Logic and Logical Philosophy
Online First Articles (2025)
DOI: 10.12775/LLP.2025.020

Andrzej Wiśniewski

Doxastic Arrogance Paradox

Abstract. The Doxastic Arrogance Paradox, `DAP`, states that the belief that a proposition is an item of knowledge implies that the proposition holds/is true. Some possible sources of `DAP`, different from the negative introspection principle for knowledge, are pointed out. The emplacement of `DAP` among related paradoxes of epistemic-doxastic logics is characterized. Finally, some profits of the identification of sources of `DAP` for the philosophical analysis of knowledge and belief are pointed out.

Keywords: knowledge; belief; epistemic-doxastic logics; paradoxes

1. Introduction

The philosophical distinction between knowledge and belief is almost as old as philosophy itself. It had already manifested in Parmenides and then Plato in their contrasting *episteme* with *doxa*. The core of the distinction between knowledge and belief lies in the idea that what is known must be true, while what is believed need not be: both true and false propositions can be subjects of beliefs.

Contemporary logics of knowledge and belief, often called epistemic-doxastic logics, try to characterize knowledge, belief, and their mutual dependencies. However, no paradigmatic, commonly accepted epistemic-doxastic logic has been elaborated so far. The existing proposals suffer from leading to paradoxical statements. These are not contradictions by themselves. To speak generally, their paradoxicality lies in ascribing to beliefs properties one usually would not be willing to ascribe, in particular features supposed to be exhibited only by knowledge. When a logic of knowledge and belief is interpreted as speaking about propositional

Received June 11, 2025. *Accepted* November 11, 2025. *Published online* November 16, 2025

attitudes of cognitive agents, the paradoxical statements ascribe to such agents powers that no human cognitive agent possesses.

In this paper I concentrate upon the formula:

$$\mathsf{BK}p \to p \tag{DAP}$$

The letters B and K stand for the belief operator and the knowledge operator, respectively, while p is a propositional variable. The acronym DAP abbreviates 'Doxastic Arrogance Paradox.' The reasons for choosing this name will be explained below. A moment's reflection reveals that formula DAP is paradoxical in the sense described above. On the other hand, it can be shown that DAP is a theorem of some epistemic-doxastic logics. This is not a big news, since it is well-known that epistemic-doxastic logics which validate the so-called negative introspection principle for knowledge, suffer, given that some additional conditions are met, from the so-called Paradox of The Perfect Believer, and this, together with the factivity of knowledge principle, leads to DAP (cf. sections 2.1.1 and 2.3 below). What is a minor news, however, is that DAP can be obtained without relying on the negative introspection principle and the factivity of knowledge principle, from assumptions that are weaker, logically an philosophically, than these principles. This will be shown below, as well as some other 'nonstandard' sources of DAP.

One may wonder if observations of this kind are worth a separate paper. My answer to this is two-fold. As for logic, identifying multiple premises which lead to a paradoxical statement is telling, since in a situation in which no paradigmatic account has been elaborated (and this is the current status of epistemic-doxastic logic), it shows where the dangers hide. As for philosophy, DAP can be viewed as a negative constraint for an adequate account of knowledge and belief, which makes search for multiple sources of DAP a worthwhile and possibly profitable enterprise.

1.1. General Assumptions, Terminology, and Notation

By a *propositional epistemic-doxastic logic* I will mean here a propositional logic in the vocabulary of which the belief operator, B, and the knowledge operator, K, occur.[1] For convenience, I assume that all the

[1] This notion may be understood more broadly, but here I am interested only in epistemic-doxastic logics with single operators of knowledge and belief. In particular, multiagent epistemic-doxastic logics are not addressed here.

Formula	Notional reading	Attitudinal reading
$\mathsf{K}\phi$	It is known that ϕ (in short: ϕ is known)	An agent knows that ϕ
$\mathsf{B}\phi$	It is believed that ϕ (in short: ϕ is believed)	An agent believes that ϕ

Table 1. Notional vs. attitudinal readings of $\mathsf{K}\phi$ and $\mathsf{B}\phi$

connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), and $\equiv$ (equivalence), are in the vocabulary. Well-formed formulas (*formulas* for short) are defined recursively in the standard way, i.e.: (i) each propositional variable is a formula, (ii) if ϕ is a formula, then $\neg\phi$ is a formula, (iii) if ϕ is a formula, then $\mathsf{B}\phi$ and $\mathsf{K}\phi$ are formulas, (iv) if ϕ and ψ are formulas, then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\phi \equiv \psi)$ are all formulas, and (v) nothing else is a formula. I will be using the letters p, q, r, s, possibly with subscripts, for propositional variables. The Greek letters ϕ, ψ, γ, again possibly with subscripts, are metalanguage variables for formulas.

A formula of the form $\mathsf{K}\phi$ can be read either as 'it is known that ϕ' (in short: ϕ is known), or as 'an agent knows that ϕ.' Observe that under the first reading, formula $\mathsf{K}\phi$ speaks about *knowledge* and says that ϕ is an item of knowledge. When the second reading comes into play, formula $\mathsf{K}\phi$ speaks about the agent's *attitude* towards ϕ, identifying this attitude as knowledge-that. A formula of the form $\mathsf{B}\phi$ can be read and analysed along similar lines. Table 1 displays the two kinds of readings. In what follows I will be treating expressions falling under the schemata: 'It is known that ϕ' and 'ϕ is an item of knowledge' as synonymous, and analogously for: 'It is believed that ϕ' and 'ϕ is a subject of belief.'

In this paper I restrict myself to propositional epistemic-doxastic logics based on Classical Propositional Logic (hereafter: CPL). Any such logic has *CPL-axioms*, that is, axioms that result from theorems of CPL by substituting formulas of the language of the logic for propositional variables occurring in these theorems. Moreover, each logic considered is assumed to be closed under *Modus Ponens* and Uniform Substitution. As for the latter, no restriction on formulas which can be substituted for propositional variables is imposed (thus formulas in which the operators K and B occur are allowed to be substituted for propositional variables). These assumptions will be accepted tacitly in all the considerations to follow. Moreover, for convenience it is assumed that the analysed logics are Hilbert-style axiomatic systems or can be presented in this format. By *theorems* I will mean axioms and all the formulas which are provable from axioms, although they are not axioms themselves.

The *doxastic component* of an epistemic-doxastic logic comprises all the theorems of the logic, in which the knowledge operator does not occur. The concept of *epistemic component* is defined analogously.[2] An epistemic-doxastic logic contains also axioms/theorems being the so-called 'linking formulas,' which claim how knowledge and belief are intertwined. Here are examples of much often used formulas of this kind:

$$\mathsf{K}p \to \mathsf{B}p \tag{KB1}$$

$$\mathsf{B}p \to \mathsf{KB}p \tag{KB2}$$

$$\mathsf{B}p \to \mathsf{BK}p \tag{KB3}$$

Formula KB1 makes propositional knowledge a species of belief: whatever is known, is also believed. In terms of propositional attitudes of cognitive agents: if an agent knows that something is the case, she also believes that it is the case. Formula KB2 ensures that a doxastic agent exhibits a kind of epistemic introspection concerning her beliefs. Formula KB3, in turn, claims that an agent who believes that something holds, also believes that she knows it. When we interpret these formulas as speaking about interrelations between concepts of knowledge and belief, we get something like: if a proposition is a subject of belief, then the proposition saying this is an item of knowledge (formula KB2), and the proposition saying that it is an item of knowledge, is a subject of belief (formula KB3).

The above formulas are far from being unanimously accepted by all researchers working in epistemic-doxastic logics; there are theorists who accept only some of them and reject the other(s). There is no room for going into details here (an interesting reader may consult, e.g., Gochet and Gribomont, 2006, or Aucher, 2014).

2. Doxastic Arrogance

Let us consider the formula:

$$\mathsf{BK}p \to p \tag{DAP}$$

[2] Observe that the mere CPL is included both in the epistemic component and in the doxastic component. This is intended.

Depending on the readings of the operators B and K involved, we get:

- *If it is believed that it is known that p, then p.*

or

- *If an agent believes that she knows that p, then p.*

When we add to this the colloquial version of the Tarski biconditional:[3]

- *p is true if, and only if p*

we arrive at consequences which are paradoxical from the intuitive standpoint, namely:

- *If it is believed that it is known that p, then p is true.*
- *If an agent believes that she knows that p, then p is true.*

Thus, if it is believed that a proposition constitutes an item of knowledge, then the proposition holds/is true. Or, speaking in terms of agent's attitudes: an agent's belief that she knows that p yields that p holds/is true. However, beliefs are not infallible and so are beliefs about being items of knowledge or having knowledge. A proposition which is not an item of knowledge may be believed to constitute an item of knowledge. An agent may believe to know something without, in fact, knowing it. Thus it can happen that a non-true proposition is believed to constitute an item of knowledge or is believed by an agent to be known. `DAP` expels these possibilities, which is paradoxical in the sense of contravening basic intuitions. To see beliefs of having knowledge as ultimate warrants of truth is, after all, a kind of arrogance. This is why the term 'Doxastic Arrogance Paradox' seems appropriate here, and why the acronym `DAP` has been chosen.

Remark 1. `DAP` is not an antinomy, but nevertheless is paradoxical from the intuitive point of view regardless of which reading, notional or attitudinal, of the expression $\mathsf{BK}p$ is chosen. An analogous effect shows up when we allow for the first-person readings of B and K. For example, consider the following substitution instance of `DAP`:

(†) *If I believe that I know that every natural number greater than 2 is the sum of two prime numbers, then every natural number greater than 2 is the sum of two prime numbers.*

and the argument based on it:

[3] Tarski himself would use the name of p at the left.

(‡) *I believe that I know that every natural number greater than 2 is the sum of two prime numbers. Therefore every natural number greater than 2 is the sum of two prime numbers.*

Although (†) is grounded in DAP, it is nonsensical to regard (‡) as a correct proof of the Goldbach Conjecture.

2.1. The Doxastic Arrogance Paradox versus Related Paradoxes of Epistemic-Doxastic Logics

The Doxastic Arrogance Paradox is akin to two well-known paradoxes of epistemic-doxastic logics: the *Paradox of The Perfect Believer* (PPB)[4] and the *Paradox of Infallibility* (PIF).[5] These are expressed by the following formulas:

$$\mathsf{BK}p \rightarrow \mathsf{K}p \qquad (\texttt{PPB})$$

$$\mathsf{B}p \rightarrow p \qquad (\texttt{PIF})$$

By and large, formula PPB claims that one cannot believe to know that something is the case without knowing that it is the case. The claim of formula PIF is: one cannot believe a false proposition. When we switch from attitudinal reading to notional reading of K and B, we get: if it is believed that a proposition is an item of knowledge, then the proposition is an item of knowledge (PPB), and if a proposition is believed, then it is true (PIF).[6] The claim of DAP is different: if it is believed that a proposition is an item of knowledge, then the proposition is true.

Looking from a purely formal point of view, one can obtain DAP from PPB or from PIF on the basis of some additional assumptions. But one can also get PPB and PIF from DAP, again by means of some assumptions. Let me analyse this in detail.

2.1.1. From PPB or PIF to DAP

Clearly, DAP is an immediate CPL-consequence of PPB and the following formula:

$$\mathsf{K}p \rightarrow p \qquad (\mathbf{T}_{\mathsf{K}})$$

[4] The authors of (Gochet and Gribomont, 2006) attribute the first proof of PPB to Voorbraak (1993).

[5] First observed by Timothy Williamson (2001).

[6] PPB and PIF are interrelated, although not equivalent. PIF is an immediate consequence of PPB, formula $\mathsf{K}p \rightarrow p$, and the linking formula KB3. On the other hand, one gets PPB from PIF by substituting $\mathsf{K}p$ for p.

which ensures the factivity of the knowledge operator and thus reflects the idea of truthfulness of knowledge.[7]

In order to get DAP from PIF, in turn, it suffices to substitute $\mathsf{K}p$ for p in PIF, and then to apply formula $\mathbf{T}_{\mathsf{K}}$.

Thus the Doxastic Arrogance Paradox can be viewed as just a corollary to the Paradox of The Perfect Believer, or to the Paradox of Infallibility. Maybe for this reason it has not attracted attention of logicians and epistemologists.

2.1.2. From DAP to PPB and PIF

There are many paths, by which one can arrive at PPB on the basis of DAP. Here are examples.

First Path. In order to get PPB from DAP, it suffices to assume that the linking formula KB2 is a theorem of the epistemic-doxastic logic in question, and that the logic is closed under the monotony rule RM_{K} for the knowledge operator K:

$$\phi \to \psi \,/\, \mathsf{K}\phi \to \mathsf{K}\psi \qquad (\mathrm{RM}_{\mathsf{K}})$$

Here is a derivation of PPB from DAP based on the above assumptions:[8]

a.1.	$\mathsf{BK}p \to p$	(DAP)
a.2.	$\mathsf{KBK}p \to \mathsf{K}p$	(a.1 RM_{K})
a.3.	$\mathsf{B}p \to \mathsf{KB}p$	(KB2)
a.4.	$\mathsf{BK}p \to \mathsf{KBK}p$	(a.3 $p/\mathsf{K}p$)
(PPB)	$\mathsf{BK}p \to \mathsf{K}p$	(a.4, a.2 CPL)

[7] Formula ($\mathbf{T}_{\mathsf{K}}$) is an epistemic version of the alethic modal formula $\Box p \to p$, usually labelled by $\mathbf{T}$. The subscript K in $\mathbf{T}_{\mathsf{K}}$ indicates that $\Box$ is understood as K (putting B instead of K would mean that $\Box$ is replaced with B). I will be applying this convention when transferring from alethic formulas having common names to their epistemic/doxastic counterparts.

[8] The notational conventions adopted below seem transparent, but, just in case: the left column comprises numbers of lines of the derivation, the middle column is the derivation itself, while the items of the right column describe where does the formula of a line come from. When 'CPL' occurs in the right column, it means that CPL-based inference rule (primitive or derivable) has been applied. I do not name the CPL-rules used; the transformations are/will be so simple that the respective rules can be easily extracted from the context. When Uniform Substitution had been applied, the item of the third column informs what was substituted for what.

Second Path. Instead of formula KB2, one can also use the linking formula KB3. In such a setting, rule RM_{K} is not needed, viz.:

b.1.	$\mathsf{BK}p \to p$	(DAP)
b.2.	$\mathsf{B}p \to \mathsf{BK}p$	(KB3)
b.3.	$\mathsf{B}p \to p$	(b.2, b.1 CPL)
(PPB)	$\mathsf{BK}p \to \mathsf{K}p$	(b.3 $p/\mathsf{K}p$)

Third Path. One can get rid of the linking formulas KB2 and KB3, and instead use the so called 'positive introspection principle' for knowledge, that is, the formula:

$$\mathsf{K}p \to \mathsf{KK}p \tag{$\mathbf{4}_{\mathsf{K}}$}$$

Assuming that the epistemic-doxastic logic in question is closed under the monotony rule RM_{B} for the belief operator B:

$$\phi \to \psi \,/\, \mathsf{B}\phi \to \mathsf{B}\psi \tag{RM_{B}}$$

we arrive at PPB from DAP as follows:

c.1.	$\mathsf{BK}p \to p$	(DAP)
c.2.	$\mathsf{BKK}p \to \mathsf{K}p$	(c.1 $p/\mathsf{K}p$)
c.3.	$\mathsf{K}p \to \mathsf{KK}p$	($\mathbf{4}_{\mathsf{K}}$)
c.4.	$\mathsf{BK}p \to \mathsf{BKK}p$	(c.3 RM_{B})
(PPB)	$\mathsf{BK}p \to \mathsf{K}p$	(c.4, c.2 CPL)

Needless to say, PIF is an immediate CPL-consequence of DAP *and* the linking formula KB3.

2.2. A Derivation of the Doxastic Arrogance Paradox Relying Neither on the Paradox of The Perfect Believer nor on the Paradox of Infallibility

Although DAP, PPB and PIF are interrelated, among others, in the ways described above, it is worth to note that DAP occupies, in a sense, a distinguished position among them. This is due to the fact that one can arrive at DAP without making use of PPB or of PIF, on the basis of assumptions that are weaker, logically and philosophically, than those which, according to the received view, are sources of these paradoxes.[9]

[9] For the received view on the origin of PPB, see footnote 10 below.

Consider the following formulas:

$$\mathsf{B}p \to \neg\mathsf{B}\neg p \qquad (\mathbf{D}_{\mathsf{B}})$$

$$\neg p \to \mathsf{K}\neg\mathsf{K}p \qquad (\mathbf{B}^*_{\mathsf{K}})$$

Formula $\mathbf{D}_{\mathsf{B}}$ is a doxastic version of the alethic formula **D**, i.e. $\Box p \to \neg\Box\neg p$. It ascribes a kind of consistency to agent's beliefs. Formula $\mathbf{B}^*_{\mathsf{K}}$ can be read as follows: if it is not the case that p, then it is known that it is not known that p.

One can easily derive DAP from formulas KB1, $\mathbf{B}^*_{\mathsf{K}}$ and $\mathbf{D}_{\mathsf{B}}$, by using CPL-means only. Here is a derivation:

d.1.	$\mathsf{K}p \to \mathsf{B}p$	(KB1)
d.2.	$\mathsf{K}\neg\mathsf{K}p \to \mathsf{B}\neg\mathsf{K}p$	(d.1 $p/\neg\mathsf{K}p$)
d.3.	$\neg p \to \mathsf{K}\neg\mathsf{K}p$	($\mathbf{B}^*_{\mathsf{K}}$)
d.4.	$\neg p \to \mathsf{B}\neg\mathsf{K}p$	(d.3, d.2 CPL)
d.5.	$\neg\mathsf{B}\neg\mathsf{K}p \to p$	(d.4 CPL)
d.6.	$\mathsf{B}p \to \neg\mathsf{B}\neg p$	($\mathbf{D}_{\mathsf{B}}$)
d.7.	$\mathsf{B}\mathsf{K}p \to \neg\mathsf{B}\neg\mathsf{K}p$	(d.6 $p/\mathsf{K}p$)
(DAP)	$\mathsf{B}\mathsf{K}p \to p$	(d.7, d.5 CPL)

The above derivation can then be extended in any of the ways presented in section 2.1.2 to obtain a derivation of PPB and/or of PIF. Observe that the 'factivity of knowledge' formula $\mathbf{T}_{\mathsf{K}}$ plays no role in the above reasoning leading to DAP, and then in the consecutive reasoning(s) from DAP to PPB and PIF. Thus none of: DAP, PPB, PIF, is rooted in the factivity of knowledge requirement.

2.3. The Doxastic Arrogance Paradox and the Negative Introspection Principle

A remarkable property of the derivation of DAP presented in the previous section is that it does employ the formula:

$$\neg\mathsf{K}p \to \mathsf{K}\neg\mathsf{K}p \qquad (\mathbf{5}_{\mathsf{K}})$$

as a premiss. This formula is an epistemic variant of the alethic modal formula **5**, i.e. $\neg\Box p \to \Box\neg\Box p$. Formula $\mathbf{5}_{\mathsf{K}}$ ascribes to an agent's knowledge a property called 'negative introspection': if an agent does not know that p, then she knows that she does not know that p.

Why is this important? It is well-known (cf., e.g., Gochet and Gribomont, 2006, p. 114) that formula PPB is provable in any epistemic-doxastic logic based on CPL which has formulas KB1, $\mathbf{D}_{\mathsf{B}}$, and $\mathbf{5}_{\mathsf{K}}$ as theorems.[10] Since DAP is an immediate CPL-consequence of PPB and the factivity formula $\mathbf{T}_{\mathsf{K}}$, at first sight it may seem that the Doxastic Arrogance Paradox is grounded, *inter alia*, in the negative introspection principle, and thus once the principle is 'blocked' or rejected, the paradox becomes an artifact with no real impact. However, this is wrong. The derivation of DAP presented in section 2.2 does not rely on formulas $\mathbf{T}_{\mathsf{K}}$ and $\mathbf{5}_{\mathsf{K}}$. Moreover, the derivations of PPB and PIF from DAP, described in section 2.1.2, do not rely on these formulas either. So the Doxastic Arrogance Paradox, DAP, constitutes a challenge even if we get rid of negative introspection. Similarly for PPB and PIF, since, as it has been pointed out in section 2.1.2, one does not need formulas $\mathbf{T}_{\mathsf{K}}$ and $\mathbf{5}_{\mathsf{K}}$ in order to derive PPB or PIF from DAP.

2.4. The Doxastic Arrogance Paradox versus the Collapse of Knowledge and Belief

One of the most striking paradoxical consequences of the Paradox of the Perfect Believer, PPB, is the collapse of knowledge and belief. Clearly, the formula which expresses the collapse, that is:

$$\mathsf{B}p \to \mathsf{K}p \tag{1}$$

is an immediate CPL-consequence of PPB and the linking formula KB3.

The Doxastic Arrogance Paradox, DAP, is equally poisonous here, viz.:

---

[10] Here is a 'textbook' derivation of PPB based on the above assumptions:

e.1.	$\mathsf{K}p \to \mathsf{B}p$	(KB1)
e.2.	$\mathsf{K}\neg\mathsf{K}p \to \mathsf{B}\neg\mathsf{K}\mathrm{p}$	(e.1 $p/\neg\mathsf{K}p$)
e.3.	$\mathsf{B}p \to \neg\mathsf{B}\neg p$	($\mathbf{D}_{\mathsf{B}}$)
e.4.	$\mathsf{B}\neg p \to \neg\mathsf{B}p$	(e.3 CPL)
e.5.	$\mathsf{B}\neg\mathsf{K}p \to \neg\mathsf{B}\mathsf{K}p$	(e.4 $p/\mathsf{K}p$)
e.6.	$\neg\mathsf{K}p \to \mathsf{K}\neg\mathsf{K}p$	($\mathbf{5}_{\mathsf{K}}$)
e.7.	$\neg\mathsf{K}p \to \mathsf{B}\neg\mathsf{K}p$	(e.6, e.2 CPL)
e.8.	$\neg\mathsf{K}p \to \neg\mathsf{B}\mathsf{K}p$	(e.7, e.5 CPL)
(PPB)	$\mathsf{B}\mathsf{K}p \to \mathsf{K}p$	(e.8 CPL)

Note that the above derivation, similarly as the derivation of DAP presented in section 2.2, uses formulas KB1 and $\mathbf{D}_{\mathsf{B}}$ as premises.

f.1.	$\mathsf{BK}p \to p$	(DAP)
f.2.	$\mathsf{B}p \to \mathsf{BK}p$	(KB3)
f.3.	$\mathsf{B}p \to p$	(f.2, f.1 CPL)
f.4.	$\mathsf{BK}p \to \mathsf{K}p$	(f.3 $p/\mathsf{K}p$)
(1)	$\mathsf{B}p \to \mathsf{K}p$	(f.2, f.4 CPL)

When we combine formula (1) and the linking formula KB1, we get:

$$\mathsf{K}p \equiv \mathsf{B}p \qquad (\mathtt{IKB})$$

that is, the indistinguishability of knowledge and belief.

According to the remarkable result of van der Hoek (1993): (a) any epistemic-doxastic logic which has formulas $\mathbf{D}_{\mathsf{B}}$, $\mathbf{5}_{\mathsf{K}}$, KB1, and KB3 as axioms, has the formula IKB as a theorem, but (b) for each proper subset of $\{\mathbf{D}_{\mathsf{B}}, \mathbf{5}_{\mathsf{K}}, \mathtt{KB1}, \mathtt{KB3}\}$ models can be build which invalidate the formula IKB. Yet, as we have seen, once formulas $\mathbf{D}_{\mathsf{B}}$, $\mathbf{B}^*_{\mathsf{K}}$, KB1 and KB3 are theorems of an epistemic-doxastic logic based on CPL, so is DAP, and DAP together with the linking formula KB3 imply the formula (1). Thus the indistinguishability of knowledge and belief shows up *also* in any epistemic-doxastic logic based on CPL whose set of theorems includes the set $\{\mathbf{D}_{\mathsf{B}}, \mathbf{B}^*_{\mathsf{K}}, \mathtt{KB1}, \mathtt{KB3}\}$. It is an open question whether the analogue of the second part of the van der Hoek result holds for the above set as well.

Remark 2. Note that one needs the linking formula KB3 to derive from DAP, in the way presented in this section, the 'collapse' formula (1) and the indistinguishability statement IKB. Moreover, formula KB3 enables an easy transition from PPB to PIF (see section 2.1.2), and from DAP to PIF. Formula KB3, however, plays no role in the derivation of DAP based on formula $\mathbf{B}^*_{\mathsf{K}}$ described in section 2.2, and in the 'textbook' derivation of PPB (see footnote 10) relying on the negative introspection principle $\mathbf{5}_{\mathsf{K}}$. Thus once the linking formula KB3 is blocked, both PIF and IKB can be regarded as temporarily blocked, but claiming that PPB and DAP are also blocked that way would be wrong.

2.5. The Status of $\mathbf{B}^*_{\mathsf{K}}$

Formula $\mathbf{B}^*_{\mathsf{K}}$ plays a key role in the derivation of DAP presented in section 2.2. So let me pay some attention to the status of this formula.

2.5.1. Formula $\mathbf{B}^*_{\mathsf{K}}$ versus the Brouwerian Axiom as well as $\mathbf{T}_{\mathsf{K}}$ and $\mathbf{5}_{\mathsf{K}}$

Looking from a purely formal point of view, one can derive from formula $\mathbf{B}^*_{\mathsf{K}}$ the following epistemic version of the Brouwerian axiom **B**, i.e. of the formula $p \rightarrow \Box\neg\Box\neg p$:

$$p \rightarrow \mathsf{K}\neg\mathsf{K}\neg p \qquad (\mathbf{B}_{\mathsf{K}})$$

by using CPL-means only, viz.:

g.1. $\neg p \rightarrow \mathsf{K}\neg\mathsf{K}p$ $(\mathbf{B}^*_{\mathsf{K}})$
g.2. $\neg\neg p \rightarrow \mathsf{K}\neg\mathsf{K}\neg p$ (g.1 $p/\neg p$)
g.3. $p \rightarrow \mathsf{K}\neg\mathsf{K}\neg p$ (g.2 CPL)

A derivation in the other direction requires applying inference rules operating with modalities, for instance rule RM_{K} or the following rule of extensionality:

$$\phi \leftrightarrow \psi \,/\, \mathsf{K}\phi \leftrightarrow \mathsf{K}\psi \qquad (\mathrm{RE}_{\mathsf{K}})$$

Even a weaker rule will do, however. For instance, the following rule of replacement:

$$\gamma[\mathsf{K}\neg\neg\phi] \,/\, \gamma[\mathsf{K}\phi] \qquad (\mathrm{R}_{\neg\neg})$$

which allows for the replacement of an occurrence of a subformula $\mathsf{K}\neg\neg\phi$ of formula γ with the formula $\mathsf{K}\phi$. Here is the essential part of the derivation:

g.4. $p \rightarrow \mathsf{K}\neg\mathsf{K}\neg p$ (g.3)
g.5. $\neg p \rightarrow \mathsf{K}\neg\mathsf{K}\neg\neg p$ (g.4 $p/\neg p$)
g.6. $\neg p \rightarrow \mathsf{K}\neg\mathsf{K}p$

The transition from g.5 to g.6 relies on RE_{K} (in this case one needs additional intermediate lines) or, alternatively, on $\mathrm{R}_{\neg\neg}$.

By the Brouwerian axiom we mean the formula $\neg\Box\neg\Box p \rightarrow p$. Its epistemic version will be:

$$\neg\mathsf{K}\neg\mathsf{K}p \rightarrow p \qquad (\mathbf{B}'_{\mathsf{K}})$$

Clearly, formulas $\mathbf{B}'_{\mathsf{K}}$ and $\mathbf{B}^*_{\mathsf{K}}$ are mutually CPL-derivable.

If the epistemic component of an epistemic-doxastic logic in question is a normal modal propositional logic[11] based on CPL, formulas $\mathbf{B}^*_{\mathsf{K}}$, $\mathbf{B}_{\mathsf{K}}$, and $\mathbf{B}'_{\mathsf{K}}$ are proof-theoretically equivalent.

Clearly, none of the above formulas yields the negative introspection principle $\mathbf{5}_{\mathsf{K}}$. If this were the case, the normal modal propositional logic

[11] That is, formula $\mathbf{K}_{\mathsf{K}}$. i.e. $\mathsf{K}(p \rightarrow q) \rightarrow (\mathsf{K}p \rightarrow \mathsf{K}q)$, is a theorem of the logic, and the logic is closed under the rule RN_{K} of necessitation for the epistemic operator K.

KB would equal the normal modal propositional logic **K5**. Yet, this does not hold. Similarly, the sole formula $\mathbf{5}_{\mathsf{K}}$ does not imply the formula $\mathbf{B}_{\mathsf{K}}$.[12] Moreover, formula $\mathbf{B}^{*}_{\mathsf{K}}$ alone does not yield the truthfulness formula $\mathbf{T}_{\mathsf{K}}$. If this were the case, the normal modal propositional logics **KB** and **KT** would coincide. They do not. For the same reason formula $\mathbf{T}_{\mathsf{K}}$ alone does not imply the formula $\mathbf{B}^{*}_{\mathsf{K}}$.

2.5.2. Abandoning the Formula $\mathbf{B}^{*}_{\mathsf{K}}$

Once it becomes known that a 'paradoxical' formula is CPL-derivable from a given set of premises, a logician who considers the formula intuitively unacceptable faces two options: (1) to give up a premise, that is, to ensure that the premise is not a theorem of the logic in question, or (2) to depart from Classical Logic. For example, since the Paradox of The Perfect Believer, `PPB`, is CPL-derivable from formulas `KB1`, $\mathbf{D}_{\mathsf{B}}$, and $\mathbf{5}_{\mathsf{K}}$ (cf. footnote 10), at least one of these formulas must be given up in order to block `PPB` when CPL-means (including the possibility of substituting epistemic and/or doxastic formulas) are supposed to be retained. One can also block `PPB` by imposing restrictions on Uniform Substitution. As a matter of fact, the actual discussion on `PPB` went along these lines, and similarly for `IKB`.[13]

A logician who wants to block `DAP` yet sticks to Classical Logic faces similar choices. A quick look at the derivation of `DAP` presented in section 2.2 reveals that there are three candidates for being given up, namely the formulas `KB1`, $\mathbf{D}_{\mathsf{B}}$, and $\mathbf{B}^{*}_{\mathsf{K}}$. If one decides to sustain formulas `KB1` and $\mathbf{D}_{\mathsf{B}}$, formula $\mathbf{B}^{*}_{\mathsf{K}}$ constitutes the only option: it must be ensured that $\mathbf{B}^{*}_{\mathsf{K}}$ it is not a theorem of the epistemic component of the logic in question. Accomplishing this, in turn, excludes not only (epistemic) **S5**, but also the logic **KTB** and all the logics from the family NEXT(**KTB**) as possible candidates for the epistemic component.[14]

Remark 3. Formula $\mathbf{B}_{\mathsf{K}}$ is almost unanimously rejected by epistemologists, and for different reasons. However, it is a theorem of the system $(\mathbf{S5})_{\mathsf{K}}$, that is, epistemically construed normal modal propositional logic

[12] As for normal modal propositional logics, one needs formula **B** *and* formula **4** in order to get formula **5**. On the other hand, in order to get **B** from **5** formula **T** is needed.

[13] An interested reader is advised to consult, e.g., (Gochet and Gribomont, 2006, pp. 114–117) and (Aucher, 2014, pp. 111–112).

[14] NEXT(**KTB**) comprises normal modal propositional logics which 'lie between' **KTB** and **S5**. For these logic, see, e.g., the monograph (Kostrzycka, 2010).

S5. Despite all well-known conceptual deficiencies of **S5** interpreted epistemically, the notorious problem of negative introspection included, the formalism of **S5** is still very often applied in logical considerations concerning knowledge, in particular knowledge dynamics and/or collective knowledge (cf., e.g., Ditmarsch et al, 2015). But when we take $(\mathbf{S5})_{\mathsf{K}}$ as the epistemic component of a, based on CPL, epistemic-doxastic logic, and the formulas `KB1` and $\mathbf{D}_{\mathsf{B}}$ remain its theorems, the logic suffers from both `PPB` and `DAP`, and possibly from `PIF` and `IKB` as well. Downsizing the epistemic component to the system $(\mathbf{KTB})_{\mathsf{K}}$ does not help in this respect.

2.6. Bypassing Negative Introspection of Knowledge and the Brouwerian Axiom on the Road to the Doxastic Arrogance Paradox

Negative introspection of knowledge is probably the most vividly criticised principle of epistemic logic. One of the results of the criticism was the switch of interest towards logics which 'lie between' $(\mathbf{S4})_{\mathsf{K}}$ and $(\mathbf{S5})_{\mathsf{K}}$, such as, for example, the systems $(\mathbf{S4.2})_{\mathsf{K}}$ or $(\mathbf{S4.4})_{\mathsf{K}}$.[15] Since no epistemic version of the Brouwerian axiom is a theorem of these logics, it cannot be said that the Doxastic Arrogance Paradox, `DAP`, *must* pertain to their epistemic-doxastic extensions. But one can arrive at `DAP` without making use of formulas $\mathbf{5}_{\mathsf{K}}$ and $\mathbf{B}^{*}_{\mathsf{K}}$ (or their equivalents), yet for a price. Let me illustrate this with two examples.

Example 1. We need the formulas:

$$\mathsf{B}(p \to q) \to (\mathsf{B}p \to \mathsf{B}q) \qquad (\mathbf{K}_{\mathsf{B}})$$

$$\mathsf{B}p \to \mathsf{BB}p \qquad (\mathbf{4}_{\mathsf{B}})$$

Let $\mathcal{L}$ be an epistemic-doxastic logic (based on CPL) such that:

---

[15] Recall the famous Lenzen's claim: "(...) the logic of knowledge must be at least as strong as system **S4.2** (...). Furthermore, the logic of knowledge must be at most as strong as system **S4.4** (...)" (Lenzen, 1978, p. 82). System $(\mathbf{S4.2})_{\mathsf{K}}$ results from the logic $(\mathbf{S4})_{\mathsf{K}}$ by adding to its axioms the following formula as a new axiom:
$(\mathbf{4.2}_{\mathsf{K}})$ $\quad \neg\mathsf{K}\neg\mathsf{K}p \to \mathsf{K}\neg\mathsf{K}\neg p$
System $(\mathbf{S4.4})_{\mathsf{K}}$, in turn, can be characterized as the extension of $(\mathbf{S4})_{\mathsf{K}}$ by the axiom:
$(\mathbf{4.4}_{\mathsf{K}})$ $\quad p \to (\neg\mathsf{K}p \to \mathsf{K}\neg\mathsf{K}p)$

Subsystems of $(\mathbf{S5})_{\mathsf{K}}$ in which formula $\mathbf{B}_{\mathsf{K}}$ is accepted, but both 'introspection' formulas, $\mathbf{4}_{\mathsf{K}}$ and $\mathbf{5}_{\mathsf{K}}$, are rejected, are rarely considered. A notable exception is Floridi's 'logic of being informed' (cf., e.g., Floridi, 2006) However, 'being informed' differs from being known, and beliefs do not enter the picture.

(i) formulas $\mathbf{T}_{\mathsf{K}}$, $\mathbf{K}_{\mathsf{B}}$, $\mathbf{D}_{\mathsf{B}}$, and $\mathbf{4}_{\mathsf{B}}$ are theorems of $\mathcal{L}$,
(ii) the following formula is a theorem of $\mathcal{L}$:

$$p \to \neg\mathsf{B}\neg\mathsf{B}p \qquad (\Diamond_{\mathsf{B}}\mathsf{B})$$

(iii) $\mathcal{L}$ is closed under the Rule of Necessitation for B, that is:

$$\phi \,/\, \mathsf{B}\phi \qquad (\mathrm{RN}_{\mathsf{B}})$$

When $\mathcal{L}$ satisfies the above conditions, `DAP` is a theorem of the logic. For a derivation of `DAP` from the above assumptions, see Appendix A.1.

It can be proven, however, that the doxastic component of an epistemic-doxastic logic which has $\Diamond_{\mathsf{B}}\mathsf{B}$ as a theorem is neither $(\mathbf{KD45})_{\mathsf{B}}$ nor any doxastic logic included in it. For the proof, see Appendix A.3.

Example 2. Assume that $\mathcal{L}$ is an epistemic-doxastic logic (based on CPL) that satisfies the following conditions:

(i) formulas $\mathbf{T}_{\mathsf{K}}$, $\mathbf{D}_{\mathsf{B}}$, and `KB1` are theorems of $\mathcal{L}$,
(ii) the following formulas are theorems of $\mathcal{L}$:

$$p \to \neg\mathsf{B}\neg\mathsf{K}p \qquad (\Diamond_{\mathsf{B}}\mathsf{K})$$

$$\mathsf{K}(p \wedge q) \to (\mathsf{K}p \wedge \mathsf{K}q) \qquad (2)$$

(iii) $\mathcal{L}$ is closed under the Rule of Necessitation for K:

$$\phi \,/\, \mathsf{K}\phi \qquad (\mathrm{RN}_{\mathsf{K}})$$

If $\mathcal{L}$ fulfils the above conditions, then `DAP` is a theorem of the logic. For a proof, see Appendix A.2.

Observe that the following:

$$p \to \neg\mathsf{K}\neg\mathsf{K}p \qquad (\Diamond_{\mathsf{K}}\mathsf{K})$$

is a theorem of $\mathcal{L}$, viz.:

h.1.	$\mathsf{K}p \to \mathsf{B}p$	(`KB1`)
h.2.	$\neg\mathsf{B}p \to \neg\mathsf{K}p$	(h.1 CPL)
h.3.	$\neg\mathsf{B}\neg\mathsf{K}p \to \neg\mathsf{K}\neg\mathsf{K}p$	(h.2 $p/\neg\mathsf{K}p$)
h.4.	$p \to \neg\mathsf{B}\neg\mathsf{K}p$	($\Diamond_{\mathsf{B}}\mathsf{K}$)
h.5.	$p \to \neg\mathsf{K}\neg\mathsf{K}p$	(h.4, h.3 CPL)

One can prove that the epistemic component of an epistemic-doxastic logic which has $\Diamond_{\mathsf{K}}\mathsf{K}$ as a theorem is neither $(\mathbf{S5})_{\mathsf{K}}$ nor any epistemic logic included in it. (For the proof, see Appendix A.4.) Thus, in particular, $(\mathbf{S4.2})_{\mathsf{K}}$ and $(\mathbf{S4.4})_{\mathsf{K}}$ do not constitute the epistemic component of the epistemic-doxastic logic in question. On the other hand, any epistemic-doxastic logic whose epistemic component is an extension of $(\mathbf{S4.2})_{\mathsf{K}}$, $(\mathbf{S4.4})_{\mathsf{K}}$, or of a logic between them, by the formula $\Diamond_{\mathsf{K}}\mathsf{K}$, suffers from the Doxastic Arrogance Paradox.

3. A Bit of Philosophy

3.1. Possibility

When we work with alethic modal propositional logic based on CPL, the possibility operator, $\Diamond$, can be defined in terms of the necessity operator, $\Box$, by the equivalence:

$$\Diamond\phi \equiv \neg\Box\neg\phi$$

By analogy, the following formulas $\neg\mathsf{K}\neg\phi$ and $\neg\mathsf{B}\neg\phi$ can be regarded as expressing epistemic possibility of ϕ, and doxastic possibility of ϕ, respectively. The first formula reads: 'It is possible, for all that is known, that ϕ' or 'It is possible, for all an agent knows, that ϕ'. This is sometimes abbreviated by saying that ϕ is not ruled out. The second formula, in turn, reads: 'It is compatible with everything that is believed that ϕ' or 'It is compatible with everything an agent believes that ϕ'.[16] The abridged readings are: 'ϕ is thought to be possible' and 'an agent thinks ϕ be possible.' In other words, formulas expressing epistemic and doxastic possibilities allow for both notional and attitudinal readings. Tables 2 and 3 summarize this.

Remark 4. It should be stressed that the above concepts of doxastic and epistemic possibility lack time dimension: one should not confuse 'is possible' with 'will happen in a future.'

3.2. Possibility of Being Believed/Known

Consider the following formula:

$$\neg\mathsf{K}\neg\mathsf{K}p$$

[16] Following the seminal work (Hintikka, 1962), I use here the term 'compatible' instead of 'consistent'; the second option also occurs in the literature.

	Formula	Notional reading
Epistemic Possibility	$\neg\mathsf{K}\neg\phi$	It is possible, for all that is known, that ϕ
Doxastic Possibility	$\neg\mathsf{B}\neg\phi$	It is compatible with everything that is believed that ϕ

Table 2. Notional readings of $\neg\mathsf{K}\neg\phi$ and $\neg\mathsf{B}\neg\phi$

	Formula	Attitudinal reading
Epistemic Possibility	$\neg\mathsf{K}\neg\phi$	It is possible, for all that an agent knows, that ϕ
Doxastic Possibility	$\neg\mathsf{B}\neg\phi$	It is compatible with everything an agent believes that ϕ

Table 3. Attitudinal readings of $\neg\mathsf{K}\neg\phi$ and $\neg\mathsf{B}\neg\phi$

Assuming that K reads 'is known', and that '$\mathsf{K}p$' is synonymous with 'p constitutes an item of knowledge', we get:

- *It is possible, for all that is known, that p is an item of knowledge.*

or briefly:

- *It is epistemically possible that p constitutes an item of knowledge.*

Now consider the formulas:

$$\neg\mathsf{B}\neg\mathsf{B}p \qquad \text{and} \qquad \neg\mathsf{B}\neg\mathsf{K}p$$

Under the notional reading of the respective phrases, we get:

- *It is compatible with everything that is believed that p is a subject of belief.*
- *Is is compatible with everything that is believed that p is an item of knowledge.*

respectively. In brief:

- *It is doxastically possible that p is a subject of belief.*
- *It is doxastically possible that p is an item of knowledge.*

Remark 5. Since the considered concepts of possibility lack time dimension, one cannot read: 'is a subject of belief' and 'is an item of knowledge' as: 'will be believed' and 'will be known,' respectively.

3.3. The Case of Formula $\mathbf{B}^*_{\mathsf{K}}$

Let me turn back to the formula $\mathbf{B}^*_{\mathsf{K}}$. Clearly, it is CPL-equivalent to the formula:

$$\neg\mathsf{K}\neg\mathsf{K}p \to p \tag{$\mathbf{B}_{\mathsf{K}}$}$$

Under the notional reading of its respective parts, formula $\mathbf{B}_{\mathsf{K}}$ says:

(∗) *If it is epistemically possible that p is an item of knowledge, then p is true.*

But p is a propositional *variable*. Thus (∗) amounts to:

(∗∗) *If it is epistemically possible that a proposition is an item of knowledge, then the proposition is true.*

When $\mathbf{B}_{\mathsf{K}}$ is a *theorem* of an epistemic logic, its claim pertain to any proposition. So if $\mathbf{B}_{\mathsf{K}}$ is a theorem of an epistemic-doxastic logic in question, the following principle is ensured:

(△) *Each proposition of which it is epistemically possible that it is an item of knowledge, is true.*

As formulas $\mathbf{B}_{\mathsf{K}}$ and $\mathbf{B}^*_{\mathsf{K}}$ are CPL-equivalent, one may conclude that the principle (△) is ensured when formula $\mathbf{B}^*_{\mathsf{K}}$ is a theorem of an epistemic-doxastic logic in question.

A philosopher can now reason as follows. Once the principle (△) is accepted, we operate with an epistemic-doxastic logic which has formula $\mathbf{B}^*_{\mathsf{K}}$ as a theorem. Assuming that the logic is based on CPL and has formulas `KB1` and $\mathbf{D}_{\mathsf{B}}$ as theorems as well, we arrive at the Paradox of Doxastic Arrogance, `DAP`, and possibly other paradoxes. But `DAP` is unacceptable to an epistemologist, in contradistinction to `KB1` and $\mathbf{D}_{\mathsf{B}}$. Therefore the principle (△) is to be rejected. However, to reject (△) it is to allow for exceptions to it. This amounts to allowing for:

(non-△) *There exists at least one proposition of which it is epistemically possible that it constitutes an item of knowledge, but the proposition is not true.*

In other words, by rejecting (△) one allows that non-true propositions may constitute items of knowledge. Since 'non-true' need not mean 'false', but may mean, for example, 'truthlike' or even 'only conventional', any argument for allowing non-true ingredients of knowledge is to be welcomed by philosophers of science and epistemologists who, for various reasons, share this view. Here the new argument is:

(I) *Once we do not allow for non-true ingredients of knowledge, then, caeteris paribus, we end up with the Doxastic Arrogance Paradox and possibly other paradoxes as well. Therefore one should allow for non-true ingredients of knowledge.*

Remark that (I) is not an argument for the existence of non-true ingredients of knowledge: what is argued for is that their existence must not be excluded.

3.4. The Case of Formula $\mathbf{5}_{\mathsf{K}}$

The negative introspection principle for knowledge is another possible source of DAP and other paradoxes. Clearly, formulas $\mathbf{5}_{\mathsf{K}}$ and

$$\neg\mathsf{K}\neg\mathsf{K}p \to \mathsf{K}p$$

are CPL-equivalent. When we analyse the theoremhood of the above formula along similar lines as in the previous section, we come to the conclusion that its theoremhood (and thus also the theoremhood of $\mathbf{5}_{\mathsf{K}}$) ensures the principle:

($\triangledown$) *Each proposition of which it is epistemically possible that it is an item of knowledge, is an item of knowledge.*

(By the way, the negative introspection claim concerning knowledge is rarely thought of in this manner.) Thus, one should leave room for the existence of propositions which may be items of knowledge, but nevertheless are not actually so; otherwise, *caeteris paribus*, DAP and possibly other paradoxes show up.

3.5. Formulas $\Diamond_{\mathsf{B}}\mathsf{B}$ and $\Diamond_{\mathsf{B}}\mathsf{K}$

Formulas $\Diamond_{\mathsf{B}}\mathsf{B}$ and $\Diamond_{\mathsf{B}}\mathsf{K}$ are another possible sources of DAP (see section 2.6). The theoremhood of $\Diamond_{\mathsf{B}}\mathsf{B}$ and $\Diamond_{\mathsf{B}}\mathsf{K}$ ensure:

($\triangleleft$) *For each true proposition, it is compatible with everything that is believed that the proposition is a subject of belief.*

($\triangleright$) *For each true proposition, it is compatible with everything that is believed that the proposition is an item of knowledge.*

respectively. Roughly, what ($\triangleleft$) says, is:

($\triangleleft'$) *There is no truth of which it is not doxastically possible to be believed.*

while the claim of ($\rhd$) is:

($\rhd'$) *There is no truth of which it is not doxastically possible to be known.*

As Example 1 illustrates, a philosopher who finds DAP unacceptable, but nevertheless accepts the principles laying behind formulas $\mathbf{T}_{\mathsf{K}}$, $\mathbf{D}_{\mathsf{B}}$, $\mathbf{4}_{\mathsf{B}}$, sticks to Classical Logic, and regards *Modus Ponens*, Uniform Substitution and RN_{B} as valid rules of inference of a doxastic logic, should then allow for exceptions to the principle ($\lhd$). Allowing for exceptions to ($\lhd$) amounts to saying that there may exist true propositions of which being believed is not compatible with everything that is believed.

Example 2 shows, in turn, that a philosopher who finds DAP unacceptable, but sticks to Classical Logic, agrees that *Modus Ponens*, Uniform Substitution and RN_{K} are valid rules of inference of an epistemic logic, and accepts the principles expressed by formulas $\mathbf{T}_{\mathsf{K}}$, $\mathbf{D}_{\mathsf{B}}$, KB1, (2), should allow for exceptions to the principle ($\rhd$). Allowing for exceptions to ($\lhd$) amounts to claiming that there may exist true propositions such that their being items of knowledge is not compatible with everything that is believed. This is not equivalent to claiming that such propositions actually exist. This is only to say that their existence should not be *a priori* excluded.

4. Summary and Final Remarks

There exist paradoxes which are antinomies and paradoxes which are not. As for the latter, some challenge basic intuitions and/or common sense, while other challenge established theories. Some challenge both. But there also exist non-antinomic paradoxes which, as a matter of fact, are illuminative: after all, a deduction is supposed to preserve truth, not intuitiveness.

The Doxastic Arrogance Paradox, DAP, is challenging rather than illuminative. The message it carries is unexpected, but also deeply puzzling. According to DAP, the belief that a proposition is an item of knowledge yields that the proposition holds/is true. How to explain this? The best explanation seems to be: beliefs of having knowledge, in contradistinction to other beliefs, are always factive, and this, by the factivity of knowledge, ensures the truth of propositions whose being items of knowledge is believed. Such an explanation, however, is more puzzling than DAP itself, not to mention its empirical inadequacy.

Having PPB and PIF among theorems is not a virtue, but a vice of a system of epistemic-doxastic logic. I think that the same holds true for DAP. As for philosophy, DAP is a *negative constraint* for an adequate philosophical account of knowledge and belief. This makes search for possible sources of DAP a worthwhile enterprise.

In this paper I pointed out three such sources, namely the formulas: $\mathbf{B}^*_{\mathsf{K}}$, $\Diamond_{\mathsf{B}}\mathsf{B}$, and $\Diamond_{\mathsf{B}}\mathsf{K}$. Neither of the above formulas is a sole source of DAP. In each case additional assumptions are needed (these were characterized in the respective sections of this paper), but the negative introspection of knowledge principle (formula $\mathbf{5}_{\mathsf{K}}$) is not among them. Of course, one can also arrive at DAP by using $\mathbf{5}_{\mathsf{K}}$ as one of the premises.

The identification of a possible source of a non-antinomic paradox carries negative information to a logician: it shows what to block in order to avoid the paradox. Since a paradox may have many sources, some of them possibly still unknown, a success achieved that way is only tentative. To a philosopher, however, each identification of a possible source of a non-antinomic paradox carries positive information, since it shows what must not be prejudged by a conceptual setting. The analysis presented in section 3 of this paper has revealed that in order to avoid DAP *exceptions* to the following claims are to be allowed:

($\triangle$) *Each proposition of which it is epistemically possible that it is an item of knowledge, is true.*

($\triangledown$) *Each proposition of which it is epistemically possible that it is an item of knowledge, is an item of knowledge.*

($\lhd$) *For each true proposition, it is compatible with everything that is believed that the proposition is a subject of belief.*

($\rhd$) *For each true proposition, it is compatible with everything that is believed that the proposition is an item of knowledge.*

assuming that the other premises which helped us to arrive at DAP in a given way (these were characterized consecutively in sections 2.2, 2.3, and 2.6) – or, to be more precise, the principles lying behind them – are regarded to be universally binding.

Any of: ($\triangle$), ($\triangledown$), ($\lhd$), ($\rhd$) provides an affirmative solution to a philosophical problem or contributes to solutions of such problems. On the other hand, DAP constitutes a negative constraint for an adequate philosophical account of knowledge and belief. A philosopher who finds DAP unacceptable can then argue in the following way (for $\odot \in \{\triangle, \triangledown, \lhd, \rhd\}$):

(II) *Once we do not allow for the existence of exceptions to (⊙), then, caeteris paribus, we end up with the Doxastic Arrogance Paradox,* DAP. *Therefore exceptions to (⊙) are to be allowed.*

Of course, allowing for exceptions is not tantamount to claiming that the exceptions actually take place. However, a philosopher who sees his enterprise as oriented towards what is necessary rather than what actually happens, would infer the negative solution to a problem from the need of allowing exceptions to its affirmative solution. To such a philosopher the analysis presented in this paper would have confirmed that the respective (⊙)'s are simply wrong. Yet, I personally would not go that far. My conclusion is: one should arrange the conceptual setting in a way that the (⊙)'s are not prejudged by it.

A. Appendices

A.1. A Derivation of DAP from $\mathbf{T}_{\mathsf{K}}$, $\mathbf{K}_{\mathsf{B}}$, $\mathbf{D}_{\mathsf{B}}$, $\mathbf{4}_{\mathsf{B}}$, $\Diamond_{\mathsf{B}}\mathsf{B}$

Since $\mathcal{L}$ is based on CPL, formula $\mathbf{K}_{\mathsf{B}}$ is a theorem of $\mathcal{L}$, and $\mathcal{L}$ is closed under rule RN_{K}, it is also closed under the rule RM_{B}, and the formula:

$$\mathsf{B}(p \wedge q) \to \mathsf{B}p \wedge \mathsf{B}q \tag{$\star$}$$

is a theorem of $\mathcal{L}$.

i.1.	$\mathsf{B}(p \wedge q) \to \mathsf{B}p \wedge \mathsf{B}q$	($\star$)
i.2.	$\mathsf{B}(\neg p \wedge \mathsf{B}p) \to \mathsf{B}\neg p \wedge \mathsf{B}\mathsf{B}p$	(i.1 $p/\neg p$, $q/\mathsf{B}p$)
i.3.	$\mathsf{B}(\neg p \wedge \mathsf{B}p) \to \mathsf{B}\neg p$	(i.2 CPL)
i.4.	$\mathsf{B}(\neg p \wedge \mathsf{B}p) \to \mathsf{B}\mathsf{B}p$	(i.2 CPL)
i.5.	$\mathsf{B}p \to \neg\mathsf{B}\neg p$	($\mathbf{D}_{\mathsf{B}}$)
i.6.	$\mathsf{B}\neg p \to \neg\mathsf{B}p$	(i.5 CPL)
i.7.	$\mathsf{B}(\neg p \wedge \mathsf{B}p) \to \neg\mathsf{B}p$	(i.3, i.6 CPL)
i.8.	$\mathsf{B}\mathsf{B}(\neg p \wedge \mathsf{B}p) \to \mathsf{B}\neg\mathsf{B}p$	(i.7 RM_{B})
i.9.	$\mathsf{B}\neg\mathsf{B}p \to \neg\mathsf{B}\mathsf{B}p$	(i.6 $p/\mathsf{B}p$)
i.10.	$\mathsf{B}\mathsf{B}(\neg p \wedge \mathsf{B}p) \to \neg\mathsf{B}\mathsf{B}p$	(i.8, i.9 CPL)
i.11.	$\mathsf{B}p \to \mathsf{B}\mathsf{B}p$	($\mathbf{4}_{\mathsf{B}}$)
i.12.	$\mathsf{B}(\neg p \wedge \mathsf{B}p) \to \mathsf{B}\mathsf{B}(\neg p \wedge \mathsf{B}p)$	(i.11 $p/\neg p \wedge \mathsf{B}p$)
i.13.	$\mathsf{B}(\neg p \wedge \mathsf{B}p) \to \neg\mathsf{B}\mathsf{B}p$	(i.12, i.10 CPL)
i.14.	$\neg\mathsf{B}(\neg p \wedge \mathsf{B}p)$	(i.4, i.13 CPL)
i.15.	$\mathsf{B}\neg\mathsf{B}(\neg p \wedge \mathsf{B}p)$	(i.14 RN_{B})
i.16.	$p \to \neg\mathsf{B}\neg\mathsf{B}p$	($\Diamond_{\mathsf{B}}\mathsf{B}$)

i.17.	$\neg p \wedge \mathsf{B}p \to \neg\mathsf{B}\neg\mathsf{B}(\neg p \wedge \mathsf{B}p)$	(i.16 $p/\neg p \wedge \mathsf{B}p$)
i.18.	$\mathsf{B}\neg\mathsf{B}(\neg p \wedge \mathsf{B}p) \to \neg(\neg p \wedge \mathsf{B}p)$	(i.17 CPL)
i.19.	$\neg(\neg p \wedge \mathsf{B}p)$	(i.18, i.15 CPL)
i.20.	$\mathsf{B}p \to p$	(i.19 CPL)
i.21.	$\mathsf{BK}p \to \mathsf{K}p$	(i.20 $p/\mathsf{K}p$)
i.22.	$\mathsf{K}p \to p$	($\mathbf{T}_\mathsf{K}$)
(DAP)	$\mathsf{BK}p \to p$	(i.21, i.22 CPL)

A.2. A Derivation of DAP from $\mathbf{T}_\mathsf{K}$, $\mathbf{D}_\mathsf{B}$, KB1, $\Diamond_\mathsf{B}\mathsf{K}$, (2)

j.1.	$\mathsf{K}(p \wedge q) \to \mathsf{K}p \wedge \mathsf{K}q$	(2)
j.2.	$\mathsf{K}(\neg p \wedge \mathsf{B}p) \to \mathsf{K}\neg p \wedge \mathsf{KB}p$	(j.1 ($p/\neg p$, $q/\mathsf{B}p$)
j.3.	$\mathsf{K}(\neg p \wedge \mathsf{K}p) \to \mathsf{K}\neg p$	(j.2 CPL)
j.4.	$\mathsf{K}(\neg p \wedge \mathsf{B}p) \to \mathsf{KB}p$	(j.2 CPL)
j.5.	$\mathsf{K}p \to \mathsf{B}p$	(KB1)
j.6.	$\mathsf{K}\neg p \to \mathsf{B}\neg p$	(j.5 $p/\neg p$)
j.7.	$\mathsf{K}(\neg p \wedge \mathsf{B}p) \to \mathsf{B}\neg p$	(j.3, j.6 CPL)
j.8.	$\mathsf{B}p \to \neg\mathsf{B}\neg p$	($\mathbf{D}_\mathsf{B}$)
j.9.	$\mathsf{B}\neg p \to \neg\mathsf{B}p$	(j.8 CPL)
j.10.	$\mathsf{K}(\neg p \wedge \mathsf{B}p) \to \neg\mathsf{B}p$	(j.9, j.7 CPL)
j.11.	$\mathsf{K}p \to p$	($\mathbf{T}_\mathsf{K}$)
j.12.	$\mathsf{KB}p \to \mathsf{B}p$	(j.11 $p/\mathsf{B}p$)
j.13.	$\mathsf{K}(\neg p \wedge \mathsf{B}p) \to \mathsf{B}p$	(j.4, j.12 CPL)
j.14.	$\neg\mathsf{K}(\neg p \wedge \mathsf{B}p)$	(j.10, j.13 CPL)
j.15.	$\mathsf{K}\neg\mathsf{K}(\neg p \wedge \mathsf{B}p)$	(j.14 RN_K)
j.16.	$p \to \neg\mathsf{B}\neg\mathsf{K}p$	($\Diamond_\mathsf{B}\mathsf{K}$)
j.17.	$\neg\mathsf{B}p \to \neg\mathsf{K}p$	(j.5 CPL)
j.18.	$\neg\mathsf{B}\neg\mathsf{K}p \to \neg\mathsf{K}\neg\mathsf{K}p$	(j.17 $p/\neg\mathsf{K}p$))
j.19.	$p \to \neg\mathsf{K}\neg\mathsf{K}p$	(j.16, j.18 CPL)
j.20.	$\neg p \wedge \mathsf{B}p \to \neg\mathsf{K}\neg\mathsf{K}(\neg p \wedge \mathsf{B}p)$	(j.19 $p/\neg p \wedge \mathsf{B}p$)
j.21.	$\mathsf{K}\neg\mathsf{K}(\neg p \wedge \mathsf{B}p) \to \neg(\neg p \wedge \mathsf{B}p)$	(j.20 CPL)
j.22.	$\neg(\neg p \wedge \mathsf{B}p)$	(j.21,j.15 CPL)
j.23.	$\mathsf{B}p \to p$	(j.22 CPL)
j.24.	$\mathsf{BK}p \to \mathsf{K}p$	(j.23 $p/\mathsf{B}p$)
(DAP)	$\mathsf{BK}p \to p$	(j.24, j.11 CPL)

A.3. The Emplacement of Formula $\Diamond_{\mathsf{B}}\mathsf{B}$

Let $Thm(\mathcal{L})$ stand for the set of theorems of a logic $\mathcal{L}$. A doxastic (epistemic) logic $\mathcal{L}^*$ constitutes the doxastic (epistemic) component of an epistemic-doxastic logic $\mathcal{L}$ just in case $Thm(\mathcal{L}^*)$ is the doxastic (epistemic) component of $\mathcal{L}$. (For the concepts of doxastic and epistemic components of an epistemic-doxastic logic see section 1.1.)

PROPOSITION A.1. *If $\mathcal{L}$ is an epistemic-doxastic logic such that formula $\Diamond_{\mathsf{B}}\mathsf{B}$ is a theorem of $\mathcal{L}$, then for each doxastic logic $\mathcal{L}^*$ such that:*

$$\mathrm{Thm}(\mathcal{L}^*) \subseteq \mathrm{Thm}((\mathbf{KD45})_{\mathsf{B}}) \tag{$\heartsuit$}$$

the logic $\mathcal{L}^*$ does not constitute the doxastic component of $\mathcal{L}$.

PROOF. Let $\mathcal{M} = \langle W, R_{\mathsf{B}}, v\rangle$ be a Kripke model such that:

- $W = \{w_1, w_2, w_3\}$,
- $R_{\mathsf{B}} = \{\langle w_1, w_2\rangle, \langle w_1, w_3\rangle, \langle w_2, w_2\rangle, \langle w_3, w_3\rangle, \langle w_2, w_3\rangle, \langle w_3, w_2\rangle\}$,
- $v(p, w_1) = \mathbf{0}, v(p, w_2) = \mathbf{0}, v(p, w_3) = \mathbf{1}$.

Clearly, R_{B} is serial, transitive and Euclidean in W. The formula $\Diamond_{\mathsf{B}}\mathsf{B}$ is not true in the world w_3 of $\mathcal{M}$, and hence is not true in $\mathcal{M}$. But each theorem of $\mathbf{KD45}_{\mathcal{B}}$ is true in any Kripke model in which the accessibility relation $R_{\mathcal{B}}$ is serial, transitive and Euclidean. Hence the formula $\Diamond_{\mathsf{B}}\mathsf{B}$ is not a theorem of $\mathbf{KD45}_{\mathcal{B}}$ and therefore is not a theorem of any doxastic logic $\mathcal{L}^*$ that fulfils the condition ($\heartsuit$). However, $\Diamond_{\mathsf{B}}\mathsf{B}$ belongs to the doxastic component of $\mathcal{L}$. It follows that no doxastic logic $\mathcal{L}^*$ that fulfils the condition ($\heartsuit$) constitutes the doxastic component of $\mathcal{L}$. $\dashv$

A.4. The Emplacement of Formula $\Diamond_{\mathsf{K}}\mathsf{K}$

PROPOSITION A.2. *If $\mathcal{L}$ is an epistemic-doxastic logic such that formula $\Diamond_{\mathsf{K}}\mathsf{K}$ is a theorem of $\mathcal{L}$, then for each epistemic logic $\mathcal{L}^*$ such that:*

$$Thm(\mathcal{L}^*) \subseteq Thm((\mathbf{S5})_{\mathsf{K}}) \tag{$\spadesuit$}$$

the logic $\mathcal{L}^*$ does not constitute the epistemic component of $\mathcal{L}$.

PROOF. Let $\mathcal{M} = \langle W, R_{\mathsf{K}}, v\rangle$ be a Kripke model such that $W = \{w, w_1\}$, $v(p, w) = \mathbf{1}$, $v(p, w_1) = \mathbf{0}$, and the accessibility relation R_{K} is universal in W. Clearly, the formula $\Diamond_{\mathsf{B}}\mathsf{K}$ is not true in the world w_1 of $\mathcal{M}$, and hence is not true in $\mathcal{M}$. But each theorem of $(\mathbf{S5})_{\mathsf{K}}$ is true in $\mathcal{M}$. Hence the formula $\Diamond_{\mathsf{B}}\mathsf{K}$ is not a theorem of $(\mathbf{S5})_{\mathsf{K}}$ and thus of any epistemic logic $\mathcal{L}^*$ that fulfils the condition ($\spadesuit$). $\dashv$

References

Aucher, G., 2014, "Principles of knowledge, belief and conditional belief", pages 97–134 in M. Rebuschi, M. Batt, G. Heinzmann, F. Lihoreau, M. Musdiol, and A. Trognon (eds.), *Interdisciplinary Works in Logic, Epistemology, Psychology and Linguistics*, Cham: Springer. DOI: 10.1007/978-3-319-03044-9_5

Floridi, L., 2006, "The logic of being informed", *Logique et Analyse* 196: 433–460. https://www.jstor.org/stable/44085232

Gochet, P., and P. Gribomont, 2006, "Epistemic Logic", pages 99–195 in D. Gabbay and J. Woods (eds.), *Handbook of the History of Logic. Vol. 7: Logic and the Modalities in the Twentieth Century*, Amsterdam: Elsevier. DOI: 10.1016/S1874-5857(06)80028-2

Hintikka, J., 1962, *Knowledge and Belief. An Introduction to the Logic of the Two Notions*, Ithaca: Cornell University Press.

Kostrzycka, Z., 2010, *On Modal Systems in the Neighbourhood of the Brouwer Logic*, Wrocław: Wydawnictwo Uniwersytetu Wrocławskiego.

Lenzen, W., 1978, "Recent work in epistemic logic", *Acta Philosophica Fennica* 30: 1–219.

van der Hoek, W., 1993, "Systems for knowledge and belief", *Journal of Logic and Computation* 3: 173–195. DOI: 10.1093/logcom/3.2.173

Ditmarsch, H., J. Halpern, W. van der Hoek, and B. Kooi (eds.), 2015, *Handbook of Epistemic Logic*, College Publications, Milton Keynes.

Voorbraak, F., 1993, "As far as I know. Epistemic logic and uncertainty", Ph.D. Thesis, Department of Philosophy, Utrecht University.

Williamson, T., 2001, "Some philosophical aspects of reasoning about knowledge", page 97 in J. van Benthem (ed.), *Proceedings of the 8th Conference TARK 2001*, San Francisco: Morgan Kaufmann.

Andrzej Wiśniewski
Department of Logic and Cognitive Science
Adam Mickiewicz University in Poznań
Poland
Andrzej.Wisniewski@amu.edu.pl
https://orcid.org/0000-0001-7820-428X