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Reaching Classicality through Transitive Closure

Abstract. Recently, Da Ré, Szmuc, Chemla and Égré (2024) showed that all logics based on *Boolean Normal monotonic three-valued schemes* coincide with classical logic when defined using a *strict-tolerant* standard (**st**). Conversely, they proved that under a *tolerant-strict* standard (**ts**), the resulting logics are all empty. Building on these results, we show that classical logic can be obtained by closing under transitivity the union of two logics defined over (potentially different) Boolean normal monotonic schemes, using a strict-strict standard (**ss**) for one and a tolerant-tolerant standard (**tt**) for the other, with the first of these logics being paracomplete and the other being paraconsistent. We then identify a notion dual to transitivity that allows us to characterize the logic TS as the dual transitive closure of the intersection of any two logics defined over (potentially different) Boolean normal monotonic schemes, using an **ss** standard for one and a **tt** standard for the other. Finally, we expand on the abstract relations between the transitive closure and dual transitive closure operations, showing that they give rise to lattice operations that precisely capture how the logics discussed relate to one another.

Keywords: three-valued logics; transitive closure; non-classical logics; strict-tolerant logic; logic of paradox; Strong Kleene logic; Weak Kleene logic

1. Introduction

Are there three-valued presentations of classical logic? Cobreros et al. (2012) answer this question affirmatively by showing that certain three-valued logics based on the Strong Kleene scheme validate precisely the same inferences as classical logic. These results were later on extended to the Weak Kleene counterparts of these logics by Szmuc and Ferguson

(2021) and Ferguson (2023). Ultimately, Da Ré et al. (2024) identified all the three-valued truth-functional logics corresponding to classical logic. In particular, they identified a subclass of these logics, characterized by a strict-tolerant (**st**) standard of argument evaluation, defined over a *Boolean normal monotonic scheme* for the connectives (BNM). One such logic is the strict-tolerant logic ST (Cobreros et al., 2012, 2013; Ripley, 2012), based on the aforementioned Strong Kleene scheme. They further proved that logics defined with a standard among **ss**, **tt**, **ts** and **ss** ∩ **tt** over a BNM scheme are necessarily weaker than classical logic. The Strong Kleene logic K_3 (Kleene, 1952), the logic of paradox LP (Asenjo, 1966; Priest, 1979), the tolerant-strict logic TS (Cobreros et al., 2012), and the logic of order KO (Makinson, 1973; also known as S_3 by Field, 2008, and RM_{fde} by Dunn, 1976) are notable examples of such logics, all based on the Strong Kleene scheme. Other such logics include those based on the Middle Kleene scheme (Beaver and Krahmer, 2001; George, 2014; Peters, 1979), and the logics based on the Weak Kleene scheme, such as K_3^w (Bochvar, 1981) and Paraconsistent Weak Kleene logic PWK (Halldén, 1949).

Nonetheless, the authors do not investigate the properties of logics defined by the union of the **ss** and **tt** standards. It is known, for instance, that the union of K_3 and LP does not align with classical logic (see, e.g., Wintein, 2016, p. 511) or equivalently ST. The relationship between K_3 and LP to classical logic has been studied by Blomet and Égré (2024), where it is shown that the set of valid inferences of ST is the relational composition of the sets of K_3- and LP-valid inferences. However, the authors do not examine the relationship between classical logic and other logics based on the **ss** and **tt** standards.

This paper first extends the aforementioned inequality result, showing that no logic defined by the union of the **ss** and **tt** standards and a BNM scheme coincides with classical logic. This naturally raises the question: what is absent from this union to recover classical logic? We will address this by showing that what is missing is *transitivity*. Once closed under transitivity, the union of an **ss**- and a **tt**-logic is classical logic.

Just as the union of an **ss**- and a **tt**-logic based on a BNM scheme fails to be classical logic, the intersection of an **ss**- and a **tt**-logic does not coincide with the aforementioned non-reflexive logic TS. Da Ré et al. (2024) show in particular that any logic with a **ts** standard defined over a BNM scheme collapses into TS. In contrast, the intersection of

an **ss**- and a **tt**-logic is reflexive, as exemplified by the logic KO. We are thus justified in asking a similar question to the one posed for the union: what is missing from the intersection to yield the logic TS? Once again, we will answer this by showing that what is missing is a relational property — specifically the property of *dual transitivity*.

We will then explore the relationship between the transitive closure operator and its dual, as well as their effects when applied to sets of validities. This analysis will ultimately enable us to present a cohesive overview of the interrelations among all the logics discussed, showing that their sets of valid inferences form a sublattice of the lattice of logics satisfying the Tarskian properties of Reflexivity, Monotonicity, Transitivity, and Structurality. We further show that their sets of antitheorems and theorems — defined here as inferences with unsatisfiable premises and inferences with an unfalsifiable conclusion — form a lattice induced by the dual transitive closure operator.

The paper is structured as follows. In Section 2, we provide all the preliminary technicalities required for the paper. In Section 3, we introduce the operations of *transitive closure* and *dual transitive closure*, and present their main properties. In Section 4 we prove that the transitive closure of the union of two **ss**- and **tt**-logics coincides with an **st**-logic (that is, with classical logic), based on possibly different BNM schemes. We then dualize this result and prove that applying the dual transitive closure to the intersection of an **ss**- and a **tt**-logic yields a **ts**-logic (i.e., the empty logic), possibly based on different BNM schemes. In Section 5, we elaborate on some remarks regarding the dual transitive closure operator. In Section 6, we provide an overview of the interrelations among all the logics discussed. Finally, in Section 7, we conclude with our final observations and remarks.

2. Preliminary definitions

DEFINITION 2.1 (Language). The propositional language $\mathcal{L}$ is built from a denumerably infinite set $Var = \{p, p', \ldots\}$ of propositional variables, using the logical constants $\neg, \vee, \wedge$ (negation, disjunction, conjunction).

Elements of $\mathcal{L}$ will be denoted by lower-case Greek letters or upper-case Latin letters, depending on the context. Subsets of $\mathcal{L}$ will be denoted by upper-case Greek letters. Given a function $f \colon Var \longrightarrow \mathcal{L}$, f extends to a *substitution* σ on $\mathcal{L}$ by letting $\sigma(\star(\phi_1, \ldots, \phi_n)) = \star(\sigma(\phi_1), \ldots, \sigma(\phi_n))$

for all logical constants $\star$. Given a formula ϕ, $\text{At}(\phi)$ denotes the set of propositional variables of ϕ. By extension, given a set of formulas Γ, $\text{At}(\Gamma) := \bigcup\{\text{At}(\gamma) : \gamma \in \Gamma\}$.

DEFINITION 2.2. An *inference* is an ordered pair $\langle \Gamma, \phi \rangle$, denoted by $\Gamma \Rightarrow \phi$, where $\phi \in \mathcal{L}$ and $\Gamma \subseteq \mathcal{L}$ is finite.

Before proceeding, a few remarks on the previous definition are in order. First, note that inferences are defined with a finite set of premises. However, this does not compromise the generality of the results in this paper. All the logics discussed here are finitely-valued, and therefore finitary (see Wójcicki, 1988, 4.1.7). Consequently, they are entirely determined by their set of inferences with finitely many premises. Second, we choose to work with single conclusions, as this framework allows for simpler and more elegant definitions of the operations introduced in the next section.[1]

Blackboard letters $\mathbb{L}$ are used to denote sets of inferences. Logics L are occasionally identified with sets of inferences $\mathbb{L}$.[2] Such an identification is more liberal than the common definition of a logic as a set of inferences satisfying the Tarskian properties of Reflexivity, Monotonicity, Transitivity, and Structurality (see, e.g., Font, 2016). Recognizing non-transitive systems like ST or non-reflexive ones like TS as logics requires adopting a broader definition. We therefore distinguish a logic simpliciter from a Tarskian logic.

DEFINITION 2.3 (Tarskian logic). A logic L is said to be *Tarskian* if it satisfies the following properties, for all $\Gamma, \Sigma, \{\phi\} \subseteq \mathcal{L}$.

(R) $\phi \Rightarrow \phi \in \mathbb{L}$.   (Reflexivity)
(M) If $\Gamma \Rightarrow \phi \in \mathbb{L}$ and $\Gamma \subseteq \Sigma$, then $\Sigma \Rightarrow \phi \in \mathbb{L}$.   (Monotonicity)
(T) If $(\exists \Delta \neq \emptyset)\, \Delta \Rightarrow \phi \in \mathbb{L}$ and $(\forall \delta \in \Delta)\, \Gamma \Rightarrow \delta \in \mathbb{L}$, then $\Gamma \Rightarrow \phi \in \mathbb{L}$.
                                                                          (Transitivity)
(S) If $\Gamma \Rightarrow \phi \in \mathbb{L}$, then $\sigma[\Gamma] \Rightarrow \sigma[\phi] \in \mathbb{L}$ for all substitutions σ.
                                                                          (Structurality)

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[1] Note that Da Ré et al. (2024) adopt a multiple-conclusion setting. However, none of their results fundamentally depend on this choice, as everything in that paper could be adapted to a single-conclusion framework. We intend to address the multiple-conclusion case in future work.

[2] Although extensionally identical, we distinguish between these two objects. This distinction is necessary since we sometimes refer to logical properties (such as the validity of an inference), while at other times, we consider set-theoretic properties (such as the union of two sets of inferences).

Notice the unconventional formulation of Transitivity (T), which requires that Δ be non-empty. This choice is motivated by two main considerations. First, the standard form of Transitivity — without this restriction — implicitly collapses into a weaker form of Monotonicity when $\Delta = \emptyset$: if $\emptyset \Rightarrow \phi \in \mathbb{L}$, then $\Gamma \Rightarrow \phi \in \mathbb{L}$. By contrast, our formulation keeps Transitivity and Monotonicity distinct. Second, this restricted version of Transitivity will later be dualized, and the constraint on Δ will play a notable role in that development (see fn. 3).

A Tarskian logic can be obtained from any set of inferences by closing it under (R), (M), (T), and (S).

DEFINITION 2.4 (Tarskian closure). For any set of inferences $\mathbb{L}$, the Tarskian closure of $\mathbb{L}$, noted $Tar(\mathbb{L})$, is the least set $\mathbb{L}' \supseteq \mathbb{L}$ satisfying (R), (M), (T) and (S).

Given a logic, we provide a semantic characterization of its set of valid inferences. A *valuation* is a function v from $\mathcal{L}$ to $\mathcal{V}$, where $\mathcal{V}$ is a set of truth values. It is important to note that, as for now, valuations are not subject to any constraints of truth-functionality. In this paper, we will consider two possible sets of values $\mathcal{V}_2 = \{0, 1\}$ and $\mathcal{V}_3 = \{0, 1/2, 1\}$.

DEFINITION 2.5. A *formula-standard* $\mathbf{z}$ is defined as $\mathbf{z} \subseteq \mathcal{V}$. A *standard* is defined as $\mathbf{xy}$, where $\mathbf{x}$, $\mathbf{y}$ are formula-standards.

DEFINITION 2.6. A valuation v satisfies a formula ϕ according to a formula-standard $\mathbf{x}$ ($v \vDash_{\mathbf{x}} \phi$), if $v(\phi) \in \mathbf{x}$. A valuation v satisfies an inference $\Gamma \Rightarrow \phi$ according to a standard $\mathbf{xy}$, denoted as $v \vDash_{\mathbf{xy}} \Gamma \Rightarrow \phi$, if the following holds:

$$\text{if } v \vDash_{\mathbf{x}} \gamma \text{ for all } \gamma \in \Gamma, \text{ then } v \vDash_{\mathbf{y}} \phi.$$

An inference is valid according to a standard $\mathbf{xy}$, denoted as $\vDash_{\mathbf{xy}} \Gamma \Rightarrow \phi$, if $v \vDash_{\mathbf{xy}} \Gamma \Rightarrow \phi$ for all v.

Two well-studied formula-standards on $\mathcal{V}_3$ are the *strict* and the *tolerant* standards, defined respectively as $\mathbf{s} = \{1\}$ and $\mathbf{t} = \{1, 1/2\}$. With these two standards, it is possible to define four possible standards: $\mathbf{ss}, \mathbf{tt}, \mathbf{st}$ and $\mathbf{ts}$ (strict-strict, tolerant-tolerant, strict-tolerant and tolerant strict, respectively). Given a standard $\mathbf{xy}$ we can define a set of inferences $\mathbb{XY}$ containing exactly the inferences valid according to $\mathbf{xy}$ by every valuation. In particular, from the aforementioned standards,
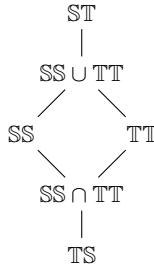
Figure 1. Six sets of inferences extracted from the standards $\mathbf{s}, \mathbf{t}$

we can extract six sets of inferences, which are ordered by inclusion in Figure 1 (see Chemla et al., 2017).[3]

Up to this point, we have imposed no constraints on the trivaluations. In this article, we will consider valuations respecting some three-valued *scheme* **X**, where **X** is a triple $(f_\neg, f_\wedge, f_\vee)$ of operations. In particular, following Da Ré et al. (2024), we will consider *Boolean normal monotonic* schemes (hereinafter referred to as BNM schemes).

DEFINITION 2.7 (Da Ré et al., 2024). An *n*-ary operation $\star$ is *Boolean normal*[4] iff for all $a_1, \ldots, a_n \in \{0, 1\}$, $\star(a_1, \ldots, a_n) = \star^{\mathsf{CL}}(a_1, \ldots, a_n)$, where $\star^{\mathsf{CL}}$ is the corresponding operation over the usual two-elements Boolean algebra. A scheme is *Boolean normal* if and only if each of its operations is.

When considering the Boolean normal bivaluations on $\mathcal{V}_2$, all the standards define what we refer to as the set of classical inferences, denoted $\mathbb{CL}$.

For *monotonicity*, we first assume that the truth values are ordered according to the order according to which $1/2 <_\mathrm{I} 0$, $1/2 <_\mathrm{I} 1$, and 1 and 0 are incomparable. The componentwise ordering induced by this relation is then defined as follows: $\langle a_1, \ldots, a_n \rangle \leq_\mathrm{I}^{comp} \langle b_1, \ldots, b_n \rangle$ if and only if $a_j \leq_\mathrm{I} b_j$ for each $1 \leqslant j \leqslant n$. We take the following definition from (Da Ré et al., 2024).

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[3] Notice that in Figure 1, the inclusion between the sets of inferences is not necessarily proper, and is independent of any particular choice of logical vocabulary. See (Chemla et al., 2017) for more details.

[4] There are alternative nomenclatures for Boolean normality. For instance, as an anonymous reviewer has pointed out, Carnielli et al. (2000, p. 129) call this property *hyper-classicality*.

DEFINITION 2.8. An n-ary operation $\star$ is *monotonic* if and only if whenever $\langle a_1, \ldots, a_n \rangle \leq_I^{comp} \langle b_1, \ldots, b_n \rangle$ then $\star(a_1, \ldots, a_n) \leq_I \star(b_1, \ldots, b_n)$. A scheme is monotonic if and only if each of its operations is.

Let us review some examples. It's straightforward to check that the usual unary operation representing negation of Strong Kleene or Weak Kleene logics, i.e. $\star_1(x) = 1 - x$, is a monotonic operation. However, we can define a nonmonotonic unary operation as follows: $\star_2(1/2) = \star_2(0) = 1$ and $\star_2(1) = 0$. Notice that $1/2 <_I 1$ but $1 = \star_2(1/2) \nleq_I \star_2(1) = 0$. For more examples of monotonic and nonmonotonic operations, see (Da Ré et al., 2024).

A valuation v is said to respect a scheme $\mathbf{X}$ if for every operation $f^\star$ of $\mathbf{X}$ and corresponding symbol $\star$ in the language, it satisfies

$$v(\star(\phi_1 \ldots \phi_n)) = f^\star(v(\phi_1) \ldots v(\phi_n)).$$

We now observe that a set of valuations, together with a standard, fully determines the semantics of a logic.

Occasionally, when working with a logic defined by some standard $\mathbf{xy}$ and a scheme $\mathbf{X}$, we denote the fact that an inference $\Gamma \Rightarrow \phi$ is satisfied by writing $v \vDash_{\mathbf{xy}}^{\mathbf{X}} \Gamma \Rightarrow \phi$. We say that $\Gamma \Rightarrow \phi$ is valid in L if it is satisfied by every valuation (symbolically, $\vDash_{\mathbf{xy}}^{\mathbf{X}} \Gamma \Rightarrow \phi$). When the context makes the scheme clear, we omit the superscript $\mathbf{X}$.

3. Transitive closure and dual transitive closure

In this section, we introduce two operators: the *transitive closure operator* and the *dual transitive closure operator*. The transitive closure operator will enable us to define the set of valid inferences of an $\mathbf{st}$ logic based on a BNM scheme. This will be achieved by closing under transitivity the union of the sets of valid inferences of two $\mathbf{ss}$ and $\mathbf{tt}$ logics, based on a BNM scheme. In contrast, the dual transitive operator will allow us to define the set of valid inferences of a $\mathbf{ts}$ logic. Specifically, applying such operation to the intersection of the sets of valid inferences of two $\mathbf{ss}$ and $\mathbf{tt}$ logics will yield the sets of valid inferences of a $\mathbf{ts}$ logic.

DEFINITION 3.1 (Transitive closure). For any set of inferences $\mathbb{L}$, the transitive closure of $\mathbb{L}$, noted $T(\mathbb{L})$, is the least set $\mathbb{L}' \supseteq \mathbb{L}$ such that

if $(\exists \Delta \neq \emptyset)$ $\Delta \Rightarrow \phi \in \mathbb{L}'$ and $(\forall \delta \in \Delta)$ $\Gamma \Rightarrow \delta \in \mathbb{L}'$, then $\Gamma \Rightarrow \phi \in \mathbb{L}'$.

We need to ensure that the operator is a closure operator.[5] This is guaranteed by the following fact.

FACT 3.1. *T is a closure operator.*

PROOF. The inclusion $\mathbb{L} \subseteq T(\mathbb{L})$ for all $\mathbb{L}$ follows directly from the definition of transitive closure.

Turning to the monotonicity of T, assume that $\mathbb{L} \subseteq \mathbb{L}'$. Then $\mathbb{L} \subseteq \mathbb{L}' \subseteq T(\mathbb{L}')$. So, $T(\mathbb{L}')$ is an extension of $\mathbb{L}$ such that for all $\Delta \neq \emptyset$, if $\Delta \Rightarrow \phi \in T(\mathbb{L}')$ and $\Gamma \Rightarrow \delta \in T(\mathbb{L}')$ for all $\delta \in \Delta$, then $\Gamma \Rightarrow \phi \in T(\mathbb{L}')$. By definition, $T(\mathbb{L})$ is the least extension of $\mathbb{L}$ satisfying such a property, so $T(\mathbb{L}) \subseteq T(\mathbb{L}')$.

To show that $T(T(\mathbb{L})) = T(\mathbb{L})$, it suffices to establish the left-to-right inclusion, as the reverse direction follows directly from the previously proven fact that $\mathbb{L} \subseteq T(\mathbb{L})$ for all $\mathbb{L}$. Trivially, $T(\mathbb{L})$ is an extension of itself and is such that for all $\Delta \neq \emptyset$, if $\Delta \Rightarrow \phi \in T(\mathbb{L})$ and $\Gamma \Rightarrow \delta \in T(\mathbb{L})$ for all $\delta \in \Delta$, then $\Gamma \Rightarrow \phi \in T(\mathbb{L})$. Now, by the definition, $T(T(\mathbb{L}))$ is the least extension of $T(\mathbb{L})$ with such a property. Hence, $T(T(\mathbb{L})) \subseteq T(\mathbb{L})$.   ⊣

We proceed to define the notion of *dual transitive closure.*

DEFINITION 3.2 (Dual transitive closure). For any set of inferences $\mathbb{L}$, the dual transitive closure of $\mathbb{L}$, noted $T^d(\mathbb{L})$, is the greatest set $\mathbb{L}' \subseteq \mathbb{L}$ such that

if $(\exists \Delta \neq \emptyset)\ \Delta \Rightarrow \phi \notin \mathbb{L}'$ and $(\forall \delta \in \Delta)\ \Gamma \Rightarrow \delta \notin \mathbb{L}'$, then $\Gamma \Rightarrow \phi \notin \mathbb{L}'$.

Equivalently, the dual transitive closure of a set $\mathbb{L}$ is the greatest $\mathbb{L}' \subseteq \mathbb{L}$ such that

if $\Gamma \Rightarrow \phi \in \mathbb{L}'$, then $(\forall \Delta \neq \emptyset)\ \Delta \Rightarrow \phi \in \mathbb{L}'$ or $(\exists \delta \in \Delta)\ \Gamma \Rightarrow \delta \in \mathbb{L}'$.[6]

Next, we show in what sense this notion of dual transitive closure is dual to the notion of transitive closure. The proposition hereinafter

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[5] A function $C \colon \mathcal{P}(A) \to \mathcal{P}(A)$ is said to be a *closure operator on A* if for all $X, Y \subseteq A$: $X \subseteq C(X)$; $CC(X) = C(X)$; if $X \subseteq Y$, then $C(X) \subseteq C(Y)$.

[6] Removing the constraint that Δ be empty, as envisioned above, causes the dual transitive of a set to reduce to a collection of inferences $\Gamma \Rightarrow \phi$ for which $\emptyset \Rightarrow \phi \in \mathbb{L}$, as the second disjunct of the consequent is vacuously false in the case where $\Delta = \emptyset$. Hence our decision to impose this restriction: it is otherwise too limiting, and the partial collapse under Monotonicity already discussed provides an additional reason motivating the constraint.

eventually expresses the duality of the transitive closure operator and the dual transitive closure operator. The two operators are interdefinable through the set-theoretical notion of absolute complement.

PROPOSITION 3.1. $T^d(\mathbb{L}) = \overline{T(\overline{\mathbb{L}})}$ and $T(\mathbb{L}) = \overline{T^d(\overline{\mathbb{L}})}$.

PROOF. For any set of inferences $\mathbb{L}$, let $P(\mathbb{L})$ express that if $(\exists \Delta \neq \emptyset)$ $\Delta \Rightarrow \phi \in \mathbb{L}$ and $(\forall \delta \in \Delta)$ $\Gamma \Rightarrow \delta \in \mathbb{L}$, it follows that $\Gamma \Rightarrow \phi \in \mathbb{L}$.

Starting with the first identity, by definition, $T(\overline{\mathbb{L}})$ is the least element of the set $\{\mathbb{L}' \mid \overline{\mathbb{L}} \subseteq \mathbb{L}', \ P(\mathbb{L}')\}$. Equivalently, $\overline{T(\overline{\mathbb{L}})}$ is the greatest element of $\{\overline{\mathbb{L}'} \mid \overline{\mathbb{L}} \subseteq \mathbb{L}', \ P(\mathbb{L}')\}$, which coincides with $\{\mathbb{L}'' \mid \mathbb{L}'' \subseteq \mathbb{L}, \ P(\overline{\mathbb{L}''})\}$. That is, $\overline{T(\overline{\mathbb{L}})}$ is the largest subset $\mathbb{L}''$ of $\mathbb{L}$ such that if $(\exists \Delta \neq \emptyset)$ $\Delta \Rightarrow \phi \notin \mathbb{L}''$ and $(\forall \delta \in \Delta)$ $\Gamma \Rightarrow \delta \notin \mathbb{L}''$, then $\Gamma \Rightarrow \phi \notin \mathbb{L}''$. Therefore, we conclude that $T^d(\mathbb{L}) = \overline{T(\overline{\mathbb{L}})}$. The other identity can be proved similarly.   ⊣

It can now be checked that T^d is an *interior operator*.[7]

FACT 3.2. T^d is an interior operator.

PROOF. 1. Since T is a closure operator, $\overline{\mathbb{L}} \subseteq T(\overline{\mathbb{L}})$, and thus $T^d(\mathbb{L}) = \overline{T(\overline{\mathbb{L}})} \subseteq \overline{\overline{\mathbb{L}}} = \mathbb{L}$, by Proposition 3.1.

Assume $\mathbb{L} \subseteq \mathbb{L}'$. Then $\overline{\mathbb{L}'} \subseteq \overline{\mathbb{L}}$, so $T(\overline{\mathbb{L}'}) \subseteq T(\overline{\mathbb{L}})$ since T is a closure operator, and hence $T^d(\mathbb{L}) = \overline{T(\overline{\mathbb{L}})} \subseteq \overline{T(\overline{\mathbb{L}'})} = T^d(\mathbb{L}')$ by Proposition 3.1.

Since T is a closure operator, $T(\overline{\mathbb{L}}) \subseteq T(T(\overline{\mathbb{L}}))$, so $\overline{T(T(\overline{\mathbb{L}}))} \subseteq \overline{T(\overline{\mathbb{L}})}$. But, by Proposition 3.1, $T^d(T^d(\mathbb{L})) = \overline{T(\overline{T^d(\mathbb{L})})} = \overline{T(T(\overline{\mathbb{L}}))}$. Therefore, $T^d(T^d(\mathbb{L})) \subseteq \overline{T(\overline{\mathbb{L}})} = T^d(\mathbb{L})$.   ⊣

With these definitions and facts at our disposal, we are equipped to offer an exact characterization of $\mathbb{ST}$ as the transitive closure of $\mathbb{SS} \cup \mathbb{TT}$, and $\mathbb{TS}$ as the dual transitive closure of $\mathbb{SS} \cap \mathbb{TT}$.

4. Characterization of $\mathbb{ST}$ and $\mathbb{TS}$

Our aim in this section is to characterize **st**- and **ts**-logics by combining **ss**- and **tt**-logics. We first establish that the union of $\mathbb{SS}$ and $\mathbb{TT}$ does not yield $\mathbb{ST}$. We then prove that the transitive closure of their union actually coincides with $\mathbb{ST}$ (and hence with $\mathbb{CL}$). A dual version of this result is then proved for the characterization of **ts**-logics: their set of

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[7] A function $I \colon \mathcal{P}(A) \to \mathcal{P}(A)$ is an interior operator if it satisfies for all $X, Y \subseteq A$: $I(X) \subseteq X$; $II(X) = I(X)$; if $X \subseteq Y$, then $I(X) \subseteq I(Y)$.

valid inferences is obtained by applying the dual transitive closure to the intersection of the sets of **tt**- and **ss**-valid inferences. Importantly, all the results in this section assume that the sets of inferences are defined using *BNM* schemes, possibly different from one logic to another. We start by restating the following results proven in (Da Ré et al., 2024):

THEOREM 4.1 (Da Ré et al., 2024). *For every BNM scheme,* $\mathbb{ST} = \mathbb{CL}$.

THEOREM 4.2 (Da Ré et al., 2024). *For every BNM scheme,* $\mathbb{TS} = \emptyset$.

The first theorem shows that any **st**-logic based on a BNM scheme collapses with classical logic. The second one that any **ts**-logic based on a BNM scheme is empty (i.e. invalidate every inference).

We now show that the union of the sets $\mathbb{SS}$ and $\mathbb{TT}$, obtained from the **ss** and **tt** standards, differs from $\mathbb{ST}$. To establish this, we provide a counterexample: an inference that is ST-valid but neither SS- nor TT-valid. First, we state the following relation between these sets.

FACT 4.1. *Let* $\mathbb{SS}, \mathbb{TT}$ *and* $\mathbb{ST}$ *be defined by (possibly different) BNM schemes. Then*

- $\mathbb{SS}, \mathbb{TT} \subseteq \mathbb{ST}$;
- $\mathbb{SS}$ *and* $\mathbb{TT}$ *are incomparable.*

PROOF. For the first bullet, take any BNM-valuation such that $v(p) = \text{\textonehalf}$ and $v(r) = 0$. It's easy to notice that $v \nvDash_{\mathbf{tt}} p \wedge \neg p \Rightarrow r$ and $v \nvDash_{\mathbf{ss}} r \Rightarrow p \vee \neg p$.[8] However, both inferences are classically valid, and thus by Theorem 4.1 they are in $\mathbb{ST}$.

As for the incomparability of $\mathbb{SS}$ and $\mathbb{TT}$, just note that, for any BNM scheme **X**, $\nvDash_{\mathbf{tt}}^{\mathbf{X}} p \wedge \neg p \Rightarrow r$, but $\vDash_{\mathbf{ss}}^{\mathbf{X}} p \wedge \neg p \Rightarrow r$, and $\nvDash_{\mathbf{ss}}^{\mathbf{X}} \emptyset \Rightarrow p \vee \neg p$, but $\vDash_{\mathbf{tt}}^{\mathbf{X}} \emptyset \Rightarrow p \vee \neg p$. ⊣

THEOREM 4.3. $\mathbb{SS} \cup \mathbb{TT} \subsetneq \mathbb{ST} = \mathbb{CL}$.

PROOF. By Theorem 4.1, we know that $\mathbb{ST} = \mathbb{CL}$. The fact that $\mathbb{SS} \cup \mathbb{TT} \subseteq \mathbb{ST}$ is trivial given the previous fact. So, let us show that $\mathbb{SS} \cup \mathbb{TT} \neq \mathbb{ST}$. In order to do so, let us consider $p \vee (q \wedge \neg q) \Rightarrow p \wedge (r \vee \neg r)$. This inference serves as a witness to the claim.[9]

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[8] Note that for every BNM scheme, if $v(p) = \text{\textonehalf}$ then $v(\neg p) = \text{\textonehalf}$, and thus $v(p \wedge \neg p) = v(p \vee \neg p) = \text{\textonehalf}$. For $v(r) = 0$, we get a **tt**-counterexample to the first inference, and for $v(r) = 1$, an **ss**-counterexample to the second.

[9] Note the similarity of this inference with the axiom *Safety* $q \wedge \neg q \Rightarrow r \vee \neg r$, which is characteristic of the intersection $\mathbb{SS} \cap \mathbb{TT}$, and thus cannot be used to substantiate the claim. We thank an anonymous reviewer for pointing this out to us.

Note first that given any BNM scheme, there is a valuation v such that $v \nvDash_{\mathbf{ss}} p \vee (q \wedge \neg q) \Rightarrow p \wedge (r \vee \neg r)$ (e.g. $v(p) = 1, v(q) = 0, v(r) = 1/2$) and there is another valuation v' such that $v' \nvDash_{\mathbf{tt}} p \vee (q \wedge \neg q) \Rightarrow p \wedge (r \vee \neg r)$ (e.g. $v'(p) = 0, v'(r) = 0, v'(q) = 1/2$). However, $\vDash_{\mathbf{st}} p \vee (q \wedge \neg q) \Rightarrow p \wedge (r \vee \neg r)$ since the inference is classically valid. ⊣

One may then ask whether the gap between the union of $\mathbb{SS}$ and $\mathbb{TT}$ logics and $\mathbb{ST}$ can be bridged. The following result shows that the transitive closure is the key to obtaining $\mathbb{ST}$ from the union of $\mathbb{SS}$ and $\mathbb{TT}$.

THEOREM 4.4. $T(\mathbb{SS} \cup \mathbb{TT}) = \mathbb{ST}$.

PROOF. ($\subseteq$) From Fact 4.1, it holds that $\mathbb{SS} \cup \mathbb{TT} \subseteq \mathbb{ST}$, so $T(\mathbb{SS} \cup \mathbb{TT}) \subseteq T(\mathbb{ST})$ since T is a closure operator. Now, $T(\mathbb{ST}) = \mathbb{ST}$, and therefore $T(\mathbb{SS} \cup \mathbb{TT}) \subseteq \mathbb{ST}$.

($\supseteq$) Assume $\Gamma \Rightarrow \phi \in \mathbb{ST}$, that is $\vDash_{\mathbf{st}}^{\mathbf{X}} \Gamma \Rightarrow \phi$. We show in particular that there is $\Delta \neq \emptyset$ such that $\vDash_{\mathbf{tt}}^{\mathbf{X}'} \Gamma \Rightarrow \delta$ for all $\delta \in \Delta$ and $\vDash_{\mathbf{ss}}^{\mathbf{X}''} \Delta \Rightarrow \phi$ with $\mathbf{X}'$ and $\mathbf{X}''$ two BNM schemes possibly different from $\mathbf{X}$. Let Δ be as follows:
$$\Delta := \Gamma \cup \{p \vee \neg p : p \in \mathrm{At}(\phi)\}$$
For all $\gamma \in \Gamma$, $\vDash_{\mathbf{tt}}^{\mathbf{X}'} \Gamma \Rightarrow \gamma$, since $\mathbb{TT}$ is reflexive. Moreover, $\vDash_{\mathbf{tt}}^{\mathbf{X}'} \Gamma \Rightarrow p \vee \neg p$ for all $p \in \mathrm{At}(\phi)$, since for any BNM valuation, $v(p \vee \neg p) \neq 0$. Turning to $\vDash_{\mathbf{ss}}^{\mathbf{X}''} \Delta \Rightarrow \phi$, assume $v(\delta) = 1$ for all $\delta \in \Delta$. By definition of Δ, it follows that $v(p \vee \neg p) = 1$ for all $p \in \mathrm{At}(\phi)$, and given that $\mathbf{X}''$ is a BNM scheme, $v(p) \neq 1/2$ for all $p \in \mathrm{At}(\phi)$. In addition, $v(\gamma) = 1$ for all $\gamma \in \Gamma$ and $\vDash_{\mathbf{st}}^{\mathbf{X}} \Gamma \Rightarrow \phi$ by assumption. By Theorem 4.1, $\vDash_{\mathbf{st}}^{\mathbf{X}''} \Gamma \Rightarrow \phi$, so $v(\phi) \neq 0$. But $v(p) \neq 1/2$ for all $p \in \mathrm{At}(\phi)$, thus $v(\phi) = 1$, since v is a BNM valuation. Now, $\Gamma \Rightarrow \delta \in \mathbb{TT}$ for all $\delta \in \Delta$ and $\Delta \Rightarrow \phi \in \mathbb{SS}$, meaning that $\Gamma \Rightarrow \delta \in \mathbb{SS} \cup \mathbb{TT}$ for all $\delta \in \Delta$ and $\Delta \Rightarrow \phi \in \mathbb{SS} \cup \mathbb{TT}$. Therefore, $\Gamma \Rightarrow \phi \in T(\mathbb{SS} \cup \mathbb{TT})$. ⊣

Note that the previous results rely heavily on the absence of truth constants for the intermediate value in the language. If we introduce a constant for this value, say λ, the closure under transitivity of the union of the valid inferences from any $\mathbf{tt}$-and $\mathbf{ss}$-logic would result in a trivial set of inferences. To see this, just note that for all Γ, ϕ, it holds that $\Gamma \Rightarrow \lambda \in \mathbb{TT}$ and $\lambda \Rightarrow \phi \in \mathbb{SS}$. As a result, $\Gamma \Rightarrow \phi \in T(\mathbb{SS} \cup \mathbb{TT})$ for any inference $\Gamma \Rightarrow \phi$, collapsing ST with the trivial logic, that is, the logic corresponding to the universal relation on $\mathcal{L}$.[10]

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[10] For characterizations of ST in languages containing λ, see (Blomet and Égré, 2024).

$\neg$		$\wedge$	1	$1/2$	0		$\vee$	1	$1/2$	0
1	0	1	1	$1/2$	0		1	1	$1/2$	1
$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$		$1/2$	$1/2$	$1/2$	$1/2$
0	1	0	0	$1/2$	0		0	1	$1/2$	0

Figure 2. Weak Kleene truth tables

$\neg$		$\wedge$	1	$1/2$	0		$\vee$	1	$1/2$	0
1	0	1	1	$1/2$	0		1	1	1	1
$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	0		$1/2$	1	$1/2$	$1/2$
0	1	0	0	0	0		0	1	$1/2$	0

Figure 3. Strong Kleene truth tables

Let us now turn our attention to the case of **ts**-logics. It is straight-forward to notice that $\mathbb{TS} \subsetneq \mathbb{TT} \cap \mathbb{SS}$, since $\mathbb{TS} = \emptyset$ while $\mathbb{TT} \cap \mathbb{SS} \neq \emptyset$ (for instance $\phi \Rightarrow \phi \in \mathbb{TT} \cap \mathbb{SS}$, for any ϕ). Thus to collapse $\mathbb{TT} \cap \mathbb{SS}$ with $\mathbb{TS}$ we need to erase all the inferences belonging to this set. The next result shows that the dual transitive closure operator does this job.

THEOREM 4.5. $T^d(\mathbb{TT} \cap \mathbb{SS}) = \mathbb{TS}$.

PROOF. ($\subseteq$) Let $\Gamma \Rightarrow \phi \in T^d(\mathbb{TT} \cap \mathbb{SS})$. Then for all $\Delta \neq \emptyset$, $\Gamma \Rightarrow \delta \in \mathbb{TT} \cap \mathbb{SS}$ for some $\delta \in \Delta$ or $\Delta \Rightarrow \phi \in \mathbb{TT} \cap \mathbb{SS}$. Meaning that for all $\Delta \neq \emptyset$, $\models_{\mathbf{tt}} \Gamma \Rightarrow \delta$ for some $\delta \in \Delta$ or $\models_{\mathbf{ss}} \Delta \Rightarrow \phi$, and in particular that for some $p \notin \text{At}(\Gamma) \cup \text{At}(\phi)$, $\models_{\mathbf{tt}} \Gamma \Rightarrow p$ or $\models_{\mathbf{ss}} p \Rightarrow \phi$. Since Γ is finite, such a p must exist. Now, $\models_{\mathbf{tt}} \Gamma \Rightarrow p$ is impossible, as for any BNM scheme, there exists a valuation v such that $v(p) = 0$ and $v(q) = 1/2$ for all $q \in Var$ with $q \neq p$, ensuring that $v \not\models_{\mathbf{tt}} \Gamma \Rightarrow p$. Likewise, $\models_{\mathbf{ss}} p \Rightarrow \phi$ is also impossible, since for any BNM scheme, there exists a valuation v' such that $v'(p) = 1$ and $v'(q) = 1/2$ for all $q \in Var$ with $q \neq p$, ensuring that $v \not\models_{\mathbf{ss}} p \Rightarrow \phi$. Therefore, $T^d(\mathbb{TT} \cap \mathbb{SS}) = \emptyset = \mathbb{TS}$.

($\supseteq$) Trivial, since $\mathbb{TS} = \emptyset$. $\dashv$

Having established how to obtain $\mathbb{ST}$ from $\mathbb{SS} \cup \mathbb{TT}$, and $\mathbb{TS}$ from $\mathbb{SS} \cap \mathbb{TT}$, we will now focus on the logics corresponding to these sets. First, note that all the results from the previous sections were stated for any BNM scheme. However, it is not necessary for the scheme used to define $\mathbb{SS}$ and $\mathbb{TT}$ to be the same. For example, we can consider the two well-known schemes from Figures 2 and 3.

$\mathbb{SS}$ and $\mathbb{TT}$ defined over the Strong Kleene scheme induce the logics K_3 and LP, respectively. Their intersection is known as S_3, KO or RM_{fde}.

In the case of the logics defined over the Weak Kleene scheme, $\mathbb{SS}$ corresponds to WK and $\mathbb{TT}$ to PWK.[11] Given these logics, Theorem 4.4 implies the following:

$$T(\mathbb{K}_3 \cup \mathbb{PWK}) = \mathbb{ST} = \mathbb{CL},$$
$$T^d(\mathbb{K}_3 \cap \mathbb{PWK}) = \emptyset.$$

The choice here of $\mathbb{K}_3$ and $\mathbb{PWK}$ was arbitrary. As mentioned earlier, provided that one logic is based on an **ss** standard and the other on a **tt** standard, the choice of a BNM scheme for each is inconsequential.

A distinctive feature of K_3 and WK, and of all the **ss**-logics is their paracompleteness: the law of excluded middle $\emptyset \Rightarrow \phi \vee \neg\phi$ fails in each (the valuation that assigns the intermediate value to ϕ serves as a counterexample). On the other hand, LP and PWK and all the **tt**-logics are paraconsistent: the rule of Explosion $\phi \wedge \neg\phi \Rightarrow \psi$ has a counterexample in each (the valuation that assigns the intermediate value to ϕ and false to ψ serves as a counterexample). Given our results, classical logic is therefore obtained by operating over the union of a paracomplete and a paraconsistent logic.

5. Additional remarks on the dual transitive closure operator

As proven in Section 3, the transitive closure operator and the dual transitive operator are interdefinable via the set-theoretical notion of absolute complement. However, one may ask how this duality manifests when applied to sets of inferences. To address this, we first need to better understand the effects of the dual transitive closure operator. From Section 3, we know that it is an interior operator, a notion dual to the notion of closure operator, which $-$ unlike its dual $-$ contracts a set rather than extends it. Could there, then, be a property analogous to transitivity that is either imposed or removed by the dual transitive closure operator?

The dual transitive closure of a set of inferences $\mathbb{L}$ is defined as the greatest subset of $\mathbb{L}$ whose complement is closed under transitivity. In light of this definition, it is easy to see that the dual transitive closure operator will not add new inferences to the set. In particular, provided

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[11] To the best of our knowledge, the intersection of the two has not yet been studied.

that L is non-trivial, the dual transitive closure operator will remove some instances of reflexivity, turning $\mathbb{L}$ into a non-reflexive relation.[12]

FACT 5.1. *Let L be structural. If L is non-trivial, then $T^d(\mathbb{L})$ is non-reflexive.*

PROOF. Assume that $T^d(\mathbb{L})$ is reflexive. Then for all $\phi \in \mathcal{L}$, $\phi \Rightarrow \phi \in T^d(\mathbb{L})$. So, $p \Rightarrow p \in T^d(\mathbb{L})$ for some arbitrary p, and thus, for all $\Delta \neq \emptyset$, $p \Rightarrow \delta \in T^d(\mathbb{L}) \subseteq \mathbb{L}$ for some $\delta \in \Delta$ or $\Delta \Rightarrow p \in T^d(\mathbb{L}) \subseteq \mathbb{L}$. This means in particular that $p \Rightarrow q \in \mathbb{L}$ or $q \Rightarrow p \in \mathbb{L}$ for some arbitrary q. In both cases $r \Rightarrow s \in \mathbb{L}$ for two different variables r and s, which, by structurality, shows that L is trivial.   ⊣

We subsequently have two dual operators — the transitive closure operator and the dual transitive operator — turning any non-trivial set of inferences into a transitive one for the first, and into a non-reflexive one for the second.

Although a relation can only be reflexive in one way, it can fail to be reflexive in numerous ways, depending on which instances of reflexivity are absent. The dual transitive closure only preserves the elements of $\mathbb{L}$ that cannot be obtained by closing its complement under transitivity. As we will see, this set corresponds precisely to the set of *theorems* and *antitheorems* of L.

DEFINITION 5.1 (Theorem and antitheorem). Given a logic L, a formula $\phi \in \mathcal{L}$ is said to be a *theorem* of L if $v \models_\mathsf{L} \phi$ for all valuations v, and an *antitheorem* of L if $v \not\models_\mathsf{L} \phi$ for all valuations v.

By extension, a set of formulas Γ is said to be a theorem if for all v and all $\gamma \in \Gamma$, it holds that $v \models_\mathsf{L} \gamma$. It is said to be an antitheorem if for all v there is $\gamma \in \Gamma$ such that $v \not\models_\mathsf{L} \gamma$.

The notions of theorem and antitheorem can be formally defined in several ways. Some of these definitions are provided in the next fact. We will find these equivalent formulations useful later.

FACT 5.2. *Let L be a logic. The following are equivalent:*

1. *Γ is an antitheorem of L;*
2. *$\models_\mathsf{L} \Gamma \Rightarrow \phi$ for all $\phi \in \mathcal{L}$;*
3. *$\models_\mathsf{L} \Gamma \Rightarrow p$ for some $p \notin \mathrm{At}(\Gamma)$.*

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[12] In this context we take a logic L to be *trivial* if $\phi \Rightarrow \psi \in \mathbb{L}$ for every ϕ, ψ.

PROOF. The only non-trivial step is the one from 3 to 1. So, assume that Γ is not an antitheorem. It follows that there is a v such that $v \models_\mathsf{L} \gamma$ for all $\gamma \in \Gamma$. Let $p \notin \mathrm{At}(\Gamma)$ and assume that v' is exactly like v except that if $v \models_\mathsf{L} p$, $v' \not\models_\mathsf{L} p$. Then $v \models_\mathsf{L} \gamma$ for all $\gamma \in \Gamma$ since $p \notin \mathrm{At}(\Gamma)$ and hence $\not\models_\mathsf{L} \Gamma \Rightarrow p$ for all $p \notin \mathrm{At}(\Gamma)$.   ⊣

By the similar proof, we obtain:

FACT 5.3. *Let* L *be a logic. The following are equivalent*:

1. ϕ *is a theorem of* L;
2. $\models_\mathsf{L} \Gamma \Rightarrow \phi$ *for all* $\Gamma \subseteq \mathcal{L}$;
3. $\models_\mathsf{L} p \Rightarrow \phi$ *for some* $p \notin \mathrm{At}(\phi)$.

Given these two facts, we will identify the set of theorems of a logic with the set of all valid inferences of the form $\Gamma \Rightarrow \phi$, where ϕ is a theorem. Similarly, the set of antitheorems of a logic will be identified with the set of inferences of the form $\Gamma \Rightarrow \phi$, where Γ is an antitheorem. For any logic L, we denote the set of its antitheorems and theorems, understood in this broader sense, by $\mathbb{L}^\star$. Specifically, we define

$$\mathbb{L}^\star := \{\Gamma \Rightarrow \phi : \Gamma \Rightarrow \psi \in \mathbb{L} \text{ for all } \psi \text{ or } \Delta \Rightarrow \phi \in \mathbb{L} \text{ for all } \Delta\}.$$

The next proposition thus expresses that for a given logic L, the operator T^d will select only the theorems and antitheorems of L.

PROPOSITION 5.1. *For any logic* L, $T^d(\mathbb{L}) = \mathbb{L}^\star$.

PROOF. ($\subseteq$) Assume $\Gamma \Rightarrow \phi \notin \mathbb{L}^\star$. Then $\not\models_\mathsf{L} \Gamma \Rightarrow p$ and $\not\models_\mathsf{L} p \Rightarrow \phi$ for all $p \notin \mathrm{At}(\Gamma \cup \{\phi\})$ by facts 5.2 and 5.3. Hence, there is $p \notin \mathrm{At}(\Gamma \cup \{\phi\})$ such that $\Gamma \Rightarrow p \in \overline{\mathbb{L}}$ and $p \Rightarrow \phi \in \overline{\mathbb{L}}$ since $\mathrm{At}(\Gamma \cup \{\phi\})$ finite, and thus $\Gamma \Rightarrow \phi \in T(\overline{\mathbb{L}})$. So, $\Gamma \Rightarrow \phi \notin \overline{T(\overline{\mathbb{L}})} = T^d(\mathbb{L})$, by Proposition 3.1.

($\supseteq$) Assume $\Gamma \Rightarrow \phi \in \mathbb{L}^\star$. Then Γ is an antitheorem or ϕ is a theorem. Assume the former. Then $\Gamma \Rightarrow \psi \in \mathbb{L}$ for all $\psi \in \mathcal{L}$. Assume further that $\Gamma \Rightarrow \chi \in T(\overline{\mathbb{L}})$ for some χ for the sake of contradiction. Consider the set $\mathbb{X} = T(\overline{\mathbb{L}}) - \{\Gamma \Rightarrow \psi : \psi \in \mathcal{L}\}$. Clearly $\overline{\mathbb{L}} \subseteq \mathbb{X} \subset T(\overline{\mathbb{L}})$ since $\Gamma \Rightarrow \psi \notin \overline{\mathbb{L}}$ for all ψ. Now, assume that for all $\Sigma \subseteq \mathcal{L}$ and all $\psi \in \mathcal{L}$, there is $\Delta \neq \emptyset$ such that $\Sigma \Rightarrow \delta \in \mathbb{X}$ for all $\delta \in \Delta$ and $\Delta \Rightarrow \psi \in \mathbb{X}$. The case is impossible for $\Sigma = \Gamma$ or $\Delta = \Gamma$, and otherwise $\Sigma \Rightarrow \delta \in T(\overline{\mathbb{L}})$ for all $\delta \in \Delta$ and $\Delta \Rightarrow \psi \in T(\overline{\mathbb{L}})$. In such a case, $\Sigma \Rightarrow \psi \in T(\overline{\mathbb{L}})$, and $\Sigma \Rightarrow \psi \in \mathbb{X}$ by definition of $\mathbb{X}$, so $\mathbb{X}$ is closed under transitivity. But $T(\overline{\mathbb{L}})$ is the least transitive superset of $\overline{\mathbb{L}}$, so this

is impossible. Therefore, $\Gamma \Rightarrow \chi \notin T(\overline{\mathbb{L}})$ for all χ and in particular $\Gamma \Rightarrow \phi \notin T(\overline{\mathbb{L}})$. So, $\Gamma \Rightarrow \phi \in \overline{T(\overline{\mathbb{L}})} = T^d(\mathbb{L})$. The latter case can be proved similarly. $\dashv$

Accordingly, the dual transitive closure will transform any non-trivial set of valid inferences of L into a non-reflexive one by preserving only the theorems and antitheorems of L.

The results from the previous section can now be reinterpreted in light of this new fact. We showed that the dual transitive closure of the intersection of $\mathbb{SS}$ and $\mathbb{TT}$ coincides with the set of TS-valid inferences. Therefore, the set of TS-valid inference corresponds to the set of theorems and antitheorems of $\mathbb{SS} \cap \mathbb{TT}$, which is known to have none.

6. Lattice operations

We now further examine the interrelation between the sets of inferences studied throughout this paper, specifically those closed under the transitive closure operator and its dual.

The relationships among the set of valid inferences of the logics ST, SS, TT, and SS∩TT, defined over the Strong Kleene scheme, are typically illustrated using a diamond-shaped Hasse diagram (Figure 4). In this diagram, $\mathbb{ST}$ (or $\mathbb{CL}$) is positioned at the top, $\mathbb{SS} \cap \mathbb{TT}$ at the bottom, with $\mathbb{SS}$ and $\mathbb{TT}$ in between. While this diagram effectively illustrates the inclusion relation holding between these logics, there is more to consider: the inclusion order actually forms a *lattice*.[13] In the case of the logics defined over the Strong Kleene scheme, this fact is well-known. The lattice of extensions of the Belnap-Dunn four-valued logic FDE — including the logics CL, K_3, LP, and KO — was studied by Rivieccio (2012) and Přenosil (2023), and shown to form a complete lattice. In this structure, CL is the least Tarskian logic extending both K_3 and LP, $K_3 \cap$ LP is the greatest Tarskian logic extended by both K_3 and LP, while K_3 and LP are incomparable. Together, these four logics constitute a sublattice of the lattice of extensions of FDE. As we shall prove, this result holds not only for logics based on the Strong Kleene scheme but generalizes to all logics defined by the **ss**, **tt**, **st**, and **ss** ∩ **tt** standards over any BNM scheme. For any two of these logics, the least Tarskian logic extended by

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[13] A *lattice* is a partially ordered structure $\boldsymbol{A} = \langle A, \leq \rangle$ such that for any pair of elements $x, y \in A$, x, y have both an infimum (join) and a supremum (meet) in A.

$$\mathbb{ST}(=\mathbb{CL})$$
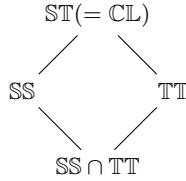
$$\mathbb{SS} \qquad \mathbb{TT}$$

$$\mathbb{SS} \cap \mathbb{TT}$$

Figure 4. Inclusion order over $\mathbb{ST}$, $\mathbb{SS}$, $\mathbb{TT}$ and $\mathbb{SS} \cap \mathbb{TT}$

both correspond to their intersection, while the greatest Tarskian logic extending both is obtained by closing their union under transitivity. Perhaps even more unexpectedly, this generalization still holds when each of the **ss**, **tt**, **st** logics are defined over a different scheme.

We begin with the following observation, which establishes that for any two Tarskian logics, the least Tarskian logic extending them corresponds to their transitive closure.

PROPOSITION 6.1. *For* $\mathbb{L}_1 = Tar(\mathbb{L}_1)$ *and* $\mathbb{L}_2 = Tar(\mathbb{L}_2)$,

$$T(\mathbb{L}_1 \cup \mathbb{L}_2) = Tar(\mathbb{L}_1 \cup \mathbb{L}_2).$$

PROOF. If $\mathbb{L}_1$ and $\mathbb{L}_2$ satisfy Reflexivity, then $\mathbb{L}_1 \cup \mathbb{L}_2$ too. By definition, $T(\mathbb{L}_1 \cup \mathbb{L}_2)$ is a superset of $\mathbb{L}_1 \cup \mathbb{L}_2$, so it also satisfies Reflexivity. Turning to Monotonicity, assume that $\Gamma \Rightarrow \phi \in \mathbb{L}_1 \cup \mathbb{L}_2$. Whether $\Gamma \Rightarrow \phi \in \mathbb{L}_1$ or $\Gamma \Rightarrow \phi \in \mathbb{L}_2$, in both cases $\Gamma, \Sigma \Rightarrow \phi \in \mathbb{L}_1 \cup \mathbb{L}_2 \subseteq T(\mathbb{L}_1 \cup \mathbb{L}_2)$. For Structurality, the case is similar. If $\Gamma \Rightarrow \phi \in \mathbb{L}_1 \cup \mathbb{L}_2$, whether $\Gamma \Rightarrow \phi \in \mathbb{L}_1$ or $\Gamma \Rightarrow \phi \in \mathbb{L}_2$, in both cases $\sigma[\Gamma] \Rightarrow \sigma[\phi] \in \mathbb{L}_1 \cup \mathbb{L}_2 \subseteq T(\mathbb{L}_1 \cup \mathbb{L}_2)$, for every substitution σ. Finally, $T(\mathbb{L}_1 \cup \mathbb{L}_2)$ is the smallest transitive superset of $\mathbb{L}_1 \cup \mathbb{L}_2$ by definition and it satisfies Reflexivity, Monotonicity and Structurality, so it is the smallest superset of $\mathbb{L}_1 \cup \mathbb{L}_2$ satisfying Reflexivity, Monotonicity, Transitivity and Structurality.   ⊣

Next, we establish that all the logics discussed thus far, except for TS, are Tarskian.

LEMMA 6.1. *For all* $\mathbb{L} \in \{\mathbb{ST}, \mathbb{SS}, \mathbb{TT}, \mathbb{SS} \cap \mathbb{TT}\}$, $Tar(\mathbb{L}) = \mathbb{L}$.

PROOF. The proofs for Reflexivity, Monotonicity and Structurality are straightforward. We thus focus solely on Transitivity. We prove that if there is $\Delta \neq \emptyset$, $\Gamma \Rightarrow \delta \in \mathbb{L}$ for all $\delta \in \Delta$ and $\Delta \Rightarrow \phi \in \mathbb{L}$, then $\Gamma \Rightarrow \phi \in \mathbb{L}$. We demonstrate first the case where $\mathbb{L} = \mathbb{SS}$, noting that the case for $\mathbb{TT}$ is similar, while the case for $\mathbb{ST}$ is straightforward. Assume

$\Gamma \Rightarrow \phi \notin \mathbb{SS}$. Then there is v such that $v(\gamma) = 1$ for all $\gamma \in \Gamma$ and $v(\phi) \neq 1$. Let $\Delta \neq \emptyset$. If for all $\delta \in \Delta$, $v(\delta) = 1$, then $v \not\models^{\mathbf{X}}_{\mathbf{ss}} \Delta \Rightarrow \phi$. If for some $\delta \in \Delta$, $v(\delta) \neq 1$, then $v \not\models^{\mathbf{X}}_{\mathbf{ss}} \Gamma \Rightarrow \delta$ for some $\delta \in \Delta$. In both cases there is no $\Delta \neq \emptyset$ such that $\Gamma \Rightarrow \delta \in \mathbb{SS}$ for all $\delta \in \Delta$ and $\Delta \Rightarrow \phi \in \mathbb{SS}$. Turning to $\mathbb{SS} \cap \mathbb{TT}$, if there is Δ such that $\Gamma \Rightarrow \delta \in \mathbb{SS} \cap \mathbb{TT}$ for all $\delta \in \Delta$ and $\Delta \Rightarrow \phi \in \mathbb{SS} \cap \mathbb{TT}$, then $\Gamma \Rightarrow \delta$ is both in $\mathbb{SS}$ and $\mathbb{TT}$ for all $\delta \in \Delta$, and similarly for $\Delta \Rightarrow \phi$. But we just proved that $\mathbb{SS} = Tar(\mathbb{SS})$ and $\mathbb{TT} = Tar(\mathbb{TT})$, so $\Gamma \Rightarrow \phi \in \mathbb{SS} \cap \mathbb{TT}$. ⊣

The join of two elements of the lattice will be defined as the Tarskian closure of their union, while their meet will be defined as their intersection. Given the identity $\mathbb{SS} \cap \mathbb{TT} = Tar(\mathbb{SS} \cap \mathbb{TT})$, which we have just proved, we could equally use the Tarskian closure of the intersection as meet, emphasizing the duality of the operations.

We now show that these operations are well-suited to define the inclusion order, as expected in a lattice. The suitability of the intersection for defining the inclusion order is straightforward, so we will focus on the transitive closure of the union.

LEMMA 6.2. *For all* $\mathbb{X}, \mathbb{Y} \in \{\mathbb{ST}, \mathbb{SS}, \mathbb{TT}, \mathbb{SS} \cap \mathbb{TT}\}$, $\mathbb{X} \subseteq \mathbb{Y}$ *if and only if* $T(\mathbb{X} \cup \mathbb{Y}) = \mathbb{Y}$.

PROOF. (Left-to-right) Assume $\mathbb{X} \subseteq \mathbb{Y}$. By Lemma 6.1, $T(\mathbb{Y}) = \mathbb{Y}$, so $T(\mathbb{X} \cup \mathbb{Y}) = \mathbb{Y}$.

(Right-to-left) Assume $T(\mathbb{X} \cup \mathbb{Y}) = \mathbb{Y}$ and let $\Gamma \Rightarrow \phi \in \mathbb{X}$. Then $\Gamma \Rightarrow \phi \in \mathbb{X} \cup \mathbb{Y} \subseteq T(\mathbb{X} \cup \mathbb{Y}) = \mathbb{Y}$. ⊣

We are now ready to prove that the sets of valid inferences of the aforementioned logics form a lattice. First, we show that, together with intersection and the binary operation of transitive closure of the union, it constitutes an algebra. Then, we show that it is a lattice.

FACT 6.1. *Let* $\mathfrak{L} := \langle \{\mathbb{ST}, \mathbb{SS}, \mathbb{TT}, \mathbb{SS} \cap \mathbb{TT}\}, \sqcap, \sqcup \rangle$, *with* $\sqcup$ *defined for all* $\mathbb{X}, \mathbb{Y} \in \mathfrak{L}$, *by* $\mathbb{X} \sqcup \mathbb{Y} := T(\mathbb{X} \cup \mathbb{Y})$ *and* $\sqcap$ *by* $\mathbb{X} \sqcap \mathbb{Y} := \mathbb{X} \cap \mathbb{Y}$. *Then* $\mathfrak{L}$ *is a lattice algebra.*

PROOF. To show that $\mathfrak{L}$ is an algebra, it is enough to show that it is closed under $\sqcap$ and $\sqcup$. The closure under $\sqcap$ is straightforward to prove since $\sqcap = \cap$. The closure under $\sqcup$ follows from Theorem 4.4, and Lemmas 6.1 and 6.2.

In order to prove that $\mathfrak{L}$ is in addition a lattice algebra, we must show that $\sqcap$ and $\sqcup$ are commutative, idempotent, associative and satisfy the absorbtion laws. The first three properties obviously hold for $\sqcap$, since

$\sqcap = \cap$. As for $\sqcup$, the first two properties are trivial to establish given the commutativity and the idempotence of union. We focus on proving associativity.

Since $\mathbb{X} \cup \mathbb{Y} \subseteq \mathbb{X} \cup \mathbb{Y} \cup \mathbb{Z}$ and T is a closure operator, $T(\mathbb{X} \cup \mathbb{Y}) \subseteq T(\mathbb{X} \cup \mathbb{Y} \cup \mathbb{Z})$. But $\mathbb{Z} \subseteq T(\mathbb{X} \cup \mathbb{Y} \cup \mathbb{Z})$, so $T(\mathbb{X} \cup \mathbb{Y}) \cup \mathbb{Z} \subseteq T(\mathbb{X} \cup \mathbb{Y} \cup \mathbb{Z})$ and hence $T(T(\mathbb{X} \cup \mathbb{Y}) \cup \mathbb{Z})) \subseteq T(T(\mathbb{X} \cup \mathbb{Y} \cup \mathbb{Z})) = T(\mathbb{X} \cup \mathbb{Y} \cup \mathbb{Z})$. On the other hand, $\mathbb{Y} \cup \mathbb{Z} \subseteq T(\mathbb{Y} \cup \mathbb{Z})$, so $\mathbb{X} \cup \mathbb{Y} \cup \mathbb{Z} \subseteq \mathbb{X} \cup T(\mathbb{Y} \cup \mathbb{Z})$ and therefore $T(\mathbb{X} \cup \mathbb{Y} \cup \mathbb{Z}) \subseteq T(\mathbb{X} \cup T(\mathbb{Y} \cup \mathbb{Z}))$. Whence $(\mathbb{X} \sqcup \mathbb{Y}) \sqcup \mathbb{Z} = T(T(\mathbb{X} \cup \mathbb{Y}) \cup \mathbb{Z}) \subseteq T(\mathbb{X} \cup T(\mathbb{Y} \cup \mathbb{Z})) = \mathbb{X} \sqcup (\mathbb{Y} \sqcup \mathbb{Z})$. The right-to-left inclusion is similar.

Turning to the absorption laws, since $\mathbb{X} = T(\mathbb{X})$ by Lemma 6.1, it holds that $\mathbb{X} = T(\mathbb{X} \cup (\mathbb{X} \cap \mathbb{Y}))$, and thus $\mathbb{X} = \mathbb{X} \sqcup (\mathbb{X} \sqcap \mathbb{Y})$. Now, $\mathbb{X} \subseteq T(\mathbb{X} \cup \mathbb{Y})$, so $\mathbb{X} = \mathbb{X} \cap T(\mathbb{X} \cup \mathbb{Y})$, meaning that $\mathbb{X} = \mathbb{X} \sqcap (\mathbb{X} \sqcup \mathbb{Y})$. ⊣

It is now easy to see that $\mathfrak{L} := \langle \{\mathbb{ST}, \mathbb{SS}, \mathbb{TT}, \mathbb{SS} \cap \mathbb{TT}\}, \subseteq \rangle$ is a lattice ordered by inclusion.

Corollary 6.1. $\mathfrak{L} := \langle \{\mathbb{ST}, \mathbb{SS}, \mathbb{TT}, \mathbb{SS} \cap \mathbb{TT}\}, \subseteq \rangle$ *is a lattice with infimum* $inf\{\mathbb{X}, \mathbb{Y}\} = \mathbb{X} \cap \mathbb{Y}$ *and supremum* $sup\{\mathbb{X}, \mathbb{Y}\} = T(\mathbb{X} \cup \mathbb{Y})$.

Proof. The proof is straightforward, given the interdefinability of a lattice order and a lattice algebra. ⊣

Importantly, $\mathfrak{L}$ forms a sublattice of the closure system $\mathfrak{T} = \{\mathbb{L} : \mathbb{L} = Tar(\mathbb{L})\}$ ordered by $\subseteq$. As a closure system, $\langle \mathfrak{T}, \subseteq \rangle$ is a complete lattice of sets of inferences closed under the Tarskian properties.[14] For any two elements of $\mathfrak{T}$, their join is the least set of inferences satisfying the Tarskian properties extending both, and their meet is the greatest such set extended by both. Consequently, the fact that $\mathfrak{L}$ is a sublattice of $\mathfrak{T}$ implies that $\mathbb{ST}$ (or $\mathbb{CL}$) is the least set closed under the Tarskian properties extending both $\mathbb{SS}$ and $\mathbb{TT}$, while $\mathbb{SS} \cap \mathbb{TT}$ is the greatest such set contained in both. Given our definition of a logic as the family of sets of its valid inferences, it follows directly that ST is the least Tarskian logic that extends both SS and TT, while SS∩TT is the greatest Tarskian logic contained in both. More interestingly, this result holds independently of

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[14] A closure system $\mathfrak{C}$ on a set A is a collection of subsets of $\mathcal{P}(A)$ such that $A \in \mathfrak{C}$, and if $\mathfrak{B} \subseteq \mathfrak{C}$ for $\mathfrak{B} \neq \emptyset$, then $\bigcap \mathfrak{B} \in \mathfrak{C}$. Every closure operator C naturally generates a closure system $\mathfrak{C} = \{X \subseteq A : C(X) = X\}$ and a partially ordered set $\langle \mathfrak{C}, \subseteq \rangle$. This partially ordered set can be proved to be a complete lattice with operations $\bigwedge F_i = \bigcap F_i$ and $\bigvee F_i = C(\bigcup F_i)$ for any $\{F_i : i \in I\} \subseteq \mathfrak{C}$ (see Font, 2016, p. 37, Proposition 1.28).
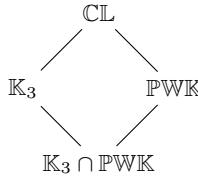
Figure 5. Lattice order over $\mathbb{CL}$, $\mathbb{K}_3$, $\mathbb{PWK}$ and $\mathbb{K}_3 \cap \mathbb{PWK}$

a choice of BNM scheme for SS, TT and ST, as illustrated by Figure 5 for the valid inferences of the logics CL, K_3, PWK, and $\mathsf{K}_3 \cap \mathsf{PWK}$.

Given the duality between the operations of transitive closure and dual transitive closure, what should we expect from the structure that arises from the dual transitive closure of the logics discussed here? It turns out that the resulting structure forms a lattice isomorphic to the one generated by the logics SS, TT, $\mathsf{SS} \cap \mathsf{TT}$, and ST. Specifically, in view of Proposition 5.1, the elements of this lattice are the sets of theorems and antitheorems of these logics. Similar to the lattice of valid inferences, the overall structure holds independently of a specific choice of scheme for SS and TT and TS.

To prove our claim, we first establish how the different sets are ordered by inclusion.

FACT 6.2. *Let* SS, TT, TS *be logics defined respectively by an* **ss**, **tt** *and* **ts** *standard over a (possibly different) BNM scheme. Then*

- $\mathbb{TS}^\star \subseteq \mathbb{SS}^\star, \mathbb{TT}^\star$;
- $\mathbb{SS}^\star$ *and* $\mathbb{TT}^\star$ *are incomparable.*

PROOF. For the first inclusion, note that $\mathbb{TS} = \emptyset \subseteq \mathbb{SS}, \mathbb{TT}$. It then follows that $T^d(\mathbb{TS}) \subseteq T^d(\mathbb{SS}), T^d(\mathbb{TT})$ since T^d is an interior operator. But $T^d(\mathbb{TS}) = \mathbb{TS}^\star$, $T^d(\mathbb{SS}) = \mathbb{SS}^\star$, and $T^d(\mathbb{TT}) = \mathbb{TT}^\star$ by Proposition 5.1. Hence, $\mathbb{TS}^\star \subseteq \mathbb{SS}^\star, \mathbb{TT}^\star$.

As for the incomparability of $\mathbb{SS}^\star$ and $\mathbb{TT}^\star$, note on the one hand that $p \wedge \neg p \Rightarrow q$ is in $\mathbb{SS}$ for every BNM scheme, so $p \wedge \neg p \Rightarrow \psi \in \mathbb{SS}$ for any ψ by Fact 5.2. Thus, $p \wedge \neg p \Rightarrow \psi \in \mathbb{SS}^\star$. On the other hand, note that $p \wedge \neg p \Rightarrow q \notin \mathbb{TT}$ for any BNM scheme (see fn. 8), and hence $p \wedge \neg p \Rightarrow q \notin \mathbb{TT}^\star$. Dually, it can be proved that $p \Rightarrow q \vee \neg q$ is in $\mathbb{TT}^\star$, but not in $\mathbb{SS}^\star$.   ⊣

We know from Proposition 5.1 that each of $\mathbb{SS}^\star$, $\mathbb{TT}^\star$ and $\mathbb{TS}^\star$ are open under dual transitivity. We show here that this is also the case of

$\mathbb{SS}^\star \cup \mathbb{TT}^\star$, which corresponds to the set of antitheorems and theorems of classical logic.[15]

FACT 6.3. $T^d(\mathbb{SS}^\star \cup \mathbb{TT}^\star) = \mathbb{SS}^\star \cup \mathbb{TT}^\star = \mathbb{ST}^\star = \mathbb{CL}^\star$.

PROOF. The left-to-right inclusion of the first identity is straightforward since T^d is an interior operator. For the right-to-left direction, note that $T^d(\mathbb{SS}^\star) = T^d(T^d(\mathbb{SS})) = T^d(\mathbb{SS}) = \mathbb{SS}^\star \subseteq T^d(\mathbb{SS}^\star \cup \mathbb{TT}^\star)$ and $T^d(\mathbb{TT}^\star) = T^d(T^d(\mathbb{TT})) = T^d(\mathbb{TT}) = \mathbb{TT}^\star \subseteq T^d(\mathbb{SS}^\star \cup \mathbb{TT}^\star)$ given Proposition 5.1 and the fact that T^d is an interior operator. So $\mathbb{SS}^\star \cup \mathbb{TT}^\star \subseteq T^d(\mathbb{SS}^\star \cup \mathbb{TT}^\star)$. Turning to the second identity, for the left-to-right inclusion, we know from Theorem 4.3 that $\mathbb{SS}, \mathbb{TT} \subseteq \mathbb{SS} \cup \mathbb{TT} \subset \mathbb{ST}$. Therefore $T^d(\mathbb{SS}), T^d(\mathbb{TT}) \subseteq T^d(\mathbb{ST})$ and hence $\mathbb{SS}^\star \cup \mathbb{TT}^\star \subseteq \mathbb{ST}^\star$. For the right-to-left direction, assume that $\Gamma \Rightarrow \phi \in \mathbb{ST}^\star$. If Γ is an antitheorem, then for all **X**-valuations v, $v(\gamma) \neq 1$ for some $\gamma \in \Gamma$, so $\Gamma \Rightarrow \phi \in \mathbb{SS}^\star$. If ϕ is a theorem, then for all **X**-valuations v, $v(\phi) \in \{1, {}^1\!/_2\}$, so $\Gamma \Rightarrow \phi \in \mathbb{TT}^\star$. Thus, in both cases, $\Gamma \Rightarrow \phi \in \mathbb{SS}^\star \cup \mathbb{TT}^\star$. The last identity is a direct consequence of Theorem 4.1.   ⊣

We then prove a lemma demonstrating the interdefinability of the inclusion order and the dual transitive closure of intersection, following a similar approach to the proof of Lemma 6.2. The interdefinability of the inclusion order and union is straightforward to prove and is omitted.

LEMMA 6.3. For any $\mathbb{X}^\star, \mathbb{Y}^\star \in \{\mathbb{SS}^\star, \mathbb{TT}^\star, \mathbb{TS}^\star, \mathbb{SS}^\star \cup \mathbb{TT}^\star\}$, $\mathbb{X}^\star \subseteq \mathbb{Y}^\star$ if and only if $T^d(\mathbb{X}^\star \cap \mathbb{Y}^\star) = \mathbb{X}^\star$.

PROOF. (Left-to-right) Assume $\mathbb{X}^\star \subseteq \mathbb{Y}^\star$. By Proposition 5.1 and Fact 6.3, $\mathbb{X}^\star = T^d(\mathbb{X}) = T^d(T^d(\mathbb{X})) = T^d(\mathbb{X}^\star)$, so $T^d(\mathbb{X}^\star \cap \mathbb{Y}^\star) = \mathbb{X}^\star$.

(Right-to-left) Assume $T^d(\mathbb{X}^\star \cap \mathbb{Y}^\star) = \mathbb{X}^\star$ and let $\Gamma \Rightarrow \phi \in \mathbb{X}^\star$. Then $\Gamma \Rightarrow \phi \in T^d(\mathbb{X}^\star \cap \mathbb{Y}^\star) \subseteq \mathbb{X}^\star \cap \mathbb{Y}^\star \subseteq \mathbb{Y}^\star$ since T^d is an interior operator.   ⊣

With this lemma at hand, we can demonstrate the following:

FACT 6.4. Let $\mathfrak{L}^\star := \langle \{\mathbb{SS}^\star, \mathbb{TT}^\star, \mathbb{TS}^\star, \mathbb{SS}^\star \cup \mathbb{TT}^\star\}, \sqcap, \sqcup \rangle$, with $\sqcap$ defined for all $\mathbb{X}^\star, \mathbb{Y}^\star \in \mathfrak{L}^\star$ by $\mathbb{X}^\star \sqcap \mathbb{Y}^\star := T^d(\mathbb{X}^\star \cap \mathbb{Y}^\star)$ and $\sqcup$ by $\mathbb{X}^\star \sqcup \mathbb{Y}^\star := \mathbb{X}^\star \cup \mathbb{Y}^\star$. Then $\mathfrak{L}^\star$ is a lattice algebra.

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[15] For $\mathbb{SS} = \mathbb{K}_3$ and $\mathbb{TT} = \mathbb{LP}$, this result was previously established by Blomet and Égré (2024).
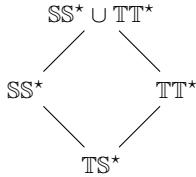
Figure 6. Inclusion order over $\mathbb{SS}^{\star} \cup \mathbb{TT}^{\star}, \mathbb{SS}^{\star}, \mathbb{TT}^{\star}$ and $\mathbb{TS}^{\star}$

PROOF. It follows from Proposition 5.1 that all the elements of $\mathfrak{L}^{\star}$ are closed under T^d. By Theorem 4.5, $T^d(\mathbb{SS} \cap \mathbb{TT}) = \mathbb{TS}$, so $T^d(T^d(\mathbb{SS} \cap \mathbb{TT})) = T^d(\mathbb{TS})$. But $T^d(T^d(\mathbb{SS} \cap \mathbb{TT})) = T^d(\mathbb{SS} \cap \mathbb{TT})$ since T^d is an interior operator and $T^d(\mathbb{TS}) = \mathbb{TS}^{\star}$ by Proposition 5.1. Thus, $T^d(\mathbb{SS}^{\star} \cap \mathbb{TT}^{\star}) = \mathbb{TS}^{\star}$. Any other pair is comparable by Fact 6.2, so the dual transitive closure of their intersection corresponds to the least of the two. Hence, the carrier set of $\mathfrak{L}^{\star}$ is closed under $\sqcap$. Evidently, the carrier set of $\mathfrak{L}^{\star}$ is also closed under $\sqcup$ since $\sqcup = \cup$. $\mathfrak{L}^{\star}$ is therefore an algebra. The remainder of the proof is dual to Fact 6.1.   ⊣

COROLLARY 6.2. $\mathfrak{L}^{\star} := \langle \{\mathbb{TS}^{\star}, \mathbb{SS}^{\star}, \mathbb{TT}^{\star}, \mathbb{SS}^{\star} \cup \mathbb{TT}^{\star}\}, \subseteq \rangle$ *is a lattice with* $\inf\{\mathbb{X}^{\star}, \mathbb{Y}^{\star}\} = T^d(\mathbb{X}^{\star} \cap \mathbb{Y}^{\star})$ *and* $\sup\{\mathbb{X}^{\star}, \mathbb{Y}^{\star}\} = \mathbb{X}^{\star} \cup \mathbb{Y}^{\star}$.

Dually to $\mathfrak{L}$, the structure $\mathfrak{L}^{\star}$ is a sublattice of the interior system $\mathfrak{T}^d = \{\mathbb{L} : \mathbb{L} = T^d(\mathbb{L})\}$ ordered by $\subseteq$. $\langle \mathfrak{T}^d, \subseteq \rangle$ is a complete lattice of sets of inferences open under dual transitivity.[16] By Proposition 5.1, $\mathfrak{T}^d$ can alternatively be viewed as a complete lattice of sets of theorems and antitheorems. For any two elements of $\mathfrak{T}$, their join is the least set of theorems and antitheorems extending both, and their meet is the greatest set of theorems and antitheorems extended by both. In consequence, the fact that $\mathfrak{L}^{\star}$ is a sublattice of $\mathfrak{T}^d$ implies that $\mathbb{CL}^{\star}$ is the least set of theorems and antitheorems extending both $\mathbb{SS}^{\star}$ and $\mathbb{TT}^{\star}$, and $\mathbb{SS} \cap \mathbb{TT}$ the greatest set of theorems and antitheorems extended by both.

Figure 6 represents the lattice of theorems and antitheorems of the $\mathsf{SS}, \mathsf{TT}, \mathsf{ST}$ and TS logics. It is easy to see that it is isomorphic to the lattice represented in Figure 4. By Fact 6.3, the top of the lattice, $\mathbb{SS}^{\star} \cup$

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[16] An interior system $\mathfrak{I}$ on a set A is a collection of subsets of $\mathcal{P}(A)$ such that $\emptyset \in \mathfrak{I}$, and if $\mathfrak{B} \subseteq \mathfrak{C}$ for $\mathfrak{B} \neq \emptyset$, then $\bigcup \mathfrak{B} \in \mathfrak{I}$. Every interior operator I naturally generates an interior system $\mathfrak{I} = \{X \subseteq A : I(X) = X\}$ and a partially ordered set $\langle \mathfrak{I}, \subseteq \rangle$. This partially ordered set can be proved to be a complete lattice with operations $\bigvee F_i = \bigcup F_i$ and $\bigwedge F_i = I(\bigcap F_i)$ for any $\{F_i : i \in I\} \subseteq \mathfrak{I}$ (see Font, 2016, p. 42).

$\mathbb{TT}^\star$, corresponds to the set of antitheorems and theorems of classical logic. Conversely, the bottom of the lattice, $\mathbb{TS}^\star$, aligns with the set of antitheorems and theorems of $\mathsf{SS} \cap \mathsf{TT}$. The duality of the transitive closure operator and the dual transitive closure operator reverberates on the meet and join of the respective lattices. In the lattice of valid inferences, as we have seen, $\mathbb{SS} \cup \mathbb{TT}$ does not coincide with $\mathbb{ST}$, the set of classically valid inferences. Dually, in the lattice of antitheorems and theorems, $\mathbb{SS}^\star \cap \mathbb{TT}^\star$ does not coincide with $\mathbb{TS}$, the set of antitheorems and theorems of $\mathsf{SS} \cap \mathsf{TT}$, for $(p \wedge \neg p) \Rightarrow (q \vee \neg q) \in \mathbb{SS}^\star \cap \mathbb{TT}^\star$. In both cases, specific operations must be applied to obtain the desired set: for the union, the transitive closure operator is used; for the intersection, the dual transitive closure operator is applied.

7. Conclusion

In this paper, we presented several results concerning the relation between **tt**- and **ss**- logics and their corresponding **st**- and **ts**-logics over three-valued BNM schemes.

On the one hand, we showed that $\mathbb{SS} \cup \mathbb{TT} \subsetneq \mathbb{ST} = \mathbb{CL}$, for any BNM scheme. We also introduced the transitive closure operator T and we proved that $T(\mathbb{SS} \cup \mathbb{TT}) = \mathbb{ST}$ (see Theorems 4.3 and 4.4). On the other hand, we proved that $\mathbb{SS} \cap \mathbb{TT} \supsetneq \mathbb{TS}$, for any BNM scheme. We introduced the dual transitive closure operator T^d, and proved that $T^d(\mathbb{SS} \cap \mathbb{TT}) = \mathbb{TS}$ (see Theorem 4.5).

We generalized and extended results from (Da Ré et al., 2024), highlighting notable relationships between **ss**- and **tt**-logics, concerning properties such as paraconsistency and paracompleteness.

We concluded with observations on the abstract relationships between the transitive closure and its dual. It was noted that, just as the operator T closes a set of inferences under transitivity, its dual T^d eliminates reflexivity in all non-trivial cases, preserving only theorems and antitheorems. Building on this, we demonstrated how the sets $\mathbb{SS}$, $\mathbb{TT}$, $\mathbb{CL}$, and $\mathbb{SS} \cap \mathbb{TT}$ form a lattice, with $\mathbb{CL}$ at the top, extending both $\mathbb{SS}$ and $\mathbb{TT}$, and $\mathbb{SS} \cap \mathbb{TT}$ at the bottom, extended by both. We concluded the previous section by observing that the respective sets of theorems and antitheorems of each of these logics notably followed the same lattice structure.

These operations can also be used to emphasize and illustrate some relations between very well-known logics, such as the ones belonging to the Strong Kleene scheme. For instance, the first lattice has $\mathbb{K}_3 \cap \mathbb{LP}$ (i.e. the logic corresponding to S_3, KO or RM_{fde}) at the bottom and $\mathbb{CL}$ at the top. This is so because RM_{fde} is the greatest Tarskian logic extended both by LP and by K_3. In a symmetric fashion, the isomorphic lattice obtained through the transitive dual operation shows that LP and K_3 share neither theorems nor antitheorems, while together they contain all the classical ones.

However, several open questions remain. All the results in this paper are specific to BNM three-valued schemes. As shown by Da Ré et al. (2024), other types of schemes — such as the Boolean Normal Truth Collapsible and Falsity Collapsible ones — are also capable of yielding classical logic under an **st** standard. Yet, logics defined over these schemes exhibit very different properties from those studied here. For example, logics over Boolean Normal Collapsible schemes with a **tt** or **ss** standard are coextensive with classical logic, and hence they are neither paraconsistent nor paracomplete. Furthermore, **ts**-logics over these schemes are not necessarily empty. Additionally, the generalization of the results in this article to multiple conclusions and metainferential logics (see, e.g., Pailos and Da Ré, 2023, for details) remains to be addressed. We are working on these questions and will address them in a subsequent article. Lastly, as an anonymous reviewer suggested, the results of the present article could be generalized by using nondeterministic semantics (as defined by Avron and Lev, 2005). In a recent article, Kadlečiková and Ferguson (2024) have introduced some non-deterministic expansions of three-valued matrices preserving counterexamples. To our knowledge, the techniques applied to build the so-called *em-nmatrices* can be applied to any BNM scheme by preserving counterexample, and thus extending the present results. However, a proof and a further exploration of this must be left to future research.

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