Logic and **L**ogical **P**hilosophy
Online First Articles (2026)
DOI: 10.12775/LLP.2026.001

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Swap Kripke Models for Deontic *LFI*s

Abstract. We present a construction of nondeterministic semantics for some deontic logics based on the class of paraconsistent logics known as *Logics of Formal Inconsistency* (*LFI*s), for the first time combining swap structures and Kripke models through the novel notion of swap Kripke models. We start by making use of Nmatrices to characterize systems based on *LFI*s that do not satisfy axiom (cl), while turning to RNmatrices when the latter is considered in the underlying *LFI*s. This paper also presents, for the first time, a full axiomatization and a semantics for the C_n^D hierarchy, utilizing the aforementioned mixed semantics with RN matrices. It includes the historical system C_1^D of da Costa and Carnielli (1986), the first deontic paraconsistent system proposed in the literature.

Keywords: deontic logic; paraconsistent logic; da Costa logics; nondeterministic semantics; Nmatrices; swap structures; moral dilemmas

1. Introduction

The pioneering work on paraconsistent deontic logic by da Costa and Carnielli (1986) proposed dealing with deontic paradoxes by changing the base logic from classical logic to a paraconsistent logic. The logic that should then be used for such an enterprise is da Costa's C_1. This logic blocks trivialization derived from conflict of obligations since the occurrence of a formula of the form $\alpha \wedge \neg\alpha$ does not trivialize the system. The modal logics which have C_1 as their propositional fragment also preserve this characteristic, thus being conjectured by the authors to be a well motivated project.

Another motivation for such an account is the fact that C_1 is the first logic in a hierarchy of logics, the da Costa's C_n. It is suggested by

Received June 6, 2025. *Accepted* December 31, 2025. *Published online* January 5, 2026

the end of the 1986 paper that the technique of adding a modal deontic operator could be applied to any logic in the C_n hierarchy, but this, in fact, was hitherto never accomplished. Indeed, the modal systems based on C_n were never given a formal (syntactic and/or semantical) treatment. Our aim is to fill these gaps in the literature.

With the passing of years, the notion of deontic paraconsistency has evolved. Many works used similar ideas as those initially presented by da Costa and Carnielli, for instance: (Beirlaen and Straßer, 2011; McGinnis, 2007; Puga and da Costa, 1987a,b; Puga et al., 1988; Coniglio and Peron, 2009; Peron and Coniglio, 2008; Coniglio, 2009).

In particular, a series of works (Peron and Coniglio, 2008; Coniglio and Peron, 2009; Coniglio, 2009) investigates the applications of deontic axioms to *LFI*s.[1] In these works, the authors present explicitly the notion of *deontic paraconsistency* which is defined as follows: a logic is *deontically paraconsistent* if it is not deontically explosive, i.e., for some α, β in the set of formulas, we have the following:

$$\mathsf{O}\alpha, \mathsf{O}\neg\alpha \nvdash \mathsf{O}\beta.$$

Moreover, a logic is a Logic of Deontic Inconsistency, *LDI* for short, if it is not deontically explosive and there is a unary connective (primitive or defined) $\bar{\boxminus}$ for which the following holds:[2]

- For some sentences $\alpha, \alpha', \beta, \beta'$,
  - $\bar{\boxminus}(\alpha), \mathsf{O}\alpha \nVdash \mathsf{O}\beta$,
  - $\bar{\boxminus}(\alpha'), \mathsf{O}\neg\alpha' \nVdash \mathsf{O}\beta'$.
- For any α, β
  - $\bar{\boxminus}(\alpha), \mathsf{O}\alpha, \mathsf{O}\neg\alpha \Vdash \mathsf{O}\beta$.

Any normal modal logic based on an *LFI* can be seen as an *LDI* simply by taking $\bar{\boxminus}(\alpha) := \mathsf{O}{\circ}\alpha$. The original semantics for *LDI*s based on *LFI*s provided in (Peron and Coniglio, 2008; Coniglio, 2009; Coniglio and Peron, 2009) was given in terms of Kripke structures together with bivaluation semantics.[3]

[1] Also to some other logics, such as Batens' CLuN (1980a; 1980b), previously named DPI.

[2] As in the case of *LFI*s, in the general case $\bar{\boxminus}(p)$ can be considered as being a set of modal formulas depending on a single propositional letter p. *LDI*s where introduced by Coniglio in (2009). Additional developments and applications of *LDI*s can be found in (Coniglio and Peron, 2009).

[3] It is worth noting that Bueno-Soler has introduced a wide class of paraconsistent modal systems based on *LFI*s, also with a semantics given by Kripke structures

The recent years have seen a rise in developments in the area of nondeterministic semantics, with prominent works (in chronological order) by Coniglio et al. (2015), Omori and Skurt (2016), Coniglio and Golzio (2019), Grätz (2021), Pawlowski and Skurt (2025), Coniglio et al. (2025) and Leme et al. (2025), among others. In general, these works show that Nmatrices and RNmatrices (i.e., restricted Nmatrices) allow for the characterization of many non-normal modal logics. These results motivated the aim to approach the semantics of deontic *LFI*s nondeterministically and the pursue to cover the whole C_n hierarchy, as envisioned by da Costa and Carnielli (1986).

It is important to note that, although the works of Coniglio and Peron, as well as Bueno-Soler, cover a portion of the *LFI*s' hierarchy, some of the *LFI*s were not studied at the time. Pertaining to the latter were the systems satisfying axiom (cl), which are not characterizable by finite Nmatrices, following the proof of the Dugundji-like theorem by Avron (2007, Theorem 11). That roadblock was moved by Coniglio and Toledo (2022), where the authors present a new possibility for a nondeterministic semantical characterization of the logics in C_n. Inspired by this work, we expand the treatment to present a characterization of modal C_n.

Having these details in mind, this paper initially presents a semantics for some deontic *LFI*s, starting with *DmbC*. The novelty at this point is that, different to the previous approaches to modal *LFI*s found in the literature, we present a semantics given by a combination between swap structures and Kripke models. We take sets of worlds and relations as a frame, where each of the worlds is nondeterministic. We then move to extensions of this logic eventually reaching *DmbCcl*, which, by previous known results, is not characterizable by finite Nmatrices. We then show, following the results presented in (Coniglio and Toledo, 2022), that the combination between RNmatrices and Kripke models allows for a characterization of this logic, as well as its extensions. In particular, we focus our attention on a few of its extensions, namely *DCila* (which is the conservative reduct of C_1^D) and the hierarchy extension C_n^D, for all C_n. Regarding the latter, we present for the first time an explicit characterization of these logics, describing their axioms and respective semantics (once again in terms of swap Kripke models), given that the

equipped with bivaluation semantics, and alternatively with a possible-translations semantics (see, e.g., Bueno-Soler, 2010).

deontic systems for the whole hierarchy were never explicitly described. We end this paper with a brief discussion of our results, applying the C_n^D hierarchy to moral dilemmas.

2. The paraconsistent deontic system *DmbC*

We give a modal account of the fundamental *LFI*, *mbC* together with a modalization proposed in (Coniglio, 2009; Peron and Coniglio, 2008; Coniglio and Peron, 2009), which is a *deontic* version of *mbC*, which we call *DmbC*.

DEFINITION 2.1. Let $\Sigma = \{\to, \neg, \vee, \wedge, \mathsf{O}, \circ\}$ be a signature for *LFI*s. The logic *DmbC* defined over Σ is the system characterized by all CPL$^+$ axioms, that is, the axioms corresponding to the positive fragment of classical propositional logic, plus the following axioms for $\neg$ and $\circ$:

$$\alpha \vee \neg\alpha \tag{EM}$$

$$\circ\alpha \to (\alpha \to (\neg\alpha \to \beta)) \tag{bc}$$

together with the following modal axioms, where $\bot_\alpha := (\alpha \wedge \neg\alpha) \wedge \circ\alpha$:

$$\mathsf{O}(\alpha \to \beta) \to (\mathsf{O}\alpha \to \mathsf{O}\beta) \tag{O-K}$$

$$\mathsf{O}\bot_\alpha \to \bot_\alpha \tag{O-E}$$

such that the only inference rules are Modus Ponens and O-necessitation.

Observe that O-necessitation is a *global* inference rule (i.e., it only can be applied to premises which are theorems). From this, the notion of *derivation from premises* needs to be adjusted in *DmbC*, as it is usually done in normal modal systems.

DEFINITION 2.2. Let $\Gamma \cup \{\varphi\} \subseteq For(\Sigma)$.

1. A *derivation* of φ in *DmbC* is a finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that $\varphi_n = \varphi$ and, for every $1 \leqslant i \leqslant n$, either φ_i is an instance of an axiom, or φ_i follows from φ_j and $\varphi_k = \varphi_j \to \varphi_i$ (for $j, k < i$) by Modus Ponens, or $\varphi_i = \mathsf{O}\varphi_j$ follows from φ_j (for $j < i$) by O-necessitation. In this case, we say that φ is *derivable* in *DmbC*, or it is a *theorem* of *DmbC*, which will be denoted by $\vdash_{DmbC} \varphi$.

2. We say that φ *is derivable from* Γ *in DmbC*, denoted by $\Gamma \vdash_{DmbC} \varphi$, if either $\vdash_{DmbC} \varphi$, or there exist formulas $\gamma_1, \ldots, \gamma_k \in \Gamma$ (for a finite $k \geqslant 1$) such that $\vdash_{DmbC} (\gamma_1 \wedge \cdots \wedge \gamma_k) \to \varphi$.

Observe that $\emptyset \vdash_{DmbC} \varphi$ iff $\vdash_{DmbC} \varphi$. We know that in *DmbC* the deduction metatheorem and proof-by-cases hold, as the results by Coniglio (2009) show.

Notice that we define $\bot_\alpha$ as being equivalent to $(\alpha \wedge \neg\alpha) \wedge \circ\alpha$. Since *mbC* is a minimal *LFI* (Carnielli et al., 2007), it contains the consistency operator $\circ$. This operator is interpreted in such a way that it indicates when a certain formula α is metatheoretically well-behaved in the system from a *logical* perspective. By taking $\sim\alpha := (\alpha \to \bot_\alpha)$, we recover classical negation and can again define permission, denoted $\mathsf{P}\alpha$, as being equivalent to $\sim\mathsf{O}\sim\alpha$. This allows us to add $(\mathsf{O}-E)$ for characterizing *DmbC* instead of the usual deontic axiom $(\mathsf{O}-D)$:
$(\mathbf{O}-D)$ $\mathsf{O}\alpha \to \mathsf{P}\alpha$

So let us take $\bot$ to be a bottom formula in *CPL*. By our definition of $\mathsf{P}\alpha$, the following result ensues:

$$\mathsf{O}\alpha \to \sim\mathsf{O}\sim\alpha \equiv \mathsf{O}\alpha \to (\mathsf{O}\sim\alpha \to \bot) \equiv (\mathsf{O}\alpha \wedge \mathsf{O}\sim\alpha) \to \bot$$

and given that $\mathsf{O}\alpha \wedge \mathsf{O}\sim\alpha \equiv \mathsf{O}(\alpha \wedge \sim\alpha)$, then we get $\mathsf{O}(\alpha \wedge \sim\alpha) \to \bot$ or, equivalently, $\sim\mathsf{O}(\alpha \wedge \sim\alpha)$ (another standard way to represent the deontic axiom). In turn, if we define $f_\alpha := (\alpha \wedge \sim\alpha)$, then the last result is equivalent to $\mathsf{O}f_\alpha \to f_\alpha$. The equivalence used to obtain the last result is target for many criticisms in deontic logics, however, it will not be within the scope of this paper to address such criticisms.

2.1. Swap Kripke models for *DmbC*

Swap structures are multialgebras of a particular kind, defined over ordinary algebras. The domains of *swap* structures are the truth values of a certain logic, but presented as finite sequences of values of the underlying algebra. These sequences, called *snapshots*, represent (semantical) states of a given formula, described by the components of the sequence. For *DmbC*, the snapshots consists of pairs over the two-element Boolean algebra with domain $2 = \{0, 1\}$ representing the semantical state of a formula and of its paraconsistent negation $\neg$. The *consistency* (or *classicality*) operator $\circ$ is defined in terms of its relation with contradiction w.r.t. the paraconsistent negation.

DEFINITION 2.3. Let $\mathcal{A}_3 := \langle A, \tilde{\wedge}, \tilde{\vee}, \tilde{\neg}, \tilde{\to}, \tilde{O}, \tilde{\circ}\rangle$ be a multialgebra with domain $A = \{T, t, F\}$. Let $\mathcal{D} = \{T, t\}$ denote the designated truth values and define $\mathcal{M}_3 := \langle \mathcal{A}_3, \mathcal{D}\rangle$ to be an Nmatrix over signature Σ.

Remark 2.1. As mentioned above, the domain of the multialgebras in which we are interested is formed by pairs over 2 intending to represent the (simultaneous) values in $2 = \{0,1\}$ assigned to the formulas φ and $\neg\varphi$. That domain is the set of truth values for such logic. Thus, we let $A \subseteq 2^2$ and define $T := (1,0)$ (φ is true, $\neg\varphi$ is false); $t := (1,1)$ (φ is true, $\neg\varphi$ is true); and $F := (0,1)$ (φ is false, $\neg\varphi$ is true). We remove from the domain the pair $(0,0)$ (φ is false, $\neg\varphi$ is false) since the paraconsistent negation is assumed to satisfy the excluded-middle law (EM) (recall Definition 2.1). From now on, we mention whenever possible the snapshots instead of their labels. Notice that $\mathcal{D} = \{(1,0),(1,1)\} = \{z \in A \ : \ z_1 = 1\}$.

DEFINITION 2.4. The modal swap structure for *DmbC* is $\mathcal{A}_3$ (cf. Definition 2.3) such that its domain is $\mathcal{B}^{DmbC}_{\mathcal{A}_3} = \{(c_1,c_2) \in A : c_1 \sqcup c_2 = 1\}$ and the multioperations $\tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\neg}, \tilde{\circ}$, as well as a special multioperator $\tilde{O} : \wp_+(A) \rightarrow \wp_+(A)$,[4] are defined as follows, for every $a,b \in A$ and $\emptyset \neq X \subseteq A$:

1. $a\tilde{\wedge}b := \{(c_1,c_2) \in A \ : \ c_1 = a_1 \sqcap b_1\}$,
2. $a\tilde{\vee}b := \{(c_1,c_2) \in A \ : \ c_1 = a_1 \sqcup b_1\}$,
3. $a\tilde{\rightarrow}b := \{(c_1,c_2) \in A \ : \ c_1 = a_1 \supset b_1\}$,
4. $\tilde{\neg}a := \{(c_1,c_2) \in A \ : \ c_1 = a_2\}$,
5. $\tilde{\circ}a := \{(c_1,c_2) \in A \ : \ c_1 \leqslant {\sim}(a_1 \sqcap a_2)\}$,
6. $\tilde{\mathsf{O}}(X) := \{(c_1,c_2) \in A \ : \ c_1 = \bigsqcap\{x_1 \ : \ x \in X\}\}$.

Remark 2.2. The symbols $\sqcap, \sqcup, \supset, \sim$ refer to the Boolean operations of meet, join, implication and Boolean complement in 2, respectively. The symbol $\bigsqcap$ is applied to a non-empty subset of 2 and denotes the meet of all the elements of that set.

Remark 2.3. $\mathcal{B}^{DmbC}_{\mathcal{A}_3} = A$ via the analytical representation of the truth values, shown in the previous remark. The non-deterministic truth-tables for the non-modal operators are displayed below, where $\mathcal{U} = \{F\}$ is the set of non-designated truth values.

$\tilde{\wedge}$	T	t	F
T	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{U}$
t	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{U}$
F	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$

$\tilde{\vee}$	T	t	F
T	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
t	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
F	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{U}$

[4] In this paper, $\wp_+(Y)$ will denote the set of non-empty subsets of a set Y.

$\tilde{\to}$	T	t	F
T	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{U}$
t	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{U}$
F	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$

	$\tilde{\neg}$
T	$\mathcal{U}$
t	$\mathcal{D}$
F	$\mathcal{D}$

	$\tilde{\circ}$
T	A
t	$\mathcal{U}$
F	A

DEFINITION 2.5. Let W be a non-empty set (of *possible worlds*), and $R \subseteq W^2$ be an *accessibility* relation on W. For each $w, w' \in W$ a function $v_w \colon For(\Sigma) \to A$ is a *swap valuation* for *DmbC* if every condition below is satisfied, for $\alpha, \beta \in For(\Sigma)$:

1. $v_w(\alpha \wedge \beta) \in v_w(\alpha) \tilde{\wedge} v_w(\beta)$,
2. $v_w(\alpha \vee \beta) \in v_w(\alpha) \tilde{\vee} v_w(\beta)$,
3. $v_w(\alpha \to \beta) \in v_w(\alpha) \tilde{\to} v_w(\beta)$,
4. $v_w(\neg\alpha) \in \tilde{\neg} v_w(\alpha)$,
5. $v_w(\circ\alpha) \in \tilde{\circ} v_w(\alpha)$,
6. $v_w(\mathsf{O}\alpha) \in \tilde{\mathsf{O}}(\{v_{w'}(\alpha) : wRw'\})$.

DEFINITION 2.6. Let W be a non-empty set of worlds, $R \subseteq W^2$ be a serial accessibility relation[5] and $\{v_w\}_{w \in W}$ a family of swap valuations for *DmbC*. We say that the triple $\mathcal{M} = \langle W, R, \{v_w\}_{w \in W} \rangle$ is a *swap Kripke model* for the logic *DmbC*.

Remark 2.4. Let $i \in \{1, 2\}$ and $w \in W$. We define $\pi_i(v_w(\alpha))$ to be the projection of the pair $v_w(\alpha)$ on its i-th coordinate. For the sake of simplicity, we adopt the notation $\alpha_{(i,w)}$ to denote $\pi_i(v_w(\alpha))$.

LEMMA 2.1. *Let $w \in W$ and $\alpha, \beta \in For(\Sigma)$. Moreover, let v_w be as in Definition 2.5. Then*

1. $(\alpha \wedge \beta)_{(1,w)} = \alpha_{(1,w)} \sqcap \beta_{(1,w)}$,
2. $(\alpha \vee \beta)_{(1,w)} = \alpha_{(1,w)} \sqcup \beta_{(1,w)}$,
3. $(\alpha \to \beta)_{(1,w)} = \alpha_{(1,w)} \supset \beta_{(1,w)}$,
4. $(\neg\alpha)_{(1,w)} = \alpha_{(2,w)}$,
5. $(\circ\alpha)_{(1,w)} \leqslant \sim(\alpha_{(1,w)} \sqcap \alpha_{(2,w)})$,
6. $(O\alpha)_{(1,w)} = \bigsqcap\{\alpha_{(1,w')} \ : \ wRw'\}$.

PROOF. Items 1 through 5 are immediate from our definitions. For item 6, consider $\tilde{\mathsf{O}}(X_{w,\alpha})$ for $X_{w,\alpha} = \{v_{w'}(\alpha) : wRw'\}$. By Definition 2.4,

$$\begin{aligned} \tilde{\mathsf{O}}(X_{w,\alpha}) &= \{c \in A : c_1 = \bigsqcap\{x_1 : x \in X_{w,\alpha}\}\} \\ &= \{c \in A : c_1 = \bigsqcap\{\alpha_{(1,w')} : wRw'\}\}. \end{aligned}$$

Thus, our result follows by item 6 of Definition 2.5. ⊣

[5] Recall that a relation $R \subseteq W^2$ is *serial* if, for every $w \in W$, there exists $w' \in W$ such that wRw'.

Remark 2.5. By the very definitions, for any $w \in W$ and for any $\alpha \in For(\Sigma)$, $v_w(\alpha) \in \mathcal{D}$ if and only if $\alpha_{(1,w)} = 1$.

Remark 2.6. We can now give a clear picture of what our models will look like. In particular, we show how any model of *DmbC* satisfy axiom (O-E). It is easy to see that $v_w(\mathsf{O}\bot_\alpha) = v_w(\bot_\alpha)$. Indeed, by item 6 of Lemma 2.1, $(\mathsf{O}\bot_\alpha)_{(1,w)} = \bigsqcap\{(\bot_\alpha)_{(1,w')} : wRw'\} = \bigsqcap\{0 : wRw'\} = 0$, given that $(\bot_\alpha)_{(1,w')} = 0$ for every $w' \in W$ (by Lemma 2.1 items 1, 4 and 5), and $\{w' \in W : wRw'\} \neq \emptyset$, since R is serial.

DEFINITION 2.7. Let $\mathcal{M} = \langle W, R, \{v_w\}_{w \in W}\rangle$ be as in Definition 2.6. For any formula $\alpha \in For(\Sigma)$, we say that α is $\mathcal{M}$-true in a world w, denoted $\mathcal{M}, w \vDash \alpha$, if $v_w(\alpha) \in \mathcal{D}$.

DEFINITION 2.8. Let $\Gamma \cup \{\alpha\} \subseteq For(\Sigma)$. We say that α is a logical consequence of Γ in *DmbC*, denoted $\Gamma \vDash_{DmbC} \alpha$, if for all $\mathcal{M}$ for *DmbC* and $w \in W$: $\mathcal{M}, w \vDash \Gamma$ implies that $\mathcal{M}, w \vDash \alpha$.

THEOREM 2.1 (Soundness of *DmbC* w.r.t. swap Kripke models). *For every $\Gamma \cup \{\varphi\} \subseteq For(\Sigma)$, if $\Gamma \vdash_{DmbC} \varphi$, then $\Gamma \vDash_{DmbC} \varphi$.*

PROOF. We first show that the theorem holds for the axioms of *DmbC*. The result for the CPL^+ axioms follows from Lemma 2.1.

For (bc), we must show that $(\circ\alpha)_{(1,w)} \tilde{\to} (\alpha \to (\neg\alpha \to \beta))_{(1,w)} = 1$. Suppose that $(\circ\alpha)_{(1,w)} = \alpha_{(1,w)} = 1$. Hence, $\sim(\alpha_{(1,w)} \wedge \alpha_{(2,w)}) = 1$, and so $(\neg\alpha)_{(1,w)} = \alpha_{(2,w)} = 0$. From this, $(\neg\alpha \to \beta)_{(1,w)} = 1$ and so $(\alpha \to (\neg\alpha \to \beta))_{(1,w)} = 1$.

For (O-K), assume $(\mathsf{O}(\alpha \to \beta))_{(1,w)} = 1$ and that $(\mathsf{O}\alpha)_{(1,w)} = 1$. We then have that $(\alpha \to \beta)_{(1,w')} = 1$ and that $\alpha_{(1,w')} = 1$ for every $w' \in W$ such that wRw'. Hence, it follows that $\beta_{(1,w')} = 1$ for every w' such that wRw', i.e., $(\mathsf{O}\beta)_{(1,w)} = 1$.

The proof for (O-E) follows from Remark 2.6.

For Modus Ponens, it is immediate to see that it satisfies the criteria, by definition of $\tilde{\to}$. For Necessitation, suppose α is a theorem. Then for every $w \in W$, $\alpha_{(1,w)} = 1$. In particular, it is the case for every $w' \in W$ such that wRw', from which it follows that $(\mathsf{O}\alpha)_{(1,w)} = 1$. ⊣

In order to prove completeness for *DmbC*, we build canonical models based on swap structures. We use the method of ψ-saturation for construction of maximal consistent sets, together with the denecessitation for the accessibility relation.

Definition 2.9. Given a Tarskian and finitary[6] logic $\boldsymbol{L}$, a set of formulas Δ is ψ-saturated in $\boldsymbol{L}$ if $\Delta \nvdash \psi$ and, if $\varphi \notin \Delta$, then $\Delta \cup \{\varphi\} \vdash \psi$.

Remark 2.7. It is well-known that any ψ-saturated set is a closed theory. Moreover, if $\Gamma \nvdash \psi$ in $\boldsymbol{L}$ then there exists a set Δ which is ψ-saturated in $\boldsymbol{L}$ and contains Γ. In particular, this property holds for *DmbC* and all the other logics to be considered in this paper.

Definition 2.10. Consider the set

$$W_{can} = \{\Delta \subseteq For(\Sigma) : \ \Delta \text{ is a } \psi\text{-saturated set in } DmbC, \\ \text{for some } \psi \in For(\Sigma)\}.$$

Definition 2.11. Let $Den(\Delta) := \{\varphi \in For(\Sigma) \ : \ O\varphi \in \Delta\}$.

Definition 2.12. Let $R_{can} \subseteq W \times W$ be given for all for $\Delta, \Theta \in W$, by:

$$\Delta R_{can} \Theta \text{ iff } Den(\Delta) \subseteq \Theta.$$

Definition 2.13. For each $\Delta \in W_{can}$, let $v_\Delta\colon For(\Sigma) \to \mathcal{A}_3$ defined as follows:

$$v_\Delta(\alpha) = \begin{cases} T, & \text{if } \alpha \in \Delta, \neg\alpha \notin \Delta \\ t, & \text{if } \alpha, \neg\alpha \in \Delta \\ F, & \text{if } \neg\alpha \in \Delta, \alpha \notin \Delta \end{cases}$$

Lemma 2.2. *For any $\Delta \in W_{can}$, the following holds:*

1. $\alpha \wedge \beta \in \Delta$ *iff* $\alpha, \beta \in \Delta$
2. $\alpha \vee \beta \in \Delta$ *iff* $\alpha \in \Delta$ *or* $\beta \in \Delta$
3. $\alpha \to \beta \in \Delta$ *iff* $\alpha \notin \Delta$ *or* $\beta \in \Delta$
4. *if* $\neg\alpha \notin \Delta$ *then* $\alpha \in \Delta$
5. *if* $\alpha \in \Delta$ *and* $\neg\alpha \in \Delta$ *then* $\circ\alpha \notin \Delta$
6. $O\alpha \in \Delta$ *iff* $\alpha \in \Delta'$ *for all* $\Delta' \in W$ *such that* $\Delta R_{can} \Delta'$.

Proof. Items 1 through 5 are immediate from the definitions and the fact that Δ is ψ-saturated, hence, by Remark 2.7, it is closed under

[6] A logic $\boldsymbol{L}$ is Tarskian if, for any set of formulas Γ, Δ and formulas φ, ψ, the following holds:

- Reflexivity: for every $\varphi \in \Gamma$, $\Gamma \vdash \varphi$;
- Monotonicity: if $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$;
- Cut: if $\Gamma \vdash \varphi$ for every $\varphi \in \Delta$ and $\Delta \vdash \psi$, then $\Gamma \vdash \psi$.

$\boldsymbol{L}$ is finitary if it satisfies

- Finiteness: $\Gamma \vdash \varphi$ implies $\Gamma_0 \vdash \varphi$, for some finite $\Gamma_0 \subseteq \Gamma$.

logical consequences. In particular, it contains any instance of the axioms of *DmbC*, and it is closed under Modus Ponens.

To prove the right-to-left direction for 6, assume that $\mathsf{O}\alpha \notin \Delta$. Since Δ is closed under logical consequences, it follows that $\nvdash_{DmbC} \alpha$, because of the O-necessitation rule. Suppose, for a contradiction, that $Den(\Delta) \vdash_{DmbC} \alpha$. By Definition 2.2, there are $\beta_1, \ldots, \beta_n \in Den(\Delta)$ (for $n \geqslant 1$) such that $\vdash_{DmbC} \beta \to \alpha$, where $\beta = \beta_1 \wedge \cdots \wedge \beta_n$. By applying necessitation and (K), $\vdash_{DmbC} \mathsf{O}\beta \to \mathsf{O}\alpha$. Observe now that $\mathsf{O}\beta_i \in \Delta$, by definition of $Den(\Delta)$, hence $\mathsf{O}\beta_1 \wedge \cdots \wedge \mathsf{O}\beta_n \in \Delta$, by item 1. But

$$\vdash_{DmbC} (\mathsf{O}\beta_1 \wedge \cdots \wedge \mathsf{O}\beta_n) \to \mathsf{O}\beta.$$

So, $\mathsf{O}\beta \in \Delta$. Using again that Δ is closed under logical consequences, we infer that $\mathsf{O}\alpha \in \Delta$, which contradicts our initial assumption.

We conclude, therefore, that $Den(\Delta) \nvdash_{DmbC} \alpha$. But then, there is some Δ' such that $Den(\Delta) \subseteq \Delta'$ and Δ' is α-saturated. Therefore, there is $\Delta' \in W_{can}$ such that $\Delta R_{can} \Delta'$ and $\alpha \notin \Delta'$.

The proof of the left-to-right direction for 6 is immediate from the definitions. Indeed, if $\mathsf{O}\alpha \in \Delta$ and $\Delta R_{can} \Delta'$ then $\alpha \in \Delta'$, given that $\alpha \in Den(\Delta)$. ⊣

PROPOSITION 2.1. *The triple $\mathcal{M} = \langle W_{can}, R_{can}, \{v_\Delta\}_{\Delta \in W_{can}} \rangle$ is a swap Kripke model for DmbC such that $v_\Delta(\alpha) \in \mathcal{D}$ if and only if $\alpha \in \Delta$ if and only if $\alpha_{(1,\Delta)} = 1$.*

PROOF. Observe first that R_{can} is serial. To see this, let $\Delta \in W_{can}$. Then, Δ is φ-saturated, for some formula φ. Suppose that $\mathsf{O}\alpha \in \Delta$, for every formula α. Then, $\mathsf{O}\alpha, \mathsf{O}\neg\alpha, \mathsf{O}{\circ}\alpha \in \Delta$ and so, since Δ is a closed theory, $\mathsf{O}\bot_\alpha \in \Delta$. This shows that $\bot_\alpha \in \Delta$, by $(\mathsf{O}{-}E)$ and so $\varphi \in \Delta$, a contradiction. From this, $\mathsf{O}\alpha \notin \Delta$ for some α. By reasoning as in the proof of item 6 of Lemma 2.2, we infer that $Den(\Delta) \nvdash_{DmbC} \alpha$ and so there exists some Δ' such that $Den(\Delta) \subseteq \Delta'$ and $\alpha \notin \Delta'$. This shows that R_{can} is serial. The rest of the proof is an immediate consequence from Remark 2.5 and Definition 2.13. ⊣

THEOREM 2.2 (Completeness of *DmbC* w.r.t. swap Kripke models). *For any set $\Gamma \cup \{\varphi\} \subseteq For_\Sigma$, if $\Gamma \vDash_{DmbC} \varphi$, then $\Gamma \vdash_{DmbC} \varphi$.*

PROOF. Suppose, to the contrary, that $\Gamma \nvdash_{DmbC} \varphi$. Thus, by Remark 2.7, there is a φ-saturated $\Delta \in W_{can}$ such that $\Gamma \subseteq \Delta$. Since $\Delta \nvdash_{DmbC} \varphi$, then $\varphi \notin \Delta$. Let $\mathcal{M}$ be the canonical swap model for

DmbC. Then, $\mathcal{M}, \Delta \vDash \Gamma$, since $\Gamma \subseteq \Delta$, but $\mathcal{M}, \Delta \nvDash \varphi$. This proves that $\Gamma \nvDash_{DmbC} \varphi$. ⊣

3. Some extensions of *DmbC*

DEFINITION 3.1. Consider the following axioms over the signature Σ:

$$\circ\alpha \vee (\alpha \wedge \neg\alpha), \tag{ciw}$$

$$\neg\circ\alpha \to (\alpha \wedge \neg\alpha), \tag{ci}$$

$$\neg\neg\alpha \to \alpha. \tag{cf}$$

The following systems can be thus defined:

- $DmbCciw := DmbC \cup \{(\text{ciw})\}$,
- $DmbCci := DmbC \cup \{(\text{ci})\}$,
- $DbC := DmbC \cup \{(\text{cf})\}$,
- $DCi := DmbCci \cup \{(\text{cf})\}$.

Remark 3.1. This section will talk about results that can be easily adaptable to each of the systems above. Let $\boldsymbol{L}$ belong to $\{DmbCciw, DmbCci, DbC, DCi\}$. The notion of derivation $\Gamma \vdash_{\boldsymbol{L}} \varphi$ in $\boldsymbol{L}$ is as in Definition 2.2 (with the corresponding set of axioms of each logic).

The multialgebras to accommodate each of the axioms are defined as follows:

DEFINITION 3.2. Let $\mathcal{A}_3$ be the swap structure for $DmbC$. The multioperators of the swap structure for DbC are defined as in Definition 2.4, with the exception of $\tilde{\neg}$, which is substituted for

$$\tilde{\neg}_1 a = \{c \in A \ : \ c_1 = a_2 \text{ and } c_2 \leqslant a_1\}.$$

The non-deterministic truth-table for $\tilde{\neg}_1$ is as follows:

	$\tilde{\neg}_1$
T	$\{F\}$
t	$\{T, t\}$
F	$\{T\}$

DEFINITION 3.3. Let $\mathcal{A}_3$ be the swap structure for $DmbC$. The multioperators of the swap structure for $DmbCciw$, $DmbCci$ and DCi are defined as in Definition 2.4, with exception of the multioperators that are mentioned in this definition, which are substituted accordingly.

1. In *DmbCciw*: $\tilde{\circ}_1 a := \{c \in A \ : \ c_1 = \sim(a_1 \sqcap a_2)\}$.
2. In *DmbCci*: $\tilde{\circ}_2 a = \{(\sim(a_1 \sqcap a_2), a_1 \sqcap a_2)\}$.
3. In *DCi*, take $\tilde{\circ}_2$ from *DmbCci* and $\tilde{\neg}_1$ from *DbC*.

The non-deterministic truth-tables for $\tilde{\circ}_1$ and $\tilde{\circ}_2$ are as follows:

	$\tilde{\circ}_1$
T	$\{T, t\}$
t	$\{F\}$
F	$\{T, t\}$

	$\tilde{\circ}_2$
T	$\{T\}$
t	$\{F\}$
F	$\{T\}$

DEFINITION 3.4. For each $w \in W$, we establish the following:

- The swap valuations for *DmbCciw*, $v_w^{DmbCciw}$, are defined as in Definition 2.5 for all operators, except for $\circ$, which satisfies the following condition:

$$v_w^{DmbCciw}(\circ\alpha) \in \tilde{\circ}_1 v_w^{DmbCciw}(\alpha).$$

- The swap valuations for *DmbCci*, v_w^{DmbCci}, are defined as in Definition 2.5 for all operators, except for $\circ$, which satisfies the following condition:

$$v_w^{DmbCci}(\circ\alpha) \in \tilde{\circ}_2 v_w^{DmbCci}(\alpha).$$

- The swap valuations for *DbC*, v_w^{DbC}, are defined as in Definition 2.5 for all operators, except for $\neg$, which satisfies the following condition:

$$v_w^{DbC}(\neg\alpha) \in \tilde{\neg}_1 v_w^{DbC}(\alpha).$$

- The swap valuations for *DCi*, v_w^{DCi}, are defined as in Definition 2.5 for all operators, except for $\neg$, which is defined using $\tilde{\neg}_1$ as in the case of *DbC* and for $\circ$, which is defined using $\tilde{\circ}_2$ as in the case of *DmbCci*.

DEFINITION 3.5. The structure $\mathcal{M} = \langle W, R, \{v_w^{\boldsymbol{L}}\}_{w \in W}\rangle$ is a swap Kripke model for the logic $\boldsymbol{L}$.

We maintain an analogous notation to the one presented in Remark 2.4. Notice that this implies that $v_w^{\boldsymbol{L}}(\alpha) \in \mathcal{D}$ if and only if $\alpha_{(1,w)} = 1$. We use this fact in the next proof.

LEMMA 3.1. *Conditions 1–6 listed in Lemma 2.1 hold for $\boldsymbol{L}$. Moreover, consider the following conditions:*

5′. $(\circ\alpha)_{(1,w)} = \sim(\alpha_{(1,w)} \sqcap \alpha_{(2,w)})$.
5*. $(\circ\alpha)_{(1,w)} = \sim(\alpha_{(1,w)} \sqcap \alpha_{(2,w)})$ *and* $(\circ\alpha)_{(2,w)} = (\alpha_{(1,w)} \sqcap \alpha_{(2,w)})$.
7. $(\neg\alpha)_{(2,w)} \leqslant \alpha_{(1,w)}$.

Then, condition 5′ holds in DmbCciw; condition 5 holds in DmbCci; condition 7 holds in DbC; and conditions 5* and 7 hold in DCi.*

PROOF. For *DbC*, condition 7 follows by Definitions 3.2 and 3.4, that is: $v_w^{DbC}(\neg\alpha) \in \tilde{\neg}_1 v_w^{DbC}(\alpha)$, and so $(\neg\alpha)_{(2,w)} \leqslant \alpha_{(1,w)}$. For *DmbCciw* and *DmbCci*, condition $5'$ and 5^* follow, respectively, by Definition 3.3 and Definition 3.4. The case of *DCi* follows from *DmbCci* and *DbC*. ⊣

DEFINITION 3.6. Let $\mathcal{M} = \langle W, R, \{\nu_w^{\boldsymbol{L}}\}_{w\in W}\rangle$ be as above. Then, a formula $\alpha \in For_\Sigma$ is said to be $\mathcal{M}$-true in a world w, denoted by $\mathcal{M}, w \vDash \alpha$, if it is the case that $\nu_w^{\boldsymbol{L}}(\alpha) \in \mathcal{D}$.

DEFINITION 3.7. Let $\Gamma \cup \{\alpha\} \subseteq For_\Sigma$. We say that α is a logical consequence of Γ in $\boldsymbol{L}$, denoted by $\Gamma \vDash_{\boldsymbol{L}} \alpha$, if for all $\mathcal{M}$ for $\boldsymbol{L}$ and $w \in W$: $\mathcal{M}, w \vDash \Gamma$ implies that $\mathcal{M}, w \vDash \alpha$.

THEOREM 3.1 (Soundness).
For every $\Gamma \cup \{\varphi\} \subseteq For(\Sigma)$, if $\Gamma \vdash_{\boldsymbol{L}} \varphi$, then $\Gamma \vDash_{\boldsymbol{L}} \varphi$.

PROOF. Consider first *DmbCciw*. Given a valuation v_w and a formula α, it must be shown that $(\circ\alpha \vee (\alpha \wedge \neg\alpha))_{(1,w)} = 1$. By the definition of the multioperator $\tilde{\vee}$ in Definition 2.4, the latter is equivalent to prove that either $(\circ\alpha)_{(1,w)} = 1$ or $(\alpha \wedge \neg\alpha)_{(1,w)} = 1$. But this is immediate, by property $5'$ of v_w given in Lemma 3.1 and the fact that $(\alpha \wedge \neg\alpha)_{(1,w)} = \alpha_{(1,w)} \sqcap \alpha_{(2,w)}$.

In *DmbCci*, it must be shown that (ci) is valid. Given v_w, assume that $(\neg\circ\alpha)_{(1,w)} = 1$. Thus, we have $(\neg\circ\alpha)_{(1,w)} = (\circ\alpha)_{(2,w)} = (\alpha_{(1,w)} \sqcap \alpha_{(2,w)}) = 1$, by property 5^* of v_w. This shows that v_w satisfies any instance of axiom (ci).

For *DbC*, the case for (cf) follows from property 7 in Lemma 3.1.

Soundness of *DCi* follows from soundness of *DmbCci* and *DbC*. ⊣

For every logic $\boldsymbol{L}$ as above, we define $W_{can}^{\boldsymbol{L}}, R_{can}^{\boldsymbol{L}}$ and $\nu_\Delta^{\boldsymbol{L}}$ following the definitions for *DmbC* and adapting to each $\boldsymbol{L}$ accordingly. Notice that each Δ is now a ψ-saturated set in $W_{can}^{\boldsymbol{L}}$ This allows us to state the following lemma:

LEMMA 3.2. *For any $\Delta \in W_{can}^{\boldsymbol{L}}$, all statements 1 through 6 in Lemma 2.2 hold. For DmbCciw, we have the following strengthening of item 5:*

5_1^+. $\alpha \in \Delta$ and $\neg\alpha \in \Delta$ iff $\circ\alpha \notin \Delta$

For DmbCci, we have an additional condition for $\circ$:

5_2^+. If $\neg\circ\alpha \in \Delta$, then $\alpha \in \Delta$ and $\neg\alpha \in \Delta$.

For DbC, we have an additional condition for $\neg$:

5_3^+. If $\neg\neg\alpha \in \Delta$, then $\alpha \in \Delta$.

For DCi, both conditions 5_2^+ and 5_3^+ are added.

PROOF. All conditions are easily proven by using the respective new axiom of $\boldsymbol{L}$, and the fact that Δ is saturated (hence it is a closed theory). ⊣

PROPOSITION 3.1. *The triple* $\mathcal{M}_{\boldsymbol{L}} = \{W^{\boldsymbol{L}}_{can}, R^{\boldsymbol{L}}_{can}, \{\nu^{\boldsymbol{L}}_{\Delta}\}_{\Delta \in W_{can}}\}$ *is a swap Kripke model for* $\boldsymbol{L}$ *such that* $\nu^{\boldsymbol{L}}_{\Delta}(\alpha) \in \mathcal{D}$ *if and only if* $\alpha \in \Delta$ *if and only if* $\alpha_{(1,\Delta)} = 1$.

PROOF. It is an immediate consequence from Lemma 3.2 and Definition 2.13. For instance, to prove that $\nu^{DmbCci}_{\Delta}(\circ\alpha) \in \tilde{\circ}_2 \nu^{DmbCci}_{\Delta}(\alpha)$, let $z := \nu^{DmbCci}_{\Delta}(\alpha)$. Suppose first that $z \in \{T, F\}$. By Definition 2.13, either $\alpha \notin \Delta$ or $\neg\alpha \notin \Delta$. By 5^+_1 and 5^+_2 of Lemma 3.2, $\circ\alpha \in \Delta$ and $\neg\circ\alpha \notin \Delta$. From this, $\nu^{DmbCci}_{\Delta}(\circ\alpha) = T \in \{T\} = \tilde{\circ}_2 z$. Now, if $z = t$ then $\alpha, \neg\alpha \in \Delta$ and so, by 5^+_1, $\circ\alpha \notin \Delta$. Hence, $\nu^{DmbCci}_{\Delta}(\circ\alpha) = F \in \{F\} = \tilde{\circ}_2 z$. In turn, in order to prove that $\nu^{DbC}_{\Delta}(\neg\alpha) \in \tilde{\neg}_1 \nu^{DbC}_{\Delta}(\alpha)$, let $z := \nu^{DbC}_{\Delta}(\alpha)$. If $z = T$ then $\alpha \in \Delta$ and $\neg\alpha \notin \Delta$. From this, $\nu^{DbC}_{\Delta}(\neg\alpha) = F \in \{F\} = \tilde{\neg}_1 z$. If $z = t$ then $\alpha, \neg\alpha \in \Delta$. From this, $\nu^{DbC}_{\Delta}(\neg\alpha) \in \{T, t\} = \tilde{\neg}_1 z$. Finally, if $z = F$ then $\alpha \notin \Delta$ and $\neg\alpha \in \Delta$. By 5^+_3 of Lemma 3.2, $\neg\neg\alpha \notin \Delta$ and so $\nu^{DbC}_{\Delta}(\neg\alpha) = T \in \{T\} = \tilde{\neg}_1 z$. The other cases are treated analogously. ⊣

The proof of the following theorem is analogous to the one for the *DmbC* case.

THEOREM 3.2 (Completeness).
For any set $\Gamma \cup \{\varphi\} \subseteq For(\Sigma)$, *if* $\Gamma \vDash_{\boldsymbol{L}} \varphi$, *then* $\Gamma \vdash_{\boldsymbol{L}} \varphi$.

4. The da Costa axiom: the case of *DmbCcl*

In this section, as well as in Sections 5, 6 and 7, we will consider axiomatic extensions of *DmbC* which include, among others, the so-called *da Costa axiom*

$$\neg(\alpha \wedge \neg\alpha) \to \circ\alpha. \qquad \text{(cl)}$$

This move has strong consequences: as it was shown in (Avron, 2007, Theorem 11), the logic *mbCcl*, obtained by adding (cl) to *mbC*, cannot be semantically characterized by a single finite *N*matrix.[7] As shown

[7] This fact is also applicable to other extensions of *mbCcl*. An important system that extends *mbCcl* is *Cila*, the version of da Costa's system C_1 in a signature with $\circ$.

in (Coniglio and Toledo, 2022), this issue can be overcome by considering a suitable (finite–valued) Nmatrix and restrict the set of permitted valuations by a (decidable) criterion, through the notion of *restricted Nmatrices* (or RNmatrices). We will adapt this technique to our swap Kripke models, in order to deal with the deontic expansions of *mbCcl*.

The logic *DmbCcl* is the extension of *DmbC* by adding axiom (cl). It is easy to show that *DmbCcl* is a proper extension of *DmbCciw*: this follows from the fact that the system *mbCcl* is a proper extension of the system *mbCciw*, which is obtained from *mbC* by adding (ciw) (Carnielli and Coniglio, 2016, Corollary 3.3.30). We thus present a swap Kripke semantics for *DmbCcl* as the corresponding one for *DmbCciw*, together with a restriction on their valuations.

DEFINITION 4.1. A swap Kripke model $\mathcal{M} = \langle W, R, \{v_w^{DmbCcl}\}_{w\in W}\rangle$ for *DmbCcl* is a swap Kripke model for *DmbCciw* such that each valuation v_w^{DmbCcl} satisfies, in addition, the following condition:

$$\text{If } v_w^{DmbCcl}(\alpha) = t, \ \text{ then } \ v_w^{DmbCcl}(\alpha \wedge \neg\alpha) = T.$$

DEFINITION 4.2. Let $\mathcal{M} = \langle W, R, \{v_w^{DmbCcl}\}_{w\in W}\rangle$ be a swap Kripke model for *DmbCcl*. We say that a formula $\alpha \in For(\Sigma)$ is $\mathcal{M}$-true in a world w, denoted by $\mathcal{M}, w \vDash \alpha$, if $v_w^{DmbCcl}(\alpha) \in \mathcal{D}$.

DEFINITION 4.3. Let $\Gamma \cup \{\alpha\} \subseteq For(\Sigma)$. We say that α is a logical consequence of Γ in *DmbCcl*, denoted by $\Gamma \vDash_{DmbCcl} \alpha$, if for all $\mathcal{M}$ for *DmbCcl* and all $w \in W$: $\mathcal{M}, w \vDash \Gamma$ implies that $\mathcal{M}, w \vDash \alpha$.

The following lemma will be useful for showing soundness of *DmbCcl* w.r.t. swap Kripke models semantics.

LEMMA 4.1. *Given the notation on Remark 2.4, let v_w^{DmbCcl} be a valuation in a swap Kripke model $\mathcal{M}$ for DmbCcl. Then, the following holds, for every formula α:*

5**. $(\circ\alpha)_{(1,w)} = (\alpha \wedge \neg\alpha)_{(2,w)}$.

Hence, any instance of axiom (cl) is true in any world of any swap Kripke model for DmbCcl.

PROOF. By Lemma 3.1, $(\circ\alpha)_{(1,w)} = {\sim}(\alpha_{(1,w)} \sqcap \alpha_{(2,w)})$. Suppose that $(\circ\alpha)_{(1,w)} = 1$. Then, $\alpha_{(1,w)} \sqcap \alpha_{(2,w)} = (\alpha \wedge \neg\alpha)_{(1,w)} = 0$. By definition of A, it follows that $(\alpha \wedge \neg\alpha)_{(2,w)} = 1 = (\circ\alpha)_{(1,w)}$. Now, suppose that $(\circ\alpha)_{(1,w)} = 0$. Then, $\alpha_{(1,w)} \sqcap \alpha_{(2,w)} = \alpha_{(1,w)} \sqcap (\neg\alpha)_{(1,w)} = (\alpha \wedge \neg\alpha)_{(1,w)}$

$= 1$. From this, $\alpha_{(1,w)} = \alpha_{(2,w)} = 1$, which means that $v_w^{DmbCcl}(\alpha) = (1,1) = t$. By Definition 4.1, $v_w^{DmbCcl}(\alpha \wedge \neg\alpha) = T = (1,0)$. Hence, $(\alpha \wedge \neg\alpha)_{(2,w)} = 0 = (\circ\alpha)_{(1,w)}$.

The latter shows that any instance of axiom (cl) is true in any world of any swap Kripke model for *DmbCcl*. ⊣

From soundness of *DmbCciw* and Lemma 4.1 we get:

THEOREM 4.1 (Soundness of *DmbCcl* w.r.t. swap Kripke models). *For every* $\Gamma \cup \{\varphi\} \subseteq For(\Sigma)$, *if* $\Gamma \vdash_{DmbCcl} \varphi$, *then* $\Gamma \vDash_{DmbCcl} \varphi$.

In order to prove completeness for *DmbCcl*, we will use canonical models, to be constructed as in the case for *DmbCciw*. Since the restriction occurs only on the valuations, we take only those valuations that are restricted appropriately. Hence, W_{can}^{DmbCcl}, R_{can}^{DmbCcl} and ν_{Δ}^{DmbCcl} are defined as in *DmbCciw*. Now, each $\Delta \in W_{can}^{DmbCcl}$ is a ψ-saturated set in *DmbCcl*. Thus we have:

LEMMA 4.2. *For any* $\Delta \in W_{can}^{DmbCcl}$, *all statements for DmbCciw stated in Lemma 3.2 hold. Besides, we add the following statement:*

7. *If* $\neg(\alpha \wedge \neg\alpha) \in \Delta$, *then* $\circ\alpha \in \Delta$.

PROOF. Given that *DmbCcl* extends *DmbCciw*, it is an immediate consequence of Lemma 3.2, axiom (cl), and the fact that Δ is a closed theory. ⊣

PROPOSITION 4.1. *The triple*

$$\mathcal{M} = \{W_{can}^{DmbCcl}, R_{can}^{DmbCcl}, \{\nu_{\Delta}^{DmbCcl}\}_{\Delta \in W_{can}^{DmbCcl}}\},$$

constructed as in the case of DmbC, is a swap Kripke model for DmbCcl such that $\nu_{\Delta}^{DmbCcl}(\alpha) \in \mathcal{D}$ *if and only if* $\alpha \in \Delta$ *if and only if* $\alpha_{(1,\Delta)} = 1$.

PROOF. Observe that each valuation ν_{Δ}^{DmbCcl} is defined according to Definition 2.13. Since *DmbCcl* extends *DmbCciw*, it follows that $\mathcal{M}$ is a swap Kripke model for *DmbCciw* such that $\nu_{\Delta}^{DmbCcl}(\varphi) \in \mathcal{D}$ if and only if $\varphi \in \Delta$ if and only if $\alpha_{(1,\Delta)} = 1$. In order to show that each ν_{Δ}^{DmbCcl} satisfies the additional condition of Definition 4.1, suppose that $\nu_{\Delta}^{DmbCcl}(\alpha) = t$. By Definition 2.13, $\alpha, \neg\alpha \in \Delta$ and so $\alpha \wedge \neg\alpha \in \Delta$. Suppose that $\neg(\alpha \wedge \neg\alpha) \in \Delta$. By Lemma 4.2, $\circ\alpha \in \Delta$. But then, by axiom (bc), $\beta \in \Delta$, for every formula β, a contradiction. From this, $\neg(\alpha \wedge \neg\alpha) \notin \Delta$, therefore $\nu_{\Delta}^{DmbCcl}(\alpha \wedge \neg\alpha) = T$, by Definition 2.13. This shows that $\mathcal{M}$ is, in fact, a swap Kripke model for *DmbCcl*. ⊣

From the previous results and the construction above, it is easy to show that completeness holds for *DmbCcl*.

THEOREM 4.2 (Completeness of *DmbCcl* w.r.t. swap Kripke models). *For any set* $\Gamma \cup \{\varphi\} \subseteq For(\Sigma)$*, if* $\Gamma \vDash_{DmbCcl} \varphi$*, then* $\Gamma \vdash_{DmbCcl} \varphi$.

5. Swap Kripke models for *DCila*

Consider the following axiom schemas for *consistency propagation* for $\# \in \{\wedge, \vee, \rightarrow\}$:

$$(\circ\alpha \wedge \circ\beta) \rightarrow \circ(\alpha\#\beta). \qquad (\text{ca}_\#)$$

We now add to *DmbC* the axioms (ci), (cl), (cf), and ($\text{ca}_\#$), obtaining a logic called *DCila*. Equivalently, *DCila* is obtained from *DmbCcl* by adding axioms (ci), (cf), and ($\text{ca}_\#$). The non-modal fragment of *DCila* is called *Cila*, and corresponds to da Costa logic C_1 presented over the signature with $\circ$ (see Carnielli et al., 2007, Section 5.2). Indeed, *Cila* is a conservative expansion of da Costa's C_1. As proved in (Avron, 2007, Theorem 11 and Corollary 6), *Cila* and C_1 are not characterizable by a single finite *N*matrix.

Based on the results presented in the previous section, as well as the characterization of *Cila* in terms of a 3-valued *RN*matrix found in (Coniglio and Toledo, 2022), in the sequel we will characterize *DCila* by means of swap Kripke models based on a suitable 3-valued *RN*matrix for *Cila*.

DEFINITION 5.1. A swap Kripke model $\mathcal{M} = \langle W, R, \{v_w^{DCila}\}_{w \in W}\rangle$ for *DCila* is a swap Kripke model for *DCi* such that each valuation v_w^{DCila} satisfies the following conditions, for $\# \in \{\wedge, \vee, \rightarrow\}$:

$$\text{If } v_w^{DCila}(\alpha) = t, \text{ then } v_w^{DCila}(\alpha \wedge \neg\alpha) = T.^{8}$$

$$\text{If } v_w^{DCila}(\alpha), v_w^{DCila}(\beta) \in \{T, F\}, \text{ then } v_w^{DCila}(\alpha\#\beta) \in \{T, F\}.$$

The notions of satisfaction of a formula α in a world w of a swap Kripke model $\mathcal{M}$ for *DCila*, denoted by $\mathcal{M}, w \vDash \alpha$, as well as the semantical consequence of *DCila* w.r.t. swap Kripke models, denoted by $\vDash_{DCila}$, are defined as in the previous cases.

The above definitions guarantee that the axioms (cl) and ($\text{ca}_\#$) hold.

[8] This condition coincides with the one for v_w^{DmbCcl}.

Lemma 5.1. *Any instance of the axioms* (cl) *and* ($\text{ca}_{\#}$) *are true in any world of any swap Kripke model for $DCila$.*

Proof. Concerning (cl), the result holds by Lemma 4.1 and Definition 5.1. Fix now $\# \in \{\wedge, \vee, \to\}$ and v_w^{DCila}. Observe that, for any α, $v_w^{DCila}(\alpha) \in \{T, F\}$ iff $\alpha_{(1,w)} \neq \alpha_{(2,w)}$ iff $\alpha_{(1,w)} \sqcap \alpha_{(2,w)} = 0$ iff $(\circ\alpha)_{(1,w)} = {\sim}(\alpha_{(1,w)} \sqcap \alpha_{(2,w)}) = 1$. From this, $v_w^{DCila}(\circ\alpha \wedge \circ\beta) \in \mathcal{D}$ implies that $(\circ\alpha \wedge \circ\beta)_{(1,w)} = 1$, which implies that $(\circ\alpha)_{(1,w)} = (\circ\beta)_{(1,w)} = 1$. As observed above, the latter implies that $v_w^{DCila}(\alpha), v_w^{DCila}(\beta) \in \{T, F\}$, and so $v_w^{DCila}(\alpha\#\beta) \in \{T, F\}$, by Definition 5.1. But this implies that $(\circ(\alpha\#\beta))_{(1,w)} = 1$, that is, $v_w^{DCila}(\circ(\alpha\#\beta)) \in \mathcal{D}$. This shows that any instance of axiom ($\text{ca}_{\#}$) is true in any world of any swap Kripke model for $DCila$. ⊣

As shown in (Coniglio and Toledo, 2022), the above characterization of $Cila$ by means of a 3-valued RNmatrix induces a decision procedure for this logic. Given that Standard Deontic Logic SDL is decidable (for instance, by tableaux systems), so is its modal extension $DCila$.

Now, soundness of $DCila$ w.r.t. swap Kripke models follows from the previous results. From soundness of DCi and Lemma 5.1 we get:

Theorem 5.1 (Soundness of $DCila$ w.r.t. swap Kripke models). *For every $\Gamma \cup \{\varphi\} \subseteq For(\Sigma)$, if $\Gamma \vdash_{DCila} \varphi$, then $\Gamma \vDash_{DCila} \varphi$.*

The proof of completeness is a straightforward adaptation of the case for $DmbCcl$, by building the canonical model as in the case for DCi, and by imposing suitable restrictions on the valuations. Thus, W_{can}^{DCila}, R_{can}^{DCila} and $\nu_\Delta^{DCila}$are defined as in DCi, but now each $\Delta \in W_{can}^{DCila}$ is a ψ-saturated set in $DCila$.

Lemma 5.2. *For any $\Delta \in W_{can}^{DCila}$, all statements for DCi stated in Lemma 3.2 hold. Besides, Δ satisfies the following statements:*

7. *If $\neg(\alpha \wedge \neg\alpha) \in \Delta$, then $\circ\alpha \in \Delta$.*
8. *If $\circ\alpha, \circ\beta \in \Delta$, then $\circ(\alpha\#\beta) \in \Delta$, where $\# \in \{\wedge, \vee, \to\}$.*

Proof. $DmbCcl$ extends DCi. From this, the result is an immediate consequence of Lemma 3.2, axioms (cl) and ($\text{ca}_{\#}$), and the fact that Δ is a closed theory. ⊣

Proposition 5.1. *The triple*

$$\mathcal{M} = \{W_{can}^{DCila}, R_{can}^{DCila}, \{\nu_\Delta^{DCila}\}_{\Delta \in W_{can}^{DCila}}\},$$

constructed as in the case of $DmbC$, is a swap Kripke model for $DCila$ such that $\nu_{\Delta}^{DCila}(\alpha) \in \mathcal{D}$ if and only if $\alpha \in \Delta$ if and only if $\alpha_{(1,\Delta)} = 1$.

PROOF. Notice that each valuation ν_{Δ}^{DCila} is defined according to Definition 2.13. Given that $DCila$ extends DCi, it follows that $\mathcal{M}$ is a swap Kripke model for DCi such that $\nu_{\Delta}^{DCila}(\varphi) \in \mathcal{D}$ if and only if $\varphi \in \Delta$ if and only if $\alpha_{(1,\Delta)} = 1$. Let us prove now that every ν_{Δ}^{DCila} satisfies the additional conditions of Definition 5.1. The first condition is proved analogously to the case for $DmbCcl$ (see the proof o Proposition 4.1). In order to prove the second condition of Definition 5.1, observe first the following:

FACT. *$\nu_{\Delta}^{DCila}(\alpha) \in \{T, F\}$ if and only if $\circ\alpha \in \Delta$.*

Indeed, suppose first that $\nu_{\Delta}^{DCila}(\alpha) \in \{T, F\}$. By Definition 2.13, either $\alpha \notin \Delta$ or $\neg\alpha \notin \Delta$. In both cases, $\alpha \wedge \neg\alpha \notin \Delta$, hence $\neg(\alpha \wedge \neg\alpha) \in \Delta$. By axiom (cl) and the properties of Δ, $\circ\alpha \in \Delta$. Conversely, suppose that $\circ\alpha \in \Delta$. By axiom (bc) and the properties of Δ, either $\alpha \notin \Delta$ or $\neg\alpha \notin \Delta$. By Definition 2.13, $\nu_{\Delta}^{DCila}(\alpha) \in \{T, F\}$.

Fix now $\# \in \{\wedge, \vee, \rightarrow\}$, and suppose that $v_{\Delta}^{DCila}(\alpha), v_{\Delta}^{DCila}(\beta) \in \{T, F\}$. By the fact, $\circ\alpha, \circ\beta \in \Delta$. By axiom ($\text{ca}_{\#}$) and by taking into account that Δ is a closed theory, $\circ(\alpha\#\beta) \in \Delta$. By the fact once again, we infer that $v_{\Delta}^{DCila}(\alpha\#\beta) \in \{T, F\}$.

This shows that $\mathcal{M}$ is, in fact, a swap Kripke model for $DCila$. ⊣

From the previous results and the construction above, it is easy to show that completeness holds for $DCila$.

THEOREM 5.2 (Completeness of $DCila$ w.r.t. swap Kripke models). *For any set $\Gamma \cup \{\varphi\} \subseteq For(\Sigma)$, if $\Gamma \vDash_{DCila} \varphi$, then $\Gamma \vdash_{DCila} \varphi$.*

6. The pioneering system C_1^D

We briefly mentioned above that $Cila$ is a conservative expansion of C_1, since it has $\circ$ in its signature. This allows $Cila$ to refer to consistency by using $\circ$ as an unary operator, while C_1 defines consistency in terms of non-contradictoriness. That is to say, C_1 is defined over the signature $\Sigma^{C_1} = \{\rightarrow, \neg, \vee, \wedge\}$ such that $\alpha^{\circ} := \neg(\alpha \wedge \neg\alpha)$, for any $\alpha \in For(\Sigma^{C_1})$. It is easy to show that if we substitute any appearance of $\circ\alpha$ in the axioms or rules for $Cila$ for α°, we get C_1. Moreover, the valid inferences in

Cila in the signature Σ^{C_1} coincide with the ones in C_1 (see Carnielli et al., 2007, Theorem 110).

Also noticeable is the treatment of strong negation in C_1, usually defined as $\sim\alpha := \neg\alpha \wedge \alpha^\circ$. Let $\Sigma_D^{C_1} := \{\to, \neg, \vee, \wedge, \mathsf{O}\}$. Because of the close relationship between *Cila* and C_1, the $\Sigma_D^{C_1}$-reduct of the swap Kripke models for *DCila* characterize the deontic expansion DC_1 of C_1, defined over $\Sigma_D^{C_1}$ by adding to C_1 the modal deontic axioms of Definition 2.1, but now by considering $\bot_\alpha := (\alpha \wedge \neg\alpha) \wedge \alpha^\circ$. Observe that, in C_1, axioms (bc) and ($\mathrm{ca}_\#$) are now replaced by the following, where $\# \in \{\wedge, \vee, \to\}$:

$$\alpha^\circ \to (\alpha \to (\neg\alpha \to \beta)) \tag{bc$'$}$$

$$(\alpha^\circ \wedge \beta^\circ) \to (\alpha \# \beta)^\circ \tag{ca$'_\#$}$$

In turn, axioms (cl) and (ci) are not considered in C_1 (since they hold by the very definition of $(\cdot)^\circ$, as well as by axiom (cf)).

One other striking fact is that the pioneering paraconsistent deontic system C_1^D (also defined over $\Sigma_D^{C_1}$) proposed in (da Costa and Carnielli, 1986) has one more axiom in addition to the ones of DC_1, namely

$$\alpha^\circ \to (\mathsf{O}\alpha)^\circ \tag{ca$'_\mathsf{O}$}$$

The system proposed, so far, however, does not validate (ca'_O). Consider the following possible model of C_1^D. Each node x shows a set Γ as a label. This indicates that for every $\varphi \in \Gamma$, $v_\Gamma^{C_1}(\varphi) \in \mathcal{D}$. We will use an analogous notation in a few more examples:

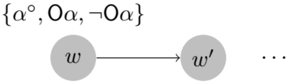


Figure 1. A representation of a counterexample to (ca'_O), when not adding the suitable restrictions to the models.

According to this model, $(\alpha^\circ)_{(1,w)} = 1$. But this does not say anything about any of the worlds w' accessible to w. Notice that the following holds: $v_w^{DC_1}((\mathsf{O}\alpha)^\circ) \notin \mathcal{D}$ if and only if $v_w^{DC_1}(\mathsf{O}\alpha) \in \mathcal{D}$ and $v_w^{DC_1}(\neg\mathsf{O}\alpha) \in \mathcal{D}$. But this is perfectly possible, since when assigning a truth value for $\mathsf{O}\alpha$, only the first coordinate of the snapshot is determined. Since the first coordinate of $v_w^{DC_1}(\neg\mathsf{O}\alpha)$ is given by reading the second coordinate of $v_w^{DC_1}(\mathsf{O}\alpha)$, then $v_w^{DC_1}(\mathsf{O}\alpha) = (1,1) = t$ is the value that falsifies the formula correspondent to the axiom. Hence, in order to give a proper semantics for the original C_1^D, we need one more restriction.

From now on, for ease of notation, we will write $v_w^{C_1}$ instead of $v_w^{C_1^D}$.

DEFINITION 6.1. A swap Kripke model $\mathcal{M} = \langle W, R, \{v_w^{C_1}\}_{w \in W}\rangle$ for C_1^D is a swap Kripke model for *DCila* (over the $\Sigma_D^{C_1}$-reduct) such that each valuation $v_w^{C_1}$ satisfies, in addition, the following condition, for $\# \in \{\wedge, \vee, \rightarrow\}$:

$$\text{if } v_w^{C_1}(\alpha) \in \{T, F\}, \text{ then } v_w^{C_1}(\mathsf{O}\alpha) \in \{T, F\}.$$

Remark 6.1. A family of maps $v_w^{C_1} \colon For(\Sigma_D^{C_1}) \to A$ satisfies Definition 6.1 iff it satisfies the following conditions, for every $\alpha, \beta \in For(\Sigma_D^{C_1})$ and $\# \in \{\wedge, \vee, \rightarrow\}$:

1. items 1–3 and 6 of Definition 2.5;
2. $v_w^{C_1}(\neg\alpha) \in \tilde{\neg}_1 v_w^{C_1}(\alpha)$, where $\tilde{\neg}_1$ is as in Definition 3.2;
3. if $v_w^{C_1}(\alpha) = t$, then $v_w^{C_1}(\alpha \wedge \neg\alpha) = T$;
4. if $v_w^{C_1}(\alpha), v_w^{C_1}(\beta) \in \{T, F\}$, then $v_w^{C_1}(\alpha \# \beta) \in \{T, F\}$;
5. if $v_w^{C_1}(\alpha) \in \{T, F\}$, then $v_w^{C_1}(\mathsf{O}\alpha) \in \{T, F\}$.

THEOREM 6.1 (Soundness of C_1^D w.r.t. swap Kripke models). *For every $\Gamma \cup \{\varphi\} \subseteq For(\Sigma_D^{C_1})$, if $\Gamma \vdash_{C_1^D} \varphi$, then $\Gamma \vDash_{C_1^D} \varphi$.*

PROOF. Let $v_w^{C_1}$ be as in Definition 6.1. Let us start by showing the following:

FACT. *$v_w^{C_1}(\alpha^\circ) \in \mathcal{D}$ if and only if $v_w^{C_1}(\alpha) \in \{T, F\}$.*

Indeed, suppose that $v_w^{C_1}(\alpha^\circ) \in \mathcal{D}$. Then, $(\alpha^\circ)_{(1,w)} = 1$. Recalling that $\alpha^\circ = \neg(\alpha \wedge \neg\alpha)$, it follows that $(\alpha \wedge \neg\alpha)_{(2,w)} = 1$, and so $v_w^{C_1}(\alpha \wedge \neg\alpha) \neq T$. By the first condition in Definition 5.1 we infer that $v_w^{C_1}(\alpha) \neq t$, hence $v_w^{C_1}(\alpha) \in \{T, F\}$. Conversely, if $v_w^{C_1}(\alpha) \in \{T, F\}$ then $\alpha_{(1,w)} \sqcap \alpha_{(2,w)} = (\alpha \wedge \neg\alpha)_{(1,w)} = 0$. From this, $(\alpha^\circ)_{(1,w)} = (\alpha \wedge \neg\alpha)_{(2,w)} = 1$ and so $v_w^{C_1}(\alpha^\circ) \in \mathcal{D}$.

Now, assume that $v_w^{C_1}(\alpha^\circ) \in \mathcal{D}$. By the fact, $v_w^{C_1}(\alpha) \in \{T, F\}$. By Definition 6.1, $v_w^{C_1}(\mathsf{O}\alpha) \in \{T, F\}$. Using the fact once again, we infer that $v_w^{C_1}((\mathsf{O}\alpha)^\circ) \in \mathcal{D}$, and so axiom ($\mathrm{ca}'_{\mathsf{O}}$) is valid w.r.t. swap Kripke models for C_1^D.

The validity of axiom (bc') follows immediately from the fact. In turn, the validity of axiom ($\mathrm{ca}'_{\#}$) is a consequence of the first condition stated in Definition 5.1 and the fact. The validity of the other axioms of C_1^D follows from the soundness of *DCila* w.r.t. swap Kripke models. ⊣

In order to prove completeness of C_1^D w.r.t. swap Kripke models, some adaptations are required in the construction of the canonical swap Kripke model and the canonical valuations.

Observe first that the φ-saturated sets in C_1^D are subsets of $For(\Sigma_D^{C_1})$ ($\circ$ is now a defined connective).

LEMMA 6.1. *Let $\Delta \subseteq For(\Sigma_D^{C_1})$ be a φ-saturated set in C_1^D. Then, it satisfies the following properties, for every $\alpha, \beta \in For(\Sigma_D^{C_1})$:*

I. *Items 1–4 and 6 of Lemma 2.2.*
II. *Item 5_3^+ of Lemma 3.2.*
III. *$\alpha^\circ \in \Delta$ iff $\alpha \notin \Delta$ or $\neg\alpha \notin \Delta$.*
IV. *If $\alpha^\circ, \beta^\circ \in \Delta$, then $(\alpha\#\beta)^\circ \in \Delta$, where $\# \in \{\wedge, \vee, \rightarrow\}$.*
V. *If $\alpha^\circ \in \Delta$, then $(\mathsf{O}\alpha)^\circ \in \Delta$.*

PROOF. Items I and II are immediate, given that C_1^D contains all the schemas of *DmbC* and *DbC* over signature $\Sigma_D^{C_1}$.

Item III: The "only if" part is a consequence of axiom (bc′). Now, suppose that $\alpha^\circ \notin \Delta$. By property 4 of item I, $\neg(\alpha^\circ) \in \Delta$. That is, $\neg\neg(\alpha \wedge \neg\alpha) \in \Delta$ and so $\alpha \wedge \neg\alpha \in \Delta$, by item II. By property 1 of item I, $\alpha, \neg\alpha \in \Delta$.

Item IV and V follow immediately from axioms ($\text{ca}'_\#$) and (ca'_O). ⊣

Define now $W_{can}^{C_1}, R_{can}^{C_1}$ and $v_\Delta^{C_1}$ as in *DCila*, but now each $\Delta \in W_{can}^{C_1}$ is a φ-saturated set in C_1^D. Observe that each valuation $v_\Delta^{C_1} : For(\Sigma_D^{C_1}) \rightarrow A$ is defined according to Definition 2.13.

PROPOSITION 6.1. *The structure $\mathcal{M} = \langle W_{can}^{C_1}, R_{can}^{C_1}, \{v_\Delta^{C_1}\}_{\Delta \in W_{can}^{C_1}}\rangle$ is a swap Kripke model for C_1^D such that, for every $\alpha \in For(\Sigma_D^{C_1})$, $v_\Delta^{C_1}(\alpha) \in \mathcal{D}$ iff $\alpha \in \Delta$.*

PROOF. Taking into account Remark 6.1, it is an immediate consequence of Lemma 6.1 and Definition 2.13. Indeed, by adapting the proofs for the previous systems, it follows that $v_\Delta^{C_1}(\alpha\#\beta) \in v_\Delta^{C_1}(\alpha)\tilde{\#}v_\Delta^{C_1}(\beta)$ (for $\# \in \{\wedge, \vee, \rightarrow\}$) and $v_\Delta^{C_1}(\neg\alpha) \in \tilde{\neg}_1 v_\Delta^{C_1}(\alpha)$. In order to prove that $v_\Delta^{C_1}$ satisfies the requirements 3–5 of Remark 6.1, suppose first that $v_\Delta^{C_1}(\alpha) = t$. Then, $\alpha, \neg\alpha \in \Delta$ and so $\alpha \wedge \neg\alpha \in \Delta$ and $\neg(\alpha \wedge \neg\alpha) = \alpha^\circ \notin \Delta$, by items I and III of Lemma 6.1. This means that $v_\Delta^{C_1}(\alpha \wedge \neg\alpha) = T$, validating requirement 3. For 4, observe first that $v_\Delta^{C_1}(\alpha) \in \{T, F\}$ iff either $\alpha \notin \Delta$ or $\neg\alpha \notin \Delta$ iff, by III, $\alpha^\circ \in \Delta$. Now, let $\# \in \{\wedge, \vee, \rightarrow\}$ and suppose that $v_\Delta^{C_1}(\alpha), v_\Delta^{C_1}(\beta) \in \{T, F\}$. By the previous observation, it follows that $\alpha^\circ, \beta^\circ \in \Delta$. By IV of Lemma 6.1, $(\alpha\#\beta)^\circ \in \Delta$. Using the observation above once again, this implies that $v_\Delta^{C_1}(\alpha\#\beta) \in \{T, F\}$. The proof for 5 is analogous. By the very definitions, $v_\Delta^{C_1}(\alpha) \in \mathcal{D}$ iff $\alpha \in \Delta$. ⊣

Completeness follows immediately, with slight adaptations, from the lemma above and completeness for *DCila*.

Proposition 6.2 (Completeness of C_1^D w.r.t. swap Kripke models). *For any set* $\Gamma \cup \{\varphi\} \subseteq For(\Sigma_D^{C_1})$, *if* $\Gamma \vDash_{C_1^D} \varphi$ *then* $\Gamma \vdash_{C_1^D} \varphi$.

7. Swap Kripke models for C_n^D

We start this section by defining extensions of notions presented in the previous section for the rest of the hierarchy C_n, for $n \geqslant 2$. For this, consider once again the signatures Σ^{C_1} for C_n and $\Sigma_D^{C_1}$ for the calculi C_n^D. We define the following notation over $For(\Sigma_D^{C_1})$:

- $\alpha^0 = \alpha$,
- $\alpha^{n+1} = \neg(\alpha^n \wedge \neg\alpha^n)$,
- $\alpha^{(n)} = \alpha^1 \wedge \cdots \wedge \alpha^n$.

We also follow the presentation of C_n given in (Coniglio and Toledo, 2022). This comprises of all axioms for CPL^+, plus (EM), (cf) and the following axioms:

$$\alpha^{(n)} \to (\alpha \to (\neg\alpha \to \beta)), \tag{bc$_n$}$$

$$(\alpha^{(n)} \wedge \beta^{(n)}) \to ((\alpha \to \beta)^{(n)} \wedge (\alpha \vee \beta)^{(n)} \wedge (\alpha \wedge \beta)^{(n)}). \tag{P$_n$}$$

Observe that the classical negation is represented in C_n by means of the formula $\sim^{(n)}\alpha := \neg\alpha \wedge \alpha^{(n)}$. From this, the new version of (D) reads

$$\mathsf{O}\alpha \to \sim^{(n)}\mathsf{O}\sim^{(n)}\alpha. \tag{D$_n$}$$

We also highlight that some decisions must be made along the way in order to get to a full axiomatization of these logics. If we want to follow the presentation of C_1^D on (da Costa and Carnielli, 1986) and extend the ideas presented there, as we did in the previous section, we need to reformulate the classicality propagation axiom, namely, $\alpha^\circ \to (\mathsf{O}\alpha)^\circ$, as follows:

$$\alpha^{(n)} \to (\mathsf{O}\alpha)^{(n)}. \tag{PO$_n$}$$

We call it the *general classicality propagation* axiom in C_n^D. Notice that when $n = 1$, $\alpha^{(n)} = \alpha^\circ = \alpha^1 = \neg(\alpha \wedge \neg\alpha)$. Also notice that this has an influence on how strong a negation has to be in order to recover classicality. For $n = 1$, strong negation is already sufficient to introduce deontic explosion back into the system, but taking $n = 2$, we have, besides $\neg$ and $\sim$, one more negation. We are in fact dealing with an increasing number of negations, or, more precisely, for each n, C_n^D has $n+1$ negations.[9]

We are now a position to characterize the family of systems C_n^D, for $n \geqslant 1$. Also notice that the case where $n = 1$ was already studied in the previous section. We also refrain in this from deeply investigating the philosophical considerations tied to these systems. We opt for a technical development of a semantics for the systems proposed, with the general propagation of classicality and a distinct version of (D). We attempt to maintain the general spirit of the system originally presentation in the paper by da Costa and Carnielli, while presenting a mix between RNmatrices and Kripke semantics.[10]

We thus follow the presentation given in (Coniglio and Toledo, 2022) to define the base system. So for each $n \geqslant 2$, the multialgebra for C_n will have domain A_n of size $n+2$, where each element of $A_n \subseteq 2^{n+1}$ is an $n+1$-tuple. Hence, the swap structures for C_n is one where the set of *snapshots* is:

$$A_n = \{z \in 2^{n+1} \ : \ (\bigwedge_{i \leqslant k} z_i) \vee z_{k+1} = 1 \ \text{ for every } 1 \leqslant k \leqslant n\}.$$

This produces exactly the following $n+2$ truth values:

$T_n = (1, 0, 1, \ldots, 1)$
$t_0^n = (1, 1, 0, 1, \ldots, 1)$
$t_1^n = (1, 1, 1, 0, 1, \ldots, 1)$
$\vdots$
$t_{n-2}^n = (1, 1, 1, 1, \ldots, 0)$
$t_{n-1}^n = (1, 1, 1, 1, \ldots, 1)$
$F_n = (0, 1, 1, \ldots, 1)$.

DEFINITION 7.1. let A_n be as in the definition above. We define the following subsets of A_n:

1. $D_n := A_n \setminus \{F_n\}$ (designated values),

[9] Although the fact is easy to observe, the argument that each one of them is, in fact, a negation will be discussed in a future paper.

[10] It is possible to construct such a system by means of restricted swap structures only, following the technique shown in (Coniglio et al., 2025). The modal operator could then be assigned one dimension in the tuple, hence its truth value being fully nondeterministic. This permits to semantically characterize logics in which the modal operator does not satisfy any of the standard inference rules or axioms assumed for such an operator.

2. $U_n := A_n \setminus D_n = \{F_n\}$ (undesignated values),
3. $I_n := A_n \setminus \{T_n, F_n\}$ (inconsistent values),
4. $Boo_n = A_n \setminus I_n = \{T_n, F_n\}$ (Boolean or classical values).

Now we can introduce the multiagebra $\mathcal{A}_{C_n^D}$:

DEFINITION 7.2. Let $\mathcal{A}_{C_n^D} = (A_n, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\neg}, \tilde{\mathsf{O}})$ be the multialgebra over $\Sigma_D^{C_1}$ defined as follows, for any $a, b \in A_n$:

1. $\tilde{\neg}a = \{c \in A_n \ : \ c_1 = a_2 \text{ and } c_2 \leqslant a_1\}$
2. $a\tilde{\wedge}b = \begin{cases} \{c \in Boo_n \ : \ c_1 = a_1 \sqcap b_1\} & \text{if } a, b \in Boo_n \\ \{c \in A_n \ : \ c_1 = a_1 \sqcap b_1\} & \text{otherwise} \end{cases}$
3. $a\tilde{\vee}b = \begin{cases} \{c \in Boo_n \ : \ c_1 = a_1 \sqcup b_1\} & \text{if } a, b \in Boo_n \\ \{c \in A_n \ : \ c_1 = a_1 \sqcup b_1\} & \text{otherwise} \end{cases}$
4. $a\tilde{\rightarrow}b = \begin{cases} \{c \in Boo_n \ : \ c_1 = a_1 \supset b_1\} & \text{if } a, b \in Boo_n \\ \{c \in A_n \ : \ c_1 = a_1 \supset b_1\} & \text{otherwise} \end{cases}$
5. $\tilde{\mathsf{O}}(X) = \{c \in A_n \ : \ c_1 = \bigsqcap\{x_1 : x \in X\}\}$, where $X \neq \emptyset$ and $X \subseteq A_n$.

Remark 7.1. Observe that the non-deterministic truth-tables for the non-modal operators of $\mathcal{A}_{C_n^D}$ are the ones displayed below, where $0 \leqslant i, j \leqslant n-1$.

$\tilde{\wedge}$	T_n	t_j^n	F_n
T_n	$\{T_n\}$	D_n	$\{F_n\}$
t_i^n	D_n	D_n	$\{F_n\}$
F_n	$\{F_n\}$	$\{F_n\}$	$\{F_n\}$

$\tilde{\vee}$	T_n	t_j^n	F_n
T_n	$\{T_n\}$	D_n	$\{T_n\}$
t_i^n	D_n	D_n	D_n
F_n	$\{T_n\}$	D_n	$\{F_n\}$

	$\tilde{\neg}$
T_n	$\{F_n\}$
t_i^n	D_n
F_n	$\{T_n\}$

$\tilde{\rightarrow}$	T_n	t_j^n	F_n
T_n	$\{T_n\}$	D_n	$\{F_n\}$
t_i^n	D_n	D_n	$\{F_n\}$
F_n	$\{T_n\}$	D_n	$\{T_n\}$

DEFINITION 7.3. Let $W \neq \emptyset$ be a set of worlds, $R \subseteq W \times W$ be a serial relation and $v_w^n = For(\Sigma_D^{C_1}) \to A_n$ for each $w \in W$, such that, for any $\alpha, \beta \in For(\Sigma_D^{C_1})$, the following holds:

1. $v_w^n(\neg\alpha) \in \tilde{\neg}(v_w^n(\alpha))$,
2. $v_w^n(\alpha\#\beta) \in v_w^n(\alpha)\tilde{\#}v_w^n(\beta)$, for $\# \in \{\wedge, \to \neg\}$,
3. $v_w^n(\mathsf{O}\alpha) \in \tilde{\mathsf{O}}(\{v_{w'}^n(\alpha) \ : \ wRw'\})$.

DEFINITION 7.4. A structure $\mathcal{M} = (W, R, \{v_w^n\}_{w \in W})$ with properties as in Definition 7.3 is said to be a *swap Kripke pre-model* for C_n^D. A formula $\alpha \in For(\Sigma_D^{C_1})$ is *true in a world* w of $\mathcal{M}$, denoted by $\mathcal{M}, w \vDash \alpha$, if $v_w^n(\alpha) \in D_n$. A formula α is *valid* in a pre-model $\mathcal{M}$, denoted by $\mathcal{M} \vDash \alpha$, if $\mathcal{M}, w \vDash \alpha$ for every $w \in W$. As it was done before, given a non-empty set Γ of formulas we will write $\mathcal{M}, w \vDash \Gamma$ to denote that $\mathcal{M}, w \vDash \alpha$ for every $\alpha \in \Gamma$.

We recall the fact that for any $\alpha \in For(\Sigma_D^{C_1})$, $w \in W$ and v_w^n, $v_w^n(\alpha) \in D_n$ if and only if $v_{(1,w)}^n(\alpha) = 1$, given a natural adaptation of the notation presented in Remark 2.4 and the definitions above.

In order to characterize C_n^D we add first the following restrictions, thus simulating the behavior of the *RN*matrix for C_n:

DEFINITION 7.5. Given a swap Kripke pre-model for C_n^D, consider the following additional restrictions on the valuations v_w^n:

1. $v_w^n(\alpha) = t_0^n$ implies $v_w^n(\alpha \wedge \neg\alpha) = T_n$,
2. $v_w^n(\alpha) = t_k^n$ implies $v_w^n(\alpha \wedge \neg\alpha) \in I_n$ and $v_w^n(\alpha^1) = t_{k-1}^n$, for every $1 \leqslant k \leqslant n-1$.

Remark 7.2. We observe that the additional restrictions in Definition 7.5 only consider valuations in which the values of $\alpha^{(n)}$ are in Boo_n, such as shown in (Coniglio and Toledo, 2022, pp. 621–622), for each world $w \in W$. Moreover, in any swap Kripke pre-model for C_n^D as in Definition 7.5, and for a fixed $w \in W$, each valuation v_w^n belongs to the set of valuations of the *RN*matrix characterizing C_n introduced in (Coniglio and Toledo, 2022). From this, all the results concerning the non-modal operators of C_n^D will hold w.r.t. the valuations of such a swap Kripke pre-models.

The restrictions on the valuations made in Definition 7.5 can be displayed by means of a very useful table (see Coniglio and Toledo, 2022, Table 1, p. 622). For the reader's convenience, Table 1 reproduces a slightly expanded version of that table, which represents the possible scenarios concerning restricted valuations for C_n (and so, for the non-modal fragment of C_n^D), according to Definition 7.5. In that table, X^* means that the value X is forced by a restriction on the corresponding valuation.

It is worth observing from Table 1 that the truth-tables of the connectives $(\cdot)^{(n)}$ and $\sim^{(n)}\alpha = \neg\alpha \wedge \alpha^{(n)}$ are as follows, for $0 \leqslant i \leqslant n-1$:

α	$\alpha^{(n)}$
T_n	$\{T_n\}$
t_i^n	$\{F_n\}$
F_n	$\{T_n\}$

α	$\sim^{(n)}\alpha$
T_n	$\{F_n\}$
t_i^n	$\{F_n\}$
F_n	$\{T_n\}$

α	$\alpha \wedge \neg\alpha$	α^1	$\alpha^1 \wedge \neg(\alpha^1)$	α^2	$\alpha^2 \wedge \neg(\alpha^2)$	$\cdots$	α^{n-1}	$\alpha^{n-1} \wedge \neg(\alpha^{n-1})$	α^n	$\alpha^{(n)}$
T_n	F_n	T_n	F_n	T_n	F_n	$\cdots$	T_n	F_n	T_n	T_n
t_0^n	T_n^*	F_n	F_n	T_n	F_n	$\cdots$	T_n	F_n	T_n	F_n
t_1^n	I_n^*	t_0^{n*}	T_n^*	F_n	F_n	$\cdots$	T_n	F_n	T_n	F_n
t_2^n	I_n^*	t_1^{n*}	I_n^*	t_0^{n*}	T_n^*	$\cdots$	T_n	F_n	T_n	F_n
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t_{n-3}^n	I_n^*	t_{n-4}^{n*}	I_n^*	t_{n-5}^{n*}	I_n^*	$\cdots$	T_n	F_n	T_n	F_n
t_{n-2}^n	I_n^*	t_{n-3}^{n*}	I_n^*	t_{n-4}^{n*}	I_n^*	$\cdots$	F_n	F_n	T_n	F_n
t_{n-1}^n	I_n^*	t_{n-2}^{n*}	I_n^*	t_{n-3}^{n*}	I_n^*	$\cdots$	t_0^{n*}	T_n^*	F_n	F_n
F_n	F_n	T_n	F_n	T_n	F_n	$\cdots$	T_n	F_n	T_n	T_n

Table 1.

We need, however, to be sure that our restricted valuations preserve validity when looking at the modal operator. It is easy to see that axioms (K) and the strong version (D_n) of (D) are valid w.r.t. the swap Kripke pre-models of Definition 7.5:

LEMMA 7.1. *Consider a swap Kripke pre-model $\mathcal{M}$ for C_n^D. Then, the following holds for any $\alpha, \beta \in For(\Sigma_D^{C_1})$, and w in $\mathcal{M}$:*

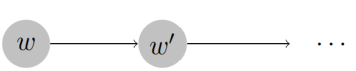
1. *If $v_w^n(O(\alpha \to \beta)) \in D_n$ and $v_w^n(O\alpha) \in D_n$, then $v_w^n(O\beta) \in D_n$.*
2. *If $v_w^n(O\alpha) \in D_n$, then $v_w^n({\sim}^{(n)} O {\sim}^{(n)}\alpha) \in D_n$, assuming that $\mathcal{M}$ is as in Definition 7.5.*

PROOF. For the first item, assume both $v_w^n(\mathsf{O}(\alpha \to \beta)) \in D_n$ and $v_w^n(\mathsf{O}\alpha) \in D_n$. This means that $v_{(1,w)}^n(\mathsf{O}(\alpha \to \beta)) = v_{(1,w)}^n(\mathsf{O}\alpha) = 1$. But then, by Definitions 7.2 and 7.3 we have that $v_{(1,w')}^n(\alpha \to \beta) = v_{(1,w')}^n(\alpha) = 1$ and so $v_{(1,w')}^n(\beta) = 1$, for every w' such that wRw'. From this, $v_{(1,w)}^n(\mathsf{O}\beta) = 1$, that is, $v_w^n(\mathsf{O}\beta) \in D_n$.

The second item follows from the nature of ${\sim}^{(n)}$ (see its truth-tables above). ⊣

Remark 7.3. Notice that, since $n \geqslant 2$, the restrictions in Definition 7.5 fail to validate (D) in its strong negation form, i.e., w.r.t. ${\sim}\alpha := \neg\alpha \wedge \alpha^1$.

As an example, consider C_2^D. Picture a model as below:



Assume that $v^2_w(\mathsf{O}\alpha) \in D_2$ and that $v^2_{w'}(\alpha) = t^2_1$. Thus, by Definition 7.5, this means that $v^2_{w'}(\alpha^1) = t^2_0$ and that $v^2_{w'}(\neg\alpha) \in D_2$. Hence, $v^2_{(1,w')}(\sim\alpha) = 1$. That implies $v^2_w(\mathsf{O}\sim\alpha) \in D_2$, so when $v^2_w(\mathsf{O}\sim\alpha) = T_2$, $v^2_w(\sim\mathsf{O}\sim\alpha) = F_2$. The argument for any $n \geqslant 2$ is similar. Besides, a similar counterexample can be found for the paraconsistent negation $\neg$.

Remark 7.4. Also notice that the version of (D) with $\sim^{(n)}$ is validated by requiring the restrictions in Definition 7.5. Without the restrictions, it is possible to construct a model and a valuation in worlds that would falsify the axiom. Picture again the model used in the previous remark, while working in C^D_2. Assume further that $v^2_w(\mathsf{O}\alpha) \in D_2$, with $v^2_{w'}(\alpha) \in I_2$. This means that $v^2_{w'}(\neg\alpha) \in D_2$, hence $v^2_{w'}(\alpha \wedge \neg\alpha) \in D_2$. Since no restriction is given to the value assigned to $(\alpha \wedge \neg\alpha)$, then it is possible that $v^2_{w'}(\alpha \wedge \neg\alpha) \in I_2$, and also that $v^2_{w'}(\neg(\alpha \wedge \neg\alpha)) = v^2_{w'}(\alpha^1) \in I_2$. These assignments allow for $v^2_{w'}(\neg(\neg(\alpha\wedge\neg\alpha)\wedge\neg\neg(\alpha\wedge\neg\alpha))) = v^2_{w'}(\alpha^2) \in D_2$. But then, $v^2_{w'}(\sim^{(2)}\alpha) \in D_2$. As we did in the previous case, taking $v^2_w(\mathsf{O}\sim^{(2)}\alpha) = T_2$ implies $v^2_w(\sim^{(2)}\mathsf{O}\sim^{(2)}\alpha) = F_2$. This is also similarly extended to any $n \geqslant 2$.

To see that indeed the restrictions guarantee that the axiom holds, assume $v^n_w(\mathsf{O}\alpha) \in D_n$. Thus for all w' such that wRw', $v^n_{w'}(\alpha) \in D_n$. Now either $v^n_{w'}(\alpha) = T_n$ or $v^n_{w'}(\alpha) \in I_n$. If the first case, then $v^n_{w'}(\neg\alpha) = F_n$. Otherwise, $v^n_{w'}(\alpha^{(n)}) = F_n$. In any case, $v^n_{w'}(\sim^{(n)}\alpha) = F_n$, hence $v^n_w(\mathsf{O}\sim^{(n)}\alpha) = F_n$, thus $v^n_w(\sim^{(n)}\mathsf{O}\sim^{(n)}\alpha) \in D_n$.

The following is easily proved, taking into consideration Remark 7.2 and Table 1:

LEMMA 7.2. *Consider a swap Kripke pre-model for C^D_n as in Definition 7.5, and let $1 \leqslant k \leqslant n$. Then, the following holds for any $\alpha \in For(\Sigma^{C_1}_D)$ and any $w \in W$:*

1. *If $v^n_w(\alpha) = T_n$, then $v^n_w(\alpha^k) = T_n$.*
2. *If $v^n_w(\alpha) = t^n_i$ for some $1 \leqslant i \leqslant k-2$, then $v^n_w(\alpha^k) = T_n$.*
3. *If $v^n_w(\alpha) = t^n_{k-1}$, then $v^n_w(\alpha^k) = F_n$.*
4. *If $v^n_w(\alpha) = t^n_i$ for some $k \leqslant i \leqslant n-1$, then $v^n_w(\alpha^k) = t^n_{i-k}$.*
5. *If $v^n_w(\alpha) = F_n$, then $v^n_w(\alpha^k) = T_n$.*
6. *If $v^n_w(\alpha) = t^n_i$ for some $0 \leqslant i \leqslant n-1$, then: $v^n_w(\alpha^k) = F_n$ iff $k = i+1$.*

LEMMA 7.3. *Let $\mathcal{M}_n$ be a swap Kripke pre-model for C^D_n as in Definition 7.5. Then, $v^n_w(\alpha^{(n)}) \in D_n$ if and only if $v^n_w(\alpha) \in Boo_n$.*

PROOF. Suppose $v^n_w(\alpha) \notin Boo_n$. Then, $v^n_w(\alpha) \in I_n$. By item 3 of Lemma 7.2 it follows that, for $0 \leqslant k < n$, $v^n_w(\alpha^{k+1}) = F_n$, hence

$v^n_w(\alpha^{(n)}) = F_n$ and so $v^n_w(\alpha^{(n)}) \notin D_n$. Now, if $v^n_w(\alpha) \in Boo_n$ then, for any $1 \leqslant k \leqslant n$, $v^n_w(\alpha^k) = T_n$, by items 1 and 5 of Lemma 7.2. Thus, $v^n_w(\alpha^{(n)}) = T_n$, so $v^n_w(\alpha^{(n)}) \in D_n$. ⊣

There is only one more modal axiom to check, namely, (PO_n). In its original formulation, that is, when $n = 1$, it was already covered in the previous section. We thus add a similar restriction in order to validate this version of the axiom:

Definition 7.6. A swap Kripke pre-model for C^D_n is said to be a *swap Kripke model for* C^D_n if the valuations satisfy, in addition, the restrictions of Definition 7.5 plus the following constraint:

$$\text{If } v^n_w(\alpha) \in Boo_n, \text{ then } v^n_w(\mathsf{O}\alpha) \in Boo_n.$$

Now we can prove the validity of (PO_n) w.r.t. swap Kripke models for C^D_n.

Lemma 7.4. *The following holds in any swap Kripke model for* C^D_n*:*

3. *If* $v^n_w(\alpha^{(n)}) \in D_n$*, then* $v^n_w((\mathsf{O}\alpha)^{(n)}) \in D_n$*.*

Proof. From Lemma 7.3, $v^n_w(\alpha^{(n)}) \in D_n$ implies that $v^n_w(\alpha) \in Boo_n$. By Definition 7.6, $v^n_w(\mathsf{O}\alpha) \in Boo_n$. By Lemma 7.3 once again, it follows that $v^n_w((\mathsf{O}\alpha)^{(n)}) \in D_n$. ⊣

Recall the notions and notation introduced in Definition 7.4, which can be also applied to swap Kripke models for C^D_n.

Definition 7.7. Given a set $\Gamma \subseteq For(\Sigma^{C_1}_D)$, we say that α is a logical consequence of Γ in C^D_n, denoted by $\Gamma \vDash_{C^D_n} \alpha$, if the following holds: for every swap Kripke model $\mathcal{M}$ for C^D_n, and for every world w in $\mathcal{M}$, if $\mathcal{M}, w \vDash \Gamma$ then $\mathcal{M}, w \vDash \alpha$.

Theorem 7.1 (Soundness of C^D_n w.r.t. swap Kripke models). *Let* $\Gamma \cup \{\varphi\} \subseteq For(\Sigma^{C_1}_D)$*. Then:* $\Gamma \vdash_{C^D_n} \varphi$ *only if* $\Gamma \vDash_{C^D_n} \varphi$*.*

Proof. The validity of the propositional (non-modal) axioms was already proven in (Coniglio and Toledo, 2022). Since our construction is similar to that, we simply refer to the proof thus given, taking into consideration Remark 7.2. The cases for the modal axioms follow from Lemmas 7.1 and 7.4. Clearly, O-necessitation preserves validity, by item 6 of Definition 2.5. ⊣

In order to prove completeness, on the other hand, we need a canonical construction that satisfies our new restrictions. Let $W^{(n)}_{can}$ be the set of all the sets $\Delta \subseteq For(\Sigma^{C_1}_D)$ such that Δ is a ψ-saturated set in C^D_n, for some $\psi \in For(\Sigma^{C_1}_D)$. The binary relation $R^{(n)}_{can}$ on $W^{(n)}_{can}$ is defined as in the previous cases. Then:

LEMMA 7.5 (Truth Lemma for C^D_n). *For any $\Delta \in W^{(n)}_{can}$, all the following statements hold, for every $\alpha, \beta \in For(\Sigma^{C_1}_D)$:*

1. $\alpha \wedge \beta \in \Delta$ *iff* $\alpha, \beta \in \Delta$.
2. $\alpha \vee \beta \in \Delta$ *iff* $\alpha \in \Delta$ *or* $\beta \in \Delta$.
3. $\alpha \to \beta \in \Delta$ *iff* $\alpha \notin \Delta$ *or* $\beta \in \Delta$.
4. *If* $\alpha \notin \Delta$, *then* $\neg\alpha \in \Delta$.
5. *If* $\neg\neg\alpha \in \Delta$, *then* $\alpha \in \Delta$.
6. *If* $\alpha \notin \Delta$ *or* $\neg\alpha \notin \Delta$, *then* $\alpha^1 \in \Delta$ *and* $\neg(\alpha^1) \notin \Delta$.
7. *If* $\alpha, \neg\alpha \in \Delta$ *then, for every* $1 \leqslant i \leqslant n$*: if* $\alpha^i \notin \Delta$, *then* $\alpha^j \in \Delta$ *for every* $1 \leqslant j \leqslant n$ *with* $j \neq i$.
8. *If* $\alpha, \neg\alpha \in \Delta$ *then there exists a unique* $1 \leqslant k \leqslant n$ *such that* $\alpha^k \notin \Delta$.
9. $\alpha^{(n)} \in \Delta$ *iff* $\alpha \notin \Delta$ *or* $\neg\alpha \notin \Delta$.
10. $\mathsf{O}\alpha \in \Delta$ *iff* $\alpha \in \Delta'$ *for all* $\Delta' \in W^{(n)}_{can}$ *such that* $\Delta R^{(n)}_{can} \Delta'$.
11. *If* $\alpha^{(n)} \in \Delta$, *then* $(\mathsf{O}\alpha)^{(n)} \in \Delta$.

PROOF. Conditions 1–5 and 10–11 are proven as in the previous cases, taken into account the axioms and rules of C^D_n.

6: Suppose that $\alpha \notin \Delta$ or $\neg\alpha \notin \Delta$. By item 1, $\alpha \wedge \neg\alpha \notin \Delta$ and so $\alpha^1 = \neg(\alpha \wedge \neg\alpha) \in \Delta$, by item 4. Since $\alpha \wedge \neg\alpha \notin \Delta$ then, by item 5, $\neg(\alpha^1) = \neg\neg(\alpha \wedge \neg\alpha) \notin \Delta$.

7: Observe that item 6 is equivalent to the following:

$$\text{If } \alpha^1 \notin \Delta \text{ or } \neg(\alpha^1) \in \Delta, \text{ then } \alpha, \neg\alpha \in \Delta. \qquad (*)$$

By induction on $1 \leqslant i \leqslant n$ it will be proven that

$$P(i) := \text{for every } \alpha, \text{ if } \alpha, \neg\alpha \in \Delta \text{ and } \alpha^i \notin \Delta, \text{ then } \alpha^j \in \Delta \text{ for every } 1 \leqslant j \leqslant n \text{ with } j \neq i$$

holds, for every $1 \leqslant i \leqslant n$ (for a given $n \geqslant 2$).

Base $i = 1$: Assume that $\alpha, \neg\alpha \in \Delta$ and $\alpha^1 \notin \Delta$. By item 6 (applied to α^1) it follows that $\alpha^2 \in \Delta$ and $\neg(\alpha^2) \notin \Delta$. By applying iteratively the same reasoning, we infer that $\alpha^j \in \Delta$ for every $1 \leqslant j \leqslant n$ with $j \neq 1$. That is, $P(1)$ holds.

Inductive step: Assume that $P(i)$ holds for every $1 \leqslant i \leqslant k \leqslant n-1$, for a given $1 \leqslant k \leqslant n-1$ (Inductive Hypothesis, IH). Let $\alpha, \neg\alpha \in \Delta$ and suppose that $\alpha^{k+1} = (\alpha^k)^1 \notin \Delta$. By item 4, $\neg((\alpha^k)^1) \in \Delta$ and so $\alpha^k, \neg(\alpha^k) \in \Delta$, by $(*)$. Since $(\alpha^k)^1 \notin \Delta$ then, by (IH) applied to α^k, it follows that $(\alpha^k)^j \in \Delta$ for every $1 \leqslant j \leqslant n$ with $j \neq 1$. Since $\alpha^k \in \Delta$, this implies that: (i) $\alpha^j \in \Delta$ for every $k \leqslant j \leqslant n$ with $j \neq k+1$.

In turn, since $\neg((\alpha^{k-1})^1) = \neg(\alpha^k) \in \Delta$, then $\alpha^{k-1}, \neg(\alpha^{k-1}) \in \Delta$, by $(*)$. By applying iteratively the same reasoning, we infer that: (ii) $\alpha^j \in \Delta$ for every $1 \leqslant j \leqslant k-1$.

From (i) and (ii) it follows that $\alpha^j \in \Delta$, for every $1 \leqslant j \leqslant n$ with $j \neq k+1$. That is, $P(k+1)$ holds.

8: It is an immediate consequence of item 7.

9: The "only if" part is immediate, by axiom (bc_n) and the fact that Δ is a closed, non-trivial theory. Now, assume that $\alpha \notin \Delta$ or $\neg\alpha \notin \Delta$. By item 6, $\alpha^1 \in \Delta$ and $\neg(\alpha^1) \notin \Delta$. By item 6 applied to α^1, and taking into account that $\neg(\alpha^1) \notin \Delta$, it follows that $\alpha^2 \in \Delta$ and $\neg(\alpha^2) \notin \Delta$. By applying iteratively the same reasoning, we infer that $\alpha^j \in \Delta$ for every $1 \leqslant j \leqslant n$, hence $\alpha^{(n)} \in \Delta$, by item 1. ⊣

Definition 7.8. For each $\Delta \in W^{(n)}_{can}$, define $\nu^n_\Delta \colon For(\Sigma^{C_1}_D) \to \mathcal{A}_n$ such that for each $\Delta \in W^{(n)}_{can}$ we have:

$$v^n_\Delta(\alpha) = \begin{cases} T_n, & \text{if } \alpha \in \Delta, \neg\alpha \notin \Delta \\ t^n_k, & \text{if } \alpha, \neg\alpha \in \Delta \text{ and } \alpha^{k+1} \notin \Delta \\ F_n, & \text{if } \alpha \notin \Delta, \neg\alpha \in \Delta \end{cases}$$

Corollary 7.1. *Let $\Delta \in W^{(n)}_{can}$. Then, the following holds:*

1. *The function v^n_Δ is well-defined.*
2. $v^n_\Delta(\alpha) \in \{T_n, F_n\}$ *iff* $\alpha \notin \Delta$ *or* $\neg\alpha \notin \Delta$ *iff* $\alpha^{(n)} \in \Delta$.
3. $v^n_\Delta(\alpha) = t^n_i$ *iff* $\alpha^{i+1} \notin \Delta$.

Proof. Item 1 is an immediate consequence of item 8 of Lemma 7.5. In turn, item 2 follows by item 9 of Lemma 7.5 and the definition of v^n_Δ. Finally, item 3 is a consequence of item 1 and the definition of v^n_Δ. ⊣

Consider now the relation $R^{(n)}_{can} \subseteq W^{(n)}_{can} \times W^{(n)}_{can}$ defined as in the previous cases.

Proposition 7.1. *The structure* $\mathcal{M}_n = \langle W^{(n)}_{can}, R^{(n)}_{can}, \{v^n_\Delta\}_{\Delta \in W^{(n)}_{can}}\rangle$ *is a swap Kripke model for* C^D_n *such that, for every* $\alpha \in For(\Sigma^{C_1}_D)$, $v^n_\Delta(\alpha) \in D_n$ *iff* $\alpha \in \Delta$.

Proof. Observe first that, by Definition 7.8, for every Δ and every α it holds: $(*)$ $v^n_\Delta(\alpha) \in D_n$ iff $\alpha \in \Delta$.

(I) Let us prove now that each function v^n_Δ satisfies the properties stated in Definition 7.3. Concerning conjunction, observe that, by $(*)$ and item 1 of Lemma 7.5, $v^n_\Delta(\alpha \wedge \beta) \in D_n$ iff $v^n_\Delta(\alpha), v^n_\Delta(\beta) \in D_n$. In turn, $z, w \in D_n$ implies that $z \tilde{\wedge} w \subseteq D_n$, and $z = F_n$ or $w = F_n$ implies that $z \tilde{\wedge} w = \{F_n\}$. Moreover, by item 2 of Corollary 7.1: $v^n_\Delta(\alpha), v^n_\Delta(\beta) \in Boo_n$ implies that $\alpha^{(n)}, \beta^{(n)} \in \Delta$ and so $(\alpha \wedge \beta)^{(n)} \in \Delta$, by (P_n), then $v^n_\Delta(\alpha \wedge \beta) \in Boo_n$. Given that $z, w \in Boo_n$ implies that $z \tilde{\wedge} w \subseteq Boo_n$, we infer from the previous considerations that $v^n_\Delta(\alpha \wedge \beta) \in v^n_\Delta(\alpha) \tilde{\wedge} v^n_\Delta(\alpha)$. Analogously, we prove that $v^n_\Delta(\alpha \# \beta) \in v^n_\Delta(\alpha) \tilde{\#} v^n_\Delta(\alpha)$ for $\# \in \{\vee, \rightarrow\}$. Concerning negation, suppose that $v^n_\Delta(\alpha) = T_n$. Then, $\alpha \in \Delta$ and $\neg\alpha \notin \Delta$, and so $v^n_\Delta(\neg\alpha) = F_n \in \{F_n\} = \tilde{\neg}\, T_n = \tilde{\neg}\, v^n_\Delta(\alpha)$. If $v^n_\Delta(\alpha) = F_n$ the proof is analogous. Now, suppose that $v^n_\Delta(\alpha) = t^n_i$. Then, $\neg\alpha \in \Delta$ and so, by $(*)$, $v^n_\Delta(\neg\alpha) \in D_n = \tilde{\neg}\, t^n_i = \tilde{\neg}\, v^n_\Delta(\alpha)$. Finally, by $(*)$, if $v^n_\Delta(\mathsf{O}\alpha) \in D_n$ then $\mathsf{O}\alpha \in \Delta$ and so $\alpha \in \Delta'$ for all $\Delta' \in W^{(n)}_{can}$ such that $\Delta R^{(n)}_{can} \Delta'$, by item 10. of Lemma 7.5. This means that $v^n_{\Delta'}(\alpha) \in D_n$ for all $\Delta' \in W^{(n)}_{can}$ such that $\Delta R^{(n)}_{can} \Delta'$, by $(*)$ once again, therefore $v^n_\Delta(\mathsf{O}\alpha) \in D_n = \tilde{\mathsf{O}}(\{v^n_{\Delta'}(\alpha) \ : \ \Delta R^{(n)}_{can} \Delta'\})$. Now, if $v^n_\Delta(\mathsf{O}\alpha) = F_n$ then $\mathsf{O}\alpha \notin \Delta$, by $(*)$, hence there exists some $\Delta' \in W^{(n)}_{can}$ such that $\Delta R^{(n)}_{can} \Delta'$ and $\alpha \notin \Delta'$, by item 10. of Lemma 7.5. This means that $v^n_{\Delta'}(\alpha) = F_n$ for some $\Delta' \in W^{(n)}_{can}$ such that $\Delta R^{(n)}_{can} \Delta'$, by $(*)$ once again. From this, $v^n_\Delta(\mathsf{O}\alpha) \in \{F_n\} = \tilde{\mathsf{O}}(\{v^n_{\Delta'}(\alpha) \ : \ \Delta R^{(n)}_{can} \Delta'\})$.

(II) Let us see now that each v^n_Δ satisfies the restrictions imposed in Definitions 7.5 and 7.6. Thus, assume first that $v^n_\Delta(\alpha) = t^n_0$. Then, $\alpha, \neg\alpha \in \Delta$ and $\alpha^1 \notin \Delta$. Hence, $(\alpha \wedge \neg\alpha) \in \Delta$, by item 1 of Lemma 7.5, and $\neg(\alpha \wedge \neg\alpha) = \alpha^1 \notin \Delta$. From this, $v^n_\Delta(\alpha \wedge \neg\alpha) = T_n$. Now, suppose that $v^n_\Delta(\alpha) = t^n_k$ for some $1 \leqslant k \leqslant n-1$. By Definition 7.8, $\alpha, \neg\alpha \in \Delta$ and $\alpha^{k+1} \notin \Delta$. From this, $(\alpha \wedge \neg\alpha) \in \Delta$ and $\neg(\alpha \wedge \neg\alpha) = \alpha^1 \in \Delta$, by items 1 and 8 of Lemma 7.5. By Definition 7.8, $v^n_\Delta(\alpha \wedge \neg\alpha) \in I_n$. Suppose that $\neg(\alpha^1) \notin \Delta$. By item 6 of Lemma 7.5 applied to α^1, it follows that $\alpha^2 \in \Delta$ and $\neg(\alpha^2) \notin \Delta$. By applying iteratively item 6 of Lemma 7.5 to α^2, α^3 and so on, we conclude that $\alpha^{k+1} \in \Delta$, a contradiction. This means that $\neg(\alpha^1) \in \Delta$. Since $\alpha^1 \in \Delta$ and $(\alpha^1)^k = \alpha^{k+1} \notin \Delta$, we conclude by Definition 7.8 that $v^n_\Delta(\alpha^1) = t^n_{k-1}$. This shows that the restrictions of Definition 7.5 are satisfied by the functions v^n_Δ. Finally, suppose that $v^n_\Delta(\alpha) \in Boo_n$. By item 2 of Corollary 7.1, $\alpha^{(n)} \in \Delta$. By (PO_n), $(\mathsf{O}\alpha)^{(n)} \in \Delta$ and so, by item 2 of Corollary 7.1 once again, $v^n_\Delta(\mathsf{O}\alpha) \in Boo_n$. This shows that the condition of Definition 7.6 are also satisfied by the functions v^n_Δ. ⊣

Theorem 7.2 (Completeness of C^D_n w.r.t. swap Kripke models).
For any set $\Gamma \cup \{\varphi\} \subseteq For(\Sigma^{C_1}_D)$, if $\Gamma \vDash_{C^D_n} \varphi$ then $\Gamma \vdash_{C^D_n} \varphi$.

Proof. Suppose that $\Gamma \nvdash_{C^D_n} \varphi$. Then, there is a φ-saturated set $\Gamma \subseteq \Delta$ such that $\varphi \notin \Delta$. From Proposition 7.1, $\mathcal{M}_n$ is a swap Kripke model for C^D_n and Δ is a world in $\mathcal{M}_n$ such that $\mathcal{M}_n, \Delta \vDash \Gamma$ but $\mathcal{M}_n, \Delta \nvDash \varphi$. This implies that $\Gamma \nvDash_{C^D_n} \varphi$. ⊣

7.1. A small addition

We briefly mention that in order to validate (D) in standard formulation, that is, using the primitive paraconsistent negation $\neg$ of C_n, we need one more restriction added to our valuations, namely:

Definition 7.9. A swap Kripke model for C^D_n is said to be *strict* if the valuations satisfy, in addition, the following constraint:

If $v^n_w(\mathsf{O}\alpha) \in D_n$ then, for every $w' \in W$ such that wRw', $v^n_{w'}(\alpha) = T_n$.

Then it is easy to see that (D) (formulated with $\neg$) is valid w.r.t. strict swap Kripke models for C^D_n.

Proposition 7.2. *Axiom schema*

$$O\alpha \to \neg O \neg\alpha \qquad (\mathsf{SD}_n)$$

is valid w.r.t. strict swap Kripke models for C^D_n.

Proof. Let $\mathcal{M}$ be a strict swap Kripke model for C^D_n, and suppose that, for some formula α and some world w in $\mathcal{M}$, $v^n_w(\mathsf{O}\alpha) \in D_n$ but $v^n_w(\neg\mathsf{O}\neg\alpha) = F_n$. The latter implies that $v^n_w(\mathsf{O}\neg\alpha) = T_n \in D_n$. By Definition 7.9 it follows that, for every w' in $\mathcal{M}$ such that wRw', it is the case that $v^n_{w'}(\alpha) = T_n = v^n_{w'}(\neg\alpha)$. But this is a contradiction, since $\tilde{\neg}\, T_n = \{F_n\}$. This shows that $\mathcal{M} \vDash \mathsf{O}\alpha \to \neg\mathsf{O}\neg\alpha$ for every strict swap Kripke model for C^D_n and for every formula α. ⊣

It also interesting to notice that these strict models collapse the two notions of permission. Whereas in the first formulation of C_D^n one would formally be able to characterize two distinct notions of permission, namely $\sim^{(n)}\mathsf{O}\sim^{(n)}\alpha$ and $\neg\mathsf{O}\neg\alpha$, in this strict formulation there is a collapse, since whenever $\mathsf{O}\alpha$ holds in w, in all the worlds accessible to w, α behaves classically and is true. It is an easy exercise to see the preservation of soundness and completeness of C_n^D plus (SD_n) w.r.t. strict swap Kripke models.

8. Applications of C_1^D and C_n^D to moral dilemmas

The motivation behind applying the logics in the C_n^D-hierarchy to moral dilemmas stems from the original work on the topic (da Costa and Carnielli, 1986) together with other works published on the topic, (Puga and da Costa, 1987a,b; Puga et al., 1988). The work by da Costa and Carnielli focuses on the idea of building a system that tolerates deontic conflicts without resulting in deontic trivialization, and trivialization as a result of O-aggregation. The others work cited diversify the topics investigated.

To give an overview on the ways they diversify the topics, we briefly mention the overall topics discussed in the aforementioned papers. In (Puga et al., 1988), the authors expand the original work to bimodal systems that satisfy instances of Kant's Law (KL) and Hintikka's Law (HL), respectively,

$$\mathsf{O}\alpha \to \Diamond\alpha, \tag{KL}$$

$$\Box\alpha \to \mathsf{O}\alpha, \tag{HL}$$

where $\Box$ and $\Diamond$ are alethic modalities, dually interdefinable. The works presented in (Puga and da Costa, 1987a,b) relate *legal* and *moral* modalities, assigning to each notion a distinct deontic modality, O_l and O_m respectively, which are independently defined and brought together by bridge axioms.

Although the aforementioned works motivate the presentation of new deontic systems on top of the ones presented in (da Costa and Carnielli, 1986), they reserve themselves to only lay the formal grounds upon which the philosopher interested in Ethics or Moral Philosophy can develop their work (Puga and da Costa, 1987b, pp. 35–36). However, as pointed out in (Vaz and Maruchi, 2025), paraconsistent deontic logics seem to be well-suited systems to deal with conflicting obligations that occur in the context of moral dilemmas. Hence, we envision that discussing the application of C_n^D to these contexts seems a fruitful enterprise. Moreover, the expansion of the original system C_1^D to bimodal systems (and potentially to multimodal systems) already present in the literature makes it reasonable to infer that the same could be done by the techniques presented in this paper and not only having C_1^D as a base logic, but any logic in the C_n^D hierarchy.

Moral dilemmas are usually stated as follows:

$$\mathsf{O}\alpha \wedge \mathsf{O}\beta \wedge \neg\Diamond(\alpha \wedge \beta),$$

where $\Diamond$ is an alethic modality. Clearly, when $\beta = \neg\alpha$, we have conflicting obligations. A standard example of moral dilemmas is Sophie's Dilemma, in which a prisoner of a Nazi camp has to decide to save either her daughter or her son, who are scheduled to be executed and if she decides not to pick between one of them, both are executed.

We allow ourselves to state a few remarks here. First, the authors in (Vaz and Maruchi, 2025) claim that conflicting obligations are the root cause of the eventual trivializations in moral dilemmas, thus relegating a secondary role to the alethic operator. Second, the negation appearing outside the scope of the alethic operator could be, as an alternative formulation, a distinct negation which behaves classically, so that it would it would render impossible the solution of a moral dilemma to be given by paraconsistent logic alone. Our focus is to solve the problem deontically, with paraconsistency being a feature of the deontic systems we are studying. In other words, we want the focus of our discussion to be formulas that are deontic and have negations only inside the scope of the modal operator.

If we look at the system C_1^D, there are a few options on how to solve moral dilemmas. The first and obvious route is to try and differentiate the negation happening in the scope of the modal operator. Thus, for example, if the formula occurring in a moral dilemma is of the form $\mathsf{O}\alpha\wedge\mathsf{O}\neg\alpha$, then this is perfectly acceptable in our model for C_1^D, although such an explanation can be deemed insufficient. We should not only point towards a *formal* solution, but also give a satisfying philosophical interpretation to the formulas so that they make sense in a deontic setting.

For example, we can interpret $\mathsf{O}\neg\alpha$ as representing a notion of *weak* prohibition. Such an interpretation would result in a different scenario than that one pictured in Sophie's Dilemma. Rather, one could think of such a prohibition to be a minor one, in which no significant results would follow when choosing either horn of the dilemma, while failing to satisfy the other.[11]

Instead, since the consequences of not saving one of the children in Sophie's Dilemma potentially result in their death, we are forced to formalize those cases as $\mathsf{O}\alpha \wedge \mathsf{O}{\sim}\alpha$, thus using a notion of *strong* prohibition[12], i.e., $\mathsf{O}{\sim}\alpha$. Under such an interpretation α behaves classically in a deontic setting and thus deontic explosion is recovered. This shows that, although C_1^D allows for weak dilemmas to occur without trivialization of the system, the same cannot be said about strong dilemmas, as those occurring in Sophie's Dilemma.

We also notice that when we start to interpret moral dilemmas in other systems that are members of the C_n^D hierarchy, we might trace finer distinctions between levels of "strength" that dilemmas might present. The C_n^D-hierarchy allows us to account for stronger dilemmas in a certain system. Since we move the classical behavior of the negation up the hierarchy, strong negation understood as $\neg\alpha \wedge \alpha^1$ can work as part of a definition of a "strong obligation", while we have room to define other kinds of "stronger obligations". In this sense, we would have the conflicting obligations in Sophie's Dilemma being assigned a designated value in the system without trivializing the system.

Another discussion that is necessary in order to bring these systems to their full potential is whether or not these distinctions between weak and strong obligations in fact play a role in the actual situations we are trying to formalize. This, however, is a discussion that the authors will delve into in further papers.

In summary, while C_1^D allows for some distinction between weak and strong prohibitions in a naive sense, it does not accommodate for

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[11] For example, picture the following scenario: you have a class on Friday night and a friend calls you offering a ticket for a concert that they can not attend anymore due to personal reasons. By attending the concert, you miss class and potentially fail your course, but it happens that is a band you really like, and might be your last opportunity to see them live, and as a big fan of art, you have a principle to always support the artists you like whenever possible. This could count as a minor dilemma, since the consequences of this act would not have big consequences, such as somebody's death or the starting of a war.

[12] We diverge from the usual talk about 'weak' and 'strong' modalities in deontic logics, usually referring to permissions, as presented in (Hansson, 2013), since our notion is heavily dependent on the kind of negation inside the scope of the modal operator, and not on the fact that the obligation is satisfied or fails to be satisfied.

a conflict between an obligation and a strong prohibition, thus limiting its usefulness to formalize and deal with moral dilemmas. On the other hand, it is just the first step of a whole hierarchy, which, in turn, allows for both cases to be formalized and dealt with, with trivialization.

9. Concluding Remarks

This paper presented a new way to give semantics to modal *LFI*s conflating possible worlds and nondeterministic endeavours via swap structures. Although our approach here is not fully deterministic, it maintains a good balance between Nmatrices, RNmatrices and Kripke semantics, showing that it is possible to mix them, by satisfactorily characterizing these logics in such a setting. In particular, we described many logics along the *LDI*s hierarchy, for the special case where $\bar{\boxminus}(\alpha) := \mathsf{O}{\circ}\alpha$ (recall Section 1). Our investigations started with $DmbC$, the minimal *LFI* equipped with the modal axioms for SDL, and walking up the hierarchy basing our propositional semantics on Nmatrices.

By adding (cl) to $DmbC$, we strengthen our systems in such a way that it becomes impossible to characterize them in terms of finite Nmatrices alone. We then resort to a reading of RNmatrices adapted to swap structures, namely, a restriction in the admissible swap valuations. This move allows us to characterize $DmbCcl$ and also stronger logics, such as $DCila$, and the whole of C_n^D hierarchy.

Regarding the latter, the developments here presented are entirely new. In the case of C_1^D, a sketch of its semantics, by means of bivaluations, was given in (da Costa and Carnielli, 1986). In this paper, we fully develop those proofs by means of the novel notion of swap Kripke structures, giving proofs for both $DCila$ and C_1^D. For C_n^D in general, it is the first time this family of systems is fully developed and semantically characterized. We also discuss briefly the different systems that can be defined given the multiple notions of negation these system are able to express. A thorough survey of such systems would require a paper on its own, and our objective here is to lay down the technical grounds upon which this discussion is allowed to be attained.

We believe this combination between nondeterministic semantics and possible worlds semantics can be fruitful in the conception of new semantics for logical systems, since it allows for the introduction of new concepts into the logic, for example, detaching modal notions from possible world semantics, thus allowing for a higher expressivity of these nondeterministic modal systems. The mix between them also allow for systems in which each world is nondeterministic and, as seen in the case for C_n^D, the modal operator that has its truth conditions based on these nondeterministic worlds inherits the nondeterministic behavior, as well as being sufficiently expressive as to accommodate for two notions of prohibition in its full capabilities. This allows a more or less fine-grained distinctions of dilemmas, depending on how far into the hierarchy the dilemma is modeled. We also point out that further investigation should look deeper into the philosophical aspects of such systems, as well how they fare when modeling other paradoxes of deontic logics.

Acknowledgments. We thank the anonymous referee for the deep and thorough comments on an earlier version of this paper. This feedback helped us to improve the overall quality of the manuscript. Vaz holds a PhD scholarship from the São Paulo Research Foundation (FAPESP, Brazil), grant 2022/16816-9, and was also financed by the German Academic Exchange Service (DAAD, Germany), under the Bi-nationally supervised/Cotutelle Doctorate degree program. Coniglio acknowledges support by an individual research grant from the National Council for Scientific and Technological Development (CNPq, Brazil), grant 309830/2023-0. All the authors were supported by the São Paulo Research Foundation (FAPESP, Brazil), thematic project *Rationality, logic and probability – RatioLog*, grant 2020/16353-3.

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