) \to ((\mathbf{\Delta}(\varphi \to \psi) \vee \mathbf{\Delta}(\psi \to \varphi)) \to \bot)$ — (8), 1, MP
4. $\Gamma^* \vdash (\mathbf{\Delta}(\varphi \to \psi) \vee \mathbf{\Delta}(\psi \to \varphi)) \to \bot$ — 3, 2, MP
5. $\Gamma^* \vdash \bot$ — 4, ($\mathbf{\Delta}$5), MP, contradicting the consistency of Γ^*. □

DEFINITION 4.1. For any maximal set Γ^* of formulas and propositional letter p, let

$$\mathfrak{v}(p) := \begin{cases} 1, & \text{if } p \in \Gamma^*; \\ 0, & \text{if } \neg p \in \Gamma^*; \\ 0.5, & \text{otherwise.} \end{cases}$$

LEMMA 4.2 (Truth lemma). *For every formula $\varphi \in \mathrm{Fm}$, we have:*

$$\mathfrak{v}(\varphi) = 1 \textit{ iff } \varphi \in \Gamma^*$$

PROOF. We prove both directions simultaneously, by induction on φ.

Base step: $\varphi := p$. Immediate by Lemma 4.2. $\varphi := \mathbf{T}$. $[[\mathbf{T}]]^{\mathcal{M}^{\Gamma^*},g} = 0.5$ by the definition of satisfaction. On the other hand, $\mathbf{T}, \neg\mathbf{T} \notin \Gamma^*$, since Γ^* is consistent (Proposition 4.1). $\varphi := \bot$. $[[\bot]]^{\mathcal{M}^{\Gamma^*},g} = 0$ by the definition of satisfaction. Because Γ^* is consistent, we also know that $\bot \notin \Gamma^*$, as required.

Inductive step: $\varphi := \psi \to \chi$. By the definition of satisfaction, if $[[\psi \to \chi]]^{\mathcal{M}^{\Gamma^*},g} = 1$, we shall distinguish three cases: $\mathfrak{v}(\psi) = 0$ iff $\neg\psi \in \Gamma^*$, (by IH) iff $\psi \to \chi \in \Gamma^*$ (by Proposition 4.3(1)).

$\mathfrak{v}(\chi) = 1$ iff $\chi \in \Gamma^*$ (by IH) iff $\psi \to \chi \in \Gamma^*$ (by Proposition 4.3(2).

$\mathfrak{v}(\psi) = 0.5$ and $\mathfrak{v}(\chi) = 0.5$ iff $\psi, \neg\psi \notin \Gamma^*$ and $\chi, \neg\chi \notin \Gamma^*$ (by IH) iff $\psi \to \chi \in \Gamma^*$ (by Proposition 4.3(3)). □

Theorem 4.1 (Completeness). *If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.*

Proof. Assume that $\Gamma \nvdash \varphi$. Obviously, Γ has to be consistent (otherwise $\Gamma \vdash \bot$ and, hence, $\Gamma \vdash \varphi$ by Proposition 3.1(4) and MP). We shall prove that $\Gamma \cup \{\neg\mathbf{\Delta}\varphi\}$ is consistent:

1.	$\Gamma \cup \{\neg\mathbf{\Delta}\varphi\} \vdash \bot$	hypothesis
2.	$\Gamma \vdash \mathbf{\Delta}(\neg\mathbf{\Delta}\varphi) \to \bot$	1, DT
3.	$\Gamma \vdash \neg\mathbf{\Delta}(\neg\mathbf{\Delta}\varphi)$	2, Definition 2.4
4.	$\Gamma \vdash \mathbf{\Delta}\varphi$	3, ($\mathbf{\Delta}$6), MP
5.	$\Gamma \vdash \varphi$	4, ($\mathbf{\Delta}$1), MP, contradicting the starting assumption

Thus, by Lemma 4.1, there is a Γ^* such that $\Gamma \subseteq \Gamma^*$, Γ^* is maximal consistent and contains $\Gamma \cup \{\neg\mathbf{\Delta}\varphi\}$. Take the valuation $\mathfrak{v}$ for Γ^* as it is defined in Definition 4.1. It follows (by Lemma 4.2) that $\mathfrak{v}(\gamma) = 1$ for all $\gamma \in \Gamma$ and $\mathfrak{v}(\neg\mathbf{\Delta}\varphi) = 1$. Hence, $\mathfrak{v}(\mathbf{\Delta}\varphi) = 0$, so $\mathfrak{v}(\varphi) \neq 1$ by Definitions 2.1 and 2.6. In consequence, by Definition 2.3, $\Gamma \not\models \varphi$. □

5. Conclusion

In the present paper, we have addressed the issue of the completeness of a particular non-classical formalism: the truth-functionally complete Ł3. Although Radzki (2017) showed that Słupecki's system was semantically incomplete, we have seen that a Completeness Theorem can be obtained for a different set of axioms. To do so, we have replaced Słupecki's 1-ary connective by a 0-ary one, included the Baaz delta operation and proved a weaker version of the Deduction Theorem. Henkin's method has been successfully adapted to this 3-valued logic.

The extension of this logic to a first-order setting is a matter of future work. We think that the intermediate truth value, 0.5, can be attributed to those statements about non-existing objects, like "Pegasus has a white hind leg". The idea is to develop a formal system for a *neutral free logic*, based upon a truth-functionally complete Ł3. Once the completeness of ŁS has been proved, solving Radzki's problem, we are one step closer to this goal.

Acknowledgements. I am grateful to David Fuenmayor for pointing me toward the incompleteness of Słupecki's axioms and for encouraging me to seek a solution. My thanks also go to the audience at the Formal Methods in Philosophy V conference for their insightful comments, especially Tommaso Flaminio and Umberto Rivieccio. Mara Manzano and Manuel Martins read an earlier draft of this paper, and their thoughtful suggestions were invaluable. I am equally indebted to the reviewers of LLP for their careful and constructive feedback.

Funding. This research was supported by projects PR17/24-31887 and PID2022-142378NB-I00 funded by UCM/CAM and MICIU/AEI/10.13039/501100011033, respectively, and by ERDF, EU.

References

Avron, A., 1991, "Natural 3-valued logics: Characterization and proof theory", *The Journal of Symbolic Logic*, 56(1): 276–294. DOI: 10.2307/2274919

Baaz, M., 1996, "Infinite-valued Gödel logics with 0-1-projections and relativizations", pages 23–33 in P. Hájek (ed.), *GÖDEL'96 – Logical Foundations of Mathematics, Computer Science and Physics*, Lecture Notes in Logic, 6, Springer Verlag.

Hájek, P., 1998, *Metamathematics of Fuzzy Logic*, Vol. 4, Springer Science & Business Media. DOI: 10.1007/978-94-011-5300-3

Malinowski, G., 2007, "Many-valued logic and its philosophy", pages 13–94 in D. M. Gabbay and J. Woods (eds.) *Handbook of the History of Logic*, Vol. 8, North-Holland, Amsterdam. DOI: 10.1016/S1874-5857(07)80004-5

Novak, V., 2005, "On fuzzy type theory", *Fuzzy Sets and Systems*, 149(2): 235–273. DOI: 10.1016/j.fss.2004.03.027

Radzki, M., 2017, "On axiom systems of Słupecki for the functionally complete three-valued logic", *Axiomathes*, 27(4): 403–415. DOI: 10.1007/s10516-016-9319-x

Słupecki, J., 1946, "The complete three-valued propositional calculus", *Annales Universitatis Mariae Curie-Skłodowska*, 1: 193–209.

Słupecki, J., 1967, "The full three-valued propositional calculus", pages 335–337 in: S. McCall (ed.), *Polish Logic 1920–1939*, Clarendon Press, Oxford.

VÍCTOR ARANDA
Department of Logic and Theoretical Philosophy
Complutense University of Madrid
Madrid, Spain
vicarand@ucm.es
https://orcid.org/0000-0003-3702-2003