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Intuitionistic Logic from a Metainferential Perspective

Abstract. This paper introduces a metainferential version of intuitionistic logic. I work on the framework proposed by some logicians of Buenos Aires, who defend that a logic should be defined in terms of inferences and metainferences of growing complexity. Three logical systems are presented and proved to be adequate from an intuitionistic point of view.

Keywords: Buenos Aires Plan; metainferential logics; intuitionism; proof-theory; substructurality

1. Introduction

In a series of articles, Barrio, Pailos and Szmuc defended the idea that a logic should not be identified with a set of logical laws, nor with a set of inferences, but with a whole collection of *metainferences* of all levels (see Barrio, Pailos and Szmuc, 2018, 2020; Barrio, Pailos and Toranzo Calderón, 2021). Nowadays there is a blooming community discussing this understanding of logical identity. It also plays a key role in a broader project in philosophy of logic dubbed 'The Buenos Aires Plan' (BA-Plan) in (Barrio, Pailos and Toranzo Calderón, 2021).

This *metainferentialist view of logical consequence* (from now on, MVLC) stemmed from the debate on substructural solutions to semantic paradoxes. The metainferential logic that Barrio and his colleagues endorse is defined over ST, a non transitive logic that was proven adequate in handling the paradoxes of Sorites (2012) and the Liar (2014) by Cobreros, Egré, Ripley and van Rooji. The logic ST and its close relatives are all based on the Strong Kleene algebra, and maybe for this reason this philosophical approach has been largely remained model-theoretic:

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although having proof-theoretical presentations, it is fair to say that the whole understanding of "the ST phenomenon" (Barrio, Pailos and Szmuc, 2020) is, for the most part, strongly alethic.

This is why the logics one usually encounters in this literature are classical logic (CL), the 3-valued Kleene logic K3 and the logic of the paradox (LP). At the same time, one logic one never encounters in this literature is intuitionistic logic (IL): and the reasons may be that, on the one hand, it is not philosophically motivated by reflections on truth, and on the other because it is a logic that *is not* an inhabitant of the Strong Kleene realm.

Yet the MVLC is broader, and thus somewhat independent, from the particular debate it stemmed from. Therefore, the purpose of this article is to take MVLC and consider IL from its perspective. This will take us to the presentation of various *metainferential intuitionistic logics*.

The paper is structured as follows. Section 2 will reconstruct the MVLC and Section 3 will briefly expound what would mean an intuitionistic understanding of it. Then in Sections 4 and 5 three metainferential logics will be presented: IC_ω, MIL and MIL^2. A final section summarizes the conclusions.

2. Logics as metainferences

Up to the first third of the XX century, logicians used to see a *logic* (or a *logical system*) as a set of formulas, the axioms and theorems of a certain calculus. To compare two logics was to compare their axioms; to choose between them was to choose which one had nicer, more intuitive or more useful theorems, etc.

Then some limitative results and new developments tackled a change of paradigm. The new perspective was not interested only in theorems and axioms but also in the properties of the *consequence relation* (between premises and conclusion(s) of an argument, or axioms and theorems of a theory). Decades forward, today we can fairly say that this is the mainstream attitude towards the identity of a logic: where $\mathfrak{L}$ is a language, a *logic* is the relation between the premises and the conclusion(s) of the *arguments* you can express with it.

In recent years, though, this picture has been challenged again. In 2012 and 2014, Cobreros, Egré, Ripley and van Rooji presented a very nice solution to (some) semantic paradoxes. The trick is ST, a logic defined over the Strong Kleene algebra that does not trivialize in the presence of a Liar sentence and allows for a straightforward result of classical recapture. *ST* induces a *failure of transitivity*, to the effect that you may have $A \vdash B$ and $B \vdash C$ but still $A \nvdash C$.

Then Barrio, Pailos and Szmuc addressed to them an unexpected critique: they seemed to be at ease with dropping transitivity, but they claimed that *ST* was not as conservative of CL as one would expect. The reason is that it preserves all classical *inferences*, but looses a lot of interesting classical *metainferences*.

What is a metainference? *An inference whose relata are also inferences.* Take the above example of transitivity. CL is usually presented in such a way that whenever $A \vdash B$ and $B \vdash C$ holds, it also holds $A \vdash C$. This claim has an obvious conditional structure:

If α holds, then β holds.

And therefore, it may be reconstructed in inferential terms $\alpha \vdash \beta$. Thus, leading us to what is commonly known as Cut:

$$(A \vdash B), (B \vdash C) \vdash (A \vdash C)$$

If CL is nothing but this, say, 'first-degree', consequence relation, then Cobreros and his colleagues are right and *ST* and CL are 'the same' logic. But Cut may be considered as a defining part of CL as well. Being Cut a, say, 'second-degree' inference, all other second-degree inferences should be taken into consideration when accounting for classical logic.

> Yet again, if this is plausible, why stop there? We can definitely consider consequence relations between [meta-inferences], and so on and so forth. It can be easily seen how this procedure can be further reproduced, giving us a whole hierarchy of inferences concerned with the logical relations between objects of the lower level(s). (Barrio, Pailos and Szmuc, 2018, p. 106) (with a slight change of terminology)

Barrio, Pailos, Szmuc and other collaborators will elaborate on this perspective in a series of articles, giving rise to the MVLC.

Let $\mathfrak{L}$ be a formal language. The set of all possible *arguments* one can express in this language will usually be $\wp(\mathfrak{L}) \times \mathfrak{L}$. These arguments will be called by the more technical name of *inferences*, or, even more precise, *meta*^{0}*-inferences*, that is, *metainferences of level 0.*[1]

[1] I do not attribute any special difference to the use of 'argument' and 'inference', but this is obviously inelegant. To the very least, 'argument' should be finite in length, and maybe 'inference' has not this constraint. For a detail on the discussion I am avoiding, see (Pailos and Da Ré, 2023, the beginning of Chapter 2).

DEFINITION 2.1. The *inferences* (*metainferences of level 0*) of the language $\mathfrak{L}$ are the elements of the set $\vdash^0 = \wp(\mathfrak{L}) \times \mathfrak{L}$.

Metainferences of all subsequent levels pile up on top of this.

DEFINITION 2.2. The *meta*$^{n+1}$*-inferences* (*metainferences of level* $n+1$) of the language $\mathfrak{L}$ are the elements of the set $\vdash^{n+1} = \wp(\vdash^n) \times \vdash^n$.

As this definitions shows, a *meta...meta-inference* is a metainference of level n, where n is the number of times you say 'meta'. It follows that $n = 0$ is the level of *inferences* (zero times 'meta') and, for completion, we will say that $n = -1$ is the level of *formulas*.

Note that we are using the symbol $\vdash$ to describe all the metainferences of a certain level. In other words, we have arguments, but not yet *valid arguments* (valid metainferences). For that purpose it is necessary to introduce the concept of *standard* (of validity) (see, e.g., Pailos, 2019; Pailos and Da Ré, 2023). The definition proposed is broad and seemingly uninformative, but the examples to follow will show its interest:

DEFINITION 2.3. A *(validity) standard* ε^n is a property of objects of metainferential level n, where $-1 \leqslant n < \omega$.

With a language and a standard of a certain level you can define a metainferential logic of that level. The logic *is* the set of metainferences of that level that conform to the standard.

DEFINITION 2.4. Let ε be a standard of level $n+1$. The *meta*$^{n+1}$*-inferential logic* that conforms to the standard ε is the set

$$\vdash_{\varepsilon}^{n+1} = \{x \mid \varepsilon(x)\} \subseteq \vdash^{n+1}$$

Piling things up we obtain a *hierarchy of metainferential logics*:

DEFINITION 2.5. Let $\varepsilon = \langle \varepsilon^n \rangle_n$, where $-1 \leqslant n < \omega$, be a series of standards. A *hierarchy of metainferential logics* is a series of metainferential logics $\langle \vdash_{\varepsilon^n}^n \rangle_n$. We call the union of all metainferences of every level in this series the *complete metainferential logic* $\vdash_{\varepsilon}^{\omega}$.[2]

At this level of abstraction the choice of the standard for every level is completely free. Yet few metainferential logics are defined in a full liberal way. Most of the metainferential logics studied in the literature have

[2] Here, as in other parts of this article, $\langle A[n] \rangle_n$ ($\{A[n]\}_n$) is a shorthand notation for the series (set) of elements A distinguished by an index n.

standards defined recursively. In fact, the customary approach is to form a standard ε^{n+1} in terms of standards σ^n and δ^n in the following way:

- if all premises of a meta^{n+1}-inference conforms to the standard σ^n, then the conclusion conforms to the standard δ^n

When this is the case, we can write $\varepsilon^{n+1} = \sigma^n/\delta^n$. In the special case in which $\varepsilon^{n+1} = \varepsilon^n/\varepsilon^n$ for every $-1 \leqslant n < \omega$, we call the standard of the complete metainferential logic $\widehat{\varepsilon}$.[3]

Summing up, we may restate the MVLC as the conjunction of the following three propositions:

1. Logic is a theory about (at least)[4] *validity*.
2. What can be *valid* are propositions, inferences (i.e., inferences between propositions), metainferences (i.e., inferences between inferences) and, in general, meta^{n+1}-inferences (i.e., inferences between meta^n-inferences).
3. To define a *logic* one has to be able to:
   (a) Provide an interpreted language in which propositions, inferences and meta^n-inferences ($0 \leqslant n < \omega$) can be expressed
   (b) Provide some *standard(s)* for the validity of each of these elements.

I call this an "abstract" presentation of MVLC because it is independent of ST_ω — the logic of Barrio and colleagues, the nontransitive solutions to the semantic paradoxes, the Strong Kleene matrixes and even the algebraic understanding of semantics. Therefore, it may cover also other ways of understanding logics and semantics, such as the constructive (intuitionistic) way. This is what the following sections of this essay will be devoted to.

3. Intuitionism and logic

In this section, I will sketch the setup that will be used, in the subsequent sections, to define various kinds of metainferential intuitionistic logics. Our main goal is to identify and characterize what an intuitionistic standard of validity may be.

[3] I borrow the notation from (Ripley, 2021), although using it in a slightly different way.

[4] As an anonymous referee pointed out, some supporters of the BA-Plan extend this definition to include also *antivalidities* and *contingencies* (see, e.g., Barrio and Pailos, 2022).

Let us begin with the metainferential level -1. This level is constituted by formulas, and formulas are meant to represent propositions. Propositions, in its turn, can be true or false. Intuitionists agree with this doctrine, but understand truth as *availability of a proof* and falsity as *the impossible availability of a proof.* Propositions can also be combined to produce new propositions of growing complexity. This is done with logical connectives. Intuitionists also agree with this idea, as far as the meaning of these connectives sticks to the *Brouwer-Heyting-Kolmogorov interpretation* (BHK):

1. A *proof* of $A \wedge B$ consists in a *proof* of A and a *proof* of B;
2. A *proof* of $A \vee B$ consists in a *proof* of A or a *proof* of B;
3. A *proof* of $A \to B$ consists in a procedure to obtain a *proof* of B from a *proof* of A;
4. Nothing is a *proof* of $\perp$.[5]

These definitions already justify the definition: $\neg A = A \to \perp$. A procedure to obtain a proof of $\perp$ out of a proof of A can exist if, and only if, there can be no proof of A. Therefore, $A \to \perp$ is true (has a proof) exactly when A is false (there is a proof that a proof of A is impossible). Incidentally, it also justifies that one can always safely claim that there is a procedure to obtain a proof of A out of a proof of $\perp$, because there is none of the latter. This justifies the acceptance of *ex falso quodlibet* among the intuitionists.[6]

In the classical setting, a proposition is *valid* when it is unconditionally true. Intuitionists will also agree to this doctrine: a proposition is *valid* when it has an unconditional proof.

What is a *proof* in this context? We do not need to delve that deep into the philosophy of intuitionism for the present purposes. Let us just say that it is something that justifies or backs up a certain piece of knowledge. We claim that *at least* mathematical objects are the kind of things that can be a proof. And among mathematical objects, formal derivations. (This will have a decisive importance in the next sections.)

[5] To be precise, I should say ‘canonical’ proof. But let me be a bit floppy with the terminology here, for the sake of readability.

[6] Yet this is not uncontroversial. Johansson’s minimal logic was originally motivated precisely as a resistance to accept this claim. See (van Dalen, 2004) for a discussion. Incidentally, the notion of “impossibility” involved in these explications of intuitionistic negation is by no means inoquous either. Yet there is no place here to account for this problem.

Let us now move to the level of inferences. In the classical setting, inferential validity amounts to *preservation of truth*: if all premises are true, then this ensures that the conclusion is also true. The intuitionist can also agree with this understanding of inferential validity, as long as the ‘preservation’ involved is understood *constructively*. And this brings us back to the sense in which we already understood conditional sentences: an inference $\Gamma \vdash A$ should be valid if there is a procedure to obtain a proof of A out of proofs of Γ.

As meta^n-inferences are just inferences between inferences, then these should be conditional propositions too, and the previous observation generalizes. This gives us the general definition of what meta^n-inferential validity ($0 \leqslant n < \omega$) should be, from an intuitionistic perspective:

- A meta^n-inference $\Gamma \vdash_n A$ ($0 \leqslant n < \omega$) is valid when it has a proof, and a proof of a meta^n-inference $\Gamma \vdash_n A$ is a *proof* of A out of *proofs* of Γ.

These are the general concepts that will shape, and in a certain extent justify, the systems to be presented in the next sections.

4. Metainferential Intuitionistic Logic I

In this section, I will sketch a formal presentation of the complete metainferential logic of standard $\widehat{IL}$; that is, the metainferential logic defined as intuitionistic validity all the way up.

Definition 4.1. The propositional language $\mathfrak{L}_1$ has signature: $\wedge$, $\vee$, $\rightarrow$, $\bot$. Well-formed formulas are defined as usual.

We use lowercase Latin letters as variables of atomic propositions, Latin uppercase letters as variables of formulas and uppercase Greek letters as variables of multisets of formulas.

Convention 4.1. The following simplifications of writing are adopted:

1. $\neg A$ is an abbreviation for $A \rightarrow \bot$.
2. After the dropping of the outermost pair of parenthesis in a formula, one of the leftmost (rightmost) opening (closing) parenthesis may be dropped and its corresponding closing (opening) parenthesis replaced by a dot.

Example: ‘$(p \rightarrow q) \rightarrow r$’ becomes ‘$p \rightarrow q. \rightarrow r$’.

Consider Ni, a Natural Deduction calculus for IL introduced in Troelstra and Schwichtenberg (2000) (Figure 1). We want to define the following standard of level -1 based on it:

DEFINITION 4.2. The *standard* IL is the following property of formulas $A \in \mathfrak{L}_1$: A has a categorical derivation in the calculus Ni.

DEFINITION 4.3. The *meta*$^{-1}$*inferential logic* $\vdash^{-1}_{MIL}$ is the set of formulas that conforms to the standard IL.

$$\dfrac{\begin{array}{c}[A]^n\\ \vdots\\ B\end{array}}{A \to B}\ \to \mathrm{I}\ (n) \qquad \dfrac{A \qquad A \to B}{B}\ \to \mathrm{E}$$

$$\dfrac{A \qquad B}{A \wedge B}\ \wedge\ \mathrm{I} \qquad \dfrac{A_1 \wedge A_2}{A_i}\ \wedge_i\ \mathrm{E}$$

$$\dfrac{A_i}{A_1 \vee A_2}\ \vee_i\ \mathrm{I} \qquad \dfrac{A \vee B \qquad \begin{array}{c}[A]^m\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[B]^n\\ \vdots\\ C\end{array}}{C}\ \vee\ \mathrm{E}\ (m, n)$$

$$\dfrac{\bot}{A}\ \bot\ \mathrm{E}$$

$$i \in \{1, 2\}$$

Figure 1. Rules for Ni. A derivation is *hypothetical* if it has open assumptions and *categorical* if not.

Here is something we already know about this logic (see Troelstra and Schwichtenberg, 2000, th. 6.1.10, and references therein):

FACT 4.1. *The Strong Normalization Theorem holds for* Ni.

The Strong Normalization Theorem states that every derivation in Ni can be brought to a single normal form through a recursive procedure. And categorical derivations in normal form exhibit one important property:

Last Rule Property: In a normal categorical derivation the last rule applied is always the introduction rule for the main connective of the formula in the conclusion.

This means that:

1. If $A \wedge B$ has a normal categorical derivation (n.c.d.) then A has a n.c.d. and B has a n.c.d.;
2. If $A \vee B$ has a n.c.d. then A has a n.c.d. or B has a n.c.d.;
3. If $A \rightarrow B$ has a n.c.d. then a n.c.d. of A can be transformed into a n.c.d. of B;
4. $\bot$ does not have a n.c.d.

As it is easy to see, this is a proper instantiation of the BHK-interpretation, with the concept of 'n.c.d.' as a precise explication of the notion of 'proof'. Therefore, it is an intuitionistically acceptable interpretation of the connectives. It also confirms the opinion of some authors (Gentzen, 1935; Martin-Löf, 1987) who claim that the introduction rules of a Natural Deduction calculus provide the meaning of the logical connectives. Moreover, a n.c.d. of A represents an unconditional proof of A, for it has no open assumptions; thus this is an acceptable account of validity for the level -1, in the sense described in the previous section.

The definition of the standard IL/IL is now straightforward.

Definition 4.4. The *standard* IL/IL is the following property of inferences $\langle \Gamma, A\rangle \in \vdash^0$: a categorical derivation of A can be constructed in the calculus Ni, on the assumption that all formulas $G \in \Gamma$ have a categorical derivation in the calculus Ni.

Note that this is a more precise way of paraphrasing the following conditional claim, understood intuitionistically: "if all formulas in Γ have a n.c.d., then A has a n.c.d.". It is also equivalent to the usual definition of *derivability*, which corresponds to what this literature calls the "absolutely global" notion of validity (see Teijeiro, 2021, for more on this).[7]

Definition 4.5. The *meta*^{0}*-inferential logic* $\vdash^0_{IL}$ is the set of inferences that conforms to the standard IL/IL.

To see why this fits the definition of the standard IL/IL, consider the following hypothetical Ni derivation:

$$\dfrac{\dfrac{A \qquad B}{A \wedge B}\,\wedge\mathrm{I}}{(A \wedge B) \vee C}\,\vee\mathrm{I}$$

[7] Thanks to an anonymous referee for pointing out the need to make this connection explicit.

As this derivation is hypothetical, its conclusion is not an intuitionistic validity. But assume that A and B are intuitionistic validities; then, there are two categorical derivations, $\mathcal{D}_1$ and $\mathcal{D}_2$, of A and B respectively. Piling up these elements together gives us the derivation:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ B\end{matrix}}{A \wedge B}\,\wedge\mathrm{I}}{(A \wedge B) \vee C}\,\vee\mathrm{I}$$

Which is, in itself, categorical, because all its assumptions are the assumptions in $\mathcal{D}_1$ and $\mathcal{D}_2$ and we already know that they are all closed. By the fact 4.1, this new derivation is normalizable, and its corresponding n.c.d. is the expected proof of its conclusion.

This example also illustrates that, as categorical derivations in Ni are normalizable, it is enough that we guarantee that the conclusion has a categorical derivation if the premises have. We can abstract from the form of the actual derivations and consider them only from the view point of their open assumptions and their conclusion. In other words, we can take

$$\dfrac{\dfrac{A \qquad B}{A \wedge B}\,\wedge\mathrm{I}}{(A \wedge B) \vee C}\,\vee\mathrm{I}$$

To be a particular instance of a more general relation between the premises A, B and the conclusion $(A \wedge B) \vee C$. This motivates the introduction of *sequents*.[8]

DEFINITION 4.6. A *sequent* is an expression $\{A_n\}_n \;\Rightarrow\; B$ $(0 \leqslant n < \omega)$.

Note that, according to this definition, sequents can be empty on their left-hand side $(n = 0)$ but cannot have more nor less than *one* formula on the right.

We want sequents to represent types (sets or species; see footnote 10) of Ni derivations, in accordance with the thought that a sequent calculus can be seen as a metacalculus for natural deduction (Troelstra and

[8] To be sure, there is at least room for disagreeing with the reading of the sequents that I am proposing here. In (Paoli, 2007), for instance, this reading is explicitly rejected in favor of one that distinguishes between the *ground* and *assumptions* of an inference (see Section 2.3 of that article). Yet, as the technicalities of my proposal are sound (and in fact you will have a correspondence between derivable sequents and derivable rules of Ni), I think my reading is, at least, well-motivated (albeit a little naïve).

Schwichtenberg, 2000, p. 28). The rules for the sequent calculus IC_0 (Intuitionistic Calculus of level 0) can be found in Figure 2.

An important fact about this calculus is the following:

FACT 4.2. *Every sequent that can be derived in IC_0 can be derived without using the rule Cut.*

PROOF. In (Troelstra and Schwichtenberg, 2000, th. 4.1.5) it is detailed a constructive proof of Cut-elimination for the calculus **G3i**. What I am going to show is that every rule of **G3i** that is not a primitive rule of IC^0 is nonetheless derivable in it (without using Cut). From this follows that the proof of Troelstra and Schwichtenberg can be adapted to my calculus. The rules of **G3i** can be found in (Troelstra and Schwichtenberg, 2000, pp. 77–78) and I will not reproduce them here.

1. The rule **G3i**-$L\bot$ is derivable in IC^0:

$$\dfrac{\dfrac{}{\Gamma, \bot \ \Rightarrow\ \bot}\ \text{Id}}{\Gamma, \bot \ \Rightarrow\ A}\ \text{EFSQ}$$

2. The rule **G3i**-$L\wedge$ is derivable in IC^0:

$$\dfrac{\dfrac{\dfrac{\Gamma, A, B \ \Rightarrow\ C}{\Gamma, A\wedge B, B \ \Rightarrow\ C}\ \wedge\text{L}}{\Gamma, A\wedge B, A\wedge B \ \Rightarrow\ C}\ \wedge\text{L}}{\Gamma, A\wedge B \ \Rightarrow\ C}\ \text{C}$$

3. For the rules **G3i**-$R\wedge$, **G3i**-$L\vee$ and **G3i**-$L\supset$ a successive application of the rule C eliminates all the repetitions of formulas in the context of the final sequent.

Given that any of these proofs uses the Cut rule, the proof for **G3i** works also for my calculus. □

COROLLARY 4.2.1. *The sequent $\Rightarrow \bot$ cannot be derived in IC_0.*

The natural step forward is to define the standard $(IL/IL)/(IL/IL)$, the calculus of *first-degree metasequents* IC_1 (see Figure 3) and the metainferential logic $\vdash^1_{\widehat{IL}}$. After all the previous explications these definitions should be self-explanatory.

DEFINITION 4.7. A *first-degree metasequent* is an expression $\{\alpha_n\}_n \Rightarrow \beta$, where $0 \leqslant n < \omega$ and $\{\alpha_n, \beta\}_n$ are either all formulas or all sequents.

Identity Group

$$\frac{}{\Gamma, A \;\Rightarrow\; A}\;\text{Id}$$

$$\frac{\Gamma \;\Rightarrow\; A \qquad \Delta, A \;\Rightarrow\; B}{\Gamma, \Delta \;\Rightarrow\; B}\;\text{Cut}$$

Structural Group

$$\frac{\Gamma, A, A \;\Rightarrow\; B}{\Gamma, A \;\Rightarrow\; B}\;\text{C}$$

Operational Group

$$\frac{\Gamma, A_i \;\Rightarrow\; C}{\Gamma, A_1 \wedge A_2 \;\Rightarrow\; C}\;\wedge\text{L} \qquad \frac{\Gamma_1 \;\Rightarrow\; A \qquad \Gamma_2 \;\Rightarrow\; B}{\Gamma_{1,2} \;\Rightarrow\; A \wedge B}\;\wedge\text{R}$$

$$\frac{\Gamma_1, A \;\Rightarrow\; C \qquad \Gamma_2, B \;\Rightarrow\; C}{\Gamma_{1,2}, A \vee B \;\Rightarrow\; C}\;\vee\text{L} \qquad \frac{\Gamma \;\Rightarrow\; A_i}{\Gamma \;\Rightarrow\; A_1 \vee A_2}\;\vee\text{R}$$

$$\frac{\Gamma_1 \;\Rightarrow\; A \qquad \Gamma_2, B \;\Rightarrow\; C}{\Gamma_{1,2}, A \rightarrow B \;\Rightarrow\; C}\;\rightarrow\text{L} \qquad \frac{\Gamma, A \;\Rightarrow\; B}{\Gamma \;\Rightarrow\; A \rightarrow B}\;\rightarrow\text{R}$$

$$\frac{\Gamma \;\Rightarrow\; \bot}{\Gamma \;\Rightarrow\; A}\;\text{EFSQ}$$

$$i \in \{1, 2\}$$

Figure 2. Rules for IC_0

DEFINITION 4.8. A rule

$$\frac{\Gamma_1 \;\Rightarrow\; A_1 \qquad \ldots \qquad \Gamma_n \;\Rightarrow\; A_n}{\Delta \;\Rightarrow\; B}$$

Is *derivable* in the calculus IC_0 iff there is a valid derivation in IC_0 of the sequent $\Delta \;\Rightarrow\; B$ from *some* of the sequents $\{\Gamma_n \;\Rightarrow\; A_n\}_n$ (taken as axioms).

DEFINITION 4.9. The *standard* $(IL/IL)/(IL/IL)$ is the following property of metainferences $\langle\{\langle\Gamma_n, A_n\rangle\}_n, \langle\Delta, B\rangle\rangle \in \vdash^1$: the rule

$$\frac{\Gamma_1 \;\Rightarrow\; A_1 \qquad \ldots \qquad \Gamma_n \;\Rightarrow\; A_n}{\Delta \;\Rightarrow\; B}$$

is derivable in IC_0.

DEFINITION 4.10. The *meta*^{1}*-inferential logic* $\vdash^1_{MIL}$ is the set of metainferences that conforms to the standard $(IL/IL)/(IL/IL)$.

Identity Group

$$\frac{}{\Gamma, \alpha \ \Rightarrow \ \alpha}\ \text{Id}$$

$$\frac{\Gamma_1 \ \Rightarrow \ \alpha \qquad \Gamma_2, \alpha \ \Rightarrow \ \beta}{\Gamma_1, \Gamma_2 \ \Rightarrow \ \beta}\ \text{Cut}$$

Structural Group

$$\frac{\Gamma, \alpha, \alpha \ \Rightarrow \ \beta}{\Gamma, \alpha \ \Rightarrow \ \beta}\ \text{C}$$

Operational Group

[The same as IC_0]

Metainferential Group

$$\frac{\begin{array}{c}[\Gamma \ \Rightarrow \ \alpha]^n \\ \vdots \\ \Delta \ \Rightarrow \ \beta\end{array}}{(\Gamma \ \Rightarrow \ \alpha) \ \Rightarrow \ (\Delta \ \Rightarrow \ \beta)}\ \Rightarrow \text{I}\ (n)$$

$$\frac{\Gamma \ \Rightarrow \ \alpha \qquad (\Gamma \ \Rightarrow \ \alpha) \ \Rightarrow \ (\Delta \ \Rightarrow \ \beta)}{\Delta \ \Rightarrow \ \beta}\ \Rightarrow \text{E}$$

Figure 3. Rules for IC_1. A, B, C are formulas and α, β are formulas or sequents.

To see why this calculus formalizes the standard $(IL/IL)/(IL/IL)$, let's look at the example in Figure 4.

The first-degree metasequent at the conclusion of this derivation is:

$$(\ \Rightarrow \ A \vee \neg A) \ \Rightarrow \ (\neg\neg A \ \Rightarrow \ A)$$

We claim that this is a proof of type $(IL/IL)/(IL/IL)$, which means that

$$\underbrace{(\ \Rightarrow \ A \vee \neg A)}_{(IL/IL)} \ \Rightarrow \ \underbrace{(\neg\neg A \ \Rightarrow \ A)}_{(IL/IL)}$$

Given our previous analysis on sequents and derivable rules of Ni, it is clear that the intended lecture of this first-degree metasequent should then be as follows:

- If $A \vee \neg A$ is taken as an axiom[9] of Ni, then the rule $\frac{\neg\neg A}{A}$ can be shown to be derivable in Ni.

[9] In the precise sense that this word has in Natural Deduction, that is, a formula

$$
\begin{prooftree}
\AxiomC{$[\Rightarrow A\lor\neg A]^1$}
\AxiomC{}\RightLabel{Id}\UnaryInfC{$A\Rightarrow A$}
\AxiomC{}\RightLabel{Id}\UnaryInfC{$A\Rightarrow A$}
\AxiomC{}\RightLabel{Id}\UnaryInfC{$\bot\Rightarrow\bot$}
\RightLabel{$\supset$ L}\BinaryInfC{$\neg A, A\Rightarrow\bot$}
\RightLabel{$\supset$ R}\UnaryInfC{$\neg A\Rightarrow\neg A$}
\AxiomC{}\RightLabel{Id}\UnaryInfC{$\bot\Rightarrow\bot$}
\RightLabel{$\bot$ R}\UnaryInfC{$\bot\Rightarrow A$}
\RightLabel{$\supset$ L}\BinaryInfC{$\neg A,\neg\neg A\Rightarrow A$}
\RightLabel{$\lor$L}\BinaryInfC{$A\lor\neg A,\neg\neg A\Rightarrow A$}
\RightLabel{Cut}\BinaryInfC{$\neg\neg A\Rightarrow A$}
\RightLabel{$\Rightarrow$ I (1)}\UnaryInfC{$(\Rightarrow A\lor\neg A)\Rightarrow(\neg\neg A\Rightarrow A)$}
\end{prooftree}
$$

Figure 4. A proof that the Excluded Middle (as an axiom) entails Double Negation Elimination (as an inference).

And this is a well-known true fact about IL:

$$
\begin{prooftree}
\AxiomC{$A\lor\neg A$}
\AxiomC{$[A]^1$}
\AxiomC{$[\neg A]^2$}
\AxiomC{$\neg\neg A$}
\RightLabel{$\to$E}\BinaryInfC{$\bot$}
\RightLabel{$\bot$E}\UnaryInfC{A}
\RightLabel{$\lor$E (1,2)}\TrinaryInfC{A}
\end{prooftree}
$$

In fact, to say that a sequent

$$\{A_n\}_n \vdash A$$

is derivable in IC_0, amounts to say that the rule

$$\frac{A_1 \qquad \dots \qquad A_n}{A}$$

Is derivable in Ni. See Lemma 4.2.

Another thing we can learn from the derivation in Figure 4 is that Cut is no longer eliminable. But we have a reasonable way to manage its use. The proof of the following theorem follows the same lines as the proof of Fact 4.2 (see Troelstra and Schwichtenberg, 2000, Section 4.5.1):

Theorem 4.1. *If a metasequent is derivable in IC^1, then it has a derivation in which the Cut rule is used at most after a supposition.*

It is evident that the derivation in Figure 4 is normalized in this sense. On the other hand, the metainferential part of the calculus is weakly normalizable, as the following theorem states:

declared to be derivable from no open assumptions; or, equivalently, as the conclusion of a rule that has no premises. Thanks to an anonymous referee for the need to clarify this point.

Theorem 4.2. *If a metasequent is derivable, then it has a derivation in which there are no detours of* $\Rightarrow$.

Proof. Applying upwards the transformation from:

$$\dfrac{\dfrac{\begin{array}{c}[\Gamma \Rightarrow A]^n\\ \mathcal{D}_1\\ \Delta \Rightarrow B\end{array}}{(\Gamma \Rightarrow A) \Rightarrow (\Delta \Rightarrow B)}\Rightarrow \mathrm{I}\ (n) \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \Rightarrow A\end{array}}{\Delta \Rightarrow B}\Rightarrow \mathrm{E}$$

To:

$$\begin{array}{c}\mathcal{D}_2\\ \Gamma \Rightarrow A\\ \mathcal{D}_1\\ \Delta \Rightarrow B\end{array}$$

Effectively eliminates all detours. An induction completes the proof. □

The theorem states that a derivation with detours can be depurated from them, but the normal derivation we end up with may not be unique (therefore it is a ‘weak’, and not ‘strong’ normalization). But this is to be expected, since sequent calculi such as IC^0 are not normalizable in general. For instance, the (meta)sequent $A \wedge B \Rightarrow A \vee B$. Has four possible derivations:

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,\mathrm{Id}}{A \wedge B \Rightarrow A}\wedge \mathrm{L}}{A \wedge B \Rightarrow A \vee B}\vee \mathrm{R} \qquad \dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,\mathrm{Id}}{A \wedge B \Rightarrow B}\wedge \mathrm{L}}{A \wedge B \Rightarrow A \vee B}\vee \mathrm{R}$$

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,\mathrm{Id}}{A \Rightarrow A \vee B}\vee \mathrm{R}}{A \wedge B \Rightarrow A \vee B}\wedge \mathrm{L} \qquad \dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,\mathrm{Id}}{B \Rightarrow A \vee B}\vee \mathrm{R}}{A \wedge B \Rightarrow A \vee B}\wedge \mathrm{L}$$

And none of them seems to be more ‘normal’ than the others.

We are now in a position to define the metainferential hierarchy $\vdash^{n}_{\widehat{IL}}$.

1. The sub-inferential logic $\vdash^{-1}_{\widehat{IL}}$ of level -1 has standard IL: it distinguishes all and only the formulas of $\mathfrak{L}_1$ that are intuitionistically valid according to the Natural Deduction calculus Ni.

2. The inferential logic $\vdash^0_{\widehat{IL}}$ of level 0 has standard IL/IL: it distinguishes all and only the inferences that are intuitionistically valid, that is, that correspond to derivable rules of Ni.

3. The metainferential logic $\vdash^1_{\widehat{IL}}$ of level 1 has standard

$$(IL/IL)/(IL/IL)$$

It distinguishes all and only the rules that are derivable in a calculus for the derivable rules of Ni.

$\vdots$

$n+1$. The meta...metainferential logic $\vdash^{n+1}_{\widehat{IL}}$ of level $n+1$ distinguishes all and only the rules that are derivable in a calculus of the derivable rules of a calculus of the derivable rules of... of a calculus of the derivable rules of Ni.

And in the end we have the sum of all stages:[10]

ω. The omega-inferential logic $\vdash^{\omega}_{\widehat{IL}}$ is the sum of every metainference distinguished in every $\vdash^{n}_{\widehat{IL}}$.

A calculus for $\vdash^{\omega}_{\widehat{IL}}$ contains a calculus for every logic in the hierarchy. And this calculus is easily obtained as a generalization of the rules of IC_1. I call this calculus IC_ω (Figure 5).[11] It obtains by liberalization from the previous concept of first-degree metasequents to metasequents of every degree (of nesting).

DEFINITION 4.11. A *metasequent* is an expression $\{\alpha_n\}_n \Rightarrow \beta$, where $0 \leqslant n < \omega$ and $\{\alpha_n, \beta\}_n$ are formulas or metasequents. The *degree* of a metasequent is defined as follows:

1. Sequents have degree 0.
2. If $\{\alpha_n\}_n \Rightarrow \beta$ is a metasequent in which the higher degree of any of the $\{\alpha_n, \beta\}_n$ is m, then its degree is $m + 1$.

DEFINITION 4.12. The calculus IC_n $(0 \leqslant n < \omega)$ corresponds to the fragment of IC_ω limited to metasequents of degree n or lower.

[10] If we are strict intuitionists, then $\vdash^{\omega}_{\widehat{IL}}$ is not a set but a *species*, a property or incomplete collection of well-determined things. This metaphysical observation has little significance in this context. For the difference between sets and species, see (Posy, 2020, Section 2.2.2).

[11] *A note on the figure:* When I say that some rules of IC_0 are also rules of IC_ω, it is meant to imply that the convention on the type of the variables is preserved from one figure to the other. The same holds for the calculus MIL (Figure 7).

Identity Group

$$\frac{}{\Gamma, \alpha \;\Rightarrow\; \alpha}\;\text{Id}$$

$$\frac{\Gamma_1 \;\Rightarrow\; \alpha \qquad \Gamma_2, \alpha \;\Rightarrow\; \beta}{\Gamma_1, \Gamma_2 \;\Rightarrow\; \beta}\;\text{Cut}$$

Structural Group

$$\frac{\Gamma, \alpha, \alpha \;\Rightarrow\; \beta}{\Gamma, \alpha \;\Rightarrow\; \beta}\;\text{C}$$

Operational Group

The same as in IC_0.

Metainferential Group

$$\frac{\begin{array}{c}[\Gamma_1 \;\Rightarrow\; \alpha_1, \ldots, \Gamma_k \;\Rightarrow\; \alpha_k]^n \\ \vdots \\ \Delta \;\Rightarrow\; \beta\end{array}}{(\Gamma_1 \;\Rightarrow\; \alpha_1), \ldots, (\Gamma_k \;\Rightarrow\; \alpha_k) \;\Rightarrow\; (\Delta \;\Rightarrow\; \beta)}\;\Rightarrow \text{I}\ (n)$$

$$\frac{\Gamma \;\Rightarrow\; \alpha \qquad \Pi, (\Gamma \;\Rightarrow\; \alpha) \;\Rightarrow\; (\Delta \;\Rightarrow\; \beta)}{\Pi, \Delta \;\Rightarrow\; \beta}\;\Rightarrow \text{E}$$

Figure 5. Rules for IC_ω. A, B, C are formulas and α, β are formulas or metasequents.

The calculus IC_ω not only allows to recover all the metainferences of a certain level, but it has also some degree of *trans-inferentiality*. What I mean by this is that it admits the expression of inferences in which the relata are not of the same inferential level.[12] Take as an example the derivation in Figure 6. This derivation expresses the following fact: if the inference $A \vee B \;\vdash^0_{IL}\; A \wedge B$ is valid, then the meta-inference

$$(A \to B \vdash^0_{IL} B) \vdash^1_{IL} (A \to B \vdash^0_{IL} A)$$

[12] In the literature these are called *mixed metainferences* (Ferguson and Ramírez-Cámara, 2021; Pailos and Da Ré, 2023; Scambler, 2020). They were foreseen by Barrio, Pailos and Szmuc in one of their very first works on metainferential logics:

> In this vein, we will also not consider *mixed* metainferences, i.e., metainferences with premises belonging to different inferential levels. Although there is nothing conceptually wrong about such metainferences, yet again, working with them will make the different proofs unnecessary complicated.
>
> (Barrio, Pailos and Szmuc, 2020, note 10)

Is valid. This true proposition about metainferences does not match with the form of a metainference, according to Definition 2.2.[13]

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A\to B \;\Rightarrow\; B]^2 \qquad \dfrac{\dfrac{}{B\;\Rightarrow\; B}}{B \;\Rightarrow\; A\vee B}}{A\to B\;\Rightarrow\; A\vee B} \qquad [A\vee B \;\Rightarrow\; A\wedge B]^1}{A\to B \;\Rightarrow\; A\wedge B} \qquad \dfrac{\dfrac{}{A,B\;\Rightarrow\; A}}{A\wedge B\;\Rightarrow\; A}}{A\to B\;\Rightarrow\; A}}{A\to B\;\Rightarrow\; B . \Rightarrow . A\to B\;\Rightarrow\; A}\,(2)}{A\vee B \;\Rightarrow\; A\wedge B\; . \Rightarrow .(A\to B\;\Rightarrow\; B)\Rightarrow (\,A\to B\;\Rightarrow\; A)}\,(1)$$

Figure 6. A transinferential proof. (Labels were omitted for a matter of space.)

One major result about IC_ω is the one to be proved next (the proof is essentially the same as before):

LEMMA 4.1. *The Theorems 4.1 and 4.2 hold for IC_ω.*

THEOREM 4.3. *Let $\{\alpha_n, \beta\}_n$ be a set of formulas or metasequents of degree m. The metasequent*

$$\alpha_1, \ldots, \alpha_n \;\Rightarrow\; \beta$$

Is derivable in IC_ω if and only if the rule

$$\frac{\alpha_1 \qquad \ldots \qquad \alpha_n}{\beta}$$

Is derivable in IC_m.

PROOF. The "if" direction is immediate. The other one has some subtleties. What we are going to show is that, given the Lemma 4.1 and what it means to be a categorical derivation, it follows that the derivation has a degree lower than $m+1$, even if it is not normalized. And from this a proof of the theorem will follow.

"If": Immediate by an application of $\Rightarrow$ I.

[13] Although a case can be made to associate it with

$$(A\vee B \;\vdash^0_{IL}\; A\wedge B), (A\to B \;\vdash^0_{IL}\; B) \vdash^1_{IL} (\,A\to B \;\vdash^0_{IL}\; A)$$

This is only one of the various derivable metasequents that fall away from Definition 2.2, but a more incontrovertible example would have produced a bigger (and less handy) tree.

"Only if": Let $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ be the metasequent described in the header of the theorem. Assume there is a categorical derivation of it in IC_ω. We are going to show that none of the sequents in this derivation has a degree greater than m.

If a derivation contains a metasequent of degree greater than its conclusion, then this metasequent is eliminated in the course of the derivation. Call these metasequents *residues*. As there are only two rules of elimination in IC_ω, we have to show that none of them may have been used to eliminate the residues.

1. *The residue is not eliminated by an application of Cut.* By Lemma 4.1 instances of Cut can be 'lifted' up to the point where one of its premises is an assumption. As the derivation is categorical, all the assumptions have to be discharged. As the residue is the middle term (for it is being eliminated), it appears in the two premises of the Cut. But then it appears in a discharged assumption, which means that it appears again in a lower part of the derivation. And this is in contradiction with it being the residue.

2. *The residue is not eliminated by an application of $\Rightarrow$ E.* Assume the residue is the minor premise in an application of the rule $\Rightarrow$ E. For the same reason as in the case of Cut, neither of these premises can be a discharged assumption. This means that the residue has a derivation.

As the major premise in the application of $\Rightarrow$ E cannot be a discharged assumption either, it must have a derivation too. There is a node in that derivation that is the first in which the residue appears in the left-hand side of a metasequent. Call this node *critical point*. Now we have two possibilities to consider:

(a) The critical point is an application of Id. The residue cannot appear in the right-hand side of the critical point, because in that case the application of $\Rightarrow$ E below would not eliminate it. Therefore, it has to be part of the context. Then the following transformation:

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \rho\end{matrix} \qquad \begin{matrix}\dfrac{}{\Gamma, \rho \;\Rightarrow\; \theta}\,\mathrm{Id} \\ \mathcal{D}_2 \\ \Gamma', \rho \;\Rightarrow\; \theta'\end{matrix}}{\Gamma' \;\Rightarrow\; \theta'}\Rightarrow \mathrm{E} \qquad \rightsquigarrow \qquad \begin{matrix}\dfrac{}{\Gamma \;\Rightarrow\; \theta}\,\mathrm{Id} \\ \mathcal{D}_2 \\ \Gamma' \;\Rightarrow\; \theta'\end{matrix}$$

Eliminates the residue from the derivation.

(b) The critical point is an application of $\Rightarrow$ I. Given that the operational rules cannot be applied to metasequents, we can transform the derivation in one in which the application of $\Rightarrow$ E that eliminates the residue occurs just after the critical point. This produces a detour, and by Lemma 4.1 this detour is eliminable. But this would deliver a derivation in which the residue is not eliminated, which is in contradiction with it being the residue.

We have concluded that the residue cannot be eliminated from the derivation. But then the residue does not exist.

Now we have shown that the metasequent $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ can be derived using only metasequents of degree m or lower. From this and Lemma 4.1 it follows that the rule

$$\frac{\alpha_1 \qquad \ldots \qquad \alpha_n}{\beta}$$

is derivable. □

This theorem is our adequacy result. The semantical content of $\vdash^{\omega}_{\widehat{IL}}$ is given by Ni and its derivable rules. The theorem proves that the calculus IC_{ω} is sound with this interpretation, and moreover that it is (at least) complete. And it is also able to express truths about the metainferences of $\vdash^{\omega}_{\widehat{IL}}$, as illustrated by the example 6. This is a feature that would be interesting to exploit further. The next section is devoted to this task, but first I want to comment one more aspect of IC_{ω}.

Just before the introduction of MVLC, when the discussion still gravitated around ST, a great deal was given to the fact that despite being non-transitive, this logic was still very close to LP. And the reason was that certain result of recapture was available. In particular, in (Barrio, Rosenblatt and Tajer, 2015) it is proved that $\mathit{\Gamma} \vdash_{ST} \mathit{\Delta}$ is derivable just in case $\bigwedge(\mathit{\Gamma}) \rightarrow \bigvee(\mathit{\Delta})$ is an LP-validity. This was received by some to mean that ST is just 'LP in lamb's clothes'. But later on Ripley made the argument that, in fact, when working with metainferential logics this is something to be expected, under certain conditions. The fact that a metainferential logic validates (a generalized form) of this result is related to a property that he calls *downward coherence* (Ripley, 2021, Definition 15).[14] Now the logic $\vdash^{\omega}_{\widehat{IL}}$ we just defined comes from a hierarchy that should be akin to those that Ripley highlights in his

[14] The work of Ripley is a little subtler than what my reconstruction makes appear. In particular, he distinguishes between inference relations and counterexample relations, and to fully agree with his framework I should be working with a model-theoretic semantic presentation of IL (Kripke's, for instance). Also, there are logics that do not validate the result that I am going to show, and are nonetheless downward

article. This is confirmed by the fact that an analogous of the collapse mentioned in (Barrio, Rosenblatt and Tajer, 2015) obtains in this logic.

DEFINITION 4.13. Let $\top = \bot \to \bot$. The *propositional avatar* θ of a metasequent is defined as follows:

1. If $A \in \mathfrak{L}_1$, then $\theta(A) = A$
2. If α is a sequent $\Gamma \Rightarrow A$ then

$$\theta(\alpha) = (\top \wedge \bigwedge_{G \in \Gamma} \theta(G)) \to \theta(A)$$

3. If $A = \alpha \Rightarrow \beta$ is an metasequent, then

$$\theta(A) = \theta(\alpha) \ \to \ \theta(\beta)$$

Using θ as a translation algorithm, the following propositions are easily proved:

LEMMA 4.2. *The rule* $\dfrac{\{A_n\}_n}{B}$ *is derivable in* Ni *iff the sequent*

$$\{A_n\}_n \ \Rightarrow \ B$$

is derivable in IC_0.

PROOF. (Troelstra and Schwichtenberg, 2000, th. 3.3.1), up to the translation between their calculus **G3i** and IC_0 already suggested in the proof of Fact 4.2. □

THEOREM 4.4. *If the metasequent* α *is derivable in* IC_ω *then* $\theta(\alpha)$ *has a n.c.d. in* Ni.

PROOF. If α is a sequent, apply Lemma 4.2. If it is a metasequent, then let $\mathcal{D}$ be its derivation. As the rules for the inferential connectives of IC_ω are the same as the rules for the connectives in Ni, the derivation obtained by replacing every metasequent α' in $\mathcal{D}$ for $\theta(\alpha')$ produces a derivation in Ni that maybe uses derived rules corresponding to applications of rules of IC_0. As these can also be transformed into complete derivations in Ni, it follows that $\theta(\alpha)$ is derivable in Ni, and fact 4.1 guarantees that a n.c.d. can be found. □

coherent in the sense of Ripley. For the time being, I am only relating positive results about my logic with the previous literature, so it is not that important that the matching is precise. I just want to give the idea that we are more or less talking about the same things.

The function θ may be seen as an intuitionistic counterpart of the function *lower* defined in (Barrio, Rosenblatt and Tajer, 2015, def. 5) and Barrio, Pailos and Szmuc (2020, def. 3.7), which is in turn a direct adaptation of the interpretation originally given by Gentzen of the sequents (Gentzen, 1935, Sec. V, §1). So there is little surprise that this result holds.

5. Metainferential Intuitionistic Logic II

The calculus IC_ω is far more expressive than IC_1, let alone IC_0. Yet this expressiveness can be improved. The link between BHK-proofs and derivable rules that we exploited in the last section can be extended to all propositional operations of intuitionistic logic. Two logics will be obtained in this way: MIL and MIL^2 (Metainferential Intuitionistic Logic).

The first of these logics can be motivated by an article of the BA-Plan (Fiore, Pailos and Rubin, 2023). In this article, the authors introduce a series of inferential connectives, following the enlightening observation that sequents are propositions of a conditional form (Fiore, Pailos and Rubin, 2023). When considered in this way, it strikes as evident that our common technical resources to formalize these propositions are disproportionately poor when compared to the object language:

> [A]t the present state of art, 'the logic of inferences' is studied in a [...] poor language. Let φ, ψ, χ and π be inferences. The current approach only considers metainferences such as "φ is valid, therefore ψ is valid", "φ is valid, ψ is valid, therefore χ is valid", if we allow multiples conclusions also "φ is valid, ψ is valid, therefore χ is valid, π is valid", and so on. The validity or otherwise of all these metainferences is determined solely by the meaning of logical consequence. The language lacks the resources to compose or negate validity claims, and so the approach restricts its analysis of metainferential validity to a 'structural level', so to say. (Fiore, Pailos and Rubin, 2023, p. 2)

This observation is also true of the system IC_ω. Take for instance a well-known true statement about intuitionistic logic such as the Disjunctive Property:

Disjunctive Property If $\vdash^0_{IL} A \vee B$ then $\vdash^0_{IL} A$ or $\vdash^0_{IL} B$.

This statement cannot be paraphrased using metasequents. But it strongly suggests the form it may have if inferential disjunctions were available in the language. Now that we have established the correspondence between derivations and proofs in the sense of the BHK-interpretation, the enrichment of the expressive resources of our calculus is straightforward.

Consider the rules for the calculus MIL in Figure 7, an alternative to IC_ω. The expressions this logic handles are called *multisequents*, a concept that subsumes the previous one (of *metasequents*) (Definitions 5.1 and 5.2).

DEFINITION 5.1. A *multisequent* is defined recursively as follows:

1. Every expression of the form $\Gamma \Rightarrow A$ is a multisequent, where Γ is a multiset of formulas or multisequents and A is either a formula or a multisequent.
2. If α and β are multisequents then
   - $\alpha \mathbin{\mathbb{A}} \beta$
   - $\alpha \mathbin{\mathbb{V}} \beta$

   Are multisequents.
3. Nothing else is a multisequent.

DEFINITION 5.2. The series of multisequents $\mathbb{L}$ is defined recursively as follows:

1. $\mathbb{L}^0 = (\emptyset \Rightarrow \bot)$
2. $\mathbb{L}^{n+1} = (\emptyset \Rightarrow \mathbb{L}^n)$

We abbreviate $\alpha \Rightarrow \mathbb{L}^n$ as $\Dashv^n \alpha$.

It is easy to see that the metainferential fragment of this calculus is just Ni.

THEOREM 5.1. *If a multisequent is derivable in MIL^ω, then it has a derivation free of detours of $\Rightarrow$, $\mathbb{V}$ or $\mathbb{A}$.*

PROOF. The same procedure that proves the fact 4.1 works in this case, with the proviso that the normalization is 'weak', because of the rules outside the metainferential group. □

In the light of this result it is now evident that:

1. A derivation that ends with an application of $\mathbb{A}$I provides a derivation of $\alpha \mathbin{\mathbb{A}} \beta$ out of a derivation of α and a derivation of β.
2. A derivation that ends with an application of $\mathbb{V}$I provides a derivation of $\alpha \mathbin{\mathbb{V}} \beta$ out of a derivation of α or a derivation of β.

Identity Group

$$\frac{}{\Gamma, \alpha \ \Rightarrow \ \alpha}\ \text{Id}$$

$$\frac{\Gamma_1 \ \Rightarrow \ \alpha \qquad \Gamma_2, \alpha \ \Rightarrow \ \beta}{\Gamma_1, \Gamma_2 \ \Rightarrow \ \beta}\ \text{Cut}$$

Structural Group

$$\frac{\Gamma, \alpha, \alpha \ \Rightarrow \ \beta}{\Gamma, \alpha \ \Rightarrow \ \beta}\ \text{C}$$

Inferential Group

The same as in IC_0.

Metainferential Group

$$\frac{\begin{array}{c}[\alpha_1, \ldots, \alpha_k]^n \\ \vdots \\ \beta\end{array}}{\alpha_1, \ldots, \alpha_k \ \Rightarrow \ \beta}\ \Rightarrow \text{I}\ (n) \qquad \frac{\alpha \qquad \Pi, \alpha \ \Rightarrow \ \beta}{\Pi \ \Rightarrow \ \beta}\ \Rightarrow \text{E}$$

$$\frac{\mathfrak{a}_i}{\mathfrak{a}_1 \ \mathbb{V}\ \mathfrak{a}_2}\ \mathbb{V}\ \text{I} \qquad \frac{\mathfrak{a}\ \mathbb{V}\ \mathfrak{b} \qquad \mathfrak{a} \ \Rightarrow \ \mathfrak{c} \qquad \mathfrak{b} \ \Rightarrow \ \mathfrak{c}}{\mathfrak{c}}\ \mathbb{V}\ \text{E}$$

$$\frac{\mathfrak{a} \qquad \mathfrak{b}}{\mathfrak{a}\ \mathbb{A}\ \mathfrak{b}}\ \mathbb{A}\ \text{I} \qquad \frac{\mathfrak{a}_1\ \mathbb{A}\ \mathfrak{a}_2}{\mathfrak{a}_i}\ \mathbb{A}\ \text{E}$$

$$i \in \{1, 2\}$$

.

Figure 7. Rules for MIL. Greek lowercase letters are variables of formulas or multisequents; gothic lowercase letters are variables of multisequents only.

3. A derivation that ends with an application of $\Rightarrow$I provides a derivation of β out of α taken as an assumption.[15]

This confirms that, at least for the positive fragment of MIL, we have a fairly acceptable instance of the BHK-interpretation of the logical connectives, where derivations in MIL play the role of proofs just as they

[15] An anonymous referee asks whether this is equivalent to the BHK-interpretation of the conditional, in the form I give earlier on (Section 5). In fact, it is true that these are not trivially equivalent. But they become equivalent after the proof of normalization (Theorem 5.1 above). For assume that you have a derivation of β out of α taken as an assumption, and assume further that you have a proof of α. The composition of these two proofs together delivers a non-normal proof that normalizes to a proof of β.

do in Ni. This extends also to negation, but in order to establish this fact we need a couple of further results. For the proof of the following lemma, see (Troelstra and Schwichtenberg, 2000, Sec. 4.5.1):

LEMMA 5.1. *Let IC_0^+ be IC_0 with a series of axiom sequents of the form $\Gamma - \{A\} \Rightarrow A$. If a sequent is derivable in IC_0^+, then it is derivable in such a way that the rule Cut is applied only to sequents that are not derivable in IC_0.*

THEOREM 5.2. *$\mathbb{L}$ cannot be derived, for all $\mathbb{L} \in \langle \mathbb{L}^n \rangle_n$.*

PROOF. Recall that $\Rightarrow \perp$ cannot be derived (Corollary 4.2.1). If $(\Rightarrow(\Rightarrow \perp))$ is derivable, then it is the result of an application of Cut (Lemma 5.1) or $\Rightarrow$I (Theorem 5.1). But it cannot be the result of an application of Cut, since then $\Rightarrow\ A$ would be derivable for the cut formula A, and $\Rightarrow\ A$ can only be derived from $\Rightarrow \perp$. But neither can it be a instance of $\Rightarrow$I, because $\Rightarrow \perp$ cannot be derived in IC_0. What we want to prove then follows by induction. □

This completes our BHK-interpretation for the inferential connectives. For all n, $\mathbb{L}^n$ cannot be derived.

This new system adds a lot of expressive power when compared with IC_ω. For instance, we can now formalize the Disjunctive Property, mentioned above (see Figure 8).

It may be argued that the conclusion of this derivation *cannot* be identified with the Disjunctive Property, because it will remain valid if IC_0 is strengthened with the rules it lacks to become CL (Classical Logic) and CL does not validates the Disjunctive Property. This is a very clever observation, and it serves to illustrate how important is to keep in mind the 'semantic' content of these calculi. Remember that Ni were adequate as building blocks for the *semantic interpretation* of the formulas of $\vdash_{IL}^{-1}$ because we have strong normalization and the Subformula Property. If we add a rule to IC_0 (for instance: $\dfrac{\Gamma \Rightarrow \neg\neg A}{\Gamma \Rightarrow A}$) the careful link between derivations, meaning and intuitionistic interpretation will be lost: sequents no longer stand as abstractions from derivable rules in Ni, and therefore multi- and metasequents no longer mean derivable rules between derivable rules, etc. This change of meaning has an impact on the meaning of the derivable multisequents. In this particular case, note that in CL

$$\vdash A \to B \iff \vdash \neg A \vee B$$

Let $\mathcal{D}_1$ be the following derivation:

$$\dfrac{\dfrac{\dfrac{[A]^3}{\Rightarrow\ A}\;{\Rightarrow}\,\mathrm{I}\ (!)}{A\ \Rightarrow\ (\Rightarrow\ A)}\;{\Rightarrow}\mathrm{I}\ (3) \qquad \dfrac{\dfrac{[\Rightarrow\ A]^2}{(\Rightarrow\ A)\ \mathbb{V}\ (\Rightarrow\ B)}\;\mathbb{V}_1\mathrm{I}}{(\Rightarrow\ A)\ \Rightarrow\ ((\Rightarrow\ A)\ \mathbb{V}\ (\Rightarrow\ B))}\;{\Rightarrow}\mathrm{I}\ (2)}{A\ \Rightarrow\ ((\Rightarrow\ A)\ \mathbb{V}\ (\Rightarrow\ B))}\;\mathrm{Cut}$$

And $\mathcal{D}_2$ be the following derivation:

$$\dfrac{\dfrac{\dfrac{[B]^4}{\Rightarrow\ B}\;{\Rightarrow}\mathrm{I}\ (!)}{B\ \Rightarrow\ (\Rightarrow\ B)}\;{\Rightarrow}\mathrm{I}\ (4) \qquad \dfrac{\dfrac{[\Rightarrow\ B]^3}{(\Rightarrow\ A)\ \Rightarrow\ (\Rightarrow\ B)}\;\mathbb{V}_2\mathrm{I}}{(\Rightarrow\ A)\ \Rightarrow\ ((\Rightarrow\ A)\ \mathbb{V}\ (\Rightarrow\ B))}\;{\Rightarrow}\mathrm{I}\ (3)}{B\ \Rightarrow\ ((\Rightarrow\ A)\ \mathbb{V}\ (\Rightarrow\ B))}\;\mathrm{Cut}$$

The full derivation then is:

$$\dfrac{\dfrac{\dfrac{[(\Rightarrow\ A \vee B)]^1 \qquad \dfrac{\mathcal{D}_1 \qquad \mathcal{D}_2}{A \vee B\ \Rightarrow\ ((\Rightarrow\ A)\ \mathbb{V}\ (\Rightarrow\ B))}\;\vee\mathrm{L}}{\Rightarrow\ ((\Rightarrow\ A)\ \mathbb{V}\ (\Rightarrow\ B))}\;\mathrm{Cut}}{(\Rightarrow\ A)\ \mathbb{V}\ (\Rightarrow\ B)}\;{\Rightarrow}\mathrm{E}}{(\Rightarrow\ A \vee B)\ \Rightarrow\ ((\Rightarrow\ A)\ \mathbb{V}\ (\Rightarrow\ B))}\;{\Rightarrow}\mathrm{I}\ (1)$$

Figure 8. Derivation of an multisequent expressing the Disjunctive Property.

And this fact conflates the meaning of the multisequent in question: *if* $A \vee B$ *is implied by* $\top$*, then either* A *or* B *is implied by* $\top$. And this is an instance of the following principle:

$$A \to (B \vee C) \vdash A \to B. \vee .A \to C$$

Which is classically valid but *not* intuitionistically valid. As a matter of fact, intuitionistic logic validates this scheme in just a few cases (in particular, and crucial to this example, when $A = \top$, taking $\top$ as an abbreviation of $\bot \to \bot$). So when the calculus changes, the *meaning* of its components changes too.

To insist on the point, the critic may observe that she does not need to add new rules to the calculus. IC_0 becomes a calculus for CL by just allowing more than one formula on the right-hand side of the sequents. This means that *the same derivation* of Figure 8 holds in some calculus for CL. Then, it cannot be said that this derivation *is* the Disjunctive Property. But the same reply applies in this case: not every conservative extension of the *calculus* wil be conservative of their *interpretations*. In this case, we cannot allow irrestrictive multiple conclusions[16] and still hold that derivable sequents correspond to derivable rules of Ni. In particular:

$$\dfrac{\dfrac{}{p \Rightarrow p, \bot}\ \text{Id}}{\Rightarrow p, \neg p}\ \supset\text{R}$$

Is an incorrect derivation from the point of view of that interpretation: there is no n.c.d. of p, and there is no n.c.d. of $\neg p$. But it is correct, as we all know and is expectable, if we interpret sequents as claims about Boolean valuations.

My defense on this point may have an holistic taste, since the meaning I am attributing to the derivation in Figure 8 and its conclusion appears to be determined by more than the explicit rules and multisequents appearing in it. In principle, I am not in disagreement with such an holistic claim. But notice that in the single-conclusion and the multiple-conclusion frameworks the rules are not strictly the same. In the first case, the conclusion is a formula; in the latter, it is a set (or a multiset, or a series). So just as A is not equivalent with $\{A\}$, a sequent-like expression with single conclusions is not equivalent to a sequent-like expression with set-like conclusions, even when these conclusions are singletons. So it seems to me that it cannot be said that the rules are exactly the same. In an alternative calculus where the conditional multisequents allow for multiple conclusions, but the rules are *exactly* the same of MIL, you will never have the chance to derive a multisequent with more than one formula on the right.

Another notable aspect of this calculus is the fact that it allows for a metainferential negation. As would be expectable, this negation does not provide a full partition of the logical space, and therefore the multisequent $⫬\, \alpha \mathbb{V} \alpha$ is not derivable. Given Cut elimination, the normal derivation of a $⫬^0$ multisequent is:

$$\dfrac{\dfrac{\dfrac{\Rightarrow A_n \qquad \dfrac{\dfrac{\Rightarrow A_1 \qquad [A_1, \ldots, A_n \Rightarrow B]^1}{A_2, \ldots, A_n \Rightarrow B}\ \text{Cut}}{\begin{matrix}\vdots\\ A_n \Rightarrow B\end{matrix}}}{\Rightarrow B}\ \text{Cut} \qquad B \Rightarrow \bot}{\Rightarrow \bot}\ \text{Cut}}{⫬^0(A_1, \ldots, A_n \Rightarrow B)}\ \Rightarrow \text{I}\ (1)$$

[16] I say 'irrestrictive' because there are multiple-conclusion sequent calculi for IL, but they usually limit the use of the rules for negation and conditional.

The formulas for which $\Rightarrow \ A$ holds are logical validities,whereas formulas for which $B \ \Rightarrow$ holds are logical *anti-validities*. Therefore, the meta-negated multisequents corresponds to what is called *global invalidities* in the literature (Barrio and Pailos, 2022; Coberos, La Rosa and Tranchini, 2021).

MIL is for sure more expressive than IC_ω, but it is still not *as expressive* as it could be. To see why, consider the derivation in Figure 9. The conclusion of this derivation pretends to establish that accepting the validity of the inference $\neg\neg A \ \Rightarrow \ A$ entails that the excluded middle, $A \vee \neg A$, can be eliminated as a premise. In very light terms this may be paraphrased as saying that accepting Full Double Negation makes the logic to collapse into CL. This is a true statement *about* intuitionistic inferences; therefore, it is desirable that an intended account of intuitionistic metainferences includes this fact. Yet this derivation has a forbidden step (labeled $*$ in the derivation):

$$\frac{\neg\neg A \ \Rightarrow \ A}{\neg\neg(A \vee \neg A) \ \Rightarrow \ (A \vee \neg A)} \ *$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Rightarrow \ \neg\neg(A \vee \neg A)\end{array} \quad \dfrac{\dfrac{[(A \Rightarrow \ \neg\neg A) \ \mathbb{A} \ (\neg\neg A \Rightarrow \ A)]^1}{\neg\neg A \ \Rightarrow \ A} \ \mathbb{A}_2\mathrm{E}}{\neg\neg(A \vee \neg A) \ \Rightarrow \ (A \vee \neg A)} \ *}{\Rightarrow \ (A \vee \neg A)} \quad [\Gamma, A \vee \neg A \ \Rightarrow \ B]^2}{\Gamma \ \Rightarrow \ B}}{(\Gamma, A \vee \neg A \Rightarrow \ B) \ \Rightarrow_1 (\Gamma \Rightarrow \ B)} \ (2)}{((A \Rightarrow \neg\neg A) \ \mathbb{A} \ (\neg\neg A \Rightarrow \ A)) \ \Rightarrow \ ((\Gamma, A \vee \neg A \Rightarrow \ B) \Rightarrow \ (\Gamma \Rightarrow \ B))} \ (1)$$

Figure 9. An intended proof that Full Double Negation entails Classical Logic. Some labels were omitted for a matter of space.

This operation of "substitution" is not derivable in MIL. Moreover, the fact that, in intuitionistic logic,

$$\neg\neg A \to A \nvdash^0_{IL} A \vee \neg A$$

Makes doubtful that this fact can be expressed in an eloquent way. So the system MIL is not as good as a logic for the metainferences of IL as it could be. As a matter of fact, Barrio, Rosenblatt and Tajer indicate the following about metainferences:

> It could be useful to clarify what a (schematic) metainference is *not*. For instance, the claim 'if $p \vDash q$, then $p \vDash p \vee q$', is not a metainference but an instance of a metainference, because p and q are formulas belonging to the object language. A metainference is a general claim to the effect that if certain *kinds* of inferences hold, then another *kind* of inference holds as well. This explains why we use schematic formulas.
>
> (Barrio, Rosenblatt and Tajer, 2015, p. 557)

So, in their terminology, MIL is a calculus only of the *instances* of the metainferences of IC_0.

There is a way to overcome this difficulty though, and transform MIL in a *true* calculus for the metainferences of IL. The trick is to consider a more powerful system than Ni as the founding base, one that can handle with quantification over propositions. This is the motivation for the definition of the last system to be introduced in this article: MIL^2.

The calculus MIL^2 is obtained by adding a propositional universal quantifier to MIL, thus obtaining a second-order intuitionistic propositional logic (see Troelstra and Schwichtenberg, 2000, Chapter 11).

Definition 5.3. A *multisequent of second-order* (2-*multisequent*) is recursively defined as follows:

1. Every $\Gamma \Rightarrow A$ is a 2-multisequent, with Γ a multiset of formulas and A a formula. The free variables in $\Gamma \Rightarrow A$ are all the propositional variables in Γ or A.
2. Every $\Gamma \Rightarrow A$ is a 2-multisequent, with Γ a multiset of formulas or 2-multisequents and A a formula or 2-multisequent. The free variables in $\Gamma \Rightarrow A$ is the union of all the free variables in Γ and A.
3. If α 2-multisequent and p is a propositional variable then $\forall p.\ \alpha$ is a 2-multisequent. The free variables in $\forall p.\ \alpha$ are all the free variables in α except for p.
4. Nothing else is a 2-multisequent.

Convention 5.1. We assume identity of formulas up to renaming of bounded variables. We also assume that the bounded variables are always chosen so that they never occur bounded and free in the same derivation.

The rules for the logic MIL^2 are depicted in Figure 10. Now we are in position to introduce real metainferences in the sense expected by Barrio, Rosenblatt and Tajer (2015) (see Figure 11).

Base Group The same as in IC_0.

Second Order Metainferential Group

$$\dfrac{\begin{matrix}[\alpha_1, \ldots, \alpha_k]^n \\ \vdots \\ \beta\end{matrix}}{\alpha_1, \ldots, \alpha_k \;\Rightarrow\; \beta} \Rightarrow \mathrm{I}\ (n) \qquad \dfrac{\alpha \qquad \Pi, \alpha \;\Rightarrow\; \beta}{\Pi \;\Rightarrow\; \beta} \Rightarrow \mathrm{E}$$

$$\dfrac{\mathfrak{a}[q/p]}{\forall\, p.\ \mathfrak{a}}\ \forall\mathrm{I} \qquad \dfrac{\forall\, p.\ \mathfrak{a}}{\mathfrak{a}[\alpha/p]}\ \forall\mathrm{E}$$

Figure 10. Rules for MIL^2. Greek lowercase letters are variables of formulas or 2-multisequents; gothic lowercase letters are variables of 2-multisequents only. p is any propositional variable; q has to be an *eigenvariable*, meaning that it does not appear in the derivation before the application of the rule.

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Rightarrow\ \neg\neg(A \vee \neg A)\end{matrix} \qquad \dfrac{\dfrac{[\forall p.(p \Rightarrow \neg\neg p) \wedge\!\!\!\wedge (\neg\neg p \Rightarrow p)]^1}{((A \vee \neg A) \Rightarrow \neg\neg(A \vee \neg A)) \wedge\!\!\!\wedge (\neg\neg(A \vee \neg A) \Rightarrow (A \vee \neg A))}\ \forall\mathrm{E}}{\neg\neg(A \vee \neg A) \;\Rightarrow\; (A \vee \neg A)}\ \wedge\!\!\!\wedge_2\mathrm{E}}{\Rightarrow\ (A \vee \neg A)}\ \mathrm{Cut} \qquad [\Gamma, A \vee \neg A \;\Rightarrow\; B]^2}{\dfrac{\Gamma \;\Rightarrow\; B}{(\Gamma, A \vee \neg A \Rightarrow B) \;\Rightarrow_1\; (\Gamma \Rightarrow B)}\ (2)}\ \mathrm{Cut}}{(\forall p.(p \Rightarrow \neg\neg p) \wedge\!\!\!\wedge (\neg\neg p \Rightarrow p)) \;\Rightarrow\; ((\Gamma, A \vee \neg A \Rightarrow B) \Rightarrow (\Gamma \Rightarrow B))}\ (1)$$

Figure 11. A correct proof that Full Double Negation entails Classical Logic.

The rest of the metaiferential connectives of MIL can be obtained from the universal quantifier and the conditional, in the same way as they can be obtained in intuitionistic second-order propositional logic (see Troelstra and Schwichtenberg, 2000, Chapter 11, Zdanowski, 2009, and Kashima, 2017, for more on this logic).

6. Conclusions

Intuitionistic logic is one of the firsts, better-known, most developed and widely applied non-classical logics. And yet the logicians working on the BA-Plan have been, for the most part, indifferent about it. The purpose of this article was to provide a first attempt to fill this gap.

In this essay I defined and worked around a metainferential logic called $\vdash_{\widehat{IL}}^{\omega}$. I defined three sound calculi for this logic: IC_ω, MIL and MIL^2. These calculi were proven to be conservative extensions of each other: MIL subsumes IC_ω, MIL^2 subsumes MIL. And the three of them are *at least* sound about $\vdash_{\widehat{IL}^{\omega}}$, in the sense that what can be derived in them can be interpreted as true facts about the logic.

This first presentation was purely proof-theoretical. It remains to be explored whether the results of the BA-Plan may be approached using Kripke models in the place of Strong Kleene matrixes and so to obtain a model-theoretic version of $\vdash_{\widehat{IL}^{\omega}}$.

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