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Logic and Logical Philosophy
Online First Articles (2025)
DOI: 10.12775/LLP.2025.028

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# Tautology Elimination, Cut Elimination and S4?

**Abstract.** The paper "Tautology elimination, cut elimination, and S5" published in this journal presents a novel method for establishing by proof analysis the admissibility of the rule of tautology elimination for certain sequent calculi. Since tautology elimination will typically imply the admissibility of cut, the method promises a new path to show the admissibility of cut for cut-free calculi on which the standard techniques within structural proof theory seem inapplicable. This paper shows that the method as presented involves an error.

**Keywords**: cut elimination; tautology elimination; proof theory for modal logic

## 1. Introduction

Following (Indrzejczak, 2017), we will refer to the following sequent calculus rule as *Tautology Elimination*:

$$\frac{A \supset A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\text{TE}$$

If the conditional $\supset$ satisfies the additive left introduction rule, then the admissibility of TE implies the admissibility of *shared-context cut*:

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\text{SC}$$

If in addition the structural rule of weakening is admissible (or $\supset$ satisfies the multiplicative left introduction rule), TE also implies the admissibility of *independent-context cut*:

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{IC}$$

*Received* April 26, 2025. *Accepted* November 17, 2025. *Published online* December 5, 2025
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Indrzejczak (2017) presents a method for establishing the admissibility of TE by proof analysis. In addition to a sequent calculus for classical first-order logic, the method is also applied on a sequent calculus for S5 for which SC and IC have previously only been shown to be admissible by semantic means and certain features of the calculus suggests it cannot be done through proof analysis with standard techniques from structural proof theory.

When speaking about establishing the admissibility of TE, it is quite tempting to say, in analogy to the method of cut-elimination for establishing the admissibility of cut by proof analysis, that we establish by tautology elimination the admissibility of tautology elimination. The two occurrences of "elimination" do not refer to the same kind of elimination. In the case of the rule, we eliminate a formula from a sequent. In the case of the proof, one aims to show that every application of TE can be eliminated from a derivation. To avoid any terminological confusion, I will refer to the method or procedure as "the method of TE-admissibility".

In any case, some cut-free sequent calculi really are such that while one can show that IC and SC are admissible by model-theoretic means through soundness and completeness theorems, the corresponding proofs by proof analysis have so far escaped us. One example is the sequent calculus for S5 investigated by Indrzejczak (2017). The method of TE-admissibility promises to solve this issue in that and other similar cases.

The aim of this paper is to show that the method of TE-admissibility does not deliver on this promise, and in particular that there is an error in the structure of the inductive argument. To that purpose, I present first a sequent calculus for which neither SC nor IC is admissible, but for which we nonetheless can show that TE is admissible by following the recipe of (Indrzejczak, 2017). It follows that not only SC but also IC should be admissible because weakening is admissible. I then explain what the error consists in.

This error occurs in both applications of the method presented by Indrzejczak (2017), that is, in the proofs of Theorem 2.6 for the sequent calculus for classical logic and Theorem 3.4 for the sequent calculus for S5. Neither proof thus works as intended.

$$p, \Gamma \Rightarrow \Delta, p$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\neg\mathrm{L} \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}\neg\mathrm{R}$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\wedge\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\wedge\mathrm{R}$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\Box\mathrm{L} \qquad \frac{\Box\Gamma \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \Box A, \Delta}\Box\mathrm{R}$$

Table 1. The sequent calculus *g3NES4*

## 2. Not exactly S4

In this section, I present a sequent calculus *g3NES4* ("Not Exactly S4") for a modal logic which includes every axiom of S4 but not every theorem of S4. It follows that the following rule of modus ponens is not admissible:

If $\Rightarrow A \supset B$ and $\Rightarrow A$ are derivable, then $\Rightarrow B$ is derivable.

Since *g3NES4* is such that weakening is admissible, and the derivability of $\Rightarrow A \supset B$ implies the derivability of $A \Rightarrow B$, it follows that neither SC nor IC is admissible for *g3NES4*.

The sequent calculus *g3NES4* is defined for a language based on $\Box$, $\neg$ and $\wedge$. $A \supset B$ is defined as $\neg(A \wedge \neg B)$. Thus, even if the topic of this paper is TE which involves a formula of the form $A \supset A$, this calculus has only primitive rules for $\wedge$ and $\neg$. I have made this choice to easier facilitate a discussion of the formula $\Box(p \wedge \neg\Box p)$ and contraction. The expression $\neg(A \wedge \neg B)$ will still act like $A \supset B$ for the purposes of the method of TE-admissibility presented by Indrzejczak (2017).

A sequent is a pair of multisets of formulas, $p$ is an arbitrary atom, $A$ and $B$ arbitrary formulas, $\Gamma$ and $\Delta$ possibly empty multisets of formulas, and $\Box\Gamma$ a possibly empty multiset of formulas of the form $\Box B$. The initial sequents and rules of *g3NES4* are presented in table 1.

The principal formula of a rule is the formula which is displayed in the conclusion-sequent in the presentation of the rule. The active formulas are those displayed in the premise-sequents. A formula is parametric in the application of a rule if it is neither active nor principal.

The height of a derivation is defined as the number of nodes in its longest branch. I will on occasion use $n \vdash \Gamma \Rightarrow \Delta$ as short for "there is a derivation with height $\leq n$ of $\Gamma \Rightarrow \Delta$". The complexity of a formula $A$ is defined inductively on its construction. Atoms $p$ are assigned 1 and complex formulas are assigned the sum of their direct subformulas +1.

A rule is height-preservingly admissible just in case, whenever there is a derivation of the premise-sequent(s) with height $\leq n$, there is also a derivation of the conclusion-sequent with height $\leq n$. For example, the following *rule of weakening* is height-preservingly admissible in *g3NES4*:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}$$

This is established by induction on the height of a derivation.

Height-preserving inversion holds for a rule just in case, whenever there is a derivation of the conclusion-sequent with height $\leq n$, there is also a derivation of each of the premise-sequents with height $\leq n$.

As required by the method of TE-admissibility by Indrzejczak (2017), height-preserving inversion holds for $\wedge$L, $\wedge$R, $\neg$L and $\neg$R. In addition, the following rule is also (height-preservingly) admissible:

$$\frac{\Gamma \Rightarrow \Delta, \Box A}{\Gamma \Rightarrow \Delta, A}$$

This follows by induction on the height of a derivation. It holds trivially if the premise is an initial sequent or $\Box A$ is parametric. If $\Box A$ is the principal formula with last applied rule being $\Box$R and $\Gamma \Rightarrow \Delta, \Box A$ is thus obtained from $\Gamma' \Rightarrow A$ where $\Gamma' \subseteq \Gamma$, then the desired result is obtained with weakening.

Yet, (height-preserving) inversion fails for $\Box$L. A simple counterexample is the sequent $\Box p \Rightarrow \Box p$ since it is not the case that $p \Rightarrow \Box p$ is derivable.

The sequent calculus *g3NES4* feels like one for S4, but isn't. The S4 axioms are derivable, but some theorems of S4 are not. Consider the following counterexample. The formula $\neg\Box(p \wedge \neg\Box p)$ is a theorem of S4, but the sequent $\Box(p \wedge \neg\Box p) \Rightarrow$ is not derivable in *g3NES4*. Let's illustrate with a little proof search:

$$\cfrac{\cfrac{\cfrac{p \Rightarrow \Box p}{p, \neg\Box p \Rightarrow}}{p \wedge \neg\Box p \Rightarrow}}{\Box(p \wedge \neg\Box p) \Rightarrow}$$

No rule is applicable on the leaf, so the sequent is not derivable.

There are multiple ways to expand *g3NES4* into a sequent calculus for S4. One way consists in replacing $\Box$L with $\Box\mathrm{L}_T$ (cf. Wansing, 2002):

$$\frac{A, \Box A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\ \Box\mathrm{L}_T$$

With this rule, we obtain a sequent calculus for which contraction is admissible, i.e. the following rules:

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

Indeed, if we replace $\Box$L with $\Box\mathrm{L}_T$ or include the contraction rules, then SC and IC become admissible.

Instead, contraction is only admissible for atoms in *g3NES4*. This can be demonstrated by induction on the height of a derivation. For initial sequents it holds that if $p, p, \Gamma \Rightarrow \Delta$ is an initial sequent, then $p, \Gamma \Rightarrow \Delta$ is an initial sequent. For the inductive step, $p$ is parametric, and we can thus eliminate the extra occurrence with the induction hypothesis. Fortunately, the method of TE-admissibility presented by Indrzejczak (2017) requires only the admissibility of contraction for atoms as opposed to contraction for arbitrary formulas.

For complex formulas, we have a counterexample. Consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{p, \neg\Box p \Rightarrow p}{p \wedge \neg\Box p \Rightarrow p}\ \wedge\mathrm{L}}{\Box(p \wedge \neg\Box p), p \Rightarrow \Box p}\ \Box\mathrm{L/R}}{\Box(p \wedge \neg\Box p), p \wedge \neg\Box p \Rightarrow}\ \neg\mathrm{L}/\wedge\mathrm{L}}{\Box(p \wedge \neg\Box p), \Box(p \wedge \neg\Box p) \Rightarrow}\ \Box\mathrm{L}$$

With $\Box\mathrm{L}_T$ we can "contract" as the last step. With $\Box$L, this is not an option. That $\Box(p \wedge \neg\Box p), \Box(p \wedge \neg\Box p) \Rightarrow$ but not $\Box(p \wedge \neg\Box p) \Rightarrow$ is derivable is thus a counterexample to the admissibility of contraction for complex formulas in *g3NES4*.

We can use this example to generate counterexamples against the admissibility of both SC and IC for *g3NES4*. If SC is admissible, then we can reason as follows:

$$\frac{\Box(p \wedge \neg\Box p) \Rightarrow \Box(p \wedge \neg\Box p) \qquad \Box(p \wedge \neg\Box p), \Box(p \wedge \neg\Box p) \Rightarrow}{\Box(p \wedge \neg\Box p) \Rightarrow}$$

This shows that if SC is admissible, then $\Box(p \wedge \neg\Box p) \Rightarrow$ is derivable. However, while we have derivations of the premise sequents in *g3NES4*, there is no derivation of the conclusion-sequent in *g3NES4*. It follows that SC is not admissible.

A little twist on that example suffices to also produce a counterexample to IC. Consider the following application of IC:

$$\frac{\Box(p \wedge \neg\Box p) \Rightarrow \Box(p \wedge \neg\Box p) \wedge \Box(p \wedge \neg\Box p) \qquad \Box(p \wedge \neg\Box p) \wedge \Box(p \wedge \neg\Box p) \Rightarrow}{\Box(p \wedge \neg\Box p) \Rightarrow}$$

We have derivations of each premise-sequent, but not the conclusion-sequent. IC is thus not admissible in *g3NES4*.

To conclude then, *g3NES4* does not define S4. We know this because $\neg\Box(p \wedge \neg\Box p)$ is a theorem of S4 but the sequent $\Box(p \wedge \neg\Box p) \Rightarrow$ is not derivable in *g3NES4*. To obtain S4, we can either replace $\Box$L with $\Box\mathrm{L}_T$, include the contraction rules, or include SC or IC.

## 3. Tautology elimination

I will now show that we can still use the method of TE-admissibility to show that TE, and thus that SC and IC are admissible in *g3NES4*. Since the latter is not the case, the reasoning in this section shows that there is an error in the method. I follow the presentation by Indrzejczak (2017) as closely as reasonable, but the reader is recommended to also confer with that presentation.

To repeat, the goal is to show that the following rule is admissible:

$$\frac{A \supset A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{TE}$$

Following (Indrzejczak, 2017), we show this by induction on the complexity of $A$. For the base case, $A$ is an atom $p$. With $A \supset B$ defined as $\neg(A \wedge \neg B)$, it follows by inversion of $\neg$L, $\neg$R and $\wedge$R that $\Gamma \Rightarrow \Delta, p$ and $p, \Gamma \Rightarrow \Delta$ are derivable. Inspired by SC, I will refer to them as the left and the right premise respectively.

We now show by a subinduction on the height of the derivation of *only* the right premise $p, \Gamma \Rightarrow \Delta$, that *if* $p, \Gamma \Rightarrow \Delta$ *then* $\Gamma \Rightarrow \Delta$. Thus, we do not, as in a standard cut-elimination proof, perform an induction on the sum of the heights of the derivations of $\Gamma \Rightarrow \Delta, p$ and $p, \Gamma \Rightarrow \Delta$ with the aim to show that $\Gamma \Rightarrow \Delta$. Instead, the method of TE-admissibility as presented by Indrzejczak (2017) involves at this stage establishing that *if* $p, \mathit{\Gamma} \Rightarrow \mathit{\Delta}$ *then* $\mathit{\Gamma} \Rightarrow \mathit{\Delta}$ by an induction on the height of the derivation of $p, \mathit{\Gamma} \Rightarrow \mathit{\Delta}$.

In the base case, the sequent $p, \mathit{\Gamma} \Rightarrow \mathit{\Delta}$ is an initial sequent. If $p \notin \mathit{\Delta}$, then we can just remove $p$ and it remains an initial sequent. If $p \in \mathit{\Delta}$, on the other hand, then the desired conclusion follows by contraction for atoms on *the left premise* $\mathit{\Gamma} \Rightarrow \mathit{\Delta}, p$. For the inductive step, $p$ is always parametric, and we can thus apply the subsidiary induction hypothesis on the premise-sequents from which $p, \mathit{\Gamma} \Rightarrow \mathit{\Delta}$ is obtained.

For the inductive step of the main induction, we distinguish cases based on the main connective. For $\neg$ and $\wedge$, we apply inversion on $\mathit{\Gamma} \Rightarrow \mathit{\Delta}, A$ and $A, \mathit{\Gamma} \Rightarrow \mathit{\Delta}$ before applying the main induction hypothesis. For $\Box$, we proceed as in the case of atoms, that is, we show by subinduction on the height of the derivation of the right premise $\Box B, \mathit{\Gamma} \Rightarrow \mathit{\Delta}$ that *if* $\Box B, \mathit{\Gamma} \Rightarrow \mathit{\Delta}$ *then* $\mathit{\Gamma} \Rightarrow \mathit{\Delta}$. If it is an initial sequent or $\Box B$ is parametric, then we simply remove it. If $\Box B$ principal with last applied rule being $\Box$L, we consider also the left premise $\mathit{\Gamma} \Rightarrow \mathit{\Delta}, \Box B$. The left premise implies $\mathit{\Gamma} \Rightarrow \mathit{\Delta}, B$ so we can apply the main induction hypothesis on that and the premise of the application of $\Box$L, namely $B, \mathit{\Gamma} \Rightarrow \mathit{\Delta}$, to obtain $\mathit{\Gamma} \Rightarrow \mathit{\Delta}$. This concludes the proof.

It follows that TE and thus SC and IC are admissible for *g3NES4*. However, we have counterexamples against the admissibility of both IC and SC. It follows that something must be wrong in the reasoning presented in this section.

## 4. The explanation

As a first step towards identifying the error, it is worth highlighting that the reasoning in the previous section really is just an argument for the admissibility of SC in disguise. After all, we can skip the introduction about TE, and just start out with the assumptions that $\mathit{\Gamma} \Rightarrow \mathit{\Delta}, A$ and $A, \mathit{\Gamma} \Rightarrow \mathit{\Delta}$ are derivable.

For readability, it is useful to introduce the following abbreviations:

- *IC(k)*: The rule IC is admissible for formulas of complexity $\leq k$.
- *SC(k)*: The rule SC is admissible for formulas of complexity $\leq k$.
- *AE(k,n)*: the rule of antecedent elimination *if* $A, \mathit{\Gamma} \Rightarrow \mathit{\Delta}$ *then* $\mathit{\Gamma} \Rightarrow \mathit{\Delta}$ is admissible for formulas of complexity $\leq k$ and for derivations of height $\leq n$.

It follows that *AE(k,n)* is the claim that *if* $n \vdash A, \Gamma \Rightarrow \Delta$ then $\exists m(m \vdash \Gamma \Rightarrow \Delta)$ where the complexity of $A$ is $\leq k$.

The proof proceeds now by induction on the complexity of $A$. In particular, we aim to show that $\forall x SC(x)$. For the base case, we are required to show that *SC(1)* since atoms have complexity 1. Clearly, $\forall x AE(k, x)$ implies *SC(k)*. It thus suffices to show $\forall x AE(1, x)$ for the base case of the proof that $\forall x SC(x)$. And we proceed to do that by a subsidiary induction on the height of the derivation of $p, \Gamma \Rightarrow \Delta$. In the base case $AE(1, 0)$ the premise $p, \Gamma \Rightarrow \Delta$ is an initial sequent. For this to be an initial sequent, it is either the case that $p \in \Delta$ or there is an atom $q$ such that $q \in \Gamma \cap \Delta$. In the latter case, we can conclude $0 \vdash \Gamma \Rightarrow \Delta$ and thus $\exists m(m \vdash \Gamma \Rightarrow \Delta)$. In the former case, we cannot. And we are in fact completely stuck. *The rule simply is not admissible.* Otherwise $\Rightarrow p$ would've been derivable, which it clearly isn't.

However, in the proof presented in the previous section and following (Indrzejczak, 2017), we introduced at this point the left premise $\Gamma \Rightarrow \Delta, p$. The desired result is now obtained by applying contraction for atoms on that sequent since $p \in \Delta$. All good. Except that we have for obvious reasons not established $AE(1, 0)$ which would licence the induction hypothesis $AE(1, n)$, the one applied by Indrzejczak (2017). Instead, we have established the following claim:

If $\exists m(m \vdash \Gamma \Rightarrow \Delta, p)$ and $0 \vdash p, \Gamma \Rightarrow \Delta$ then $\exists m(m \vdash \Gamma \Rightarrow \Delta)$

Thus, we have established the base case for a proof of *SC(1)* by induction on the height of the right premise $p, \Gamma \Rightarrow \Delta$. It follows that (Indrzejczak, 2017) applies the wrong subsidiary induction hypothesis in the proofs of Theorems 2.6 and 3.4. Whereas Indrzejczak (2017) employs $AE(1, n)$ as induction hypothesis, the correct induction hypothesis for the subsidiary induction is the following:

If $\exists m(m \vdash \Gamma \Rightarrow \Delta, p)$ and $n \vdash p, \Gamma \Rightarrow \Delta$ then $\exists m(m \vdash \Gamma \Rightarrow \Delta)$

As it turns out, this induction hypothesis is not very useful, neither for us in the case of *g3NES4*, nor for Indrzejczak (2017) in the case of the particular sequent calculus for S5 under consideration.

In our case, we cannot use the correct induction hypothesis to establish the desired result for $n+1$. Consider the case in which $\Gamma$ in $p, \Gamma \Rightarrow \Delta$ is $\Box B, \Gamma'$ and last applied rule is $\Box$L with premise $p, B, \Gamma' \Rightarrow \Delta$, so we have $n \vdash p, B, \Gamma' \Rightarrow \Delta$. In order to apply the induction hypothesis, we must first convert, or strictly speaking, invert, $\Box B, \Gamma' \Rightarrow \Delta, p$ into

$B, \Gamma' \Rightarrow \Delta, p$ before we can apply the induction hypothesis because it requires shared contexts. Moreover, it wouldn't do us any good to attempt to prove the base case for IC rather than SC since the inductive step for complex formulas will fail in both cases with regard to *g3NES4*.

In the case of the sequent calculus for S5 under consideration by Indrzejczak (2017), the reader may easily verify that the proof of Theorem 3.4 requires for complex formulas of the form $\Box A$ the induction hypothesis $\forall x AE(k, x)$ rather than $SC(k)$ or $IC(k)$ due to certain features of the calculus under consideration. Indeed, it is precisely because neither $SC(k)$ nor $IC(k)$ is applicable in the case of complex formulas of the form $\Box A$ that Indrzejczak (2017) explored other venues, including the method of TE-admissibility.

To conclude, the error consists in that the base case one establishes for the subsidiary induction on the height of the right premise, and what is thus available as subsidiary induction hypothesis, is not the one for AE. Instead, it is the one for SC. This is because we had to use not only the assumption that $p, \Gamma \Rightarrow \Delta$ is derivable with height 0, but also that $\Gamma \Rightarrow \Delta, p$ is derivable.

**Acknowledgments.** Research for this paper was supported by a PNRR grant, under the European Union's NextGenerationEU research and innovation programme. The author would also like to thank the referee for their valuable suggestions and helpful comments to improve this paper.

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