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Paolo Maffezioli

Proof Theory for Intuitionistic Stable Theories

Abstract. In this paper we show how to extend the standard cut-elimination procedure from first-order intuitionistic stable logic to a class of intuitionistic stable theories. Building on previous works by Negri and von Plato, we aptly modify the underlying calculus for first-order intuitionistic logic so as to preserve the admissibility of all the structural rules, including cut, in the presence of a restricted version of the rule of classical *reductio ad absurdum* and of a special case of universal rules.

Keywords: stable logic; intuitionistic logic; proof theory

1. Introduction

In constructive mathematics a relation is *stable* when it satisfies the law of double-negation elimination, whereas it is *decidable* when it satisfies the law of excluded middle. Interestingly, on the background of intuitionistic logic, a relation may be stable without being decidable, although any decidable relation is also stable. Examples include the intuitionistic theory of equality as defined within Brouwer's theory of apartness: given the axioms governing the apartness relation, if an equality $a = b$ is defined as the intuitionistic negation of apartness $\neg a \neq b$, the stability of equality $\neg\neg a = b \supset a = b$ follows; the decidability of equality $a = b \vee \neg a = b$, however, does not.[1] While the proof theory of apartness has been extensively investigated in [5], stable equality and, more generally, stable intuitionistic theories have hardly been considered at all in proof theory.

[1] The axioms for the apartness relation, firstly introduced in [4], are irreflexivity and the principle of split: $a \neq b \supset a \neq c \vee b \neq c$.

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A preliminary step towards filling this gap has been taken by Negri and von Plato, who in [6] introduced a sequent calculus for *stable logic*. Such a calculus extends the single-succedent sequent calculus for propositional intuitionistic logic $\mathsf{G3ip}$ of [8] with an inference rule corresponding to the classical law of *reductio ad absurdum* restricted to atomic formulas.

$$\frac{\neg P, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow P}\ \text{raa}$$

Since in $\mathsf{G3ip}$+raa the sequent $\Rightarrow \neg\neg P \supset P$ is derivable, whereas $\Rightarrow P \vee \neg P$ is not, $\mathsf{G3ip}$+raa qualifies as an intermediate calculus between the intuitionistic and the classical one. Importantly, Negri and von Plato showed that $\mathsf{G3ip}$+raa admits an entirely standard cut-elimination procedure. Since in this work we are primarily interested in first-order theories we shall extend Negri and von Plato's result in the presence of quantifiers, namely to stable logic as based on a calculus for first-order intuitionistic calculus logic such as $\mathsf{G3i}$. Indeed, Theorem 1 below establishes that $\mathsf{G3i}$+raa, too, admits an entirely standard cut-elimination procedure.

In order to characterize stable intuitionistic theories there remains to extend $\mathsf{G3i}$+raa beyond stable logic. As we have seen, intuitionistic equality is not just any relation satisfying the law of double-negation elimination, but one which also satisfies the axioms of equivalence relations. And since these axioms are clearly not derivable in $\mathsf{G3i}$+raa alone, the question naturally arises as to whether can we extend $\mathsf{G3i}$+raa without jeopardizing the admissibility results, especially cut elimination. To be sure, in a series of works starting from [7] Negri and von Plato has shown how to extend $\mathsf{G3i}$ with certain rules in such a way that the admissibility of the structural rules still holds.[2] In particular, they show how to recover cut elimination for $\mathsf{G3i}+\mathbf{R}$, where $\mathbf{R}$ is a set of rules following the so-called universal rule scheme.

The next step towards a satisfactory proof-theoretic analysis of stable intuitionistic theories is to see whether universal rules preserve cut elimination when the underlying logical calculus is not just $\mathsf{G3i}$ but rather its extension $\mathsf{G3i}$+raa. To put it more precisely, building on previous work by Negri and von Plato (as well as on Theorem 1 below) we know that in both $\mathsf{G3i}$+raa and $\mathsf{G3i}+\mathbf{R}$ all the structural rules are admissible. Are they admissible, too, in the combined calculus $\mathsf{G3i}+\text{raa}+\mathbf{R}$?

[2] In fact, the underlying calculus for intuitionistic logic considered in [7] and other works is the multi-succedent calculus $\mathsf{G3im}$. This issue will be addressed later.

In this paper I firstly show that the answer to this question is negative: not any set of rules following the universal rule scheme yields an extension of the basic calculus for stable logic to which the standard cut-elimination procedure can be applied. Secondly, I suggest a simple modification on the underlying calculus for intuitionistic logic and on the syntactic form of a universal rule in order to recover the standard procedure of cut elimination. This permits to generalize Negri and von Plato's original approach from intuitionistic stable logic to intuitionistic stable theories. Finally, I discuss the prospect of extending the present approach to multi-succedent calculi.

2. Preliminaries

Let $\mathcal{L}$ be a first-order language (without identity) containing infinitely many constants, variables, n-ary functions and predicates ($n \geqslant 0$), the nullary connective $\bot$ (falsity), the binary connectives $\wedge$ (conjunction), $\vee$ (disjunction), and $\supset$ (implication) as well as the quantifiers $\forall$ (universal) and $\exists$ (existential). A term t is either a constant or a variable or else an application of n-ary function to n terms, whereas an atom P is an application of a n-ary predicate to n terms. Finally, a formula A is built up from atoms and $\bot$ in the usual way. Notice that $\bot$ is not an atom, though it is a formula. We agree that $\neg A$ is an abbreviation for $A \supset \bot$. The set of the free variables of a term and of a formula are defined as usual and so is the operation of substitution of a variable in a term and in a formula (bearing in mind that we shall assume throughout the principle of α-conversion). Also the notion of a term being free for a variable in a formula is the standard one. A sequent is $\Gamma \Rightarrow A$, where Γ is finite, possibly empty, multi-set of formulas and A is a formula. We shall refer to Γ and A as the antecedent and the succedent of the sequent $\Gamma \Rightarrow A$, respectively. Let G3i be the single-succedent sequent calculus for intuitionistic logic from [8] (see Table 1).

As usual, the variable y does not occur free in the conclusion of $\mathrm{R}_\forall$ and $\mathrm{L}_\exists$. A derivation in G3i is a tree of sequents which grows according to its rules and whose leaves are initial sequents or conclusions of $\mathrm{L}_\bot$. A derivation of a sequent is a derivation concluding that sequent and a sequent is derivable when there is a derivation of it. The height h of a derivation is defined inductively as follows: the height of an initial sequent or of a conclusion of $\mathrm{L}_\bot$ is 0, the height of a derivation of a con-

$$P, \Gamma \Rightarrow P \qquad \frac{}{\bot, \Gamma \Rightarrow C}\,\mathrm{L}_\bot$$

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\,\mathrm{L}_\wedge \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\,\mathrm{R}_\wedge$$

$$\frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\,\mathrm{L}_\vee \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\,\mathrm{R}_\vee \quad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\,\mathrm{R}_\vee$$

$$\frac{A \supset B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \supset B, \Gamma \Rightarrow C}\,\mathrm{L}_\supset \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\,\mathrm{R}_\supset$$

$$\frac{A(t/x), \forall x A, \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}\,\mathrm{L}_\forall \qquad \frac{\Gamma \Rightarrow A(y/x)}{\Gamma \Rightarrow \forall x A}\,\mathrm{R}_\forall$$

$$\frac{A(y/x), \Gamma \Rightarrow C}{\exists x A, \Gamma \Rightarrow C}\,\mathrm{L}_\exists \qquad \frac{\Gamma \Rightarrow A(t/x)}{\Gamma \Rightarrow \exists x A}\,\mathrm{R}_\exists$$

Table 1.

clusion of a rule with one premise is the derivation height of its premise plus 1, and the derivation height of a derivation of a conclusion of a rule with two is the maximum of the derivation heights of its premises plus 1. A sequent is h-derivable if it is derivable with a derivation of height at most h. A rule is admissible if the conclusion is derivable whenever the premises are derivable; a rule is height-preserving admissible if the conclusion is h-derivable whenever the premises are h-derivable. As is well known, in G3i all the structural rules

$$\frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\,\mathrm{wk} \qquad \frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\,\mathrm{ctr} \qquad \frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\,\mathrm{cut}$$

are admissible. In fact, weakening and contraction are also height-preserving admissible, whereas cut is just admissible.

3. A calculus for first-order stable logic

We shall now consider two ways of extending G3i. The first one consists of adding on top of G3i the rule of classical *reductio ad absurdum* restricted to atoms.

$$\frac{\neg P, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow P}\,\mathrm{raa}$$

The proof theory for the quantifier-free part of G3i+raa, namely the calculus referred to as G3ip+raa, has been extensively investigated by Negri and von Plato in [6]. In particular they showed that in G3ip+raa all the admissibility results of G3ip are preserved (cf. Theorem 7.2.2 in [6]). It is easy to see that this holds even in the presence of quantifiers, namely in G3i+raa.

THEOREM 1. *In* G3i+raa *weakening and contraction are height-preserving admissible and cut is admissible.*

Moreover, Negri and von Plato showed that the set of derivable sequents in G3ip+raa properly includes the set of derivable sequents in G3ip and is properly included in the set of derivable sequents of G3cp, where G3cp is the sequent calculus for classical propositional logic from [8]. In other words, G3ip+raa is a calculus for an intermediate logic, called "stable logic" (cf. Theorem 7.2.1 in [6]). This result essentially hinges upon the fact that the rule raa is restricted to atomic formulas. Indeed, had an arbitrary propositional formula A been allowed to be principal of raa, the calculus would be equivalent to G3cp. Nevertheless, the rule raa with a $\vee$-free A instead of P is admissible in G3ip+raa (cf. Theorem 7.2.3 in [6]). This too can be generalized to quantifiers.

THEOREM 2. *If A is $\vee\exists$-free, then in* G3i+raa *the rule* raa *with A as principal is admissible.*

PROOF. By induction on A. If A is $\bot$, then we need to find a derivation of $\Gamma \Rightarrow \bot$ from $\neg\bot, \Gamma \Rightarrow \bot$. Consider the following derivation:[3]

$$\dfrac{\dfrac{\dfrac{}{\bot \Rightarrow \bot}\,\mathrm{L}_\bot}{\Rightarrow \neg\bot}\,\mathrm{R}_\neg \qquad \neg\bot, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow \bot}\,\mathrm{cut}$$

If A is an atom P, then the rule to be proved admissible is just raa itself. If A is $A \wedge B$, then consider the following derivation of $\Gamma \Rightarrow A \wedge B$ from $\neg(A \wedge B), \Gamma \Rightarrow \bot$, where the double inference line denotes an application of the inductive hypothesis IH and the sequents $\neg A \Rightarrow \neg(A \wedge B)$ and $\neg B \Rightarrow \neg(A \wedge B)$ are derivable:

[3] We shall freely use the rules for negation to shorten some derivations.

$$\dfrac{\dfrac{\dfrac{\neg A \Rightarrow \neg(A \wedge B) \quad \neg(A \wedge B), \Gamma \Rightarrow \bot}{\neg A, \Gamma \Rightarrow \bot}\text{cut}}{\Gamma \Rightarrow A}\text{IH} \qquad \dfrac{\dfrac{\neg B \Rightarrow \neg(A \wedge B) \quad \neg(A \wedge B), \Gamma \Rightarrow \bot}{\neg B, \Gamma \Rightarrow \bot}\text{cut}}{\Gamma \Rightarrow B}\text{IH}}{\Gamma \Rightarrow A \wedge B}\text{R}_\wedge$$

If A is $A \supset B$, then we need to show how to derive $\Gamma \Rightarrow A \supset B$ from $\neg(A \supset B), \Gamma \Rightarrow \bot$. Consider the following derivation, where the sequents $\neg B, A \Rightarrow \neg(A \supset B)$ and $\neg B, \neg\neg B \Rightarrow \bot$ are easily derivable:

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg B, A \Rightarrow \neg(A \supset B) \quad \neg(A \supset B), \Gamma \Rightarrow \bot}{\neg B, A, \Gamma \Rightarrow \bot}\text{cut}}{A, \Gamma \Rightarrow \neg\neg B}\text{R}_\neg \qquad \dfrac{\neg B, \neg\neg B \Rightarrow \bot}{\neg\neg B \Rightarrow B}\text{IH}}{A, \Gamma \Rightarrow B}\text{cut}}{\Gamma \Rightarrow A \supset B}\text{R}_\supset$$

Finally, if A is $\forall x A$, then we need to derive $\Gamma \Rightarrow \forall x A$ from $\neg\forall x A, \Gamma \Rightarrow \bot$ and this can be done by the following derivation, where y does not occur free in conclusion of $\text{R}_\forall$. Notice that the sequent $\neg A(y/x) \Rightarrow \neg\forall x A$ is derivable.

$$\dfrac{\dfrac{\dfrac{\neg A(y/x) \Rightarrow \neg\forall x A \quad \neg\forall x A, \Gamma \Rightarrow \bot}{\neg A(y/x), \Gamma \Rightarrow \bot}\text{cut}}{\Gamma \Rightarrow A(y/x)}\text{IH}}{\Gamma \Rightarrow \forall x A}\text{R}_\forall$$

All applications of the cut rule are legitimate by Theorem 1. □

4. A calculus for first-order stable theories

The second way to extend G3i is by adding rules corresponding to axioms of a mathematical theory. Building on previous works starting from [7] such an extension consists of G3i plus **R**, where **R** is a finite and non-empty set of rules, called universal rules, of the form:

$$\dfrac{Q_1, P_1, \ldots, P_n, \Gamma \Rightarrow C \quad \cdots \quad Q_m, P_1, \ldots, P_n, \Gamma \Rightarrow C}{P_1, \ldots, P_n, \Gamma \Rightarrow C}\text{r}$$

Intuitively, a rule in **R** is a way to express in sequent calculus an axiom of the form $\forall\bar{x}(P_1 \wedge \cdots \wedge P_n \supset Q_1 \vee \cdots \vee Q_m)$, called a universal formula.

Notice that if $n = 0$, the universal formula is just $\forall \bar{x}(Q_1 \vee \cdots \vee Q_m)$ and the corresponding universal rule is:

$$\frac{Q_1, \Gamma \Rightarrow C \quad \cdots \quad Q_m, \Gamma \Rightarrow C}{\Gamma \Rightarrow C}\ \mathrm{r}$$

On the other hand, if $m = 0$ the universal formula is $\forall \bar{x}\neg(P_1 \wedge \cdots \wedge P_n)$ and the corresponding universal rule is:

$$\frac{}{P_1, \ldots, P_n, \Gamma \Rightarrow C}\ \mathrm{r}$$

Clearly many axioms of mathematical theories are universal formulas.

In [5] Negri showed that for a specific $\mathbf{R}$, the one that includes the rules corresponding to the axioms of apartness, it is possible to preserve the admissibility results, whereas the generalization to an arbitrary $\mathbf{R}$ in the full language of first-order logic has been given recently in [3]. Thus, $\mathsf{G3i}+\mathbf{R}$, too, satisfies the admissibility results of weakening, contraction, and cut (cf. Theorem 8 in [3]).

Alas, the extension of $\mathsf{G3i}$ by both raa and $\mathbf{R}$ is problematic as it does not appear to admit a standard cut-elimination procedure. Consider, for example, the case of an application of cut in which the left premise has been derived by raa and the right premise by a universal rule r with one premise.

$$\frac{\dfrac{\neg P, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow P}\ \mathrm{raa} \qquad \dfrac{Q, P, \Delta \Rightarrow C}{P, \Delta \Rightarrow C}\ \mathrm{r}}{\Gamma, \Delta \Rightarrow C}\ \mathrm{cut}$$

If the derivation height of the left (right) premise of cut is $n > 0$ ($m > 0$, respectively), then the cut-height (namely the sum of the derivation height of the two premises of cut) is $n+m$. According to the standard cut-elimination procedure, in order to eliminate such an instance of cut we should find a derivation of $\Gamma, \Delta \Rightarrow C$ by "pushing cut upwards", in the sense that in the derivation of $\Gamma, \Delta \Rightarrow C$ all cuts, if any, need to be on a formula with lower weight than P or need to be of a less cut-height. A close inspection reveals, however, this cannot be done using only the premises of raa and r. Perhaps the closest we can get is to use sequent

$P \Rightarrow Q$, which is derivable by r, and to conclude $\Gamma, \Delta \Rightarrow C$ with two cuts and contraction as follows:

$$\dfrac{\Gamma \Rightarrow P \qquad \dfrac{\dfrac{\dfrac{\dfrac{Q, P \Rightarrow Q}{P \Rightarrow Q}\,\mathrm{r} \qquad Q, P, \Delta \Rightarrow C}{P, P, \Delta \Rightarrow C}\,\mathrm{cut}}{P, \Delta \Rightarrow C}\,\mathrm{ctr}}{\Gamma, \Delta \Rightarrow C}\,\mathrm{cut}$$

Assuming that contraction is height-preserving admissible, since the derivation-height of the sequent $P \Rightarrow Q$ is 1, the upper-most cut has a cut-height of m, hence it is admissible by the inductive hypothesis. However, since $m > 0$ by hypothesis, the height of the derivation of $P, \Delta \Rightarrow C$ is m and the lowest cut has cut-height $n + m$ as the original one; and this is problematic since the cut formula P has the same weight in the two cases. This example, of course, does not count as counter-example to cut elimination for $\mathsf{G3i}$+raa+$\mathbf{R}$. For this would require to find a sequent which is derivable with cut but not derivable without. What the example does show, however, is that the standard cut-elimination procedure fails for $\mathsf{G3i}$+raa+$\mathbf{R}$. The calculus may very well be cut-free but there is no known cut-elimination procedure. In general, the main obstacle towards finding a standard cut-elimination procedure for $\mathsf{G3ip}$+raa+$\mathbf{R}$ is that the same atom can be the conclusion of raa and the conclusion of a rule in $\mathbf{R}$. One way to overcome this obstacle is to change the shape of universal rules in such a way that all active and principal atoms may only occur in the succedent.

An entirely standard procedure of cut elimination for stable intuitionistic theories can be found by considering an alternative, albeit equivalent, calculus for intuitionistic logic. Firstly, we shall consider a variant $\mathcal{L}'$ of $\mathcal{L}$ where $\perp$ is an atom. Thus, in $\mathcal{L}'$ an atom P is either $\perp$ or an application of a n-predicate to n terms. Let $\mathsf{G3i}'$ be $\mathsf{G3i}$ where the rule $\mathrm{L}_\perp$ is removed and the following rule of *ex falso quodlibet* is added instead:

$$\dfrac{\Gamma \Rightarrow \perp}{\Gamma \Rightarrow C}\,\mathrm{efq}$$

Notice that in $\mathsf{G3i}'$ the sequent $\perp, \Gamma \Rightarrow \perp$ is initial since it is a special case of $P, \Gamma \Rightarrow P$. Since the calculus $\mathsf{G3i}'$ is not entirely standard in the literature, we need to show that it is indeed a calculus for intuitionistic logic. To this end we show that $\mathsf{G3i}'$ and $\mathsf{G3i}$ are equivalent.

THEOREM 3. *A sequent is derivable in* G3i′ *if and only if it is derivable in* G3i.

PROOF. It suffices to show that $\mathrm{L}_\perp$ is derivable in G3i′ and efq as well as the initial sequent $\perp, \Gamma \Rightarrow \perp$ are derivable in G3i. As for the first claim, consider the following derivation.

$$\dfrac{\perp, \Gamma \Rightarrow \perp}{\perp, \Gamma \Rightarrow C}\ \text{efq}$$

As for the second claim, since $\perp, \Gamma \Rightarrow \perp$ is an instance of $\mathrm{L}_\perp$, we only need to show that efq is derivable in G3i. Consider the following derivation.

$$\dfrac{\Gamma \Rightarrow \perp \quad \dfrac{}{\perp \Rightarrow C}\ \mathrm{L}_\perp}{\Gamma \Rightarrow C}\ \text{cut}$$

Notice that the application of cut is legitimate since G3i is cut-free. □

We now focus on universal rules. Since we are only considering single-succedent calculi we need to give up on some generality and restrict the notion of universal formula as to only allow those with at most one atom in the consequent of the implication. More precisely, let a quasi-universal formula be universal formula where $m \leqslant 1$. Thus, a quasi-universal formula is of the form $\forall \bar{x}(P_1 \wedge \cdots \wedge P_n \supset Q_m)$. A quasi-universal formula corresponds to a quasi-universal rule of the form:

$$\dfrac{\Gamma \Rightarrow P_1 \quad \cdots \quad \Gamma \Rightarrow P_n}{\Gamma \Rightarrow Q_m}\ \mathrm{r}$$

As above when $n = 0$, then a quasi-universal formula is just $\forall \bar{x} Q_m$ and the corresponding quasi-universal rule is a rule with no premise:

$$\dfrac{}{\Gamma \Rightarrow Q_m}\ \mathrm{r}$$

Moreover, when $m = 0$, a quasi-universal formula is $\neg(P_1 \wedge \cdots \wedge P_n)$ and the corresponding quasi-universal rule is:

$$\dfrac{\Gamma \Rightarrow P_1 \quad \cdots \quad \Gamma \Rightarrow P_n}{\Gamma \Rightarrow \perp}\ \mathrm{r}$$

It is clear that an application of a quasi-universal rule may introduce $\perp$ in the succedent. This is precisely why we have considered the calculus G3i′ instead of the admittedly much more standard G3i. For G3i+$\mathbf{R}^{\mathrm{q}}$, where $\mathbf{R}^{\mathrm{q}}$ is a set of quasi-universal rules, is not cut-free. To see this, consider a cut in which the left premise introduces $\bot$ in the succedent by an instance of r and the right premise introduces $\bot$ by an instance of $\mathrm{L}_\bot$. Such a cut can hardly be eliminated.

$$\dfrac{\dfrac{\Gamma \Rightarrow P_1 \quad \cdots \quad \Gamma \Rightarrow P_n}{\Gamma \Rightarrow \bot}\,\mathrm{r} \qquad \dfrac{}{\bot, \Delta \Rightarrow C}\,\mathrm{L}_\bot}{\Gamma, \Delta \Rightarrow C}\,\mathrm{cut}$$

However, if we consider the calculus $\mathsf{G3i}'$ instead of $\mathsf{G3i}$, then the sequent $\Gamma, \Delta \Rightarrow C$ can be concluded via efq and several applications of weakening directly from the left premise of cut:

$$\dfrac{\dfrac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow C}\,\mathrm{efq}}{\Gamma, \Delta \Rightarrow C}\,\mathrm{wk}$$

As a concrete example of such a quasi-universal rule one may consider the irreflexivity of the strict partial order $<$, i.e. the formula $\forall x\, \neg x < x$. The corresponding quasi-universal rule is:

$$\dfrac{\Gamma \Rightarrow t < t}{\Gamma \Rightarrow \bot}\,\mathrm{r}_1$$

Clearly, $\mathsf{G3i}+\mathrm{r}_1$ is not cut-free. For the sequent $t < t \Rightarrow C$ can certainly be derived with cut as follows:

$$\dfrac{\dfrac{t < t \Rightarrow t < t}{t < t \Rightarrow \bot}\,\mathrm{r}_1 \qquad \dfrac{}{\bot \Rightarrow C}\,\mathrm{L}_\bot}{t < t \Rightarrow C}\,\mathrm{cut}$$

However, it is clear that there is no cut-free derivation of it. On the other hand, in $\mathsf{G3i}'+\mathrm{r}_1$ such a sequent has the following cut-free derivation:

$$\dfrac{\dfrac{t < t \Rightarrow t < t}{t < t \Rightarrow \bot}\,\mathrm{r}_1}{t < t \Rightarrow C}\,\mathrm{efq}$$

We won't provide a proof of the admissibility of the structural rules for the extensions of $\mathsf{G3i}'$ with quasi-universal rules, since this result immediately follows from the admissibility of the structural rules for the extensions of $\mathsf{G3i}'$ with quasi-universal rules and the rule raa.

5. Cut elimination

We shall now show that $\mathsf{G3i}'$ extended with both the rule raa and quasi-universal rules admits an entirely standard cut-elimination procedure. We first need some preparatory results. Let $\mathbf{R}^{\mathrm{q}}$ be a set of quasi-universal rules.

THEOREM 4. *In* $\mathsf{G3i}'$+raa+$\mathbf{R}^{\mathrm{q}}$ *the sequent* $A, \Gamma \Rightarrow A$ *is derivable.*

PROOF. By induction on A. If A is an atom P, the claim holds since $P, \Gamma \Rightarrow P$ is initial. If A is $B \wedge C$, we know that by the inductive hypothesis $B, C, \Gamma \Rightarrow B$ and $B, C, \Gamma \Rightarrow C$ are derivable. By applying the rules of conjunction we obtain $B \wedge C, \Gamma \Rightarrow B \wedge C$. If A is a disjunction or an implication or a quantified formula, the proof is entirely similar. □

Next, we shall prove the height-preserving admissibility of the rule of substitution. Let t be free for x in Γ, C. The rule of substitution is:

$$\frac{\Gamma \Rightarrow C}{\Gamma(t/x) \Rightarrow C(t/x)}$$

THEOREM 5. *In* $\mathsf{G3i}'$+raa+$\mathbf{R}^{\mathrm{q}}$ *the rule of substitution is height-preserving admissible.*

PROOF. By induction on the height n of the derivation of the premise. If $n = 0$, then $\Gamma \Rightarrow C$ is either an initial sequent or the conclusion of a quasi-universal rule with no premise and so is the conclusion of the substitution rule. If $n > 0$, then we only consider the case in which the premise of the substitution rule is concluded by efq or by raa. In the first case we apply the inductive hypothesis on the premise of efq $\Gamma \Rightarrow \bot$ so as to obtain $\Gamma(t/x) \Rightarrow \bot(t/x)$. Since $\bot(t/x)$ is just $\bot$, from $\Gamma(t/x) \Rightarrow \bot$ we can conclude $\Gamma(t/x) \Rightarrow C(t/x)$ by efq. Analogously, if $\Gamma \Rightarrow C$ is concluded by raa, then C is P and we apply the inductive hypothesis on the premise of raa so as to obtain $\neg P(t/x), \Gamma(t/x) \Rightarrow \bot(t/x)$. Once again, since $\bot(t/x)$ is $\bot$, from $\neg P(t/x), \Gamma(t/x) \Rightarrow \bot$ we can conclude $\Gamma(t/x) \Rightarrow P(t/x)$ by raa. □

THEOREM 6. *In* $\mathsf{G3i}'$+raa+$\mathbf{R}^{\mathrm{q}}$ *weakening is height-preserving admissible.*

PROOF. By induction on the height n of the derivation of the premise. If $n = 0$, then the premise of weakening $\Gamma \Rightarrow C$ is initial or the conclusion of a rule $r \in \mathbf{R}^{\mathrm{q}}$ with no premise. In the first case, $P \in \Gamma$ and C is P, for some atom P. Thus, the conclusion of weakening $A, \Gamma \Rightarrow C$ is initial. In the second case, C in the premise of weakening is an atom P and the conclusion of weakening is a conclusion of r. If $n > 0$, then we reason by cases on the last rule R applied in the derivation of the premise of weakening. In each case we apply the inductive hypothesis on the premise of R and then R, unless R is a quantifier rule with variable restriction; in this case we also need to apply the rule of substitution in order to make R applicable (such an application is legitimate by Theorem 5). □

With height-preserving admissibility of weakening we can show that some of the rules are height-preserving invertible, a result commonly known as "inversion lemma".

THEOREM 7. *In* $\mathsf{G3i}'$+raa+$\mathbf{R}^{\mathrm{q}}$ *the rules* $\mathrm{L}_\wedge$, $\mathrm{L}_\vee$, *and* $\mathrm{L}_\exists$ *are height-preserving invertible, whereas the rule* $\mathrm{L}_\supset$ *is height-preserving invertible with respect to the right premise.*

PROOF. We only consider the case of $\mathrm{L}_\supset$. We need to show that:

$$\frac{A \supset B, \Gamma \Rightarrow C}{B, \Gamma \Rightarrow C}\ \mathrm{iL}_\supset$$

is height-preserving admissible and we proceed by induction on the derivation-height n of the premise of $\mathrm{iL}_\supset$. If $n = 0$, then $A \supset B, \Gamma \Rightarrow C$ is a initial or the conclusion of a rule $r \in \mathbf{R}^{\mathrm{q}}$ with no premise. In the first case, $P \in \Gamma$ and C is Q, for some atom P, and the conclusion of $\mathrm{iL}_\supset$ is also initial. In the second case, C is an atom Q and the conclusion of $\mathrm{iL}_\supset$ is a conclusion of r. If $n > 0$, then we distinguish two cases according to whether $A \supset B$ is principal of the last rule R applied in the derivation of the premise of $\mathrm{iL}_\supset$ or it is not. If $A \supset B$ is the principal formula of the last rule R applied in the derivation, then R is $\mathrm{L}_\supset$ and its right premise is the conclusion of $\mathrm{iL}_\supset$ and $B, \Gamma \Rightarrow C$ is derivable by the inductive hypothesis. If $A \supset B$ is not the principal formula of R, then we reason by cases according to R. In each case we apply the inductive hypothesis on the premise of R and then R. The proof of the height-preserving admissibility of $\mathrm{L}_\wedge$ and $\mathrm{L}_\vee$ is entirely similar and hence left to the reader, whereas for $\mathrm{L}_\exists$ the proof does not substantially differ from the standard one (cf. Lemma 4.2.3 in [6]). □

We now consider the height-preserving admissibility of contraction.

THEOREM 8. *In* $\mathsf{G3i}'$+raa+$\mathbf{R}^{\mathrm{q}}$ *contraction is height-preserving admissible.*

Proof. By induction on n. If $n = 0$, then the premise of contraction $A, A, \Gamma \Rightarrow C$ is a initial or the conclusion of rule $r \in \mathbf{R}^{\mathrm{q}}$ with no premise. In the first case, either $P \in \Gamma$ and C is P or A and C are both P, for some atom P. In each case, the conclusion of contraction $A, \Gamma \Rightarrow C$ is initial. In the second case, C is an atom P and the conclusion of contraction is a conclusion of r. If $n > 0$, then either none of the occurrences of A is principal of the last rule R applied in the derivation or exactly one is. (The case of both occurrences of A being principal of R does not arise). In the first case, we proceed by cases on R. In each case, we apply the inductive hypothesis on the premise of R and then R. We only show the case of R being either efq or raa or else a rule $r \in \mathbf{R}^{\mathrm{q}}$ with at least one premise. If R is efq, then the premise of efq is $A, A, \Gamma \Rightarrow \bot$ and the conclusion of contraction $A, \Gamma \Rightarrow C$ follows by the inductive hypothesis and efq. If R is raa, then C in the premise of contraction is an atom P and the premise of raa is $\neg P, A, A, \Gamma \Rightarrow \bot$. On such a premise we apply the inductive hypothesis and raa again to conclude $A, \Gamma \Rightarrow P$ as desired. Finally, if R is a rule $r \in \mathbf{R}^{\mathrm{q}}$ with at least one premise, then C in the premise of contraction is an atom Q and the premises of r are $A, A, \Gamma \Rightarrow P_1$ and ... and $A, A, \Gamma \Rightarrow P_n$, for $n \geqslant 1$. On each premise of r we apply the inductive hypothesis and then r again to conclude $A, \Gamma \Rightarrow Q$. In the second case, i.e. if exactly one occurrence of A is principal of R, we proceed by cases on R. If R is $\mathrm{L}_\wedge$ or $\mathrm{L}_\vee$ or $\mathrm{L}_\supset$ or else $\mathrm{L}_\exists$, we apply the inversion lemma (Theorem 7) on the premise of R, the inductive hypothesis and, finally R again. If R is $\mathrm{L}_\supset$ we apply directly the inductive hypothesis of its premise and then $\mathrm{L}_\supset$ again. □

Finally, to prove the admissibility of cut we proceed by induction on the weight of the cut formula A with sub-induction on the cut-height. The proof follows the pattern of the proof of Theorem 2.4.3 from [6].

Theorem 9. *In* $\mathsf{G3i}'$+raa+$\mathbf{R}^{\mathrm{q}}$ *cut is admissible.*

Proof. We consider the following cases: (i) the left premise of cut is initial or the conclusion of a rule $r \in \mathbf{R}^{\mathrm{q}}$; (ii) the right premise of cut is initial or the conclusion of a rule $r \in \mathbf{R}^{\mathrm{q}}$; (iii) the cut formula A is not principal in the left premise of cut; (iv) A is principal only in the left premise; (v) A is principal in both premises.

We start from (i). If $\Gamma \Rightarrow A$ is initial or the conclusion of a rule $r \in \mathbf{R}^{\mathrm{q}}$, then we consider the following cases.

(i.1) $P \in \Gamma$ and A is P, for some atom P. Then, the conclusion of cut is concluded from the right premise by weakening.

(i.2) A is an atom Q. Thus, the original cut is

$$\dfrac{\dfrac{}{\Gamma \Rightarrow Q}\,\mathrm{r} \qquad Q, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\,\mathrm{cut}$$

We need to consider the right premise of cut. If it is initial or the conclusion of a rule $r' \in \mathbf{R}^{\mathrm{q}}$, then we have the following cases:

(i.2.a) $P \in \Delta$ and C is P, for some atom P. Then, the conclusion of cut is initial.

(i.2.b) C is Q. Then, the conclusion of cut is concluded from the left premise by weakening.

(i.2.c) C is an atom Q'. Then the conclusion of cut is the conclusion of r'.

If the right premise of cut is not initial nor the conclusion of a rule $r' \in \mathbf{R}^{\mathrm{q}}$, then we proceed by cases according to the last rule R applied in its derivation. Notice that the cut formula Q can never be principal in R. If R is a propositional rule or quantifier rule without variable restriction, apply the inductive hypothesis on the left premise of cut and the premise of R, and then R again. For example, if R is raa, then C is an atom P and the original cut is:

$$\dfrac{\dfrac{}{\Gamma \Rightarrow Q}\,\mathrm{r} \qquad \dfrac{\neg P, Q, \Delta \Rightarrow \bot}{Q, \Delta \Rightarrow P}\,\mathrm{raa}}{\Gamma, \Delta \Rightarrow P}\,\mathrm{cut}$$

The conclusion of cut is concluded as follows:

$$\dfrac{\dfrac{\dfrac{}{\Gamma \Rightarrow Q}\,\mathrm{r} \qquad \neg P, Q, \Delta \Rightarrow \bot}{\neg P, \Gamma, \Delta \Rightarrow \bot}\,\mathrm{IH}}{\Gamma, \Delta \Rightarrow P}\,\mathrm{raa}$$

If R is a quantifier rule with variable restriction, we proceed similarly but we additionally need to apply the rule of substitution, which is admissible by Theorem 5, in order to make R applicable. This completes the proof of case (i).

Concerning case (ii), if $A, \Delta \Rightarrow C$ is initial or the conclusion of a rule $r \in \mathbf{R}^{\mathrm{q}}$, then we have the following cases.

(ii.1) $P \in \varDelta$ and C is P, for some atom P. Then, the conclusion of cut is an axiom.
(ii.2) A and C are P. Then, the conclusion of cut is concluded from the left premise by weakening.
(ii.4) C is an atom Q. Then, the conclusion of cut is conclude by r.

We now consider the case (iii). If the cut formula A is not principal in the left premise of cut, then the last rule R applied in its derivation is either $\mathrm{L}_\wedge$ or $\mathrm{L}_\vee$ or $\mathrm{L}_\supset$ or $\mathrm{L}_\forall$ or $\mathrm{L}_\exists$. The conclusion of cut is found by applying the inductive hypothesis on the premise(s) of R and the right premise of cut, and then R again (unless R is $\mathrm{L}_\exists$ because in this case we also need an application of the substitution rule admissible by Theorem 5).

As for case (iv), if the cut formula A is principal only in the left premise of cut, then the right premise is concluded by a rule R where A is not principal. We proceed by cases on R. In each case the conclusion of cut is found by applying the inductive hypothesis on the left premise of cut and the premise(s) of R, and then by R again. We consider only the cases of R being either efq or raa or else a rule $r \in \mathbf{R}^{\mathrm{q}}$ with at least one premise. If R is efq, then the original cut is:

$$\dfrac{\varGamma \Rightarrow A \qquad \dfrac{A, \varDelta \Rightarrow \bot}{A, \varDelta \Rightarrow C}\,\text{efq}}{\varGamma, \varDelta \Rightarrow C}\,\text{cut}$$

The conclusion of cut is found as follows:

$$\dfrac{\dfrac{\varGamma \Rightarrow A \qquad A, \varDelta \Rightarrow \bot}{\varGamma, \varDelta \Rightarrow \bot}\,\text{IH}}{\varGamma, \varDelta \Rightarrow C}\,\text{efq}$$

If R is raa, then C is an atom P and the original cut is:

$$\dfrac{\varGamma \Rightarrow A \qquad \dfrac{\neg P, A, \varDelta \Rightarrow \bot}{A, \varDelta \Rightarrow P}\,\text{raa}}{\varGamma, \varDelta \Rightarrow P}\,\text{cut}$$

The conclusion of cut is found as follows:

$$\dfrac{\dfrac{\varGamma \Rightarrow A \qquad \neg P, A, \varDelta \Rightarrow \bot}{\neg P, \varGamma, \varDelta \Rightarrow \bot}\,\text{IH}}{\varGamma, \varDelta \Rightarrow P}\,\text{raa}$$

If R is rule $r \in \mathbf{R}^{\mathrm{q}}$ with at least one premise, then C is an atom Q and the original cut is ($n \geqslant 1$):

$$\dfrac{\Gamma \Rightarrow A \qquad \dfrac{A, \Delta \Rightarrow P_1 \quad \cdots \quad A, \Delta \Rightarrow P_n}{A, \Delta \Rightarrow Q}\,\mathrm{r}}{\Gamma, \Delta \Rightarrow P}\,\mathrm{cut}$$

The conclusion of cut is found as follows:

$$\dfrac{\dfrac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow P_1}{\Gamma, \Delta \Rightarrow P_1}\,\mathrm{IH} \qquad \cdots \qquad \dfrac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow P_n}{\Gamma, \Delta \Rightarrow P_n}\,\mathrm{IH}}{\Gamma, \Delta \Rightarrow Q}\,\mathrm{r}$$

Finally, as for case (v), we proceed by induction on the weight $w(A)$ of the cut formula. Notice that the case in which $w(A) = 0$ does not arise because A can neither be $\perp$ or P since these formulas cannot be principal of any rule deriving the right premise of cut. If $w(A) > 0$, then A is a compound formula and we distinguish two cases according to whether the last rule R applied in the derivation of the left premise of cut is efq or it is not. If R is efq, then the original cut is:

$$\dfrac{\dfrac{\Gamma \Rightarrow \perp}{\Gamma \Rightarrow A}\,\mathrm{efq} \qquad A, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\,\mathrm{cut}$$

The conclusion of cut is concluded by efq and weakening as follows:

$$\dfrac{\dfrac{\Gamma \Rightarrow \perp}{\Gamma \Rightarrow C}\,\mathrm{efq}}{\Gamma, \Delta \Rightarrow C}\,\mathrm{wk}$$

If R is not efq, then the proof is the same as in Theorem 2.4.3 of [6]. □

6. Towards a multi-succedent calculus

The most obvious generalization of the present approach is to consider multi-succedent calculi. After all, a multi-succedent calculus for intuitionistic logic G3im has been introduced already in [1] and its extension with universal rules has been extensively investigated ever since [7].

A multi-succedent version $\mathsf{G3im}'$ of the single-succedent calculus $\mathsf{G3i}'$ can be obtained from $\mathsf{G3im}$ by replacing the rule $\mathrm{L}_\perp$ with the following multi-succedent rule of *ex falso quodlibet* from [2]:

$$\frac{\Gamma \Rightarrow \Delta, \perp}{\Gamma \Rightarrow \Delta}\ \text{efq}$$

In the multi-succedent calculus it is not possible to formulate the rule of classical *reductio* for atoms with an arbitrary succedent. Indeed, if we extend $\mathsf{G3im}'$ with the following, fully multi-succedent, rule:

$$\frac{\neg P, \Gamma \Rightarrow \Delta, \perp}{\Gamma \Rightarrow \Delta, P}\ *$$

it is clear that we would end up with a calculus for classical logic. Instead of giving a proof of the equivalence with standard calculi for classic logic such as $\mathsf{G3c}$ we simply notice that in the presence of the rule of classical *reductio* for atoms the sequent $\Rightarrow P \vee \neg P$ would be derivable as the following derivation shows:

$$\frac{\dfrac{\neg P \Rightarrow \neg P, \perp}{\Rightarrow P, \neg P}\ *}{\Rightarrow P \vee \neg P}\ \mathrm{R}_\vee$$

Thus, it is necessary to impose a single-succedent restriction of the premise of the rule and consider the following rule of classical *reductio* for atoms with no Δ in the premise:

$$\frac{\neg P, \Gamma \Rightarrow \perp}{\Gamma \Rightarrow \Delta, P}\ \text{raa}$$

On the other hand, the extensions of $\mathsf{G3im}'$ with multi-succedent non-logical rules is more problematic. Firstly, notice that a multi-succedent sequent calculus is *a priori* open to the possibility of considering fully universal rules of the form:

$$\frac{\Gamma \Rightarrow \Delta, Q_1, \ldots Q_m, P_n \quad \cdots \Gamma \Rightarrow \Delta, Q_1, \ldots Q_m, P_n}{\Gamma \Rightarrow \Delta, Q_1, \ldots Q_m}\ \text{r}$$

This would be a significant generalization of the single-succedent approach developed so far which only work for quasi-universal rules. However, it appears in the multi-succedent calculus with universal rules we cannot apply the standard cut-elimination procedure. To see this recall that the cut rule in the multi-succedent calculus is:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\text{ cut}$$

Now, consider a cut where the left premise is concluded by a universal rule r with no premise in which the cut formula A is principal and the right premise is concluded by raa. So, Δ contains the atoms $Q_1, \dots, Q_{m-1}$ and the cut formula A is Q_m, whereas Σ contains P.

$$\dfrac{\dfrac{}{\Gamma \Rightarrow \Delta', Q_1, \dots Q_{m-1}, Q_m}\text{ r} \quad \dfrac{\neg P, Q_m, \Pi \Rightarrow \bot}{Q_m, \Pi \Rightarrow \Sigma', P}\text{ raa}}{\Gamma, \Pi \Rightarrow \Delta', Q_1, \dots Q_{m-1}, \Sigma', P}\text{ cut}$$

In this case it is not clear how to eliminate such a cut. For if a cut is applied in the left premise and the premise of raa, so as to obtain the sequent $\Gamma, \neg P, \Pi \Rightarrow \Delta', Q_1, \dots Q_{m-1}, \Sigma', \bot$ then the presence of the context $\Delta, Q_1, \dots Q_{m-1}$ in the succedent would make raa inapplicable. This strongly suggests that even in the multi-succedent calculus we should consider only single-succedent quasi-universal rules. This is a serious obstacle towards a multi-succedent calculus for stable theories.

7. Conclusions

In this paper we have shown how to extend the standard cut-elimination procedure from intuitionistic stable logic to a class of intuitionistic stable theories. Building on previous works by Negri and von Plato, we aptly modified the underlying single-succedent calculus for intuitionistic logic so as to preserve the admissibility of all the structural rules, including cut, in the presence of a restricted version of the rule of classical *reductio ad absurdum* and of a special case of universal rules.

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Paolo Maffezioli
Departamento de Lógica y Filosofía Teórica
Universidad Complutense de Madrid
pmaffezi@ucm.es
https://orcid.org/0000-0001-9832-924X