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Exploring Essential Occurrence

Abstract. An expression occurs *essentially* in a formula (or sentence) when it occurs in every formula equivalent to the given formula, taking equivalence as logical equivalence relative to the logic in play in the discussion. Setting aside various niceties, this amounts to provable equivalence if that logic is presented via some proof system, and to valid equivalence if the salient characterization is couched in semantic terms. This notion of essential occurrence, or an informal analog thereof, has found its way into numerous philosophical discussions over the past seventy or more years, and here we tease out some issues of specifically logical interest it presents, stretching that description somewhat so as to subsume under it the frequently mooted connection between the essential occurrence of a singular term in a sentence and that sentence's being *genuinely about* what the term denotes. This connection, stressed originally by Nelson Goodman, is touched on in several sections in the main body of the paper, but especially in §4, where it is contrasted with an alternative suggestion due to R. Demolombe and L. Fariñas del Cerro. Some issues raised by this and other parts of the discussion are also treated in several longer notes (referred to by means of letters A, B, . . . , K) which are postponed to an Appendix (§5) of roughly the same length as the main body of the paper. This enables readers with a special interest in one or more topics to consult them selectively, while allowing those with no such interest to avoid involvement with the further details supplied in the associated longer note(s).

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1. Introduction

An expression e will be taken, for present purposes, to occur essentially in a sentence, relative to a logic, if e occurs in every sentence equivalent, according to that logic, to the given sentence (or formula: this latter

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term will be used from now on when a formal language is at issue); a more precise definition will be given shortly (Definition 2). For simplicity, we consider mainly logics in which provable equivalence suffices for the replacement property — equivalent formulas freely interreplaceable in longer formulas without affecting provability — so instead of saying "equivalent" one might equally well say "synonymous" (to use the terminology of Smiley [160, p. 116]). More detail on the logics of current concern is given at under the heading 'Default Background Conditions' below. For the most part, these are various propositional logics, and the expressions e of interest will mostly be the propositional variables, or sentence letters — these terms taken to be interchangeable — of the languages of these logics, with the same countable stock of such variables, any three of which we use we use p, q, r to stand for; when longer lists are needed, $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ are used, again with a presumption of distinctness (i.e., if $i \neq j$, then $p_i \neq p_j$ and $q_i \neq q_j$). As metalinguistic variables over arbitrary formulas ('schematic letters'), we mostly use A, B, C, ... (with no such distinctness presumption in this case). For some applications (esp. Section 4 and Appendix F) it is individual constants in first-order languages serving as the expressions whose essential occurrence is at issue; mention is occasionally made of predicate letters in this connection. Note that in each case e is an expression without internal structure.

Since each formula is equivalent to itself, whenever e occurs essentially in a formula, it does indeed occur in that formula.[1] If e occurs

[1] In this respect "essentially" behaves here as it does in metaphysical parlance, an object's having the property P essentially implying that it does indeed have the property P, despite this logical use of the adverb's close relationship with talk of the inessential 'variables', 'arguments', or 'inputs' — this last being the term used in Shtrakov and Denecke [158] to avoid obvious distractions — of a function (mapping, operation,...). These inputs on which, for any way of fixing the other variables, the value of the function does not depend. This leads to talk of an n-ary function's being essentially m-ary (where $m < n$), to mean that $n - m$ of the inputs are inessential. Since on this usage, a ternary operation can be essentially binary, being *essentially* m-ary does not entail being m-ary. Yet another terminology for the inputs is evidenced in this sentence from Berman and McKenzie [6, p. 8, italics added] defining what it is for f to be essentially m-ary: "there are exactly m distinct *coordinates* on which f depends." For all its use–mention infirmities, the term *variables* remains the most commonly used for the 'inputs'. (An n-ary connective #'s being essentially m-ary, where $m \leq n$, can be defined in terms of the corresponding function/operation in the associated Lindenbaum algebra, or more directly when exactly m variables occur essentially in the formula $\#(p_1, \ldots, p_n)$, though this rough and ready gloss needs

in a formula but does not occur essentially in it, we say that e occurs inessentially in the formula, and for an occasionally more convenient formulation, e's not occurring essentially in a formula e will be described as e's occurring *at most* inessentially in the formula — i.e., it occurs either not at all or else only inessentially in the formula.

These and related concepts figure in the articulation of numerous philosophical notions and theses in the work of Goodman, Quine, Nakhnikian, and others also. Quine used the term *vacuous* in place of *inessential*. The terminology has been used with an additional relativization to inferences, as in Pigden [133],[2] discussing the issue ('Hume's Law') of the deducibility of moral conclusions from non-moral premises; the second ellipsis represents material between the pages cites; the first is for bibliographical references to Quine — concerning which, see Appendix B below; the definition of essential occurrence from Quine recalled here is not quite the same as that of our opening sentence above and articulated more carefully in Def. 2 below, but the equivalence of which to that definition will be made explicit in Section 3; these passages are from [133, pp. 134, 136], respectively:

> According to Quine (...), an expression appears vacuously in a sentence if it can be uniformly replaced without prejudice to the sentence's truth-value. [...] Having reminded ourselves of the concepts of consequence and vacuous occurrence, we can now combine the two. I define *inference-relative vacuity* as follows:
>
> An expression (predicate, propositional variable) φ occurs vacuously in the conclusion of a valid inference $K \Vdash X$, iff under any interpretation of $K \cup \{X\}$ such that both K and X come out true, we can uniformly substitute for φ any expression ψ of the same grammatical type, without prejudice to the truth of the resulting X'.

The current reminder of material in this genre, rounded out by a further quotation from Schurz [155] in Appendix C, has as its point only the comment that such relativizations of talk of essential occurrence to the expressions appearing *qua* premise- or conclusion-parts of particular inferences will not be under consideration here — just the simple notion

to be relativized to the logic L under consideration, and even, unless the default background conditions given below are satisfied, to exactly which L-applicable notion of equivalence is in play. Further aspects of essentiality, related to our main theme, are discussed in Appendix A.)

[2] See also [135], for a more recent statement of Pigden's approach to these matters.

introduced in our opening paragraph.[3] What is being set to one side here is not the concentration on inferences, as Pigden's use of the term "inference-relative" may suggest[4] — since we could just as easily (and in a precisely parallel way) define an expression's occurrence essentially in a *sequent*, but the fact that it is specifically the *conclusion* of the inference (the r.h.s. or 'succedent' of the sequent) in question that is singled out in the second paragraph quoted from Pigden above.

One particular aspect of the topic of essential occurrence which prompted the present discussion of the area concerns essential-occurrence preserving contexts, or *e.o.-preserving* contexts for short. By this is meant 1-ary contexts $C(\cdot)$ with the feature that any expression occurring essentially in A occurs essentially in $C(A)$.[5] The extreme example of a context lacking this property is a *constant* context, i.e., a $C(\cdot)$ for which, according to the logic of interest, $C(A)$ and $C(B)$ are equivalent for all formulas A, B. For example, if that logic is classical propositional logic and $C(A)$ is taken as $A \wedge \bot$ or instead as $A \vee \top$ then $C(\cdot)$ is a constant context and no expression in A occurs essentially in $C(A)$. Here, as already mentioned, we address mainly sentential logic(s) and for present purposes the paradigmatic example of an expression concerning which the question of its occurring essentially or otherwise in a formula is a propositional variable (or sentence letter).[6]

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[3] There is indeed a further relativization involved, made explicit there, but that is to the *logic* in play in the discussion, rather than the role of the expression concerned as a premise or a conclusion component of this or that inference. (Notice also, apropos of Appendix C, that Schurz writes of the inessentiality of the — occurrence of — the whole formula Op, where O is the deontic Obligation/*Ought* operator, rather than reserving such terminology for the occurrence of unstructured expressions, as here.)

[4] And for which the term *argument* appears in the passage quoted from Pap [125] in Appendix B

[5] A more explicit notation is suggested in Williamson [177]: such a context is an ordered pair comprising a formula and a sentence letter $\langle C, q\rangle$ and putting the formula A into this context is a matter of substituting A for all occurrences of q in C; obviously a more general version to cover other than 1-ary contexts would replace q with a sequence of sentence letters). For present purposes most 1-ary contexts of interest will be even simpler, in that q will be the only variable in C.

[6] For much of the discussion of essential occurrence in the literature, a good deal of it stimulated by Quine, there is no restriction to sentential logic or to the expressions whose essential occurrence is at issue being drawn from the non-logical vocabulary, as here. In fact Quine's discussion concerns a rather differently formulated notion of essential occurrence, as is explained Appendix B and Section 3 below.

Except where otherwise indicated we think of logics primarily as consequence relations satisfying certain conditions, but since most of the literature on the present topic was produced in a period when logics were thought of primarily as sets of formulas, for the sake of continuity, most formulations here make use of characterizations in the latter style. To simplify things, the "certain conditions" alluded to and assumed to be satisfied in the absence of an explicit indication to the contrary, are the following

Default Background Conditions:

> $\vdash_{\mathsf{L}}$ is a finitary congruential substitution-invariant consequence relation on a language whose connectives include two written $\to$ and $\leftrightarrow$ satisfying the conditions that for all formulas A, B and sets Γ of formulas:
>
> $$\Gamma, A \vdash_{\mathsf{L}} B \text{ if and only if } \Gamma \vdash_{\mathsf{L}} A \to B;$$
> $$A \to B, B \to A \vdash_{\mathsf{L}} A \leftrightarrow B \quad A \leftrightarrow B \vdash_{\mathsf{L}} A \to B \quad \text{and}$$
> $$A \leftrightarrow B \vdash_{\mathsf{L}} B \to A.$$

Because of these inset conditions, the condition of congruentiality alluded to above, and taken to be part these Default Conditions, can be spelled out equivalently either as the condition that for all formulas $A, B, C(q)$ in the language of L: $A \dashv\vdash_{\mathsf{L}} B$ implies $C(A) \dashv\vdash_{\mathsf{L}} C(B)$, or as the condition that for all such formulas, $\vdash_{\mathsf{L}} A \leftrightarrow B$ implies $\vdash_{\mathsf{L}} C(A) \leftrightarrow C(B)$.[7] This is the requirement alluded to in the opening paragraph that (provably) equivalent formulas should be synonymous, or more explicitly, that L-equivalent formulas should be L-synonymous. The "provably" is optional, since, although we read $\vdash_{\mathsf{L}} A$ (alias $\varnothing \vdash_{\mathsf{L}} A$) as saying that A is provable in L, it makes no difference whether we think of the logic as presented via a proof-system of some kind or, instead, in terms of some semantically specified notion of validity.

If there are several distinct notions of equivalence in play, for example if for the L under consideration, the four relations our Default Background Conditions require to coincide do not in fact coincide — relations holding between A and B according as *(i)* $\vdash_{\mathsf{L}} A \leftrightarrow B$, (ii) $A \dashv\vdash_{\mathsf{L}} B$, (iii) $\vdash_{\mathsf{L}} C(A) \leftrightarrow C(B)$ for all $C(\cdot)$, or *(iv)* $C(A) \dashv\vdash_{\mathsf{L}} C(B)$ for all $C(\cdot)$, then the opening sentence of the present section gives rise, for the L

[7] This last version of congruentiality is satisfied by the formula-logic AL of Example 3(ii) below.

in question, to potentially distinct notions of what it is for e to occur essentially in a sentence (or formula). We illustrate with the case in which e is a sentence operator (a 1-ary connective):

Example 1. Suppose we expand the language of monomodal propositional modal logic with an additional 1-ary connective, here written "$\mathcal{A}$" (heuristic reading: "actually"), and also expand the models — for simplicity, omitting any accessibility relation for $\Box$ and taking this to be the universal relation — with a distinguished point (heuristically: representing the *actual* world), the truth of the formula B at which is necessary and sufficient for the truth of $\mathcal{A}B$ at any point in the model. Then we face a choice: deem validity (on the frame of the model) to consist in truth at each point (in all models on that frame), *general validity*, or instead, deem it to consist in truth at the distinguished point in (in all models on the frame): *real-world validity.* On the former choice, the formulas valid on every frame constitute a normal bimodal logic, but not so on the latter option. Relative to the logic implementing that approach, $\mathcal{A}$ occurs inessentially in $\mathcal{A}p$, in that this formula is equivalent to the $\mathcal{A}$-free p. These formulas are not, however synonymous, yielding non-equivalent formulas when spliced into the context $\Box(\cdot)$. Indeed, no $\mathcal{A}$-free formula is *synonymous* with $\mathcal{A}p$, though, as Hazen [56] observes, every formula has an *equivalent* $\mathcal{A}$-free formula — which as he also notes, is not a result that extends to any plausible quantificational extension of the logic. (Warning: p, q, ... in [56] are used as schematic letters for formulas, rather than, as here, restricted to sentence letters/propositional variables. There are several further notational differences as well.) For further information on this topic, to which we return in the discussion after Corollary 16 below, see Humberstone [77] (note 177 on p. 219, and Remark 4.7.1 on p. 305) and French [43]. $\lhd$

Other matters arising in view of the potential plurality of candidate notions of equivalence bearing on the informal notion of essential occurrence, illustrated here by the contrast between provable equivalence and synonymy, are taken up in Appendix A. We proceed next with some simple illustrations (Examples 3) of the main concept introduced above, in connection with two logics, the second of which involves the suspension of the Default Background Conditions adumbrated above. But for clarity, we first give a precise definition of that concept, making its logic-relativity explicit by hyphenating a reference to the logic concerned to the word *essential(ly)*:

DEFINITION 2. For a logic L, a sentence letter q occurs L-*essentially* in a formula A of the language of L if and only if for all formulas B for which $\vdash_{\mathsf{L}} A \leftrightarrow B$, q occurs in B.

Note that if $\vdash_{\mathsf{L}} A$ for L satisfying the Default Background conditions, no sentence letter occurs L-essentially in A, for any q occurring in A and any q-free B for which $\vdash_{\mathsf{L}} B$ (alias $\varnothing \vdash_{\mathsf{L}} B$) we have $A \vdash_{\mathsf{L}} B$ by the 'weakening' or 'monotonicity' condition on consequence relations, and so $\vdash_{\mathsf{L}} A \to B$, so by the 'only if' direction of the first-listed of the displayed Default Background Conditions. Similarly, since also $\vdash_{\mathsf{L}} A$, $\vdash_{\mathsf{L}} B \to A$, so $\vdash_{\mathsf{L}} A \leftrightarrow B$ (and indeed for all $C(\cdot)$: $\vdash_{\mathsf{L}} C(A) \leftrightarrow C(B)$).

The following examples mention three logics, IL, CL and AL — respectively intuitionistic (propositional) logic, classical logic, and the Abelian logic of Meyer and Slaney [108, 109]. More specifically, they involve implicational formulas of the — we may assume — common language of these logics, and even more specifically, "left-stacking" such formulas:

Examples 3. (i) The variable q occurs IL-essentially in the formula $(p \to q) \to p$, and CL-inessentially in this same formula. For the first claim, the methods used in the Examples of the following section show that the existence of a q-free IL-equivalent $A_1 = (p \to q) \to p$ would imply that $A_2 = (p \to \top) \to p$ was also equivalent to A_1. But that is not the case, since the implication A_1 to p is the famously IL-unprovable Peirce's Law, whereas $(A_2 \to p) \to p$ is IL-provable. For the second claim, we just appeal to the fact that, by contrast, Peirce's Law is CL-provable, in addition to its (this time IL-provable) converse.

(ii) The variable q occurs CL-essentially in the formula $(p \to q) \to q$, and AL-inessentially in this same formula. For the first claim, recall that the formula in question is CL-equivalent to $p \vee q$, in which q occurs essentially. (If this is not obvious, the methods of the following section's examples will again supply a proof.) For the second claim, note that $(p \to q) \to q$ is AL-equivalent to the conspicuously q-free formula p. $\lhd$

As already remarked, Example 3(ii) involves a suspension of the Default Background Conditions, in the absence of which we do not have such conditionals as: "if $\vdash_{\mathsf{L}} B$ then $\vdash_{\mathsf{L}} A \to B$" as were mentioned immediately after Definition 2, which are not correct for $\mathsf{L} = \mathsf{AL}$. (See note 7, assuring us that we do not need to worry about AL-equivalence and AL-synonymy coming apart.) Moreover, there is no uniquely natural consequence relation having as the consequences of $\varnothing$ exactly the theorems of the implicational fragment of AL, as is shown in Butchart and Rogerson

[14], and those that do satisfy this '$Cn(\varnothing)$' condition — Butchart and Rogerson describe two such relations — cannot, for the reason already given, satisfy the Default Background Conditions above.

Examples 3 raise a question to which it would be interesting to know the answer. For a logic L in a language with $\to$ among its connectives, let $\mathsf{L}_\to$ be the implicational fragment of L. Now suppose A is a formula in which $\to$ is the only connective appearing. If q occurs $\mathsf{L}_\to$-essentially in A, does it follow that q occurs L-essentially in A? Of course, this question could also be asked about fragments, more generally. Note that $\mathsf{L}_\to$, as a logic, for (almost) any choice of L violates the letter of the Default Background Conditions, in that, with $\leftrightarrow$ absent, we need to reformulate those conditions involving that connective, and also replace the "$\vdash_\mathsf{L} A \leftrightarrow B$" part of Definition 2 with "$\vdash_\mathsf{L} A \to B$ and $\vdash_\mathsf{L} B \to A$" — which we continue to describe by saying that A and B are L-equivalent. These purely implicational logics provide a natural pond in which to fish for cases of what Humberstone and Meyer [82] call the *relevant equivalence property*. This is the property possessed by those L for which any two L-equivalent formulas are constructed from exactly the same propositional variables. Putting the matter in terms of our present theme: a logic L with the relevant equivalence property is one for which every variable that occurs in a formula occurs L-essentially in that formula. Some well-known implicational logics are shown to enjoy this property in [82], among which the best-known is perhaps $\mathsf{R}_\to$, the implicational fragment — alias BCIW[8] — of the relevant/relevance logic R. The latter enjoys only the 'Belnap relevance property': if A provably implies B in L, then some propositional variable occurs both in A and in B — but not the relevant equivalence property, as the discussion immediately following Example 6 below explains, with the aid of an example involving both $\wedge$ and $\vee$ (see also Jago [85, p. 88*f.*]).

To conclude this section, let us emphasize that our main concern here is, as the title indicates, essential *occurrence* and not with essential *occurrences*. That is, the title features the abstract noun referring to the phenomenon of an expression-type occurring one or more times in a sentence, rather the plural form of the count noun which invites us to distinguish one such occurrence from another with possible differences in 'essentiality' between such different expression-tokens. The latter is

[8] For the sources of this nomenclature, see Humberstone [74, pp. 164–170], where the label "*BCIW* logic" is used.

not an unnatural adaptation of the terminology — just not something of current concern; for instance, to illustrate with a case from the Peircean neighbourhood of Example 3(i), one might want to say of the underlined occurrences of q in

$$(((p \to q\,\blacksquare \to p) \to p) \to \underline{\underline{q}}) \to \underline{\underline{q}},$$

that they are IL-essential though not CL-essential, to convey that this formula is CL-provable and remains provable however the underlined occurrences are replaced, provided that they are replaced by the same formula, while the in the IL case, in which the formula is already provable ("Wajsberg's Law") all three occurrences would have to be subjected to the same replacement (which would then just be a case of uniform substitution). Evidently this way of speaking would require quite some care over the definition of the terminology employed, and not just because, we are talking about provable formulas in logics meeting the Default Background Conditions, the remark after Definition 2 notwithstanding. Without attempting such a refinement of the idea of essential occurrence to cover particular occurrences of a given variable, the topic of numbers of occurrences of a given variable itself will come up briefly below in Section 3, in the paragraph (containing the sentence) to which note 40 is appended, as well as in the paragraph after Corollary 27 in Appendix A, with its reference to Humberstone [80]. Compare also Schurz [154, pp. 82, 84] for the use of underlining to single out particular occurrences of sentence and predicate letters in a formula, and terminology sensitive to whether all or only some such occurrences are "relevant" (approximately, 'essential' — see Appendix C), and see also Shtrakov [157, Def. 2.11 and subsequent discussion], for an occurrence- or position-relativized treatment of essentiality in equational logic.

2. Contexts Destroying Essential Occurrence

Here we take up the matter of 'essential occurrence'-preservation alluded to in the preceding section.

Definition 4. A context $C(\cdot)$ of the language of a logic L is *e.o.-preserving for* L just in case for all formulas A, and all sentence letters q, if q occurs L-essentially in A, then q occurs L-essentially in $C(A)$.

Strictly, it would be more accurate to say that it is *embedding a formula* in a certain context that is or isn't e.o.-preserving, but we stick with the more concise form introduced in Definition 4, attributing this property to the context itself. In our section title, *destroying* just means *not preserving*.[9]

In part (iv) of the following unpacking of aspects of Definition 2, CL stands, as in Examples 3 for classical non-modal propositional logic, though for definiteness let us now add: in some functionally complete set of primitive connectives.

PROPOSITION 5. (i) *If q occurs L-essentially in A and $\vdash_{\mathsf{L}} A \leftrightarrow B$, then q occurs L-essentially in B.*

(ii) *If q occurs L-essentially in $\#A$ for a 1-ary (primitive or derived) connective $\#$ of the language of L, then q occurs essentially in A; more generally, if q occurs L-essentially in $\#(A_1, \ldots, A_n)$ for n-ary $\#$, then q occurs essentially in at least one A_i $(1 \leq i \leq n)$.*

(iii) *If $\mathsf{L}_0 \subseteq \mathsf{L}_1$, then for any formula A in the language of L_0, if q occurs L_1-essentially in A, then q occurs L_0-essentially in A.*

(iv) *If q occurs L-essentially in A, for any $\mathsf{L} \supseteq \mathsf{CL}$, then q occurs L_0-essentially in $\neg A$.*

PROOF. (i)–(iii) are more or less immediate from Definition 2. For (iv), saying that over a classical background (i.e., for extensions of CL), $\neg$ is e.o.-preserving: by (i), if q occurs essentially in A then, q occurs essentially in $\neg\neg A$, and so, by (ii), taking $\#$ as $\neg$, in $\neg A$. $\dashv$

By contrast with Prop. 5(iv), we shall in due course (Example 10), however, that there are logics weaker than CL of considerable importance, in which negation — or at least something happily going by that name — does not preserve essential occurrence in this way.

As an initial step in that direction, we notice that the context $C(A) = \Box A$ of a normal modal logic such as $\mathsf{S4}$ is not e.o.-preserving (for occurrences of sentence letters); the reasoning assumes a familiarity with $\mathsf{S4}$:

Example 6. Consider the formula $A = \Box(p \vee q) \wedge p$. The variable q (1): occurs $\mathsf{S4}$-essentially in A, but (2): does not occur $\mathsf{S4}$-essentially in $\Box A$. To establish (1), suppose otherwise: there is a q-free formula B with

$$\vdash_{\mathsf{S4}} (\Box(p \vee q) \wedge p) \leftrightarrow B \qquad (*)$$

[9] So, because of the negation involved, an e.o.-destructive context is a context, embedding within which leads to *some* variable no longer occurring essentially — as opposed to a constant context, for which this “some” would be “every”.

Since q does not occur in B, uniform substitution of p for q in this equivalence affects only its l.h.s., yielding $\vdash_{\mathsf{S4}} (\Box(p \vee p) \wedge p) \leftrightarrow B$, and hence $\vdash_{\mathsf{S4}} (\Box p \wedge p) \leftrightarrow B$, and so $\vdash_{\mathsf{S4}} \Box p \leftrightarrow B$. Combining this with $(*)$ itself, we should have

$$\vdash_{\mathsf{S4}} (\Box(p \vee q) \wedge p) \leftrightarrow \Box p \qquad (**)$$

But the $\rightarrow$ direction of $(**)$ is easily seen to fail since the l.h.s. is evidently an S4-consequence of [= is S4-provably implied by] $\Box q \wedge p$, while the r.h.s. is not.[10]

Turning our attention to (2), to see that q does not occur essentially in $\Box A = \Box\big((\Box(p \vee q) \wedge p)\big)$, we distribute the $\Box$ across the conjunction, getting $\Box\Box(p \vee q) \wedge \Box p$, and simplify the left-hand conjunct: $\Box(p \vee q) \wedge \Box p$ and now "undistribute" the $\Box$: $\Box((p \vee q) \wedge p)$. In the scope of $\Box$ is now a formula amenable to simplification ("absorption laws") to plain p, so $\vdash_{\mathsf{S4}} \Box A \leftrightarrow \Box p$, revealing q to occur only S4-inessentially in $\Box A$. (A dual form of this reasoning shows that $\Diamond$ is not e.o.-preserving in S4, with q occurring S4-essentially in $B = \Diamond(p \wedge q) \vee q$ but not in $\Diamond B$.) $\lhd$

The key role of the absorption laws stating the equivalence of A with $A \wedge (B \vee A)$ as well as with $A \vee (A \wedge B)$ in "dummying in," as he puts it, a new (foreign to A) variable as the B here, was especially well emphasized in the first ten lines of p. 353 in Dunn [32], along with its greater applicability for producing inessentially occurring variables since it applies even in the case of Anderson–Belnap style relevant logics, such as R, where other dummying-in moves fail. Those moves include exploiting the CL-equivalence of q-free A with $A \wedge (q \vee \neg q)$ or the IL-equivalence of such an A with $A \vee (q \wedge \neg q)$, both of which would fail in a relevant setting, with $\neg$ as the usual De Morgan negation of the Anderson–Belnap tradition.[11] In view of those De Morgan properties, either $\wedge$ or $\vee$ can be defined in terms of the other and $\neg$. so the $\{\wedge, \neg\}$

[10] The consequence relation relevant to seeing S4 is the formula logic satisfying the Default Background Conditions of Section 1 is the local consequence relation rather than the global consequence relation associated with S4, characterized semantically in terms of preserving truth at each point in any Kripke model (on a transitive reflexive frame), rather than preserving the property of being true-at-all-points in the model.

[11] With the logic of first-degree entailment in mind, we may think of the sole connectives as conjunction, disjunction and negation, and consider the consequence relation $\vdash_{\mathsf{FDE}}$ as holding between Γ and D when there the conjunction of finitely many $C \in \Gamma$ 'first degree' entails D, suspending the connection between logics as sets of formulas and logics as consequence relation embodied in the Default Background

and $\{\vee, \neg\}$ also permit 'dummying in'. Somewhat further afield, Olson et al. [120, p. 468, Remark 6.6] observe that — in what they call **CL**, which is not our CL, but a version of (classical) linear logic, "if a signature contains $\rightarrow, \wedge$ or $\rightarrow, \vee$ we can always arrange that a formula uses *all* of a given finite set of variables, because p is logically equivalent over **CL** to $p \wedge ((p \rightarrow q) \rightarrow q)$ and also to $((q \rightarrow q) \rightarrow p) \vee p$." (This reference to [120] updates that given at p. 177 of [82], when [120] was still in a pre-publication form.)

Digression on Hart's Desiderata. In Section 3 and Appendix F below, we shall benefit from the discussion in Hart [53] of Goodman [50] and reactions to it, but for the moment, pause to note that the above paragraph makes a comment on another aspect of [53] timely. Had Hart, when working on [53], been aware of the absorption laws or any similarly dummying-in options, he might have proceeded differently. Section 2 of its opening chapter lists some provisional conditions of adequacy for an account of *aboutness*, and though he goes on to observe that the fourteen conditions he comes up with are not jointly consistent, the problem sets in at a more immediate level than the discussion recognises.[12] There is an Equivalence Condition which says that (logically) equivalent sentences are about the same things. The logic by which equivalence is judged is taken to be classical predicate logic, and Hart uses "logically indeterminate" (Carnap-style) to mean *neither provable nor refutable* with the aid of that logic in formulating a further condition: logically indeterminate sentences are each about something. And there is a 'Composition Condition' to the fact that if a sentence is about a thing, then any logically indeterminate sentence of which it is a truth-functional component is also about that thing. We can set aside the remaining eleven proposed adequacy conditions to note that if logically indeterminate S_1 is about something, as all such sentences must be, let us take some such thing, s_1,

Conditions of Section 1. Equivalence is then taken as mutual $\vdash_{\mathsf{FDE}}$ consequence. Alternatively, we may consider a richer language with the relevant implication connective $\rightarrow$ present and construe equivalence as the provability of an implication and its converse, in which case again the Default Background Conditions need to be violated for the same reason as was given apropos of AL after Example 3(ii).

[12] A similar procedure of articulating candidate conditions and then reconsidering them in the light of their mutual inconsistency is found in another Goodman-oriented discussion of aboutness, Niebergall [118], which contains a list of the main secondary literature on Goodman — though with no mention of the concise but very telling footnote 20 in Lewis [99].

say, and consider the compound $S_2 \wedge S_1$, where S_2 is any other sentence which is consistent with S_1. The Composition Condition tells us that this is also about s_1 (whatever else it may also be about), since $S_2 \wedge S_1$ is not refutable, S_2 having been chosen as consistent with S_1, and not provable either, since otherwise S_1 itself would be; thus $S_2 \wedge S_1$ is logically indeterminate. Invoking that condition again, we infer that $S_2 \vee (S_1 \wedge S_2)$ is also about s_1, and so by the Equivalence Condition (and the tacit choice of logic, endorsing the absorption laws), S_2 is about s_1. So whatever any logically indeterminate sentence is about, every sentence consistent with it is also about. In particular, all contingently true sentences are about the same things. If "Dresden was subjected to fire-bombing in World War II" is about Dresden, then so is "Pumpkins are often on display at Halloween."[13] Within a decade, Lewis [99] was to provide a systematic account of which ('partializing') variations on the relation of being *entirely about* satisfied which of the several conditions in play — as in Hart's list of desiderata. In the course of this, Lewis diagnosed various problems that had been raised for proposals about cognitive significance in the literature on logical positivism. Some further connections with the topic of Hume's Law (touched on in Section 1 and Appendix C of the present paper) can be found in Humberstone [66], especially Figure 3 on p. 135 and the imaginary diagram modifying it as described on p. 134.

End of Digression

Still on the absorption theme, the L-equivalence, for a good many L — though non-absorptive logics continue to attract a certain amount of attention[14] — of A with $A \wedge (A \vee B)$, puts paid to the conjecture that, at least if we concentrate on L's $\{\wedge, \vee\}$-fragment, any variable occurring L-essentially in at least one of two formulas C, D, occurs L-essentially in $C \wedge D$. After all, in the case of p and $p \vee q$, with q not occurring L-essentially in $p \wedge (p \vee q)$, q does not occur, let alone occur essentially, in the first conjunct. This suggests a modification: what about the question of whether any variable occurring L-essentially in *each of* two

[13] If we throw in Hart's 'Limited Negation Condition' which says that logically indeterminate sentences are about the same things as their negations are about, we can go further and conclude that all logically indeterminate sentences are about the same things. When Hart acknowledges that his list of conditions is inconsistent, he [53, p. 15] is inclined to question one of them not isolated above — a *Condition of Adequacy with regard to Consequences*: "if φ implies ψ, then φ is at least 'as strong as' ψ. Hence, if ψ is about k, then φ is about k as well."

[14] For example: Fine [37], French [44]; see also note 20.

formulas C, D, occurs L-essentially in $C \wedge D$? Perhaps surprisingly, still restricting attention to $\{\wedge, \vee\}$-formulas,[15] the answer is still negative. Consider $C = (p \vee q) \wedge r$; $D = (p \vee r) \wedge q$. The conjunction is equivalent to $q \wedge r$, so p does not occur essentially in $C \wedge D$. In fact, this is really a somewhat concealed appeal to absorption, as one sees by writing out the conjunction $C \wedge D$ explicitly:

$$((p \vee q) \wedge \underline{r}) \wedge ((p \vee r) \wedge \underline{\underline{q}})$$

and interchange the underlined conjuncts, exploiting the associativity and commutativity of $\wedge$, so as to invoke absorption on the first conjunct of the resulting formula and then again on the second conjunct. (Another 'concealed' absorption example, from Demolombe and Fariñas del Cerro [26, p. 92] in classical predicate logic: $Fa \wedge \exists x(Fx \vee Gx)$, equivalent to its G-free first conjunct.)

We return to Example 6 itself, first to view it from a slightly different perspective — that of describing the 'inverse images' of $\Box$-formulas in modal logic — and then to proceed to an analogue of that example for the case of intuitionistic logic (Example 10 below). The *inverse image* (under $\Box$) of a formula A, relative to a congruential modal logic L, we denote by $\Box^{-1}[A]^{\mathsf{L}}$, which is defined to be $\{B \mid\ \vdash_{\mathsf{L}} A \leftrightarrow \Box B\}$ (as in [75, p. 1034]). If A is not L-equivalent to any $\Box$-formula, then of course this set is empty, and it may happen that for A of the form $\Box B_0$ (or -equivalent to such a formula) $\Box^{-1}[A]^{\mathsf{L}}$ contains many elements, all of which are S-equivalent,[16] in which case one can conservatively extend S by expanding it language with a left-inverse forming operator, L, say, and considering the smallest congruential (or indeed monotone, regular, or normal in respect of $\Box$) modal logic extending S in this new language and containing all formulas of the form $L\Box A \leftrightarrow A$.[17] But the typical

[15] And L — sometimes left implicit in our formulations here — to being a logic agreeing with CL on the logical relations among these formulas. We could let in $\neg$ too, by passing to FDE, which shares the same $\{\wedge, \vee\}$-fragment but excludes the $\neg$-involving counterexamples such as $C = p \vee q$, $D = p \vee \neg q$ (since their conjunction is not FDE-equivalent to the q-free p).

[16] This happens when L is closed under the Cancellation Rule for $\Box$: from $\Box A \leftrightarrow \Box B$ to $A \leftrightarrow B$ — a converse of the Congruentiality Rule for the same operator.

[17] This option is explored (for the $\Box$-normal case), in Humberstone and Williamson [83], where right inverses, R, are also considered, with the characteristic schema $\Box RA \leftrightarrow A$. If congruential L provides a left inverse operator L for one of its 1-ary connectives O, then O is e.o.-preserving for L in the sense of Definition 4.

situation is that there are many non-L-equivalent formulas in the inverse image of a given $\Box B_0$. Indeed, for the case of $\mathsf{L} = \mathsf{S4}$, with a bit of elaboration, Example 6 itself provides a countably infinite set of pairwise non-equivalent elements of $\Box^{-1}[\Box p_1]^{\mathsf{S4}}$, choosing the p of the example as the variable p_1 in our official enumeration of the sentence letters from the opening paragraph of Section 1, similarly take q there to be p_2 and now consider the formulas $B_i = \Box(p_1 \vee p_{i+1}) \wedge p_1$. B_1 is then the formula A of Example 6 in which p_2 is shown to occur $\mathsf{S4}$-essentially, but not to occur essentially in $\Box B_1$, the latter being $\mathsf{S4}$-equivalent to $\Box p_1$. By the same reasoning, for each $i > 1$, p_{i+1} occurs $\mathsf{S4}$-essentially in B_i but not so in $\Box B_i$, which is again $\mathsf{S4}$-equivalent to $\Box p_1$. Thus $\{B_i \mid 1 \leq i < \omega\} \subseteq \Box^{-1}[\Box p_1]^{\mathsf{S4}}$, while for no distinct B_i, B_j in the former set does either imply the other, since, with any $\Box(p_1 \vee p_i) \wedge p_1$ and $\Box(p_1 \vee p_j) \wedge p_1$ with $i \neq j$, $\Box p_i \wedge p_1$ has the former as an $\mathsf{S4}$-consequence but not the latter.

The following Remark is prompted by the discussion of inverses and inverse images for $\Box$ in modal logic uses the terms *functor* and *argumentor* in the sense introduced into categorial grammar in Potts [138]; in particular, there is no (non-etymological) connection between this use of the word *functor* and the use of the same word as used in category theory. (Likewise of course, for the word *category* itself.) One may prefer the terminology *operator/operand* over of *functor/argumentor*, as Potts himself later came to, in [139].

Remark 7. Suppressing any logic-relativity here, for simplicity, while $\Box^{-1}[\,\cdot\,]$ applies to a formula to deliver the set of — in particular, mutually non-equivalent — expressions in the argumentor category yielding results equivalent to that of applying the functor category expression $\Box$ to the formula in question, we could also envisage the process that applies to a formula to deliver the set of expressions in the functor category

For suppose q does not occur essentially in OA. Thus $\vdash_{\mathsf{L}} OA \leftrightarrow B$ for some q-free B. By congruentiality we can prefix L to each side of this biconditional, and then by appeal to the left-inverse schema $LOA \leftrightarrow A$ infer that $\vdash_{\mathsf{L}} A \leftrightarrow LB$, showing A to be L-equivalent to the q-free formula LB. This is the reason we have negation e.o.-preserving (Proposition 5(iv)) for classical but not intuitionistic logic — as we shall see in Example 10; in the classical case, negation has itself as a left inverse while the addition of a (congruential) left inverse operator to IL immediately turns it into an extension of CL. (See Example 4.21.1(i), Humberstone [74, p. 540], for this familiar fact, along with 4.21.1(ii) showing that the same goes for any proposed *right* inverse for intuitionistic negation.)

yielding results equivalent to that of applying a given such expression to an argumentor expression — for example, considering all the ways the monadic predicate letter F could be varied so as to yield something non-equivalent to F (or $\lambda x.\,Fx$, to be more explicit) but yielding equivalent predications for the given argumentor expression. Humberstone [69] makes a few moves in that direction. (One might consider thinking of F as itself the argumentor, type-promoting — "Montague raising/lifting" — the former: $\lambda\varphi.\varphi a(F)$.) ◁

Rounding out the discussion of Example 6, let us attend briefly to an equivalence relation between formulas suggested by it:

DEFINITION 8. For formulas A, B of the language of a logic L, we define $A \equiv^{eo}_{\mathsf{L}} B$ iff A and B have the same sentence letters occurring L-essentially in them.

For brevity, borrowing a term from biology and chemistry, we might like to read $A \equiv^{eo}_{\mathsf{L}} B$ as saying that A and B are L-*congeneric*, suppressing the "L" when it is clear from the context of the discussion.[18] In view of Proposition 5(i), if $\vdash_{\mathsf{L}} A \leftrightarrow B$ then $A \equiv^{eo}_{\mathsf{L}} B$ for any formulas A, B, of the language of L.[19] Thus, for formulas A, B, there arise exactly three possibilities:

(i) $\vdash_{\mathsf{L}} A \leftrightarrow B$ (& so $A \equiv^{eo}_{\mathsf{L}} B$);

(ii) $\nvdash_{\mathsf{L}} A \leftrightarrow B$ & $A \equiv^{eo}_{\mathsf{L}} B$;

(iii) $\nvdash_{\mathsf{L}} A \leftrightarrow B$ & $A \not\equiv^{eo}_{\mathsf{L}} B$.

Note that for logics L with the *relevant equivalence property* mentioned towards the end of Section 1, $A \equiv^{eo}_{\mathsf{L}} B$ iff the same variables occur in A and B.

Example 9. Illustrating the possibilities (i)–(iii) just distinguished: $p \vee q$ and $p \wedge q$ stand, for any consistent L one is likely to encounter, in relation

[18] This terminology has a particular resonance for logics as sets of formulas when we think of the Lindenbaum algebras of fragments with different sets of generators but the same fundamental operations (corresponding to the primitive connectives), since even if p_1, p_2 and p_3 occur in A and p_1, p_2 and p_4 in B, with p_3 and p_4 L-inessentially in A, B, respectively. The L-equivalence classes of A, B happily cohabit in the Lindenbaum algebra of L's two-variable fragment with generators the equivalence classes of A, B.

[19] Recall that, by the Default Background Conditions imposed to simplify the discussion in Section 1, this implies that for all contexts C: $\vdash_{\mathsf{L}} C(A) \leftrightarrow C(B)$ (or, equivalently, by those Conditions, $C(A) \dashv\vdash_{\mathsf{L}} C(B)$).

(ii): congeneric though not equivalent, as are $p \vee q$ and $\neg p \to q$ for $\mathsf{L} = \mathsf{IL}$, though for $\mathsf{L} = \mathsf{CL}$ this pair stand in relation (i). $\triangleleft$

In this terminology, what Example 6 shows is that for suitably chosen A, different formulas in the set $\Box^{-1}[A]^{\mathsf{S4}}$ may not only fail to be $\mathsf{S4}$-equivalent, exhibiting even greater diversity than that description conveys: they need not even be congeneric (relation (iii), in other words). For any A, we have A and $\neg A$ related as in (ii) for $\mathsf{L} = \mathsf{CL}$ (by Prop. 5(ii), (iv)). Example 10 below will show that for some choices of A, we have the non-equivalent A, $\neg A$ not even congeneric.[20] We now set off in the direction of that Example.

While Example 6 provides a perfectly clear illustration of how modal contexts in normal modal logics can fail to be e.o.-preserving, and the case does seem to be of some interest in its own right, it may not seem especially surprising. But $\mathsf{S4}$ was chosen to illustrate the phenomenon with a certain ulterior motive in mind, foreshadowed in the paragraph after Proposition 5, namely that supplied by the analogy made especially vivid by the Kripke semantics (for these logics) between $\mathsf{S4}$ and intuitionistic (propositional) logic — or IL for short.[21] Certain differences between the two cases, however, and particularly the fact that every formula in the IL case, once true (at a point in a model) remains true at all accessible points, rather than, as in the $\mathsf{S4}$ case, having this persistence (or 'heredity') property only for certain (specifically, $\Box$-) formulas, make

[20] Goodman [49, Example 61, p. 643] defines a binary relation $\sim$ between formulas that are logically equivalent and have the same constants occurring in them — "occurring" rather than "essentially occurring"; and *constant* is apparently intended to cover (non-logical) predicates as well as individual constants. The 'agglomerative algebras' discussed in the paper are a generalization of Boolean algebras in which the absorption laws are observed to fail — Prop. 7, p. 634 — and the $\sim$-equivalence classes of formulas are seen in Example 61 to constitute such an algebra. (Note that although the intended application is to aboutness, this is not a reference to *Nelson* Goodman.) See also Brast-McKie [10].

[21] This is not to suggest that $\mathsf{S4}$ is required for illustrating failure of e.o.-preservation under $\Box$, or even for the particular illustration of that failure in Example 6. Restricting attention to normal modal logics, one can check that part (1) of the argument there goes through for any such logic in which $p \to \Box p$ is not provable; indeed, that for (normal modal L) q occurs L-essentially in the formula A of Example 6 iff $\mathsf{L} \not\supseteq \mathsf{KT}_c$. (We use the Chellas notation, as in [77], so T_c is the converse, $A \to \Box A$, of the T-schema, with $\mathsf{T!}$ being the corresponding biconditional form.) As for part (2) of the argument, to the effect that q occurs L-inessentially in $\Box A$, the reasoning does not require the full force of the combined T and $\mathsf{4}$ schemas, and goes through already in $\mathsf{K4!}$.

the analogy somewhat indirect. The modalized p in the first conjunct of the left-hand side of biconditional in ($*$) above was weakened to an unmodalized p in the second conjunct, whereas below the p in the first conjunct of (†)'s left-hand formula — no explicit modality in sight — is weakened to $\neg\neg p$ in the second conjunct. Some familiarity with IL is assumed, and, in particular, we make use of the following consequence of Glivenko's Theorem, with CL being non-modal classical propositional logic: whenever $\vdash_{\mathsf{CL}} C \leftrightarrow D$, we have $\vdash_{\mathsf{IL}} \neg C \leftrightarrow \neg D$.

Example 10. Consider the formula $A = (p \vee q) \wedge \neg\neg p$. The variable q (1): occurs IL-essentially in A, but (2): does not occur IL-essentially in $\neg A$. Checking (1), suppose for Reductio that we have a q-free formula B with

$$\vdash_{\mathsf{IL}} ((p \vee q) \wedge \neg\neg p) \leftrightarrow B \tag{†}$$

As in Example 6, we exploit the fact that q does not occur in B to invoke closure under uniform substitution (p for q) to conclude that $\vdash_{\mathsf{IL}} ((p \vee p) \wedge \neg\neg p) \leftrightarrow B$, and hence $\vdash_{\mathsf{IL}} (p \wedge \neg\neg p) \leftrightarrow B$ and thus $\vdash_{\mathsf{IL}} p \leftrightarrow B$. Then, appealing again to (†), we should have

$$\vdash_{\mathsf{IL}} ((p \vee q) \wedge \neg\neg p) \leftrightarrow p \tag{††}$$

But the $\rightarrow$ direction of ($**$) is easily seen to fail since the l.h.s. is evidently an IL-consequence of $q \wedge \neg\neg p$ while the r.h.s. is not.

Now addressing (2), as at the corresponding point in Example 6, we reach for the absorption equivalence of $(p \vee q) \wedge p$ with p, or more accurately for a CL-available reformulation thereof:

$$\vdash_{\mathsf{CL}} ((p \vee q) \wedge \neg\neg p) \leftrightarrow p.$$

By the above-noted corollary to Glivenko's Theorem we have

$$\vdash_{\mathsf{IL}} \neg((p \vee q) \wedge \neg\neg p) \leftrightarrow \neg p.$$

Since the l.h.s. here is $\neg A$ for the current choice of A, while q does not occur on the right, this establishes (2). ((1) and (2) here could be shown to hold in the case of Minimal Logic no less then Intuitionistic logic, but (a) that would be more time-consuming since Glivenko's Theorem (equivalently, the corollary used here) is not correct for Minimal Logic above, despite occasional claims to the contrary in the literature — an example is cited in Remark 8.33.11 of Humberstone [74, p. 1271] — and (b) the connection with contrariety and contradiction touched on briefly below and more fully in Appendix D would then be lost.) $\lhd$

Here we have given Example 10 first because of the close affinity to Example 6, both exploiting the absorption equivalences and the former needing a little help from Glivenko. In the intuitionistic case, we can offer a simpler example — also requiring that same assistance, though this time with no hint of absorption:

Example 11. Consider the formula $A = q \vee \neg q$. The variable q (1): occurs IL-essentially in A, but (2): does not occur IL-essentially in $\neg A$. As before, we observe, for (1), that if A has a q-free IL equivalent, we can pass via it using Uniform Substitution to the conclusion that $\vdash_{\mathsf{IL}} (q \vee \neg q) \leftrightarrow (r \vee \neg r)$, which is easily seen not to be the case, since $\nvdash_{\mathsf{IL}} q \to (r \vee \neg r)$. On the other hand, for (2), by the Glivenko corollary used before, $\vdash_{\mathsf{IL}} \neg(q \vee \neg q) \leftrightarrow \neg(r \vee \neg r)$ (the negation of each side of this $\leftrightarrow$ being IL-provable) — so q does not occur essentially in $\neg A$. $\lhd$

Pausing for a moment, now that we have the law of excluded middle before us as it fares in IL, to interrupt the discussion of e.o.-preservation and its absence, we take the opportunity to recall also the intuitionistic unprovability of the 'weak' law of excluded middle, $\neg q \vee \neg\neg q$, made famous as the characteristic axiom of an intermediate (i.e., consistent superintuitionistic) logic in the 1960s (Dummett and Lemmon [31], Jankov [86]), called KC in [31]. Of course q occurs KC-inessentially in $\neg q \vee \neg\neg q$, since no variable occurs L-essentially in an L-provable formula (as noted after Definition 2, and with the Default Background Conditions assumed taken as satisfied). But as in the case of the law of excluded middle itself (Example 11), q occurs IL-inessentially in $\neg q \vee \neg\neg q$, since otherwise this formula would be IL-equivalent to $\neg p \vee \neg\neg p$ and so $\neg q \vee \neg\neg q$ would be provably IL-implied by $\neg p$, meaning that, by substituting $\perp$ for p, we would be able to conclude that $\vdash_{\mathsf{IL}} \neg p \vee \neg\neg p$ — which is not the case: KC is a *properly* intermediate logic.[22] We can now proceed to

Example 12. As an alternative (though CL-specific) definition of q's occurring essentially in $A(q)$, consider the following, which presumes the availability of the connective $\neg$ in the language of L:

q *occurs Körner-*L*-essentially in* $A(q)$ if and only if $A(q)$ is not L-equivalent to $A(\neg q)$.

The appropriateness of the nomenclature here can be gleaned from remarks on p. 63 of Körner [87], recalling and generalizing an earlier pas-

[22] As we can easily check using the Kripke semantics for IL, or by recalling that IL has the Disjunction Property.

sage (p. 24*f.*) in the same book, but, for a fuller appreciation, see §3 of Schurz [156] and the references there to an earlier book by Körner and to a later article by J. P. Cleave, as well as Schurz's own discussion of Körner's criterion of essential occurrence. The criterion was, incidentally, never proposed as offering an explication of essential occurrence for variable L as above, and indeed for the case Körner had in mind — of L as CL — this alternative version L-essential occurrence coincides with L-essential occurrence as in Definition 2. Readers to whom the point is familiar may prefer to skip the following paragraph justifying this assertion.

If q occurs CL-inessentially in A, $\vdash_{\mathsf{CL}} A(q) \leftrightarrow B$, for some q-free B, in which case, by Uniform Substitution, $\vdash_{\mathsf{CL}} A(\neg q) \leftrightarrow B$, so $\vdash_{\mathsf{CL}} A(q) \leftrightarrow A(\neg q)$ and thus q occurs Körner-CL-inessentially in $A(q)$. For the converse, suppose that q occurs Körner-CL-inessentially in $A(q)$: $\vdash_{\mathsf{CL}} A(q) \leftrightarrow A(\neg q)$. Without loss of generality we can suppose that other than q, the sentence letters in A are $p_1, \ldots, p_n$, and we write α for the Boolean function (truth-function) induced by the formula $A(q)$ as a function of the truth-values of these $n+1$ variables — though we leave $\neg q$ intact rather than absorbing that negation into the effects of α — so that for all v this supposition means that

$$\alpha\big(v(p_1), \ldots, v(p_n), v(q)\big) = \alpha\big(v(p_1), \ldots, v(p_n), v(\neg q)\big). \tag{Kö}$$

Then, where r is (for simplicity) a new sentence letter, we claim that for all v:

$$\alpha\big(v(p_1), \ldots, v(p_n), v(r)\big) = \alpha\big(v(p_1), \ldots, v(p_n), v(q)\big),$$

and thus that $\vdash_{\mathsf{CL}} A(q) \leftrightarrow A(r)$, showing that q occurs CL-inessentially in A. The correctness of the claim is clear for any v for which $v(r) = v(q)$. (The same function applied to the same arguments yields the same value.) On the other hand, if $v(r) \neq v(q)$, then $v(\neg q) = v(r)$, and we appeal to (Kö) to draw the same conclusion.

We now get to the Example proper, illustrating the fact that, by contrast, outside of CL, Körner essentiality and the standard notion cannot be counted on to coincide. Although, as we saw at the end of the paragraph after Example 11, q occurs IL-essentially in $\neg q \vee \neg\neg q$, q does not occur Körner-IL-essentially in this formula since we do have $\vdash_{\mathsf{IL}} (\neg q \vee \neg\neg q) \leftrightarrow (\neg\neg q \vee \neg\neg\neg q)$, since the second disjunct on the right-hand side simplifies intuitionistically to $\neg q$ ("the law of triple negation") and we can then appeal to the IL-commutativity of $\vee$. As in some other cases we have already seen, a similar example could be given from (classically based) modal logic, where the notion of contingency — ∇A for "it is contingent whether A" is often taken as a defined connective (∇A as $\Diamond A \wedge \Diamond\neg A$) and occasionally as a primitive (see the discussion and references in Humberstone [80]), and in most modal logics of interest q occurs essentially in ∇q though not Körner-essentially since we'll have ∇q and $\nabla\neg q$ provably equivalent. Note also that as well as failing — outside of the confines of CL — to capture the intuitive notion of essential occurrence, it has no application to the question of essential occurrence in a sentence for expressions of a syntactic type for which there is no plausible notion of negation available, such as individual constants in predicate logic. (One can naturally, if derivatively, apply it to predicates, as well quantifiers, so whether a predicate letter occurs essentially in a sentence could be given a Körnerian interpretation, and again we would find it not to live up to expectations, even in the setting of *classical* predicate logic. The contingency example just considered has the obvious analogue: $\exists x(Fx) \wedge \exists x(\neg Fx)$, classically equivalent to the result of replacing Fx by $\neg Fx$, but not susceptible of reformulation without the aid of F.) $\lhd$

We can think of Körner-essentiality as specifically suited to the case of CL, then, in the sense that it supplies a *criterion* of L-essential occurrence in the general sense provided by Definition 2 for the case of $\mathsf{L} = \mathsf{CL}$. One would like to see it apply to various fragments of classical logic, in which $\vdash_{\mathsf{CL}}$ would be taken as understood as the restriction of the classical consequence relation to the fragment concerned, for which, since $\leftrightarrow$ and $\rightarrow$ may be absent, the Default Background Conditions of Section 1 may have to be waived.[23] And whenever $\neg$ is absent (and not definable in terms of the primitive connectives of the fragment), the substitutions of $\neg q$ for q and so on, in Example 12, are not available, so it is convenient to have a less linguistic formulation of a Körner-style criterion, sometimes encountered in the literature on Boolean functions (as the local incarnation of the concept of being an essential argument/variable of a function: see note 1).[24]

[23] The pure $\leftrightarrow$-fragment of CL, considered as a consequence relation, here denoted by $\vdash_{\leftrightarrow}$, has the following distinctively classical feature, observed in Humberstone [67, Prop. 4.5(ii)]: for all A, C in this fragment, if $\nvdash_{\leftrightarrow} C$ while $A \vdash_{\leftrightarrow} C$, then for all B (in the fragment) such that, putting it lazily, $A \vdash_{\leftrightarrow} B \vdash_{\leftrightarrow} C$, the formulas A, B, and C all have the same variables occurring essentially in them (are all classically congeneric/$\equiv^{eo}_{\mathsf{CL}}$-equivalent, as per Definition 8).

[24] The specifically Boolean case, with functions taking arguments and values from

A useful presentation of such a treatment, for present purposes, is given in Pietruszczak [131].[25] It is particularly pertinent for present purposes because of the connections it makes with the ideas of information and, more especially, aboutness, which we attend to after Example 13 (as well as in §4 and Appendix F). The elaborate treatment of information in [131] will not occupy us in what follows, though a brief comment seems in order concerning some of the introductory remarks about it that are slightly puzzling, possibly because of issues of translation. For example, Pietruszczak writes [131, p. 91] concerning a relation $\models_{\mathrm{i}}$ — "i" for information — as follows:[26]

> In Section 5 of this paper we will define the consequence relation $\models_{\mathrm{i}}$ that preserves information:
>
> $\varphi \models_{\mathrm{i}} \psi$ iff (a) neither φ is a contradiction no ψ is a tautology
> (b) information contained in ψ is a part of information contained in φ.

What seems a little odd about this is that "preserves information" suggests that any information provided xby φ, on the left (of "$\models_{\mathrm{i}}$"), is also provided by ψ, on the right: information is not lost on passing from left to right. But if no information is lost in the transition from φ to ψ that all the information provided by φ is provided by ψ, whereas the formulation in (b) makes it sound as though it is the other way around. Further clarification can be found in the account of information provided in [131] already alluded to above, invoked in the semantic explanation consequence-like relation in question. The motivating considerations are

a single two-element set, presents further special features. By way of example, quoting from Zakharova and Yablonskii [179, p. 435] "An important property of two-valued functions is that under substitution of some function for an essential argument of another function, all essential arguments of the internal function prove to be essential arguments of the whole superposition." Linguistically, then: if q occurs CL-essentially in $B(q)$ and p occurs essentially in a q-free formula $A(p)$, then q occurs essentially in $A(B(q))$, where this last is the result of substituting B, alias $B(q)$, for p in $A(p)$. Still further special observations arise when restrictions are imposed on the stock of two-valued functions under consideration, as illustrated in note 23 above.

[25] Thanks are due to a *Logic and Logical Philosophy* referee, who drew this paper — from an earlier volume of the journal — to my attention, on which see also note 29.

[26] Somewhat loosely, Pietruszczak calls this a consequence relation, though it is a binary relation between formulas rather than between sets of formulas and formulas; we could convert it into a consequence relation proper in the manner described in note 11 for the case of First-Degree Entailment à la Anderson and Belnap.

those associated in the Anglophone world with W. T. Parry, R. B. Angell, and the *analytic implication* tradition.[27] Pietruszczak's own references are to the (respectively) Russian and German publications of Alexandr Zinov'ev and Horst Wessel, and he is interested in improving the latter's attempt to improve on the former's account on how the idea of checking that appropriately occurring propositional variables on the right (in the consequence formula) occur appropriately on the left (of which is to be considered a consequence).[28] Appropriate occurrence is explicated as essential occurrence, and essentiality is taken as CL-essentiality, which brings us to the promised, less language-dependent, variation on Körner-essentiality:[29]

Example 13. We first explain that by Boolean valuation is meant an assignment of truth-values (T, F) to all formulas of a language, which respects the conventional association of truth-functions with those connectives that for which such conventions exist (or are stipulated to be in force) — so, if $\wedge$ is a connective of the language, then a Boolean valuation v must obey the constraint that $v(A \wedge B) = T$ iff $v(A) = v(B) = T$ for all A, B; and so on. for $\neg$, $\rightarrow$, $\vee$, etc. For all the connectives of the usual language of CL and its fragments, there are such associated functions/constraints, so $v(A)$ is uniquely determined for a Boolean valuation v by $v(p_1)$, …, $v(p_n)$ where the indicated p_i exhaust the variables used

[27] A full discussion of this tradition, with ample references, is provided by Ferguson [35]; the papers by Fine and by French cited in note 14 are also relevant, Ferguson's coverage includes mention of Zinov'ev, mentioned in the next sentence. For an aboutness-oriented focus, in particular, see the discussion and references in Ferguson [36]. Along with essentiality of occurrence Pietruszczak [131] attends to matching in respect of positive vs. negative occurrences, a continuing theme of current work on variable-inclusion logics — as in Randriamahazaka [145] and Rubin and Szmuc [150].

[28] This Horst Wessel is not to be confused with the Horst Wessel associated with early *Nationalsozialismus*, who died before the HW of current concern was born. A second warning: our secondary reference, Pietruszczak [132, e.g., p. 127], makes use of the two forms of the same lower case Greek letter *phi* (φ and ϕ) — for two different things.

[29] The account given here is not claimed to be original in [131], though the adaptation of it to what are there called 'alternatives of information' is. The presentation that follows undoes this adaptation and applies the notion directly to *formulas*, following a suggestion — though not the suggested notation — of the referee mentioned in note 25. (The suggested notation used special metalinguistic variables ranging over truth-value assignments to the sentence letters, as is done in [131] itself, rather than sticking with the current valuations, assigning truth-values to all formulas.)

in the construction of A. If v is such a valuation, let $v^{\pm p_i}$ be the unique Boolean valuation which "flips" p_i's truth value leaving other variables alone, i.e.

$$\text{For all } j\text{: } v^{\pm p_i}(p_j) = v(p_j) \text{ if and only if } i \neq j.$$

Then p_i is defined to occur essentially in A just in case there is some Boolean valuation v for which $v(A) \neq v^{\pm p_i}(A)$. In other words, we get the effect on v of negating q, as in (Kö) of Example 12 of keeping the valuation the same and changing the formula, by instead keeping the formula the same and changing the valuation (from v to $v^{\pm q}$). This then makes sense even if the only available object-language connectives are, say, $\wedge$ and $\rightarrow$. ◁

We return to matters of e.o.-preservation and our earlier Examples 10 and 11 of how intuitionistic negation lacks this property. The moral of these examples is potentially disconcerting for an intuitionistically inclined advocate of any approach to aboutness similar to that mooted by Goodman [50, p. 5*f.*]: a sentence is about whatever is referred to by any term occurring essentially in it.[30] In view of our current setting, adapted to the case of predicate logic, since there is no model-independent notion of reference for the terms of a formal language, we would need to introduce not only the logic-relativity in play in the above discussion but also a model-relativity to get from the (now first-order) language (of the logic classical predicate logic in Goodman's case) to what it is interpreted in terms of, in particular specifying which elements are the denotations of which individual constants. That is certainly more trouble than it's worth for present purposes, so let's just keep it at the linguistic level and ask whether this or that individual constant occurs essentially in the formulas under discussion. (This delicate use–mention issue will come up again in Section 4.)

That said, let us now consider a predicate-logical version of Example 10, taking the language to have in its vocabulary a monadic predicate letter F and individual constants a, b, with A as $(Fa \vee Fb) \wedge \neg\neg Fa$. So, adapting Example 10, b occurs in A IL-essentially, meaning, according to

[30] In fact Goodman only uses the term twice in the paper, in both occurrences as part of the word "non-essentially", which itself appears only in scare quotes, and he favours what in Section 3 is called a *generality* explication of the notion, the gist of which can be gleaned from this remark: "Now we must seriously raise the question whether a statement can properly be regarded as saying about any particular thing what it says about everything else." [50, p. 5].

the proposal under consideration, then relative to $\mathcal{M}$ and IL, A is about whatever b is taken to denote, while $\neg A$ is not. This seems surprising: you say something about an individual, but when I deny it by asserting its negation, what I say is apparently *not* about that individual. Uhh?[31] This certainly gives a whole new meaning to the Quinean slogan about 'denying the doctrine and (thereby) changing the subject'.[32]

One might just say: "So much the worse for intuitionistic logic," or "So much the worse for the proposal," or again "But intuitionistically available alternative negations are ten a penny — or 'a dime a dozen' — and who's to say that the conventionally notated $\neg$ is the right one to pick for present purposes?" The third response might be fleshed out with an invitation to consider some form of strong negation/constructible falsity, or perhaps dual intuitionistic negation. These particular suggestions are open to the objection that their logical behaviour goes against the spirit of IL,[33] which would perhaps leave a lot resting on the first of the above "so much the worse" reactions. As for the second 'reject the proposal' response, the proposal under consideration is perhaps no longer felt to be a candidate account of aboutness in view of the subsequent dominance of the account in Lewis [99, 100], and later refinements thereof, as in

[31] Most accounts of aboutness in the literature are wedded to the idea that the negation of a statement about something is another statement about the same thing, and because they tend to be couched in terms of classical logic, it is $\neg$ as interpreted in CL that they have in mind, though Niebergall [118, p. 151 disagrees] taking *Paris is a city* to be about Paris but $\neg$(*Paris is a city*) not to be about Paris because the latter "can also be true if Paris does not exist". The subject of aboutness is quite hard enough, one would have thought, to work one's way through, without additionally having to worry about extraneous complications arising from admitting non-denoting terms, or (if you *must*) terms denoting things — cities even — that don't exist. An adjacent (though distinct) range of questions arise over aboutness in fiction, imagination, etc., which have received recent considerable attention but are again not on our current menu: Berto [7, 8], Lamarque [91], and perhaps with fictional cases in mind Osorio-Kupferblum [123, §3], where aboutness is described as "a relation internal to language."

[32] The relevant passages from Quine are cited and discussed in Morton [115].

[33] For example, IL extended by dual intuitionistic negation lacks the disjunction property, running against its constructive *raison d'être*, while the extension via strong negation, although conservative at the level of provability, does not conserve synonymy: formulas free of strong negation can be synonymous in IL without being so in the extended logic. These considerations can be found in Gabbay [46] and Humberstone [74, p. 1236], respectively. Both of these reactions are conveniently summarized in Standefer [162, §4)], along with further pertinent discussion.

Yablo [178, especially §2.5].[34] For one thing the non-linguistic relatum is different (as is emphasized in [178, §§2.1–2.2]): for Goodman, as for most others discussing aboutness before the appearance of Lewis [99], statements are about such things as people, places and other such things one might encounter in the world, perhaps plurally conceived, as when people complained that "All ravens are black" is about ravens rather than being about non-black things — an example to which we return in note 70 (and elaborate on in Appendix K); see also the remark from Ladd-Franklin quoted in Appendix A. (In Goodman's treatment of such examples, this emerges as being about the *class* of ravens, thought of as a potential candidate k — see Appendix F — falling within the range of first-order quantifiers.[35]) For Lewis, by contrast, the things statements are about are *subject-matters*, conceived as partitions of the set of possible worlds (or the corresponding equivalence relations[36]) — a suggestion

[34] For commentary, see Osorio-Kupferblum [121]; Fine [39] provides some discussion of Yablo's account and its relation to his own preferred treatment; see also Hawke [54] and Giordani [48] on the latter. Indeed these two papers provide useful critical summaries of the recent literature on aboutness, with Osorio-Kupferblum [122] providing additional historical background and Hawke, Hornischer and Berto [55] on aspects of hyperintensionality in this field.

[35] As has no doubt been observed by many (and is certainly pointed out in the text to which note 5 is appended in Niebergall [118, p. 139]), this does not seem to sit well with Goodman's more typically nominalistic tendencies; in any case, as Tichý [165] points out, if abstract entities are going to feature in such cases, properties (and relations-in-intension) would be more suitable candidates than sets. (Tichý was actually making this point about the treatment of predicate letters in Putnam's account of aboutness in [141], before proceeding to other problems emerging in the courser of trying to fix that account.) Many issues arising from aboutness questions for sentences with plural noun phrases ("S is about Fs") are discussed extensively in Hart [53], with a focus on Goodman and commentators, as well as in Vranas [169], which also supplies many useful references to more recent literature. There is an excellent discussion of problems and possible oversights in Goodman's extension of his aboutness proposals from singular term position to predicate position in Patton [130, pp. 316–325]. Our own discussion has not ventured into this mire of issues, many of them not specifically relevant to essential occurrence — well-known extensionality questions (stories about unicorns vs. stories about mermaids, etc.), the 'primitive plurality vs. sets or other aggregates' question, etc. Yet further issues concerning aboutness for predicates, quantified noun phrases, and other expressions are raised, along with options for dealing with them, in Simchen [159, pp. 71–85].

[36] A sentence is, on Lewis' proposal, *entirely about* a given subject matter when it takes the same truth-value relative to any worlds standing in that equivalence relation. Yablo [178, p. 36*f.*] voices qualms about the requirement of transitivity built into the notion of an equivalence relation, opting at this point, instead, for similarity relations

we return to in Appendix J. The syntactic side of this distinction is emphasized in Carlson [17, p. 249] in terms of the category to which the object of the preposition *about* belongs, a distinction he illustrates there with a children's verse from Lewis Carroll. There is more to be said about the status of $\neg$ in IL, which underlines the problematic side of its not being e.o.-preserving, but for the moment we proceed with the formal development — rather than any philosophical implications — of the above themes.[37]

3. The Approach via Generality

The transitions from ($*$) to ($**$) in Example 6 and from (†) to (††) in Example 10, proceeded via substitutions that affect only the side of the biconditionals involved that contain the inessentially occurring variable. Similar moves can be put a somewhat different gloss on essential occurrence which is sometimes presented by way of a definition of this notion as an alternative to Definition 2 or a version of that definition which replaces the reference to q by one for arbitrary expressions e (as in Section 1); the point is obvious enough and no doubt widely familiar, but we spell out the proof anyway:[38]

Proposition 14. *Let* L *be any logic meeting the Default Background Conditions of Section 1. The following are equivalent: (1)* q *occurs at most* L*-inessentially in* A*; (2) For all formulas* C, $\vdash_{\mathsf{L}} A(q) \to A(C)$.

Proof. (1) $\Rightarrow$ (2): Suppose (1), so there is a q-free formula B for which $\vdash_{\mathsf{L}} A(q) \leftrightarrow B$. Let p be some variable distinct from q and occurring in neither A nor B. Since $\vdash_{\mathsf{L}} A(q) \leftrightarrow B$, substituting p uniformly for q gives $\vdash_{\mathsf{L}} A(p) \leftrightarrow B$, so we have $\vdash_{\mathsf{L}} A(q) \leftrightarrow A(p)$, and again by uniform substitution, $\vdash_{\mathsf{L}} A(q) \leftrightarrow A(C)$, so in particular $\vdash_{\mathsf{L}} A(q) \to A(C)$.

(2) $\Rightarrow$ (1): Suppose (2), and choose as C some variable $p \neq q$ with p not occurring in $A(q)$, so we have $\vdash_{\mathsf{L}} A(q) \to A(p)$. By simultaneously substituting q for p and p for q, $\vdash_{\mathsf{L}} A(p) \to A(q)$. Thus $A(p)$ is a q-free formula L-equivalent to $A(q)$. $\dashv$

(though confusingly writing that "similarity is intransitive," a much stronger — and here inappropriate — claim than the denial that it is transitive, when foreshadowing this move at p. 5 of [178]).

[37] A discussion of such implications is deferred to Appendix D.

[38] As in Humberstone [71, note 9, p. 75*f.*].

The following, with "$\leftrightarrow$" in place of "$\rightarrow$", is more of an alternative formulation of Prop. 14 that a Corollary proper, as it could just as easily have been stated first, its equivalence to Prop. 14 is evident in the proof:

COROLLARY 15. *Let* L *be any logic meeting the Default Background Conditions of Section 1. The following are equivalent: (1)* q *occurs at most* L*-inessentially in* A*; (2) For all formulas* C, $\vdash_{\mathsf{L}} A(q) \leftrightarrow A(C)$.

Another simple moral we can draw from the working in Examples 6 we state here as follows, for any L meeting the Default Background Conditions of Section 1, by choose B as $A(C)$ in Coro. 15:

COROLLARY 16. *If* q *does not occur* L*-essentially in* $A(q)$ *then there is not just some* q*-free formula* B *or other that is* L*-equivalent to* $A(q)$*, but more specifically such a* B *can be found which is a substitution instance of* $A(q)$ *itself.*

Coro. 16 involves a claim in general stronger than the content of essential occurrence from the informal discussion of Section 1, in that e's occurring L-inessentially in A need not imply the existence of a substitution instance (now understanding this relative to e's syntactic category) of A always being available to serve a witness to this inessentiality. Example 18 below, for instance, mentions in passing the fact that propositional quantifiers $\exists q$ and $\forall q$ in a second-order classical propositional logic never occur essentially in a formulas since that can always be replaced by suitable disjunctions or conjunctions (reducing $\exists q(A(q))$, $\forall q(A(q))$, to $A(\top) \vee A(\bot)$, $A(\top) \wedge A(\bot)$, resp.), these sentences are not substitution instances of the originals. Another case that might be cited is that arising in Example 1, taking L to be the (substitution-invariant though not congruential) modal actuality logic of Hazen [56]: $\mathcal{A}$ occurs L-inessentially in $\Box(p \vee \mathcal{A}q)$—e.g., since this is equivalent to $p \vee \Box q$. Clearly, however, every substitution instance of this formula, taking q as the substituend, still have $\mathcal{A}$ occurring in it. More to the point, taking $\mathcal{A}$ as the substituend and taking its category to be that of a 1-ary sentence operator, so such substitution will remove q from the scope of $\Box$ and end up equivalent to the original (or to the $\mathcal{A}$-free equivalent just given). Even for the logics (and choices of e) we are principally concerned with here, though, the increased specificity of Coro. 16 need not be regarded an improvement on the original criterion of inessential occurrence, if what we are interested in is simplicity. Any substitution instance of a formula has at least the complexity (= number of occurrences of connectives, for present purposes) of the original formula, and we may be more interested in reducing that complexity, as we saw in the case of the absorption equivalences in Section 2: when we notice that $p * (q \circ p)$ is equivalent to p, for $*$, $\circ$ as $\wedge$, $\vee$, respectively, or as $\vee$, $\wedge$, resp., it may be the reduction of complexity that impresses us rather than the reduction in diversity among the surviving occurrences of sentence letters.[39] If anything useful and general can be said on the complexity-reducing front, that would be welcome, but the observations made in the present discussion do not seem particularly to help there. The $\{\circ, *\}$-formulation of absorption just given prompts the inclusion of a question or conjecture often arising in formal studies and calling for the deployment of the notion of essential occurrence, in connection with a related condition:

Example 17. First let us recall that $\{\wedge, \vee\}$ absorption suggests, in view of the commutativity of these connectives (according to most familiar logics) that, generalizing to cover cases which are different in this respect, that any of the equivalence of any of the following with p be regarded as an absorption principle for binary connectives $*, \circ$:

$$p * (q \circ p),\ p * (p \circ q),\ (q \circ p) * p,\ (p \circ q) * p,$$

and similarly with $*$ and $\circ$ interchanged in the given formula, making $\circ$ rather than $*$ the main connective. Similarly, we may regard as an *extraction* principle (for $*, \circ$), the equivalence of any of these formulas with, not p but instead q. Take the last one listed. The equivalence, for any L meeting the Default Background Conditions, with binary $\circ$ and $*$ (primitive or defined binary) connectives in the language of L, which could be recorded either as

$$(p \circ q) * p \dashv\vdash_{\mathsf{L}} q \qquad\qquad \vdash_{\mathsf{L}} ((p \circ q) * p) \leftrightarrow q,$$

[39] Just to make the notion explicit (as it applies to propositional languages) we define the *diversity* of a formula A to be n, where there are distinct sentence letters $p_1, \ldots, p_n$ where a sentence letter occurs at least once in A if and only if it is one of these p_i. Of course, one could introduce a similar diversity measure for connectives, which tracks the number of distinct connectives that occur in a formula rather than the number of *occurrences* of connectives, as complexity does, but we have no need of such a measure here. On the other hand, we shall re-encounter the notion of sentence-letter diversity just defined — towards the end of Appendix A in Section 5 in the discussion of what we call the 'removability' of sentence letters, which is a matter of reducing a formula's diversity while keeping it equivalent in a certain respect to the original formula (putting things very loosely: keeping it interdeducible with the original formula).

thus guaranteeing, in view of those Conditions that for any A, B, the formulas $(A \circ B) * A$ and B are L-synonymous: we can 'extract' B from any compound $A \circ B$ in which has been embedded, by application of our second connective $*$ which 'undoes' that embedding, when fed that compound and the formula A again as arguments. Since we get back to where we started by means of the latter application, extraction is one analogue for binary connectives of the (left) inverse connective theme for unary connectives discussed in note 17 and the text to which it is attached. (For further information, see the index entries in Humberstone [74] under "reciprocal function," as well as p. 707 there for the connections with logical subtraction, as in Remark 19(iii) below.)

With the form of extraction illustrated in the equivalence inset above (formulated twice over), we see that for $\mathsf{L} = \mathsf{CL}$ the following illustrations come to mind: (i) take $\circ$ and $*$ both as material equivalence ($\leftrightarrow$), (ii) take both as exclusive disjunction (however one prefers to notate it), (iii) take $\circ$ as a binary connective for the left projection ("projection to the first coordinate") truth-function and $*$ for the right or (second coordinate) projection. It is well known that cases (i) and (ii) for correspondingly notated connectives do not hold for $\mathsf{L} = \mathsf{IL}$, though of course there is considerable slack as to which connective in the intuitionistic case is involved. The convention use of $\leftrightarrow$ in discussion of IL gives an obvious candidate, since $\{p \leftrightarrow q\}$ is a set of *equivalence formulas* in the sense of abstract algebraic logic (see Appendix G) for IL though in the exclusive disjunction case, there are several discriminable options worthy of consideration [see 74, p. 786*f.*]. We return to asking about how option (iii) fares in for IL, after returning briefly to (i). A relatively familiar line of reasoning quickly shows that taking $\circ$ as $\leftrightarrow$ there is no way of finding a suitable candidate for $*$ That is, there is no definable or even (conservatively addable) $*$ for which $(p \leftrightarrow q) * p \dashv\vdash_{\mathsf{IL}} q$. If there, we would have the following implication (for all formulas):

$$A \leftrightarrow B \dashv\vdash_{\mathsf{IL}} A \leftrightarrow C \implies B \dashv\vdash_{\mathsf{IL}} C,$$

since we could '$*$-multiply' both sides of the antecedent here (by A), so that we had $(A \leftrightarrow B) * A$ and $(A \leftrightarrow C) * A$, which would simplify, using appropriate substitution instances of $(p \leftrightarrow q) * p \dashv\vdash_{\mathsf{L}} q$, to B and C, respectively. But no such cancellation conditional holds in IL, since, as we see by taking A as $\bot$, that this would then be a cancellation principle for intuitionistic negation ($\neg B \dashv\vdash \neg C$ implying $B \dashv\vdash C$, that is), which is notoriously not in fact correct (e.g., take B and C as $\neg\neg p$ and p, respectively: see note 17, where this is put in terms of the nonexistence of a left inverse for intuitionistic $\neg$). Stepping beyond the evidence presented by this example, let us venture a

Conjecture: IL provides no essentially binary connectives (primitive or defined) $\circ$, $*$, for which $(p \circ q) * p \dashv\vdash_{\mathsf{IL}} q$, or indeed any of the four extraction principles mentioned at the start of the present Example.

The case numbered (iii) above, using left and right projection connectives for $\circ$ and $*$ is now handled by the restriction to *essentially* — or more, explicitly IL-essentially binary connectives — where this is defined as in note 1. Another way of putting the conjecture, or rather, a slight strengthening of it, which emphasizes $*$ as the connective that 'does the extracting', is to say that there is no formula $C(p, q)$ in which p, q both occur essentially, and for which IL provides a binary connective satisfying: $C(p, q) * p \dashv\vdash_{\mathsf{IL}} q$. Though with a very differently behaving $\neg$ connective, we might venture for the case of FDE, introduced after Example 6, and with only $\wedge, \vee, \neg$ as primitive connectives in its language, a similar 'non nontrivial extraction'

Conjecture: FDE provides no essentially binary connectives (primitive or defined) $\circ$, $*$, for which $(p \circ q) * p \dashv\vdash_{\mathsf{FDE}} q$ (or, again, any of the earlier variants).

Both IL and FDE allow for the dummying in of new variables via absorption, as do all strengthenings of the $\{\rightarrow, \wedge\}$ and $\{\rightarrow, \vee\}$ fragments of (classical or intuitionistic) linear logic — see the paragraph after Example 6 above. In any such logic the projection connectives can be defined, for instance (illustrating with the absorption case, and tailored to the present discussion), using $\#^i_j$ for the i-ary projection connective to the j^{th} coordinate ($j \leq i$), and sticking with our $i = 2$ case: $A \#^2_1 B$ defined as $A \wedge (A \vee B)$, and $A \#^2_2 B$ as $B \wedge (B \vee A)$. At this point, one may react by saying that we can always define such projection connectives without worrying about the available primitive connectives are and how they behave accord to the logic under consideration, *simply by stipulating* that for any A, B, the formula $A \#^2_1 B$ is to be the formula A, and $A \#^2_2 B$, the formula B. The discussion this reaction merits is deferred to Appendix E. $\lhd$

Before launching into Example 17 we noted that the observations made to that point did not seem particularly helpful over the question of reducing complexity as opposed to reducing diversity to help there. One last such observation before returning to our 'generality' theme: the reference just made to the diversity among the surviving occurrences of sentence letters is to the number of token occurrences of sentence letters, rather than the number of types. If we take the case of $A(p_1, \ldots, p_n, p_{n+1})$ in which p_{n+1} occurs inessentially relative to the logic under discussion, then since this can be replaced by any C to give something equivalent, we can choose one of the $p_i \leq n$, as long as $n \neq 0$, thereby reducing the diversity in question.[40] This applies also to the discussion in Example 12 apropos of choosing a new variable r, meaning one distinct from the p_i as well as q there; this is why the parenthetical words "for simplicity" appear in the line after the equation (Kö) there — avoiding the more involved and potentially obfuscating reference to re-using an earlier p_i. In the case of the absorption example, the 're-use' strategy yields as a q-free equivalent to $p * (q \circ p)$, the formula $p * (p \circ p)$, with lower diversity but equal complexity, as opposed to the more obvious p, lowering both complexity and diversity.

We turn now to the historical setting in which the talk of generality in this connection arose. Whereas in (2) of Proposition 14 (or Coro. 15) the universal quantification governing the consequent remains of the implication figuring there remains in the metalanguage ("for all formulas D"), the generality theme as it figured in discussion emanating from Goodman [50] tended to be object-linguistic. So, in particular, with regard to the essential occurrence of singular terms rather than of sentence letters, and taking L — for definiteness — as classical first-order predicate logic, instead of saying something like "for all terms t: $\vdash_{\mathsf{L}} A(\delta) \to A(t)$," where δ is the term whose essentially was at issue, one had "$\vdash_{\mathsf{L}} A(\delta) \to \forall x.A(x)$."[41] The simplest example is perhaps the following excerpt from Appendix F, quoting from a recapitulation in Hart ([53], p. 29) of a (supervisorial) suggestion of H. E. Hendry:[42]

[40] And if there are nullary connectives available in the language (such as $\top, \bot$), we can use one of them in the $n = 0$ case.

[41] Or alternatively $\vdash_{\mathsf{L}} \forall x(A(\delta) \to A(x))$, where x is chosen so as not to occur free in the antecedent.

[42] Here the ellipsis omits the words "any designator part of", the obscurity of which has been noted by more than one commentator: see the last two lines of Niebergall [118], p. 143, and note 14 there. Concerning this 'designator part' aspect of Goodman's account, Tichý ([165], p. 89, line 7) writes that it "seems to me intuitively rather undermotivated," though he concedes it may help with a version of (what is now often called) the 'Slingshot' argument in the course of mounting an objection to

> Let us say that a designator δ *occurs essentially* in a sentence ψ if and only if ψ does not imply its own generalization with respect to [...] δ.

This is representative of Goodman and the immediately post-Goodman literature, conveniently and comprehensively reviewed in Hart [53], a small fraction of which is touched on in Appendix F below, where a slightly extended version of the passage just quoted appears. It is clear from the literature concerned that "implying its own generalization" means "implying its own universal generalization," in these publications.

Going back to Proposition 14, the analogous object-linguistic version would use propositional quantifiers.[43] Going back to the part of the (1) $\Rightarrow$ (2) proof of Prop. 14 where we had "$\vdash_{\mathsf{L}} A(q) \leftrightarrow A(p)$," we extract the forward direction, $\vdash_{\mathsf{L}} A(q) \rightarrow A(p)$, and since p does not occur in the antecedent, infer by uncontroversial rules governing (propositional) "$\forall$": $\vdash_{\mathsf{L}} A(q) \rightarrow \forall p \,.\, A(p)$. Now, we could equally well have extracted the other half of our biconditional instead, and from the fact that $\vdash_{\mathsf{L}} A(p) \rightarrow A(q)$ proceeded equally to the conclusion that $\vdash_{\mathsf{L}} \exists p \,.\, A(p) \rightarrow A(q)$ (i.e. $\vdash_{\mathsf{L}} \exists p(A(p)) \rightarrow A(q)$). So instead of saying "implying its own universal generalization," we could equally well have said "implied by its own existential generalization." Since both implications automatically imply their own converses, under the assumptions in force regarding the occurrence of p and q here, what we have is that the three formulas:

$$A(q) \qquad \exists p \,.\, A(p) \qquad \forall p \,.\, A(p),$$

are all L-equivalent, given the conditions on L in Proposition 14, which are satisfied by, for example, IL with propositional quantifiers (as formalized, natural deduction style, in Zdanowski [180],[44] for example). For (non-modal) propositionally quantified CL, on the other hand, it is a standard observation that $\exists p \,.\, A(p)$ and $\forall p \,.\, A(p)$ are already definable, as

$$A(\top) \vee A(\bot) \qquad \text{and} \qquad A(\top) \wedge A(\bot), \tag{‡}$$

Goodman's account. Since we are concerning ourselves here with essential occurrence for *unstructured* expressions, these complex issues can be avoided here.

[43] A comprehensive discussion of this topic is provided by Fritz [45]; such quantifiers figure prominently alongside other higher-order apparatus in several parts of Bacon [3] (e.g., Chapter 6 there).

[44] The title of [180] notwithstanding, rules for both $\exists$ and $\forall$ are given. The presentation uses a sequent-to-sequent formulation of the rules but is, as Zdanowski says, a natural deduction system rather than a sequent calculus (i.e., with operational rules in which material does not disappear from the premise-sequents on passage via the rules to the conclusion-sequent).

respectively. When the disjunction and the conjunction of two formulas are equivalent, so are the formulas themselves, so we have $\vdash_{\mathsf{CL}} A(\top) \leftrightarrow A(\bot)$, either side of which will do as a p-free formula witnessing its at most CL-inessential occurrence in $A(p)$, by which stage we have gone a considerable distance from a characterization in terms of generality, quantificationally construed — whether universally or existentially — in the object language itself.

There are also various conceptual complications raised by this latest turn the discussion has taken. For example, one might prefer to say not that $\exists p \centerdot A(p)$ and $\forall p \centerdot A(p)$ are definable, but that formulas behaving *like these* are already definable, if one takes the object-linguistic as opposed to the metalinguistic view of definition. The former view conceives of definition as adding vocabulary to the object language with the requirement that the *definiens* and *definiendum* are to be synonymous in the logic thereby extended in this expanded language, while the latter view thinks of definition as not changing the object language at all, but adding vocabulary to the metalanguage for referring to the formulas in a more convenient way.[45] For the object-linguistic approach, $\exists p \centerdot A(p)$ as introduced above, has for its principal or main logical symbol an existential propositional quantifier. For the metalinguistic approach, $\vee$ is the principal symbol — or main connective, in this case — is disjunction. This bears on this issue of which vocabulary occurs essentially in a formula because it bears on the more basic question of which vocabulary occurs at all in a formula. But there is also a further complication bearing specifically on essentiality, occasioned by the fact that quantifiers are under discussion, because of the relation between the terminology of "inessentially occurring" variables and that of "apparent variables". The opening paragraph of Leblanc [96] is helpful on this matter:

> The old distinction between an *apparent variable* and a real one was never too clearly drawn. Passages from *Principia Mathematica* and earlier logic treatises suggest, though, that an individual variable X

[45] Both views can be found in the literature on defining items of logical vocabulary when discussing *logics*; the object-linguistic view is more common in the literature on defining items of non-logical vocabulary when discussing (non-logical) *theories*. See the discussion starting in bottom paragraph of p. 176 of Humberstone and Meyer [82], and including note 21 on the following page (and references there supplied), as well as p. 57 of Humberstone [65] or subsection 3.16 of Humberstone [74]. In the course of his discussion of Tarski, Leśniewski, and Łukasiewicz's writings on definition, Hodges [62, p. 102], also describes these contrasting approaches to definition.

> *apparently* occurs in a formula A if X occurs in A just for form, i.e., if X can be replaced *salvo sensu* in A by some individual variable foreign to A, and that X *really* occurs in A if X does not *apparently* occur in A. Thus, 'x' *apparently* occurs in Russell's '$(\forall x)f(x)$' (= '$(x) \centerdot \varphi(fx)$', since '$x$' can be replaced *salvo sensu* in '$(\forall x)f(x)$' by 'y' whereas 'x' *really* occurs in '$f(x)$'. The distinction between a bound variable and a free one, which eventually displaced that between an apparent variable and a real one, does not match it, all assertions to the contrary notwithstanding. An individual variable may — by current standards — occur free in a given formula, and yet not *really* occur therein by the above criterion. 'x', for example, though it occurs free in '$(\forall x)f(x)$ & $f(x)$', does not *really* occur in '$(\forall x)f(x)$ & $f(x)$', since it can be replaced *salvo sensu* in '$(\forall x)f(x)$ & $f(x)$' by any one of 'y' [...] or — as we prefer to put it — since '$(\forall x)f(x)$ & $f(x)$' is semantically equivalent to any one of '$(\forall y)f(y)$ & $f(y)$', '$(\forall z)f(z)$ & $f(z)$', and so on.

Leblanc goes on to contrast a syntactic bound/free distinction and a semantic bound/free division to sort things out, though perhaps a preliminary hygienic measure would consist in notationally distinguishing free and bound variables, as Gentzen was perhaps the first to insist on, calling the former *parameters*, though for purposes of pure logic where there are no non-logical axioms and there is no particular interpretation (structure, model, ...) — or class thereof — in focus, there is really no distinction between such expressions and individual constants. Accordingly, when predicate logic is at issue (as in Appendix F, for example) we use x, y, ... as (bound) variables and a, b, ... as individual constants/parameters. This practice is not common in the case of second-order propositional logic, however, so it seems potentially burdensome to impose it in the present discussion. But just temporarily enforcing the distinction by using upper case italics for the bound variables and lower case bold italics for the propositional parameters, we are no longer tempted, as we might be with the notation of "$p \to q$" and "$\forall p(p \to q)$", to think that the propositional quantifier is e.o.-destructive, with p occurring essentially in the former and inessentially in the latter, once these are rewritten as $\boldsymbol{p} \to \boldsymbol{q}$ and $\forall P(P \to q)$. That bears particularly on the option of having propositional quantification in the object-linguistic (as opposed to the metalinguistic) option canvassed above. A case free from this complication is provided in Example 18, in which it is very clear that the author quoted intends q not to occur in the object-language formula denoted by the metalinguistic description — choosing A as $p \to q$, for example — "$\mathsf{del}_q(p \to q))$," despite the double appearance of "q" in that description:

Example 18. Weber [173, p. 339] defines the *deletion of q from $A(q)$*, which is intended to provide a way to "delete all information on a particular atom [here q] from a formula":

$$\text{Definition 3.4 } \mathsf{del}_q(A(q)) \;=\; A(\top) \vee A(\bot).$$

Weber points out that (i) the sentence letters occurring in $\mathsf{del}_q(A(q))$ are exactly those other than q occurring in $A(q)$, (ii) $A(q) \vdash_{\mathsf{CL}} \mathsf{del}_q(A(q))$, and (iii) that for any formula B in which only those sentence letters occur, if $A(q) \vdash_{\mathsf{CL}} B$ then $\mathsf{del}_q(A(q)) \vdash_{\mathsf{CL}} B$. Weber also notes that any formula A' satisfying the conditions (i)–(iii) say are satisfied by $\mathsf{del}_q(A(q))$, then $A' \dashv\vdash_{\mathsf{CL}} \mathrm{del}_q(A(q))$. Darwiche and Marquis [22] make the further observation that, so understood, $\mathsf{del}_q(A(q))$ is in effect, the background logic being CL: $\exists q.A(q)$.[46] Returning to the disjunctive form Still further, is more or less explicit in (i)–(iii), the disjunctive form $A(\top) \vee A(\bot)$, we recall that this is the form used in Kreisel's proof [89, p. 5] of *interpolation* for classical propositional logic, successive applications for all the q_i occurring in the antecedent but not the consequent of an implication for which we have $\vdash_{\mathsf{CL}} A(p_1, \ldots, p_\ell, q_1 \ldots, q_m) \to C(p_1, \ldots, p_\ell, r_1 \ldots, r_n)$ to obtain an interpolant B with $\vdash_{\mathsf{CL}} A \to B$ and $\vdash_{\mathsf{CL}} B \to C$ constructed from at most sentence letters (here the p_i) common to A and C. More recently, it is common to observe instead of pruning away the antecedent A's variables not occurring in C to obtain an interpolant B, we could equally well have started with the consequent $C(p_1, \ldots, p_\ell, r_1 \ldots, r_n)$ and pruned away the r_j instead (or indeed, as well), in this case using substitutions dual to those described by Weber, and illustrated in the $n = 1$ case here, starting with:

$$\vdash_{\mathsf{CL}} A(p_1, \ldots, p_\ell, q_1 \ldots, q_m) \to C(p_1, \ldots, p_\ell, r)$$

passing to:

$$\vdash_{\mathsf{CL}} A(p_1, \ldots, p_\ell, q_1 \ldots, q_m) \to (C(p_1, \ldots, p_\ell, \top) \wedge C(p_1, \ldots, p_\ell, \top)),$$

the consequent here serving as the interpolant B implied by A and implying C and not constructed from variables common to A and C. From a syntactic (or perhaps we might say, morphological) perspective, this

[46] Note that we are talking about arbitrary sentence letters q, not specifically about those occurring inessentially in $A(q)$; [22] uses a slightly different notation. We refer back to the present example from Appendix A.

Latin to Greek	Greek to Latin
$\tau(p) = \pi$	$t(\pi) = p$
$\tau(q) = \pi \leftrightarrow \rho$	$t(\rho) = p \leftrightarrow q$
$\tau(\#(A_1, \ldots, A_n)) = \#(\tau(A_1), \ldots, \tau(A_n))$	$t\#(\alpha_1, \ldots, \alpha_n) = \#(t(\alpha_1), \ldots, t(\alpha_n))$

Table 1. Two Translations

second style 'deletes' all occurrences of the unshared variables from the consequent, just as the first deletes all of the antecedent's unshared variables, though semantically it would not warrant Weber's attention because it doesn't delete r-pertinent *information* from C as the disjunctive form weakens A by deleting q-pertinent information from it, constructing instead a formula which is in general *stronger* than C. (This contrast between working from the antecedent and working from the consequent to find an interpolant arose in the work by Andrew Pitts in the setting of propositionally quantified intuitionistic logic, in which the simple equivalence to Boolean compounds of formulas with suitably placed $\bot, \top$ is not available but there is a corresponding antecedent-based *vs.* consequent-based distinction. A clear discussion with relevant references can be found in Iemhoff [84]. Also worth bearing in mind is the observation, in Połacik [137], that Pitts' interpretation of intuitionistic propositional quantifiers is not the only game in town.)

Returning to Weber's del operation, to illustrate what might be considered a case of unwelcome language-dependence on the part of the del operation, we enlist a simplified translation argument in the style of Max Black and David Miller.[47] We have two interpreted propositional languages, each with only two sentence letters, between which we will keep track of the difference by writing the sentence letters of one as p, q and of the other as π, ρ, understood in such way that p and π are accurate translations of each other, whereas q is translated from the first language into $\pi \leftrightarrow \rho$ in the second, and conversely, ρ is translated in the other direction as $p \leftrightarrow q$, as recorded in Table 1. (The logical independence of p, $p \leftrightarrow q$, is only fitting given that they are translations of distinct — and thus logically independent — sentence letters π, ρ.) In the table, the translation into the second language, with its Greek sentence letters, is

[47] See Miller [112], and, for Black's discussion as well as further relevant publications by Miller, consult the index entries in [74] under 'language-dependence objections'. Several criticisms of Miller's use of these considerations specifically in connection with verisimilitude are cited and responded to in [113, 114].

called τ, and that into the first language, with its Latin sentence letters, t. The third row of the Table 1 pertains to any Boolean connectives (such as $\leftrightarrow$), over which we use $\#$ as a variable (for an n-ary connective), which we assume to be common to both languages, and to be presumed by speakers of both of them to have their logical properties as specified by CL. Writing $\equiv_{\mathsf{CL}}$ for CL-equivalence, under these conditions, with both the Latin-based language and the Greek-based language, the two translations are mutually inverse, in the sense that for any formula A of the former language we have $t((\tau)A) \equiv_{\mathsf{CL}} A$ and for any formula α of the latter, $\tau(t(\alpha)) \equiv_{\mathsf{CL}} \alpha$, as we illustrate with $A = q$, showing that $t(\tau(q)) \equiv_{\mathsf{CL}} q$. This holds because, since $\tau(q) = \pi \leftrightarrow \rho$, we have

$$t(\tau(q)) = t(\pi \leftrightarrow \rho) = t(\pi) \leftrightarrow t(\rho) = p \leftrightarrow (p \leftrightarrow q) \equiv_{\mathsf{CL}} q.$$

Now observe that applying Weber's del operation on the Latin-based side we get $\mathsf{del}_p(p \wedge q) = q$, and similarly on the Greek-based side, $\mathsf{del}_\pi(\pi \wedge \rho) = \rho$. But then $\mathsf{del}_{\tau(p)}(\tau(p) \wedge \tau(q)) = \mathsf{del}_\pi(\pi \wedge (\pi \leftrightarrow \rho)) \equiv_{\mathsf{CL}} \mathsf{del}_\pi(\pi \wedge \rho) = \rho \not\equiv_{\mathsf{CL}} \tau(q)$ since $\tau(q)$ is, instead, $\pi \leftrightarrow \rho$. $\lhd$

Apropos of this Example, we add:

Remarks 19. (i) In later publications, such as Lin and Reiter [101] and Lang et al. [93], "Delete" (or del) becomes "Forget." Here is a passage from p. 392*f.* of the latter paper:

> Two notions play a central role in this paper. The first one, (*semantical*) *formula-variable independence* (FV-independence for short) tells [us] that a propositional formula Σ is independent from a given set V of variables if and only if it can be rewritten equivalently as a formula in which none of the variables in V appears. The second one is the notion of *forgetting a given set V of variables in a formula* Σ. It is intimately linked to the notion of formula-variable independence because, as we show, the result of forgetting a set of variables V in a formula Σ can be defined as the strongest consequence Σ independent from V.

"Equivalently" here means "CL-equivalently." Thus, when $V = \{p_i\}$ the authors' talk of a formula's being independent from V amounts to p_i's not occurring CL-essentially in the formula in question. The formulation addressing the general case of V is suggestive of the possibility that there might be several variables which collectively have the property of occurring — one or another of them — in any equivalent of the formula in question without any of them individually occurring essentially in that formula. A discussion of the possibility such a 'disjunctive' version of essential occurrence is postponed to Appendix G.

(ii) Unwelcome or otherwise, the language-dependence considerations just seen in the case of del affect the notion of essential occurrence itself: from the fact that a sentence letter occurs essentially in a formula, we can't conclude that that sentence letter's translation into an intertranslatable language occurs essentially in the translation of that formula. One sees easily that p occurs CL-essentially in $p \leftrightarrow q$, whereas $\tau(p) = \pi$ does not occur CL-essentially in $\tau(p \leftrightarrow q) = \tau(p) \leftrightarrow \tau(q) = \pi \leftrightarrow (\pi \leftrightarrow \rho) \equiv_{\mathsf{CL}} \rho$, and of course π does not occur at all in the distinct sentence letter ρ. (See Note 28 in Humberstone [70], where, in different notation, the case presented in Example 18 appears in a different notation, and the point is made that the choice of sentence letters whose $\equiv_{\mathsf{CL}}$-classes freely generate the Boolean algebra of equivalence classes of formulas does not represent an 'algebraically objective' choice from among numerous alternative candidate sets of free generators. As [70] mentions, the situation is rather different with IL and Heyting Algebras.)

(iii) The language-dependence issue is also a potential threat to one of two possible ways of taking the notion of *logical subtraction*.[48] In the course of Example 18, it was noted that $\mathsf{del}_p(p \wedge q)$, or, if preferred, $\mathsf{forget}_p(p \wedge q)$ was q — just as one one expect from an account of logical subtraction (notated with the aid of "$-$") $(A \wedge B) - A$ was equivalent provided that A, B have no propositional variables in common. Without the proviso, inconsistency results (given that we require the subtraction to provide a congruential context), as it does if the proviso is weakened to "provided that A, B are logically independent". (Take the background logic to be CL for definiteness, though the point is more general that this might suggest.) The first of four decisions to make about logical subtraction in subsection 5.22 of [74, p. 680] was headed "Is a connectival treatment desirable?" and an affirmative answer was preferred there: since '$\_ \wedge B$' typically adds requirements for the truth of $A \wedge B$ to those required for the truth of A, so the envisaged reciprocal operation '$\_ - B$' should typically remove such truth-requirements, so if either is given a connectival treatment — which is to say, is treated as a (fundamental or derived) operation in the (absolutely) free algebras of formulas — then so should the other be. More recent work on logical subtraction has

[48] For background, see Humberstone [74, subsection 5.21], [74, pp. 677–680], or [68, §3].

gone for the non-connectival option, a function taking a pair of formulas to a formula without necessarily having those formulas as immediate subformulas — or indeed a partial function. (See the index entries for subtraction in Yablo [178] — though Yablo's discussion is exploratory rather than committed to a particular account. The distinction between connectival and non-connectival operations on formulas is related to, but does not coincide with, the distinction between the two conceptions of definition raised above in the text to which note 45 is appended.) $\lhd$

4. Aboutness *chez* Demolombe and Fariñas del Cerro.

The title of the preceding section spoke of approaching the subject "via generality," and although our main subject is essential occurrence, rather than aboutness, we have seen the connection proposed by Goodman between the things a sentence is about and the essential occurrence in that sentence of references to those things. These considerations together suggest the value of looking at a more recent development in theorizing on aboutness and comparing it with Goodman's own proposal (and variants thereof, mainly touched on in Appendix F). This more recent work is represented in Demolombe and Fariñas del Cerro [26–28], which appeared in 2000, 2009, and 2010 respectively, and will usually be referred to in what follows by via a reference to the relevant year.[49] — the focus of Demolombe and Jones [29], an issue to which return at the end of Appendix J. Oversimplifying the situation somewhat, 2000 paper considers first-order languages without function symbols or identity, but with, in addition to the usual $\exists$ and $\forall$ quantifiers, a special-purpose restricted quantifier $\exists u \neq t$ (and definable dual $\forall u \neq t$) and t is a individual constant or an individual variable, but no independent predicate symbol $=$ (or $\neq$), creating the appearance, which in these papers the authors warn is deceptive, that identity is present.[50]

[49] These papers deal with being about an object, individual or, as the authors tend to say, an entity, rather than being about a topic, or subject-matter

[50] Note 1 of [26], for example, reads: "Here $x \neq c$ is used as a notation to denote restricted quantifiers; it is not taken as a sentence with an occurrence of [the] equality predicate." On the other hand, we take it that c *does* occur in formulas constructed with these restricted quantifiers. "$\neq$" can be regarded as an expression which takes a quantifier and a term and yields a corresponding quantifier ranging over all elements of the domain other than the current interpretation (denotation) of that term.

The respects in which the above description oversimplifies concern the status of these restricted quantifiers, which has been generalized in more than one way in that description. In fact, in the 2000 and 2009 papers, Demolombe and Fariñas del Cerro do not allow this 'exceptive' quantifier $\exists u \neq t$ to feature a variable in the role of t here, requiring an individual constant. Not only that, but the language they discuss has, among its individual constants, a single *distinguished* constant c (the language being called L_c in fact, to highlight this role) to figure as t, so they always write "$\neq c$" in this restricted quantifier construction. (The term 'exceptive' just used is based on the corresponding universal construction, where we can read "$\forall x \neq c \,.\, Px$" as everything except for c is P," as long as this is understood as meaning "everything with the possible exception of c is P," since we do not want to exclude the truth of Pc. For this reason, the word *exemptive* would perhaps be more suitable than *exceptive*, as an alternative to "restricted" with its suggestion that the restriction involves what would otherwise be a self-standing open formula — "$x \neq c$."[51]) Apropos of logical subtraction, mentioned in Remark 19(iii), one might express "$\forall x \neq c \,.\, Px$" with its aid as "$(\forall x \,.\, Px) - Pc$" (as in Humberstone [68, p. 67]).

Remark 20. One casualty of the presence of the $\neq$-restricted quantifiers in the absence of $=$ (or $\neq$) as a predicate symbol would appear to be the interpolation property, last seen in a propositional setting, in Example 18. Here we can put this by calling a formula true in every interpretation/model (the details of which follow shortly) *valid* and consisting in the existence for all formulas A, C with $A \to C$ valid of a formula B with no non-logical vocabulary except that appearing in both A and C, for which the implications $A \to B$ and $B \to C$ are both valid. (Complications for certain more refined versions of interpolation — involved with tracking positive and negative occurrences of such vocabulary in the sense of Example 24 and note 88 — arise for first-order logic with identity/equality, treated in Motohashi [116], but we are concerned here neither with those refinements nor with the presence of identity.) For example, with P, Q monadic predicate letters, consider $A = \neg Pb \wedge \forall x \neq c \,.\, Pc$ and $C = Qb \to Qc$. The only common non-logical vocabulary from which to construct the desired B consists of the constants b, c, but the current language is missing the obvious would-be

[51] The phrase *exemptive quantifier* is used in Humberstone [64, p. 179] with a somewhat different meaning.

interpolant $b = c$ (Adding identity as a self-standing predicate to the language is problematic, as is explained in Demolombe and Fariñas del Cerro [28]; see also note 107 of Humberstone [81]). It would be interesting to know what classes of structures are definable using a language with the currently envisaged resources, intermediate in terms of expressive power between first-order logic with identity and (what is normally called) first-order logic without identity. $\lhd$

As for disallowing variables in the post-"$\neq$" position, it seems unduly restrictive (now using that word in its everyday sense) to disallow the inference from $\exists x \neq c \centerdot Fx$ to $\exists y \exists x \neq y \centerdot Fx$. In fact in the 2010 paper, where function symbols, but still not identity — identity proper and unbridled — are added to the vocabulary, the exceptive quantifies have disappeared from the language, along with the distinguished individual constant, the language now being called L rather than L_c. So we have lost expressive power along that dimension. Also lost is the opportunity convey a significant feature of the earlier language, which is highlighted by the authors and which we will get to presently — more specifically: in the 'non-necessity' part of the proof of Proposition 22. The presentation of this material on Demolombe and Fariñas del Cerro in Appendix B of Humberstone [81] stays closer to the original "distinguished constant" version, in order to compare this with the use of models with distinguished elements (one per model) in play in Russell [151] — such as Kripke models with a distinguished ('actual') world, and models for indexical logic singling out a domain element as representing the current speaker, to provide a denotation for first person pronouns, etc.

Demolombe and Fariñas del Cerro [28] contains material similar, insofar as our current discussion is concerned, to that in the same authors' earlier published [27] — the 2010 and 2009 papers, respectively — though there are some notation differences. In particular the "*NAbout*" of the earlier paper (and which we shall use here) appears as simply "*NA*" in the later one. That dyadic metalinguistic predicate is defined in a way that will be explained shortly, it having proved more convenient to do things this way round, starting with this N(egated) Aboutness relation first (p. 23): "We say that a **formula** F **is about the entity** c, if it is not the case that $NAbout(F, c)$." (The definiendum is then abbreviated to $About(F, c)$.[52]) Here, for the sake of continuity with the discussion in

[52] The 2000 paper contains a potentially confusing typo near the base of p. 1236 — "We say that a formula F is about the entity t, if it is not the case that it is about the

Demolombe and Fariñas del Cerro's papers, we use $F, G, \ldots$ rather than $A, B, \ldots$ as schematic symbols for (open or closed) formulas.

Somewhat adapting the presentation in especially the first two of these papers, we think of a model (or interpretation) for a first-order language as incorporating an assignment of values to individual variables, rather than as needing supplementation by such an assignment relative to which truth is defined for the closed formulas of such a language, L. As noted in the second paragraph of this section, Demolombe and Fariñas del Cerro work instead with a language L_c with a single distinguished individual constant c, for which a model provides an element of its domain to interpret this constant. The adaptation alluded to is that we do not single out a constant and a domain element, but simply take a model M to comprise a domain or universe D_M and functions $\|\cdot\|^M_{\mathsf{vbl}}$, $\|\cdot\|^M_{\mathsf{cns}}$ and $\|\cdot\|^M_{\mathsf{prd}}$, which assign appropriate values to the individual variables, individual constants and predicate letters of L.[53] That is, for each individual variable v, $\|v\|^M_{\mathsf{vbl}} \in D_M$, for each individual constant c, $\|c\|^M_{\mathsf{cns}} \in D_M$, and for each n-ary predicate letter P:

$$\|P\|^M_{\mathsf{prd}} \subseteq \underbrace{D_M \times \cdots \times D_M}_{n \text{ times}}$$

Definitions 21. (i) For an individual variable v, models M_1, M_2 are *v-variants* iff $D_{M_1} = D_{M_2}$, $\|c\|^{M_1}_{\mathsf{cns}} = \|c\|^{M_1}_{\mathsf{cns}}$ for all constants c, $\|P\|^{M_1}_{\mathsf{prd}} = \|P\|^{M_2}_{\mathsf{prd}}$ for all predicate letters P, and for all variables v' other than v, $\|v'\|^{M_1}_{\mathsf{vbl}} = \|v'\|^{M_1}_{\mathsf{vbl}}$. We write $M_1 \approx_v M_2$ when M_1 and M_2 are v-variants.

(ii) For an individual constant c, models M_1, M_2 are *c-variants* iff $D_{M_1} = D_{M_2}$, $\|v\|^{M_1}_{\mathsf{vbl}} = \|v\|^{M_1}_{\mathsf{vbl}}$ for all variables v, $\|P\|^{M_1}_{\mathsf{cns}} = \|P\|^{M_2}_{\mathsf{cns}}$ for all predicate letters P, and $\|c'\|^{M_1}_{\mathsf{cns}} = \|c'\|^{M_1}_{\mathsf{cns}}$ for all constants c' other than c. We write $M_1 \approx_c M_2$ when M_1 and M_2 are c-variants.

(iii) For models M_1, M_2 with $D_{M_1} = D_{M_2} = D$ and $d \in D$, we say that M_1 and M_2 are *d-variants* when, as well as having $D_{M_1} = D_{M_2}$, we have $\|v\|^{M_1}_{\mathsf{vbl}} = \|v\|^{M_1}_{\mathsf{vbl}}$ for all variables v, $\|c\|^{M_1}_{\mathsf{cns}} = \|c\|^{M_1}_{\mathsf{cns}}$ for all constants c, and for all n-place predicate letters P (for each n) $\langle d_1, \ldots, d_n\rangle \in \|P\|^{M_1}_{\mathsf{prd}} \Leftrightarrow$

entity t"—which appears in the intended form in the 1999 paper (using the predicate *NAbout*, which needs to do the work of the second occurrence of "about" in the sentence just quoted).

[53] A treatment closer in approach to that of Demolombe and Fariñas del Cerro is given in Appendix B of [81] in order to focus on parallels with parts of Russell [151] in which the models come equipped with distinguished elements.

$\langle d_1, \ldots, d_n\rangle \in \|P\|^{M_2}_{\mathsf{prd}}$ whenever $d \notin \{d_1, \ldots, d_n\}$. We write $M_1 \approx_d M_2$ when M_1 and M_2 are d-variants.

A more explicit notation would, for $M_1 \approx_v M_2$, $M_1 \approx_c M_2$ and $M_1 \approx_d M_2$, write respectively:

$$M_1 \overset{\mathsf{vbl}}{\approx}_v M_2, \quad M_1 \overset{\mathsf{cns}}{\approx}_c M_2, \quad M_1 \overset{\mathsf{prd}}{\approx}_d M_2 \tag{‡‡}$$

with the superscript indicating which vocabulary is subject to reinterpretation as we pass between variant models, but we forgo the additional clutter here.[54] Note that in the first and second cases of the more explicit notation, the superscripted indicator simply reflects the syntactic type already indicated in the subscript, but in the third case it does not: we use, instead, an *element of the domain* — not any kind of linguistic expression — so as to indicate how to vary the interpretation of a *predicate letter*. The first these three equivalence relations is just the usual device needed for the inductive definition of truth/satisfaction to handle the case of quantifiers: $M \models \exists v \mathbin{\text{.}} \varphi$ iff for some $M' \approx_v M$, we have $M' \models \varphi$.[55] Similar clauses in the case of the Boolean connectives we may safely omit here, as also for the universal quantifier, but the special exceptive/exemptive quantifier "$\exists u \neq t$" mentioned above deserves have its clause spelled out, though, where t (for 'term') is either an individual constant or an individual variable (in this generalized version of Demolombe and Fariñas del Cerro), and $\|t\|^{M'}$ means $\|t\|^{M'}_{\mathsf{cns}}$ in the former case and $\|t\|^{M'}_{\mathsf{vbl}}$ in the latter:

$$M \models \exists v \neq t \mathbin{\text{.}} \varphi \text{ iff for some } M' \approx_u M, \|v\|^{M'}_{\mathsf{vbl}} \neq \|t\|^{M'} \text{ and } M' \models \varphi.$$

The third equivalence relation is the key conceptual novelty introduced in the work of Demolombe and Fariñas del Cerro mentioned above. In

[54] On the subject of notation, it should be noted that Demolombe and Fariñas del Cerro use "$i_M(v)$", "$i_M(c)$", and "$i_M(P)$" for the interpretations in a model M of the expressions concerned, rather than the present "$\|\cdot\|^M$" notation. Note that here the *interpretation* of such-and-such a symbol is not intended to mean anything more than its *semantic value* in the model in question, rather than to buy into anything like the interpretational vs. representational construals of model theory featuring in Etchemendy [34].

[55] This is what is called the *model-changing* semantic treatment of quantifiers in p. 256*f.* of Humberstone [73], as distinct from the more commonly encountered *model-fixed* treatment there described, though the two come to the same thing for defining semantic consequence if attention is restricted to closed formulas. (This note is continued in Appendix J.)

their [28, p. 1234], the authors provide a helpful gloss on their proposed identification of "is not about" (reproduced two paragraphs below as (⊛)):

> The intuitive idea in the semantical characterization of sentences that are not about the entity denoted by a term t is that the truth value of such sentences should remain unchanged if we change the truth value of the atomic facts that are about t.

For Demolombe and Fariñas del Cerro this talk of facts is purely heuristic and informal, and may strike some as involving a quaint throwback to the days of logical atomism, so it is worth recalling that the standard model-theoretic approach to first-order languages is indeed atomistic in the relevant respects, and can be recast, if desired, at a level which explicitly invokes (relative to a given model) fact-like entities in the inductive definition of truth (in a model) — as in McKinsey [106], van Fraassen [168]. But, as Demolombe and Fariñas del Cerro show, their idea does not itself require this departure from the standard presentation of the semantics.[56]

The idea is promising, though it may need some minor tweaking because of the issue raised in the paragraph following Example 10: the fact that, independently of a specified model (or class thereof, alike in this respect), there is no such thing as *the denotation* — or, more generally, the interpretation — of an item of non-logical vocabulary. Another way of putting this point (and others like it) is to warn of a danger of use–mention confusion in discussions of aboutness, a cautionary note sounded in Niebergall [118, pp. 141 and 148] apropos of the Goodman-related discussion of the subject.[57] The role of the third relation in (‡‡), $\approx_d$, lies in specifying a binary relation whose complement will be the desired

[56] If one were to make such a departure, it would probably be best not to speak, as our authors do, of the "truth value of the atomic facts," as though the facts were themselves atomic sentences, and to formulate matters in terms of *obtaining* (relative to a model) rather than *being true* (in the model).

[57] The title of Humberstone [69] can be accused of exhibiting the same confusion: In "What Fa says about a" the first occurrence of a is mentioned, while the second is used. But if this is taken as the schematic representation of such sentences as "Athens is a city," the — as Carnap called it — *autonymous* use of language is harmless, at least when empty names are avoided, since the second occurrence simply denotes what the denotation of the first occurrence itself denotes: in the present case, the first occurrence denoting the name "Athens," the second denotes the city of Athens itself. We can't do quite that in the present model-theoretic setting, since there is no privileged "what such-and-such denotes" to be had. Incidentally, more than once in the papers under discussion here, Demolombe and Fariñas del Cerro cite Carnap [15]

aboutness relation, and which is notated as *NAbout* in the 2010 paper, the notation used here.[58]

To give the definition of this *NAbout* relation given on p. 23 of the 2009 paper and on p. 1236 of the 2010 paper using the notion of a 'ground term' (as computer science logicians call what are elsewhere called closed terms), which, seeing as we are not here considering function symbols, we reproduce as "individual constant," avoiding further complications induced by the presence of function symbols in the 2010 paper and using the $\approx_d$ relation where the d in question is the interpretation of the constant c in the definiendum — we get to problems with this way of speaking, in the following paragraph — we say that for any sentence F and constant c, we have:[59]

$$NAbout(F, c) \text{ iff } \forall M \forall M' \big((M \approx_{\|c\|^M_{\mathsf{prd}}} M' \ \&\ M \models F) \Rightarrow M' \models F\big). \quad (\circledast)$$

Note that we could equally well have written "$\|c\|^{M'}_{\mathsf{prd}}$" in place of "$\|c\|^{M}_{\mathsf{prd}}$", as a subscript to the "$\approx$", since, as Definition 21(iii) tells us, any models related by the $\approx_d$ relation only in respect of which tuples of entities from their common domain the models must agree on for the interpretation of predicate letters — namely those tuples in which d does not appear; in particular they agree on the interpretation of the constants, so here, where the relevant d is $\|c\|^{M}_{\mathsf{prd}}$, d is also $\|c\|^{M'}_{\mathsf{prd}}$. Note that although the individual constant c figures prominently in specifying the relevant variance relation here, the subscripts have been kept in place in the $\|\cdot\|$ notation, to emphasize, with the "prd" (rather than "cns") that the only differences between the variants M and M' in ($\circledast$) pertain to the semantic values of the predicate letters, not of individual constants such as c. Finally, $About(F, c)$ is defined as: not $NAbout(F, c)$, and it is this last notion that is the proposed formal explication of what it is for a sentence to be about an object.

All is well but for one detail, somewhat ironically given the inclusion of references to Carnap in Demolombe and Fariñas del Cerro's publications (see note 57): what we have been given is an account of what it

in connection with aboutness, though, in the absence of any specific section or page numbers, it is not always clear exactly what they have in mind.

[58] The notation *NA* is used use 2000 paper and $N\,About$ — with a space — in the 2009 paper.

[59] Recall, though, that while we are contemplating the prospect of several such constants as in their 2010 paper, Demolombe and Fariñas del Cerro have just one in mind in the earlier papers, the distinguished constant of L_c.

is for a sentence to be about an individual constant (or more generally, a closed term), rather than of what it is for a sentence to be about an individual (or object, or 'entity') in general: the definition of *NAbout* is in the metalanguage and the "c" in "$NAbout(F, c)$" refers to the constant c (of one or other of the object languages under discussion, L_c or L, depending on which paper is considered).[60] As already remarked, this point was briefly raised in connection with Goodman in the paragraph after Example 11.

Definition 5 of the 2000 paper [26, p. 90] is representative:

> Let F be a sentence of language L_c. We say that F is *not about* an object named by the constant symbol c iff for every interpretation M, we have $M \models F$ iff for every interpretation M' in M^c we have $M' \models F$.

The authors write "M^c" for the set of models M' such that — as we have been putting it here — $M' \approx_{\|c\|^M_{\mathsf{prd}}} M$ (or, equivalently, such that $M \approx_{\|c\|^M_{\mathsf{prd}}} M'$), and should be thought of as defining *NAbout*, or *NA* as they were notating it in the 2000 paper (though we reproduce their remarks using the longer expression here), since an explicit definition should ideally follow the protocol of defining only unstructured vocabulary, if only for the sake of reinforcing habits of methodological hygiene in respect of such matters. We could fix this by simply shifting the "not" in "not about" over the right of the main (here the first occurrence of) "iff," though the authors' route is slightly different, writing "The fact that F is not about object c is denoted by $NAbout(F, c)$": so here we

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[60] Our authors become even more confused on the subject of aboutness in their 2010 paper's Definition 4 [28, p. 1235], trying to define a notation for the "set of elements of D_M which are about the term t": aboutness relates a linguistic (or more abstract but still representational) item to something in the realm being represented, so if anything was going to be about anything here, it would be the term that was about an element of D_M.[61] And in any case, it is sentences (or the propositions thereby expressed) that are in the relevant linguistic or representational category — or at best predicates (and their propositional analogues),[62] and certainly not names or terms — that can be described as being *about* things. (Why, you might ask, even if they have got things the wrong way round here, do the authors speak of the "set of elements of D_M" instead of "the element of D_M," since an individual constant c is assigned a unique denotation relative to M, called here $\|c\|^M$, and called by the authors $i_M(c)$? The answer is that here they are permitting function symbols, so a term can have proper subterms, which can have proper subterms again,... — meaning there is no upper limit to the number of domain elements that can referred to in the construction of a single term.)

have passed from “object named by the constant symbol c” of the passage inset above, “object c.”[63]

This masks the issue of whether the aboutness (or more precisely, the ‘nonaboutness’) relation has been defined for linguistic expressions or not (as the second relatum). To repeat for the present setting the point already made: the initial formulation doesn’t strictly make sense, because the words “an object named by [...] c” occur on the left in on the left of the “iff,” but the quantification over models only begins on the right, so we are not considering the constant c with reference to any model/interpretation and there is no such thing as an object named by c: that would have to be $\|c\|^M$ for some given M, which we have precisely not yet been given.

One might consider remedying the defect and defining *NAbout* (and so *About*) to relate sentences F to potential domain elements rather than to individual constants that might be interpreted as naming them, simply enough; here the formulation stays as close as possible to that quoted above, even to extent of addressing L_c with its distinguished constant c, and suboptimal use of “not about” on the left:

> Let F be a sentence of language L_c. For any object d, we say that F is *not about* d iff for every interpretation M with $\|c\|^M = d$, we have $M \models F$ iff for every interpretation M' in M^c we have $M' \models F$.

This does not seem very promising, though, since the only properties of objects (d here) that matter for model-theoretic purposes are the properties that are given by this or that model/interpretation — at least insofar as questions of validity and satisfiability are concerned (in respect of which there is no role to be played by any notion of an ‘intended model’). A rough-and-ready alternative response would be to replace “about” when followed by a reference to a term t to with a new word, “about*”, say, taking the use in a particular discursive context of

Sentence S is about* “Babylon”

to mean that S is about whatever, relative to that context, “Babylon” refers to.[64] In the kind of conversational setting envisaged by Carnap (see note 57 above, and Appendix H below), this would be — setting

[63] The double occurrence of “iff” in this definition may initially seem puzzling, and its relation to the formulation in the earlier (⊛) unclear; the situation is clarified in Appendix I.

[64] There is no intended connection between the asterisk in “about*” here — or “*About**” below — and the famous use of “*” in David Lewis’s discussion (in [100])

aside Carnap's own qualms — the city of that name, though in a different setting, it might be the horse that just came in second in race that has just finished at Kenilworth, South Africa. Similarly in the case of a formal language a sentence is about* t in a context in which a model M is salient, amounts to its being about $\|t\|^M$ — making "whatever t refers to" a context-shifting device of the decried by Kaplan, though, to the contrary, many of us were reassured early on by our mothers that there weren't really any monsters to be afraid of.[65] (The formulation "whatever t refers to" is used here in preference "what t refers to" as a gloss on "the contextually salient denotation of t" to by-pass a temptation to think of the latter as a concealed question — a suggestion with some *prima facie* appeal made and later rebutted in the linguistic literature on *about*-phrases.[66]) We can retain the definition ($\circledast$) of *NAbout* provided by Demolombe and Fariñas del Cerro, but relabelling the definiendum as "*NAbout**" and its negation as "*About**" to make it clear that the relation defined is one between a sentence and a term rather than between the sentence and what that term is (relative to a given model) interpreted as denoting — though of course the definiens concerns itself with the latter potentially varying denotations of that term.

With that out of the way, we can attend to an issue Demolombe and Fariñas del Cerro raise in several places, introducing it in [26] with the following words:

> As pointed out by Goodman in [50], the fact that a sentence mentions an object c does not occur in a sentence F does not imply that F is not about object c. [...] First, the fact that c does not occur in a sentence

of aboutness in relevant logic - a playful allusion to the "Routley star" semantic treatment of De Morgan negation (from Routley and Routley [148], originally).

[65] Our mothers, and in this case, others too: Schlenker [153] started the ball rolling, *q.v.* for the relevant Kaplan references; see also Humberstone [73, p. 260], for connections with some other themes of the current discussion. Subsequent theoretical discussions can be found in Briciu [11] and work by Santorio and by Rabern and Ball in [11]'s bibliography; relevant empirical findings for specific natural languages are given in, for example, Anand and Nevins [2], Sudo [163], Deal [25], and Major [102].

[66] See Frana [40], or, for a brief treatment supplying the relevant details and references, discussion on pp. 18–20 of Frana [41]. The phase "contextually salient" is deliberately vague, with context not to be understood as specifically the ambient *linguistic* context — whether that is taken as the sentential context, as in the title of Section 2, or the context of utterance, for resolving the interpretation of indexicals, and its appropriate explication a matter on which disagreement is to be expected. Partee [128] helpfully aired many relevant considerations with no claim to have settled everything in this area definitively.

> F does not imply that F is not about c. [...] Second, the fact that c occurs in a sentence F does not imply that F is about object c.

In the remark about Goodman, all is well on the use–mention front, aside from the choice of the letter "c" (reserved for individual constants), since a non-linguistic object is the right kind of thing, typically, for a sentence to mention and also to be about. In the second and third remarks, things go off the rails, because that kind of thing is *not* the right kind of thing to 'occur' in a sentence, for which we need to take the choice of "c" seriously as appropriately suggesting an individual constant. Clearing this up, then, the question becomes one of what implications hold between c's occurring in F on the one hand, and F's being about* c — in the technical sense that we have $About^*(F, c)$ — on the other; we use the "F is about* c" formulation.[67] Given our present concerns, however, a more pertinent version of the question arises as to the implications between c's occurring *essentially* in F and F's being about* c. As the authors observe in the passage marked by the second ellipsis in the quotation about, we can always 'dummy in' any extraneous constant by such devices as the absorption equivalences of classical predicate logic, called QCL below (as in Appendix F) for "Quantified CL" — which is the appropriate L with respect which to take this reference to essentiality as L-essentiality.[68] Since these involve *inessential* occurrence, they leave open the question of whether c can occur essentially in F without F's being about* c.

As it happens, the authors address this last question in an aside to the way they fill the gap in our *first* ellipsis from the passage quoted, so we include both points in this in the following distillation of relevant discussion in Demolombe and Fariñas del Cerro [26, p. 94]:

PROPOSITION 22. *An individual constant c's occurring QCL-essentially in a sentence F is neither necessary nor sufficient for F's being about* c.*

PROOF. Non-necessity: It is clear that c occurs QCL-essentially in, for example, $\forall x \neq c(Px)$, where P is a monadic predicate letter, but since models M and M' with $M \approx_{\|c\|^M} M'$, as far as $d = \|c\|^M$ P is concerned differ at most on whether $d' \neq d$ is in $\|P\|^M$ $(= \|c\|^{M'})$, M and M' cannot differ over the truth of $\forall x \neq c(Px)$. (Here we are identifying $\langle d \rangle$ with

[67] Reminder: in Demolombe and Fariñas del Cerro [26], *About* and *NAbout* of the successor papers, to which the asterisk is here appended, appear simply as A and NA.

[68] The authors also note at this point such examples as $Pc \vee \neg Pc$, in which c occurs but which are not (as we would say) about* c.

d, and taking it that c does indeed occur in this formula — see note 50 — and not just in some metalinguistic expression denoting the formula.)

Non-sufficiency: Switching from restricted to unrestricted quantification, we get this counterexample to the hypothesis that F's being about* c implies that c occurs essentially in F: $\forall x(Px)$. We must supply models M, M' agreeing on everything with the possible exception of how P is interpreted and differing on the evaluation of $\forall x(Px)$. We can arrange for this to happen by letting $\|P\|^M$ be the (common) domain D of M, M' and $\|P\|^{M'}$ be $D \smallsetminus \{\|c\|^{M'}\}$ (recalling that $\|c\|^{M'} = \|c\|^M$). This gives $M \models \forall x(Px)$ while $M' \not\models \forall x(Px)$. In a similar way, we see that $\exists x(Px)$ is likewise about* c, as is $\forall x \exists y(Rbxy)$. In all these examples, P is a 1-adic predicate letter, and in the last, R, a 3-adic predicate letter (and b, a constant distinct from c). ⊣

We should pause to compare the definition of $NAbout(F, c)$ from Demolombe and Fariñas del Cerro, given as (⊛) above, repeated here a similarly styled definition of occurring at most inessentially in a formula — we'll write this as $NEssOcc(F, c)$ (listed here as (⊛⊛)); for readability we write the main metalinguistic biconditional (formerly "iff") as a stretched version of the (metalinguistic) biconditional "⇔" featuring on the right-hand sides of the variants (⊛′) and (⊛⊛′) given below this pair:

$$NAbout(F, c) \text{ iff } \forall M \forall M' \big((M \approx_{\|c\|^M_{\mathsf{prd}}} M' \;\&\; M \models F) \Rightarrow M' \models F\big), \quad (\circledast)$$

$$NEssOcc(F, c) \Longleftrightarrow \forall M \forall M' \big((M \approx_{\|c\|^M_{\mathsf{cns}}} M' \;\&\; M \models F) \Rightarrow M' \models F\big) \quad (\circledast\circledast)$$

One may prefer a 'supervenience' formulation of such notions — agreement in respect of the relevant ≈-equivalence class implying agreement in respect of verifying F:[69]

$$NAbout(F, c) \Longleftrightarrow \forall M \forall M' \big(M \approx_{\|c\|^M_{\mathsf{prd}}} M' \Rightarrow (M \models F \Leftrightarrow M' \models F)\big) \quad (\circledast')$$

$$NEssOcc(F, c) \Longleftrightarrow \forall M \forall M' \big(M \approx_{\|c\|^M_{\mathsf{cns}}} M' \Rightarrow (M \models F \Leftrightarrow M' \models F)\big) \quad (\circledast\circledast')$$

[69] The most well-known appearance of a supervenience account in the aboutness literature is perhaps that provided in Lewis [99]; see note 36 above, where the discussion is couched in terms of possible worlds rather than models. Note, however, that Lewis gives a supervenience account of aboutness rather than of "not-aboutness" (NAboutness) — quite a contrast, at least on the face of it. Appendix J takes up this issue in the course of a general discussion of proposed 'possible worlds' adaptations of (or appeals to) Demolombe and Fariñas del Cerro's work.

The equivalence of the right-hand sides of (⊛) and (⊛′), and therefore of (⊛) and (⊛′) themselves is entirely routine and so is postponed to the end of Appendix I. Likewise for the case of (⊛⊛) and (⊛⊛′).

Regardless of formulation, the question arises as to which of the two metalinguistic predicates defined here might be the more appropriate to have its negation count as a candidate explication of aboutness (or, strictly, about*ness). For F to stand in the relation concerned to c, both of course pay close attention to the truth of F as it fares relative to each of a pair of models defined with special reference of the denotation of c in the two models. The negated version of *NEssOcc* looks for a difference, in F's truth-value, between a pair of models differing at most w.r.t. what that the denotation of that constant is (whence the "cns" subscript), while the negated version of *NAbout* keeps the denotation the same but allows for arbitrary differences over which atomically expressed (with the "prd" as a reminder) relations that denotation stands in to itself and other objects (or: which sets it belongs to, in the 1-ary case). *Prima facie*, each has a good claim, if informal discourse is anything to go by. In support of *NAbout*: someone who has so far read the applications of nine of the ten applicants for a vacancy and makes this clear — in order to block any conversational implicature than might otherwise arise — may say "All the candidates other than c [= the tenth candidate] satisfy the advertised criteria for the position" and legitimately claim to have said nothing about c, and avow an intention not to do so until after reading that tenth application. On the other hand, to grasp the content of what has been said, one does need to see that it is specifically *that individual* whose application is being exempted from the claim made about the various applicants: concerning c it is being said that everyone *else* (= other than c) has met the required conditions.

The question of which of the above candidate explications of aboutness should be endorsed over the other may accordingly be a question to reject, rather than one to answer. What a sentence — or perhaps more appropriately, an utterance — is *about* is a highly fluid and interest-sensitive matter, as is stressed in the remarks by Strawson and Partee (note 75) and in the passage from Bar-Elli [4] quoted near the end of Appendix H. We may be after nothing more than mere mention, satisfied with Goodman's differential aboutness without the "differential." Focalization, topicalization, or some other form of highlighting may be required for aboutness: Appendix A, starting with the quotation there from Christine Ladd-Franklin. Still more demandingly, we move in the

Goodman–Quine direction of *essential* mention, and the neither-weaker-nor-stronger notion (Proposition 22) isolated in the work of Demolombe and Fariñas del Cerro.[70]

The reason for saying, above, that choosing between the two notions of aboutness in play in Proposition 22 *may* be a choice one should reject rather than to make is the following. Not only might each candidate in principle be appropriate to explicate aboutness for this or that purpose, and nor has anything been said to rule out further candidate explications, but: one might feel that one or other of the options — Demolombe-aboutness and Goodman-aboutness, as they are called in Humberstone [81, Appendix B] — is not really a serious contender. In particular, Demolombe aboutness (i.e., about*ness, the complement of *NAbout*-ness) would seem, in view of the non-sufficiency examples in the proof of Prop. 22, to be at best a notion of potential aboutness, perhaps in better shape if transformed into a model-relative or model-specific notion.[71] A strong candidate in this vein would be the complement of the following: relative to a model M, F is *NAbout** c iff, now removing from $(\circledast)'$ the first universal quantifier over models, and suitably adjusting the remainder:

$$\forall M'\big(M \approx_{\|c\|^M_{\mathsf{prd}}} M' \;\Rightarrow\; (M \models F \Leftrightarrow\; M' \models F)\big)$$

Thus, relative to an M for which (for monadic P) we have $\|P\|^M = \{\|b\|^M, \|c\|^M\}$, with $\|b\|^M \neq \|c\|^M$, the sentence $\exists x(Px)$ is not about* c, because no change to M in respect of the $\|c\|^M$-involving 'atomic facts' of M — in particular of whether or not $\|c\|^{M'}$ (alias $\|c\|^M$) is removed from the extension of $\|P\|^M$ to get $\|P\|^{M'}$ — can change the M-relative truth-value of $\exists x(Px)$: it remains true because of the surviving witness $\|b\|^M$.

[70] The example of "Everything other than c is G" suggests that attention might usefully be paid to "All non-Fs are G" should be susceptible of a treatment not permitting contraposition to being about c on the latter account is reminiscent of the idea that Hempel's "All ravens are black" is about ravens and not about, for instance, and about non-black things (Hempel [58, p. 12*ff.*]) — a sentiment already alluded to in Section 2 above — might be treated with the aid of a theory of restricted quantification not taken as a mere notational abbreviation. See further, Appendix K.

[71] A similar move is urged in [79, pp. 1424–1427], [81, p. 220] (and for a world-relativized Lewis-style aboutness relation [66, Section IV]). Note that Demolombe and Fariñas del Cerro are well aware of the examples of such unrestrictedly quantified sentences, and explicitly venture the tentative conjecture (p.94) that the sentences which are, as they put it, about c, are exactly those that are equivalent to sentences in the language with standard but no $\forall x \neq c$-restricted quantifiers.

On the other hand, if we make the additional stipulation that $D_M = \{\|b\|^M, \|c\|^M\}$, then this same removal changes $\forall x(Px)$ from being true to being false, so this universal sentence is, relative to M, 'genuinely' about* c in this respect: its truth-value hangs on how things stand with its denotation — despite its not mentioning that denotation. The reader will have no difficulty illustrating the converse case in which it is a sentence's initial falsity in M that cannot or can turn into truth in a suitably related M' (i.e., related by the relation defined in Def. 21(iii) and appearing as the third equivalence relation in (‡‡) just before that definition).

5. Longer Notes as Appendices

A. "Essential" — for What? Here we continue the point made after Example 1 about the fact that even relative to a single logic, there may be several candidate equivalence relations for which the occurrence of expression e in a sentence may be counted as inessential when a suitably equivalent e-free sentence can be found. This phenomenon arises, also, for more informal talk of essential occurrence in natural language sentences comes up for discussion, as we shall be particularly keen to illustrate here (until the paragraph leading up to Definition 23 below, where discussion returns to a more formal setting).

If we think of being in the same equivalence class of some such relation as being alike in respect of this or that feature of interest, then the further relativity just noted to can be approached by asking which transformations of the sentence preserve the feature in question. This way of putting things is emphasized in Brożek [12], mainly considering the subject from the perspective of natural language. For example, one transformation of this kind might consist simply in *removing* the expression e from the sentence, in the case in which this results in another sentence (as opposed to a sentence-like string with a gap in it).[72]

[72] Compare Hösli [63, p. 202] describing a certain three-valued logic: "A formula X is a tautology [of the matrix in question] if every subformula of X — arisen by discarding some variables — is a classical tautology." This would need further clarification before it was intelligible. (Contrast the case of *deleting* a variable from a formula defined precisely in Example 18, though the present case one perhaps expects an inductive definition rather than a single equation.) For instance, if q and r are 'discarded' from $(p \wedge q) \vee (p \rightarrow (\neg r \wedge s))$ the result is presumably not intended to be the non-formula $(p \wedge \_) \vee (p \rightarrow (\neg \_ \wedge s))$, with blanks at the underlined spaces, but

Such an expression is inessential in the most obvious sense imaginable: it is completely redundant, from the perspective of the feature we are interested in. (As this reminds us, what is essential for achieving one goal may be redundant in respect of another, and likewise the other way round.) If we think of the sentences of such a language as being in an informal sense logically equivalent — from the truth of either, the truth of the other can be inferred *a priori* — the inessentially occurring e may be so in virtue of its removal yielding an alternative way of expressing the same proposition. It can sometimes fail to be obvious when such redundancy is present — for example, whether this is so for the last six words of the parenthetical sentence just written, beginning with "As this". Notice, in any case, how far we have strayed considerably from our main business here, since such a candidate expression is replete with internal structure rather than being semantically atomic, and even if we were to restrict the discussion the case of unstructured expressions, the issue of redundancy here pertains more naturally to individual token occurrences of an expression type within a sentence rather than the type as such, whereas, as remarked at the end of Section 1, our main focus on essential occurrence does not isolate a notion of essential occurrences.

A variation on the 'same proposition' relation just informally characterized arises when Brożek [12, §6.4] considers separately the preservation of merely the denotation rather than the connotation (using the Polish cognates of these words), of the expressions concerned, taking "connotation" as amounting approximately to Frege's "sense" (*Sinn*) and more exactly to Carnap's "intension", in such a way that if b has exactly one sister then *b's sister* can replace *b's sole sister* within a sentence in a denotation-preserving way, making the word "sole" occur inessentially in any such sentence, when sameness of denotation rather than sameness of connotation is at issue.[73] Concerning these transformation-based for-

rather, at $p \vee (p \to s)$. Here I have followed the heuristic: "delete and sew up," but a precise general account is needed. The example just given uses all of [63]'s primitive connectives, but what if all we had was a single ternary primitive connective # and wanted to 'discard' q from $\#(p, q, r)$? And, even for Hösli's language, what formula is the result of discarding q and r from $q \to \neg r$?

[73] As it happens, this example illustrates further noteworthy aspects of Brożek's discussion, such as the focus on expression tokens rather than types, since other occurrences of "sole" in the same sentence need not be be similarly removable, and also the fact that she distinguishes inessentiality from the special case involved here of direct *removability*, the literal deletion of the inessentially occurring expression rather than any more indirect transformation of the sentence into one equivalent in

mulations, note that the relation R obtaining between S_1 and S_2 when S_2 arises from S_1 via such a transformation (such as simply removing e from S_1) is not in general an equivalence relation, the associated such relation being the transitive symmetric closure of R.

Similarly, transforming "President Macron, like everyone else, needs oxygen to live" into "Everyone needs oxygen to live" removes the name "President Macron" while preserving the proposition expressed, from which perspective that name occurs inessentially in the first sentence.[74] But if what is to be preserved is more demanding than this and requires specification of which aspects of the world someone uttering the sentence will be taken as actively directing attention to, then this feature is not preserved by the transformation just described, and in addition, "Marine le Pen, like everyone else, needs oxygen to live" directs our attention in a quite different direction, while expressing the same abstract 'proposition'. If we set aside some of the finer distinctions involved (presupposed/asserted, given/new...) in such contrasts, we can approximate the distinction formally by contrasting $Fa \wedge \forall x(x \neq a \to Fx)$ with $Fb \wedge \forall x(x \neq b \to Fx)$, and coupling this with the 'attentional content'

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the relevant respects but from which in which that expression is absent. (A different notion of removability will be treated at the end of this Appendix: Definition 23 onward.) A further complication about this 'sole sister' example: Brożek may be taking *b's sister* as a Russellian definite description with the uniqueness condition built into its meaning; alternatively she may not be assuming that, but be envisaging a case in which a in fact has exactly one sister, so we are relying on the *de facto* given co-denotation of *a's sole sister* and *a's sister*. The text makes it clear that it is the latter considerations that drive Brożek's (autobiographically inspired) discussion here, in which, in fact, the brother and sister pair appear, not as b and, say, a, but as *Bartosz* and *Anna*, Anna being the only sister of Bartosz Brożek, the philosopher of law and co-editor of (as well as contributor to) the Templeton Foundation supported Brożek et al. [13]. Because of his theory of descriptions, we can't quite say that what is being preserved is the Russellian proposition expressed; in terms of Lemmon's distinction between statements and propositions in [97] it is the *statement* expressed (by a sentence in a given context), rather than the *proposition* expressed, that is preserved. Since Lemmon-style 'statements' survive the replacement of a term by any co-referential term, there may be danger from the 'slingshot' argument here, presented in §5 of Tichý [165] as an objection to Goodman's account of aboutness.

[74] The simpler conception of the proposition expressed by S as the set of worlds in which S is true, will suffice for present purposes. (If the presence of the proper name *Macron* is felt to be too much of a complication either with the characterization in terms of *a priori* inference or — for those taking its bearer to exist contingently — that in terms of possible worlds, another example might be used to make the point, such as "Mammals with tails, like all other mammals, need oxygen to live.")

semantics of Roelofsen [147]. Issues in this vicinity were nicely encapsulated long ago in the following remark from Ladd(-Franklin), [90, p. 27], though the use of "proposition" differs from that just in play:

> The propositions, "no men are mortal," and "there are no mortal men," convey the same information; but the first offers it by way of information about men, and the second by way of a description of the universe.

Perhaps, rounding out the picture, she would have had a similar attitude in respect of "No mortals are men," — same information again, this time offering it as information about mortals.

There has been considerable empirical linguistic research on such attentional devices: see Gildea [47], and, critically surveying much early work, McNally [107] as well as the more recent Endriss [33], and the considerably earlier Van der Auwera [167] (briefly touched on in Appendix H). Hinterwimmer [61] and Roberts [146], adjacent entries in the same encyclopedia/handbook, helpfully discuss potentially conflatable distinctions; at p. 1876 of the former, on the subject of the topic-comment and focus-background contrasts we have:

> (1) a. Tell me something about John. b. John married BERTHA.
> In the case of (1b), the subject noun phrase John is given, and furthermore has been made the aboutness topic (see below) by the preceding utterance (1a). On the other hand, the verb phrase *married Bertha* is not only new, but also focal, which is indicated by the main stress on the object noun phrase Bertha [. . .].[75]

Or again, setting aside matters of focus and the topic/comment contrast, one might follow the lead of Fine [38], distinguish between truth in the same possible worlds and the narrower equivalence relation of having the same ('exact') truthmakers, with conjunctions made true by the fusions of truthmakers for the separate conjuncts. Goodman [50, p. 10] considers

[75] See also §4.1 in Lambrecht [92], including the useful quotation there from Strawson: "There is great variety of possible types of answer to the question what the topic of a statement is, what a statement is 'about' [. . .] and not every such answer excludes every other in a given case." In a similar vein, we have this in Partee [127, p. 153] (in the section headed 'Reply to Perry'): "Perry's claims notwithstanding, our pretheoretic intuitions do not seem to delimit a unique notion of aboutness in a clear and distinct way." §§1–3 of Chapter 5 of Carlson [17] provides an extensive discussion of the various topic/comment, theme/rheme, etc., style distinctions made in the history of linguistic semantics and their relation to aboutness. A more recent venture into this territory is provided by Plebani and Spolaore [136], with a direct representation of contrastive stress in the formalized object-language.

cases — in particular "Maine and everything else prospers" like those in the text, saying of it that it is not about Maine despite mentioning Maine "and is not logically true or contradictory and yet, since it yields no consequence differentially with respect to Maine, is not absolutely about Maine."

One might also consider here *connotation* in the more popular sense of Fregean 'colouring' (*Farbung*, or, in Dummett's rendering, *tone*): a pejorative expression in a sentence occurs inessentially there if it can be replaced, for instance, by a neutral expression with the same sense if it is just the proposition expressed (understood as above) that is to be preserved, though not if one is concerned with the preservation also of such — let us, for the sake of the example, assume — non-truth-conditionally relevant overtones.

In any discussion of essential occurrence, especially down at the informal end of the spectrum, it is accordingly always worth asking "essential for (preserving) *what*?" Let us look at some illustrations.

We begin with he case of Donnellan, who, having introduced (in [30]) his distinction between attributive and referential uses of definite descriptions, writes [30, p. 235]:

> In the first case the definite description might be said to occur essentially, for the speaker wishes to assert something about whatever or whoever fits that description; but in the referential use the definite description merely one tool for doing a certain job — calling attention to a person or thing — and in general any other device for doing the same job, another description or a name, would do as well.

Discussing this remark of Donnellan's, Martinich [105, p. 160], observes that "it is important to know what he means by an essential occurrence of an expression. Since Donnellan unfortunately does not explain what he means by this notion, it is necessary for us to try to provide at least some explanation of it." While not wanting to deter anyone from consulting Martinich's attempts in that direction, I would like to suggest that Donnellan is not best interpreted here as suggesting that any *particular* expression — such as "Smith's murderer" — occurs essentially in his examples, but that what is essential is that some expression *of a given type* occur in them, namely a descriptive expression (it might just as suitably be "Smith's killer," "the person who did this to Smith,"...), and what it is essential for is picking whichever individual satisfies the description, as opposed to some independently identified specific individual presumed to satisfy it.[76]

Intermission on Copular Sentence Taxonomy. Martinich's point about proper names, and not just definite descriptions, being capable of being used on both sides of the referential/attributive distinction, had, as it happens, been made some years before, in Stalnaker [161], and some of that discussion is cited in the virtuoso dissertation (later published as) Higgins [60]. Some mention of the issues involved is called for because of their connection not only with reference but with aboutness (as in Section 4).

The passage from Stalnaker quoted in Higgins [60, p. 222] reads as follows:

> Proper names, for example, are normally used to refer, but can be used in a way resembling the attributive use of definite descriptions. When you ask, "Which one is Daniels?" you are not referring to Daniels, since you do not presuppose of any one person that he is Daniels. When I answer "Daniels is the bald one" I am using "the bald one" referentially, and the name *Daniels* attributively. I am telling you not that Daniels is bald, but that he is Daniels.[77]

After Higgins has introduced his fourfold typology of copular sentences[78] as Identity, Identificational, Specificational or Predicational Sentences, he returns to the above theme, explaining that the status of the distinction between the first two types is somewhat unclear, writing [60, p. 263] with emphasis as in the original:

> Clearly, Identity sentences are close to Identificational sentences, and perhaps if one abstracts from "conditions of use" may be analyzed as identical with them. Thus, a sentence like:
>
> (146) That man over there is Joe Bloggs.
>
> is normally used to teach someone a name, and it does not seem to me that the name is used Referentially in such sentences — nothing is said *about* Joe Bloggs.

[76] An example of a somewhat similar kind occurs on p. 88 of Tichý [165], where, concerning an identity statement labelled "(5)", we read "Note that the right-hand side of (5) constitutes an essential occurrence of a definite description fitting Chicago."

[77] Stalnaker [161, p. 393*f*.]. Here I have italicized the name as it occurs near end of the second last sentence, just for use/mention hygiene.

[78] Really, no doubt, one should say — as in Mikkelsen — 'clauses' rather than 'sentences' here, though the latter is more common in the literature.

This leaves a bit of a puzzle. Suppose that (146) is said to me, with a suitable pointing gesture, perhaps, in reply to my asking at a party who it is over by the bar playing two harmonicas at once. One thing I seem to have learned that night — assuming conditions were favourable for the transmission of knowledge by testimony (it really was Joe Bloggs, I believed what I was told, etc.) — that night was that Joe Bloggs was at the party, which is hard to explain, if nothing had been said to me about Joe Bloggs. Heller and Wolter [57] have their own doubts abouts the first two types on Higgins' list, wanting to subsume Identificational sentences under the 'Predicational' rubric.[79] The publications of Ilaria Frana cited in note 66 are also relevant to sorting out this evidently challenging terrain, also treated in Mikkelsen [110], or more briefly in Mikkelsen [111]; another chapter of the same collection containing the Partee paper cited in note 66, namely considers related material with different methodological priorities in mind: [129]. ***End of Intermission***

To extract the nub of this discussion before the intermission: the point hangs on a scope ambiguity in saying, where E is a class of expressions, that some expression in E occurs essentially in a sentence S may mean (1) that some $e \in E$ occurs essentially in S in the sense in which we have been principally concerned — S has no e-free equivalent — but it may also mean (2) that S has no equivalent in which no $e \in E$ occurs (modulo precisification of "equivalent" in both cases). The suggestion is that Donnellan had the latter in mind with E as the class of descriptive expressions — though this stretches the understanding of "e" in Section 1 as ranging over unstructured expressions. Another such case arises in Zimmerman's discussion [181, p. 122], of a remark from [119, p. 263]:

> Nozick's claim that [a certain group of agents] coerce Z only if they do not act within their rights is striking in that it makes "coercion" out to be an essentially moral concept, in the sense that its conditions

[79] To illustrate the difference between the Specificational and Predicational types, consider the sentence, uttered, say, by the father of a child who is tapping repeatedly on the underside of the dining room table: "What you are doing is annoying your mother." This can be taken predicationally (now dropping Higgins' capitalization convention — as in Mikkelsen [110]) if it is taken as saying that whatever is that the child is doing — as it happens, in the present case, tapping the table — satisfies the predicate "is annoying your mother". But it could also be taken specificationally, which can be encouraged by giving it a 'colon' intonation: "What you are doing is (the following): annoying your mother."

> of application contain an ineliminable reference to moral rightness or wrongness.[80]

Here it is evident that essentiality is tied to eliminability not in the sense that any specific item of moral vocabulary (*ought*, *wrong*, *permissible*,. . .) is ineliminable but that there is no relevantly equivalent formulation from which *all* such items have been eliminated.

We turn to a passage from Castañeda [18, p. 206*f.*], which is at the same time highly suggestive and potentially confusing in threatening to undermine the connection between essential and ineliminable occurrence. (The subscripted i is a variable ranging over the sources of the obligations, etc., involved.)

> There are cases in which a propositional function in the scope of a deontic operator cannot, preserving equivalence, be brought outside that scope. To see this consider deontic propositions of the form:
>
> (13) It is $permitted_i$ that everybody who did (does) B do A.
>
> Here the propositional function expressed by the clause 'who did (does) B' cannot be brought out of the operator *it is permitted that*. The statements normally made with a sentence of form (13) are not equivalent to the statements normally made with the corresponding sentences of the form
>
> (14) Everybody who did (does) B is $permitted_i$ to do A.
>
> The statements made with a sentence of form (13) entail, but are not entailed by, the statements made with the corresponding sentence of form (14). Suppose that our statements of form (13) are about a club that has only two members say, a and b. Hence, (13) would amount to:
>
> (13a) It is $permitted_i$ that both if a did (does) B, he do A and if b did (does) B, he do A.
>
> On the other hand, (14) would amount to:
>
> (14a) If a did (does) B, he is $permitted_i$ to do A, and if b did (does) B, he is $permitted_i$ to do A.
>
> Clearly, (13a) entails (14a). But (14a) does not entail (13a): each

[80] Clarifying the issue in note 4 of the following page of [181], Zimmerman writes: "When I urge that 'coercion' is not an essentially moral concept, I trust it is clear that I am making a more-or-less conceptual point and in no way denying that substantive moral judgments can be made about coercion. In fact, I am insisting on the conceptual point precisely because I believe that this is the only way to be clear about the basis of the substantive moral judgment."

> person can have a right or permission to do something while lacking a right or permission to do it jointly.
> Thus, the act B or the doing of B appears in (13) and in (13a) *ineliminably* in the scope of the deontic operator *it is *permitted*$_i$ that*, in spite of the fact that it does not occur essentially, so to speak, in such a scope. In a statement made with sentence (13), or (13a), doing B is conceived of only as a circumstance, not as an act in that peculiar sense in which acts are the subject matter of deontic considerations, or, as we shall say, as an act prescriptively or practitionally considered.

Readers familiar with the literature connected with Example 1 may recognise in (13) a deontic version of the example "It is possible that everything which is (actually) red is shiny," from that literature. Humberstone [72, §4] discusses the relation between the 'actuality operator' approach and Castañeda's practition-based or, more generally, subjunctivity-based approach (and more recent subjunctive ventures[81]) to making good the expressive weakness here noted. The point of present interest lies in the unexpected juxtaposition: some material appears *inessentially*, though still *ineliminably*, in the scope of a deontic operator. By "inessentially" here, Castañeda means that, despite its appearance in the scope of such an operator, no reference is being made to the permissibility or otherwise of doing B: the permissibility (or "permittedness$_i$") concerns only doing A, though under circumstances specifiable by reference to having done B.[82] Although this use of "essential" again involves

[81] See Wehmeier and Rückert [174] and references therein.

[82] And, crucially, since what is (being said to be) permissible is that all of those those who have done B, we need the universal quantifier — and therefore the reference to B — in the scope of the permissibility operator — as in (13), rather than the other way round, as in (14), where *everybody* amounts to *anybody*. When translated into the language of modal actuality logic, this illustrates the point attributed to Hazen [56] under Example 1 above: the result that every formula has a real-world equivalent $\mathcal{A}$-free formula does not extend from propositional to predicate logic. If we were to take the "ifs" in Catañeda's (13a) and (14a) as material implications ("$\rightarrow$" in what follows) then we would presumably disagree with his "ineliminably" as it bears on (13a) in the opening sentence of the final paragraph quoted here. After all, representing "x does A" as Ax, and similarly for B, and with "!" and "$\mathcal{P}$" as our subjunctive/practition marker and permissibility operator respectively, (13a) amounts to the conjunction of $Ba \rightarrow \mathcal{P}!Aa$, $Bb \rightarrow \mathcal{P}!Ab$ and $(Ba \wedge Bb) \rightarrow \mathcal{P}(!Aa \wedge !Ab)$, via their respective equivalence with $\mathcal{P}(Ba \rightarrow !Aa)$, $\mathcal{P}(Bb \rightarrow !Ab)$ and $\mathcal{P}((Ba \wedge Bb) \rightarrow (!Aa \wedge !Ab))$ — thereby successfully extricating "B" from the scope of "$\mathcal{P}$." But Castañeda has an elaborate discussion of *if/then* in [18, Ch. 3 (esp.§4), Ch. 4 (again §4] suggesting that no such 'material' interpretation is intended. Indeed, no explicit formal semantics is provided,

the idea of needing a specific type of linguistic formulation to convey what is intended, it is evidently further still from the use of this term in the main body of our discussion than those sampled earlier in this appendix. The next — and final — essentiality-of-occurrence like concepts to be considered here lends itself to a more formal treatment.

Definition 0.1(i) of Humberstone [80, p. 30] ran approximately as follows, introducing a notion of *removability* distinct from that in play in the discussion of Brożek above. The background is that we are thinking of logics as sets of formulas of some fixed language and a lattice Λ of such logics[83] with ordering $\subseteq$ and, where A is a formula of that language, write "$\mathsf{L} \oplus^{\Lambda} A$" for the least $\mathsf{L}' \in \Lambda$ such that $\mathsf{L} \cup \{A\} \subseteq \mathsf{L}'$. The definition reads:

Definition 23. With Λ, L, and A as just described and $A = A(q_1, \ldots, q_{n-1}, q_n)$ we say that q_n is *removable from* A (as a candidate axiom for extending L to $\mathsf{L} \oplus^{\Lambda} A$) if there is a formula $B = B(q_1, \ldots q_{n-1})$ with $L \oplus^{\Lambda} A = L \oplus^{\Lambda} B$.

Thus 'removing' q_n from A does not just consist in passing to a q_n-free formula B yielding from L the same logic in Λ as adding A yields: we could do this simply by substituting uniformly for q_n some other variable distinct from all of $q_1, \ldots, q_{n-1}$ — at least, we can to this if each logic in Λ is closed under Uniform Substitution.[84] Rather more demandingly, the variables occurring in B must comprise just the remaining variables in A.

Example 24. For an example in the style of those in [80], consider the language of propositional (mono)modal logic, with primitive connectives $\perp$, $\rightarrow$ and $\Box$ of arities 0, 1, 2 respectively, and the lattice (= Λ for the present example) of monotonic modal logics in this language, which is to say: those containing all CL-tautologies and closed under Uniform Substitution, Modus Ponens and the 'monoton(icit)y rule,' licensing the transition from $A \rightarrow B$ to $\Box A \rightarrow \Box B$. More generally, a context $C(\cdot)$ is monotone (according to L) or antitone (according to L) if $C(A) \vdash_{\mathsf{L}} C(B)$ whenever $A \vdash_{\mathsf{L}} B$ or, respectively, whenever $B \vdash_{\mathsf{L}} A$. We identify the

the remarks just made having been made in the light of the semantics sketched in §4 of [72], and so perhaps not quite what Castañeda might have had in mind.

[83] For example, in the case of the language of monomodal propositional logic: the lattice of all modal logics, the lattice of all monotonic modal logics, the lattice of all normal modal logics, etc.

[84] As mentioned in note 106, this condition is not always imposed, so care is needed over the question of whether, in particular, congruential, monotone, or normal modal logics are required to be closed under Uniform Substitution.

occurrence of sentence letter q's being positive, or being negative, in a formula $C(q)$ — in which there may be occurrences of further sentence letters — with the 'Williamson context' (as in note 5) $\langle C(q'), q'\rangle$, being respectively monotone or antitone, and $C(q')$ results from q' replacing the single occurrence of q in question where q' is a variable not occurring in $C(q)$. (The consequence relation $\vdash_{\mathsf{L}}$ here can be thought of in terms of the set of provable formulas of L in accordance with: $\Gamma \vdash_{\mathsf{L}} A$ iff for some $C_1, \ldots, C_n \in \Gamma$, we have $C_1 \to (C_2 \to \ldots (C_n \to A) \ldots)$ as one such formula.) ◁

DEFINITIONS 25. (i) We say that q occurs *homotonically* in $C(q)$ if all occurrences of q in $C(q)$ are positive or else all occurrences of q in in $C(q)$ are negative.[85]

(ii) A *homotonic* formula is one in which every variable that occurs, occurs homotonically in it.

(iii) A formula is *uniformly* homotonic if every variable occurring in it occurs positively, or else every variable occurring in it occurs negatively.

(iv) For $n \geq 1$, an n-ary primitive connective $\#$ in the language of L is *homotonic* (*uniformly homotonic*) if the formula $\#(p_1, \ldots, p_n)$ is homotonic (uniformly homotonic).

Thus, for instance, for the L introduced in Example 24 in $p \to q$ is homotonic but not uniformly so since p and q occur (resp.) negatively and positively. In $(p \to q) \to q$ the first occurrence of q is negative and the second is positive, and the sole occurrence of p is positive, so p occurs homotonically and q does not. On the other hand, in the L-equivalent (because CL-equivalent) formula $(p \to \bot) \to q$ both variables occur only positively, so this formula is uniformly homotonic. Thus, if $\vee$ is introduced on the basis of the primitive Boolean (i.e., non-modal) connectives of Example 24 by taking $A \vee B$ as the formula $(A \to B) \to B$, and we tried to extend Definition 25(iv) from primitive to encompass also defined connectives, $\vee$ would not count as a homotonic connective, whereas if we took $A \vee B$ instead to be the formula $(A \to \bot) \to B$ then $\vee$ would emerge

[85] Formulas all of whose variables occur homotonically in them are called *normal formulas* in Sidorenko [152, p. 57] but here we need to attend to whether this or that particular variable occurs homogeneously with respect to 'tonicity type' (monotonic/positive or antitonic/negative). Blackburn et al. [9] use the word *uniformly* for *homotonically* in this sense, but for present purposes it is more convenient to reserve this adverb for to mark a different kind of homogeneity ((iii) below).

as (uniformly) homotonic.[86] If we included $\leftrightarrow$ as a primitive Boolean connective it would not be homotonic, since neither p nor q occurs homotonically in $p \leftrightarrow q$, making such a choice of primitives an unpropitious starting point for results such as Proposition 26 below. The same goes for certain candidate modal primitives, such as the noncontingency operator discussed in [80] — or its negation, the "∇ operator, touched on in Example 12 above, or indeed $\Box$ in the lattice of all congruential modal logics. This is the reason EM rather than E (using the nomenclature of Chellas [19]) appears in Example 24: positive, resp. negative, occurrences of a variable in a formula B remain positive, resp. negative, in $\Box B$.[87] One might suspect that $\leftrightarrow$ (or its complementary form, XOR, say) could never in fact appear, other than redundantly, in a functionally complete set of Boolean connectives, but that would be wrong: $\{\wedge, \leftrightarrow, \bot\}$, for example, is an irredundant functionally complete set.

Returning to *removability* in the sense of Definition 23, we give a simple illustration of a sufficient condition for its applicability in the connection with L as EM and Λ as the lattice of all monotonic modal logics, as in Example 24:

PROPOSITION 26. *With Λ as just recalled, for any A, n with $A = A(q_1, \ldots, q_{n-1}, q_n)$ in which q_n occurs homotonically (according to EM), $\mathsf{EM} \oplus^{\Lambda} A((q_1, \ldots, q_{n-1}, q_n)) = \mathsf{EM} \oplus A(q_1, \ldots, q_{n-1}, \top)$, or $A((q_1, \ldots, q_{n-1}, q_n)) = \mathsf{EM} \oplus A(q_1, \ldots, q_{n-1}, \bot)$.*

PROOF. Since q_n occurs homotonically in $A = A(q_1, \ldots, q_{n-1}, q_n)$, all of its occurrences in A are positive or all are negative.

In the former case:

- $\mathsf{EM} \oplus A(q_1, \ldots, q_{n-1}, q_n) = \mathsf{EM} \oplus A(q_1, \ldots, q_{n-1}, \bot)$,

since $A(q_1, ..., q_{n-1}, \bot)$ is a substitution instance of $A(q_1, ..., q_{n-1}, q_n)$, so

- $\mathsf{EM} \oplus A(q_1, \ldots q_{n-1}, \bot) \subseteq \mathsf{EM} \oplus A(q_1, \ldots, q_{n-1}, q_n)$,

while for the converse, since $\vdash_{\mathsf{EM}} \bot \to q_n$ and q_n occurs positively in $A(q_1, ..., q_{n-1}, q_n)$, we get $\vdash_{\mathsf{EM}} A(q_1, ..., q_{n-1}\bot) \to A(q_1, ..., q_{n-1}, q_n)$, so

- $\mathsf{E} \oplus A(q_1, \ldots, q_{n-1}, q_n) \subseteq \mathsf{EM} \oplus A(q_1, \ldots, q_{n-1}, \bot)$.

[86] Here we are taking the metalinguistic approach to defined logical vocabulary, as explained in note 45 and the paragraph to which it is appended.

[87] See, further, the first sentence of note 88 below.

In the latter case: a similar argument handles the case of all occurrences of q_n in A being negative, using $\top$ (defined, say, as $\bot \to \bot$, to stick with the announced primitives). $\dashv$

COROLLARY 27. *With q_n occurring homotonically in $A = A(q_1, \ldots, q_{n-1}, q_n)$, q_n is removable from A (as a candidate axiom for extending* EM *to* $\mathsf{EM} \oplus^{\Lambda} A$*).*

PROOF. Proposition 26 showed how to remove such a homotonically occurring q_n from A leaving a formula in which only the remaining variables $q_1, \ldots, q_n$ occur. $\dashv$

Proposition 26 corresponds to Theorem 2.2 in Humberstone [80], though the latter result explicitly mentions only the case of formulas in which the variable to be 'removed' — here q_n, there p — occurs *exactly once* in A. If the variable in question only occurs once in A, and the primitive (n-ary for $n \geq 1$) connectives are all homotonic, then it must occur homotonically in A. (As far as connectives are concerned, then, Boolean primitives like $\leftrightarrow$ as well as modal primitives like ∇, mentioned in Example 12, need to be avoided for such purposes.[88])

The removability of a propositional variable from a candidate axiom for extending a logic L clearly has some affinities with the notion of a variable's occurring L-inessentially in a formula. So we should devote some attention to the resemblance by asking the informal question suggested by the title of this Appendix: what is or isn't essential (in the sense of *needed*) and for what might it be being said *not* to be essential in the present case? In the case of L-essential the occurrence of a sentence letter in a formula, what was mainly at issue amounted to being (informally speaking) 'essential' for any other formula to be L-equivalent the given formula — and we mostly set aside the potential alternative option of being L-synonymous with the given formula, by concentrating on cases in which these two equivalence relations did not coincide. But what might the equivalence relation be, if we can find one, in the present case?

For L-equivalence in the general case this was a matter of mutual consequence by a consequence relation $\vdash_{\mathsf{L}}$ associated with L in such

[88] The exposition in [80] keeps a greater distance than here between the positive/negative and the monotone/antitone distinctions, bridging the resulting gap by spelling out their relationship with the aid of an inductive argument (the proof of Lemma 2.1 there); more discussion of ∇ and its dual (which coincides with its negation), $\triangle$, in this connection can be found in the same paper.

formulations as that of the Default Background Conditions in Section 1. And there is also a consequence relation involved in considering the results of extending L with a new candidate axiom: theorems of the extended system are "consequences" of the new axiom in the sense of being, as it is sometimes put, *deducible* from the new axiom and all the theorems of L. But this requires us to know what rules are to govern such deductions: these are rules of *proof* rather than rules of *inference*, in terms of Smiley's famous distinction (e.g., [160], note 3 and the text on which bears), and so B's being deducible "in the field of L" from $A_1, \ldots, A_n$ cannot be expected to coincide with its being the case that $A_1, \ldots, A_n \vdash_{\mathsf{L}} B$. This is where the relativity to Λ as the background lattice of logics (L included) under consideration comes in, the rules in question being those under which the logics in Λ are required to be closed. The consequence relation at issue, which we may call $\vdash^{\Lambda}_{\mathsf{L}}$ can be introduced as follows (cf. Definition 23 above and the preamble to it):

DEFINITION 28. For formulas $A_1, \ldots A_n, B$ in the language of Λ and a logic $\mathsf{L} \in \Lambda$:

$$A_1, \ldots, A_n \vdash^{\Lambda}_{\mathsf{L}} B \text{ iff for every } \mathsf{L}' \in \Lambda \text{ such that } \mathsf{L} \cup \{A_1, \ldots, A_n\} \subseteq \mathsf{L}', B \in \mathsf{L}'.$$

The suggestion is, then, that the desired equivalence relation is mutual consequence by the consequence relation of Definition 28. To say that formulas A, B stand in this relation, we write $A \dashv\vdash^{\Lambda}_{\mathsf{L}} B$. Then, continuing to fix on Λ as the lattice of $(\Box)$-monotone extensions of EM, and with $A = A(q_1, \ldots, q_{n+1})$ we want to say something along the lines of "for all B, if $A \dashv\vdash^{\Lambda}_{\mathsf{EM}} B$, then each of $q_1, \ldots q_{n+1}$ occurs in B." Something along those lines — but not, of course, exactly that, since, as already noted, we could trade in all of $q_1, \ldots, q_{n+1}$ for $n+1$ new pairwise distinct sentence letters. What would remain of A after such transformations would be the unreduced *number* of such letters, so this is the feature that any massaging of Coro. 27 into the current format must specify as invariant: what is 'essential' to standing in the currently pertinent equivalence relation to A is being constructed from at least as many sentence letters as $A = A(q_1, \ldots, q_{n+1})$: For all B, if $A \dashv\vdash^{\Lambda}_{\mathsf{EM}} B$, then no fewer than $n + 1$ propositional variables occur in B.

B. Quine and Others. As mentioned in note 6, talk of *essential occurrence* (and *occurs essentially*) gained currency through Quine's use of it in a sense different from that with which this terminology is used here, most conspicuously — in addition to the respects of difference remarked on in that note — in its being applied to the occurrence of vocabulary in sentences of an interpreted language. And even with that in mind, the explanation on p. 2 of Quine [143] was presumably far from what is intended, a footnote on that page directing the reader to a "somewhat more careful formulation" in Quine [144]. The definition offered in [143] runs like this: "A word may be said to occur *essentially* in a statement if replacement of that word by another is capable of turning the statement into a falsehood," bizarrely implying that an expression can't occur essentially in any statement that is false to begin with. When we turn to the authorially approved formulation in [144, p. 73], we find that things have become very complicated, though the problem just noted is addressed for the in explanation of "vacuously" (Quine's preferred adverb instead of "inessentially" — for which Goodman [50] uses "non-essentially") with the disjunctive formulation *truth or falsehood*:

> An expression will be said to occur *vacuously* in a given statement if its replacement therein by any and every other grammatically admissible expression leaves the truth or falsehood of the statement unchanged. Thus for any statement containing some expressions vacuously there is a class of statements, describable as *vacuous variants* of the given statement, which are like it in point of truth or falsehood, like it also in point of a certain skeleton of symbolic make-up, but diverse in exhibiting all grammatically possible variations upon the vacuous constituents of the given statement. An expression will be said to occur *essentially* in a statement if it occurs in all the vacuous variants of the statement, i.e., if it forms part of the aforementioned skeleton. (Note that though an expression occur non-vacuously in a statement it may fail of essential occurrence because some of its parts occur vacuously in the statement.)

The correction already remarked on is made with less clutter (but, curiously, no mention of Quine) in [117, p. 179], offering the following, in which W and S range over words and sentences respectively:[89]

[89] Rather more helpfully, Rowe [149, p. 78] cites both Quine and Nakhnikian, writing there (emphasis added here to highlight the 'modal emaciation' complained of below): "A word occurs essentially in a sentence provided that its replacement in every one of its occurrences by some other word *will change* the truth-value of the sentence. Thus 'lions' in the true statement 'There are lions in Africa' occurs essentially since its replacement by the word 'tigers' yields the false statement 'There are tigers in Africa.' " (Probably we should interpret the latter to mean that tigers are native/endemic to Africa.)

> W occurs essentially in S = Df. (1) W occurs in S (2) by replacing every occurrence of W in S with a word W', it is possible to produce a sentence S', and (3) S and S' have opposite truth-values.

Now, while it seems fair enough to have W here be a word, to the extent that the most convenient things to make substitutions for what in a formal setting would be atomic expressions of whatever syntactic category is at issue and we can think of the restriction to individual words as playing that kind of role in the natural language setting, there seems no reason to impose this restriction on W': the potential replacements should surely be any phrases of the relevant category.

There remains a more serious anomaly: on the supposition that everything is less than 1000 light years away from a star, these definitions would have it that in "Neptune is less than 1000 light years away from a star," the name "Neptune" occurs inessentially/vacuously. The apparent modality signalled by "possible" here in Nakhnikian's revision seems not to get at the noncontingency (or perhaps more appropriately, *apriority*) one would like to see — and which is presumably avoided by Quine on ideological grounds.[90] It creeps in with the casual formulation in which Quine writes of (italics added) "replacement of that word by another *is capable of* turning the statement into a falsehood" but in the official formulation of the definition (of the complementary concept of vacuous occurrence) quoted from [143] above, the relevant condition is that replacement of the expression in question "leaves the truth or falsehood of the statement unchanged," and from here the modality has been purged. Of course, one could simply recognise both notions as legitimate by a further relativization of the concept of occurrence, informally perhaps a relativization to worlds (and taken in particular w.r.t. the actual world), and formally by one to models; this aspect of as is brought out by the following quotation from Pap, who in [125, p. 507] writes of:

> the distinction between logical constants, as terms on whose specific meanings the validity of an argument depends, and descriptive constants which occur *inessentially* (to borrow Quine's term) with respect to the validity of the argument–although they may occur essentially

[90] A desire to keep the discussion extensional, that is; but there is also a circularity issue: one can hardly define logical truths as truths in which the only essentially occurring vocabulary is logical vocabulary if essential occurrence is itself defined in terms of logical truth (the logical truth of no biconditional linking a sentence containing the vocabulary item in question with one in which that item does not occur).

with respect to the factual truth of propositions entering into the argument.

In the notion of essential occurrence with which the main body of our discussion has been concerned, what corresponds to this modal element is provability in the logic at issue. Expressed more semantically, this is a matter this is matter of truth on all of the valuations in a class of valuations w.r.t. a given $\vdash_{\mathsf{L}}$ is sound and complete — such as the Boolean valuations (as defined in the course of Example 13) for the case of $\mathsf{L} = \mathsf{CL}$.

Quine's (and Nakhnikian's) interest in these concepts lies in their use to go on to characterize analyticity and logical truth, given a specification of which expressions comprise the logical vocabulary, with logical truths characterized as those truths in which the only vocabulary occurring essentially is logical vocabulary. In connection with this venture, it is customary to note the striking similarities with (and occasional differences from) earlier moves in a similar direction by Bolzano.[91] Nakhnikian takes as given, a division of the vocabulary into logical and non-logical vocabulary and explicitly includes the identity predicate on the former list, prompting a tweak of Quine's account by the addition of conjunct (1) in the definition in question (from [117, p. 180], of S's being logically true:

(1) S is true regardless of the number of objects in the universe and
(2) Only logical words occur essentially in S.

It is interesting to see the intensional construction "regardless of the number" appearing here. Nakhnikian writes, "We need this restriction about the number of objects in the universe. The sentence 'There is exactly one object' is equivalent to a sentence in which only logical words occur essentially."

[91] See Bar-Hillel [5], Hale and Wright [52]; see also Lapointe [94, pp. 68–71] or Lapointe [95]. The following from p. 329 of Hale and Wright: "If we say, in accordance with a well-established terminology, that an expression occurs *essentially* in a statement if and only if uniformly replacing it throughout that statement may result in a statement that differs in truth-value from the original one, and give a parallel explanation of an idea's occurring essentially in a proposition, then we can see that Bolzano's definition of logical analyticity is virtually the same as Quine's definition of logical truth: for Bolzano, a proposition is logically analytically true if and only if it is true and only logical ideas or concepts occur in it essentially; while for Quine, a statement is logically true iff it is true and contains only logical expressions essentially." This formulation in terms of uniform replacement rather than the existence or otherwise some equivalent free of the expression, is the theme of Section 3 below.

C. More from the 'Hume's Law' literature. In a somewhat similar vein to the discussion quoted from Pigden [133] in Section 1, Schurz [155], having presented three sample arguments, or inferences, bearing on this same *is/ought* issue ((I-1)–(I-3))[92], writes as follows [155, p. 202*f.*]:

> So what is it that gives a mixed conclusion or premise in an argument an inferentially non-trivial normative content? An answer to this question is provided by the replacement criterion of inessentiality of subformulas. In the inference (I-1), $p \vee Oq$ does not have relevant ethical content because the normative subformula Oq in the conclusion of this inference is completely *inessential*, which means that it can be replaced by any other formula whatsoever, *salva validitate* of the inference, i.e. without this replacement making the inference invalid. In the inference (I-2), the disjunction $p \vee Oq$ has a relevant ethical content, because here the subformula Oq of the second premise *is* essential; a replacement of it by an arbitrary new formula would make the inference invalid. Likewise, in the inference (I-3) the conclusion $p \vee Oq$ has relevant ethical content because here Oq is an essential subformula, i.e., it cannot by replaced in an arbitrary manner without making the argument invalid.

One issue here is that the idea of inessentiality as the existence of an equivalent formula in which the expression concerned does not occur does not quite work as it stands, since we can replace Oq by $O(q \vee q)$ or $O\neg\neg q$, for instance. One might consider adapting Definition 2, where A' plays the role played by q there, along the following lines (taking the Default Background Default Conditions to be satisfied, in the interests of simplicity):

> For a logic L, a formula A' occurs L-*essentially* as a subformula of A if and only if for all formulas B for which $\vdash_{\mathsf{L}} A \leftrightarrow B$, B has a subformula L-equivalent to A'.

Observe that the point made in the text to which note 1 is appended, about essential occurrence implying occurrence, doesn't apply to this modified form: as it stands, the modification counts $O(q \vee q)$ as occurring L-essentially as a subformula of $p \wedge Oq$ (for L a normal deontic logic), even though $O(q \vee q)$ is not in fact subformula of $p \wedge Oq$. Further, would we want to say that pq occurs essentially $(p \wedge q) \wedge r$? The suggestion just mooted would not allow us to, since no subformula of the suitably equivalent $p \wedge (q \wedge r)$ is equivalent to $p \wedge q$; we might be able to cobble up

[92] (I-1) has the premise p and conclusion $p \vee Oq$; (I-2) has premises $p \vee Oq$ and $\neg p$, and conclusion Oq; (I-3) has premises $\neg p \rightarrow r$ and $r \rightarrow Oq$, and conclusion $p \vee Oq$.

a contextually relativized notion of equivalence according to which p is in that context equivalent to $p \wedge q$ (or that q is, for that matter). For present purposes we do not need to resolve any such issues here, since only the essential occurrence of unstructured expressions is under consideration.[93]

After the passage quote above from [155], Schurz goes on to remark that "This replacement criterion of inessentiality, or of irrelevance as I call it, has an interesting history," which he traces back to 1947 work by Stefan Körner, which the interested reader can follow up by consulting [155], and — the original stimulus for much of this debate and the source of the inferences alluded to in note 92 — Prior [140]. What would be called Körner-inessentiality in the terminology of Example 12 is a matter of a very specific replacement criterion: is the sentence equivalent to the result of replacing the expression (whose essential occurrence in the sentence is at issue) by the negation of that expression.

Prior himself wrote, in this connection, not of inference-relative vacuity as Pigden does, but of *contingent* vacuity, which is of a piece with the move to relativizing the moral/non-moral distinction among statements to worlds or to models, a recurrent theme of Humberstone [79] surveying this and other reactions to Prior's discussion; see also Russell [151] for a monograph-length treatment of the topic, revising the earlier Restall–Russell account criticized in §3 of [79], as well as including some comparative remarks on, *inter alia*, Pigden, Schurz and the earlier Restall–Russell account [151, pp. 3–7]; Pigden [135] is a direct reply to Schurz [155]. Prior himself [140, p. 200] did also make use of the simpler notion of essential occurrence in play in the present discussion (as will be clearer in Section 3), echoing a faulty formulation of Quine's: "It is also necessary, for example, that at least one of the ethical expressions which are present should occur 'essentially', i.e., should not be just replaceable by any expression whatsoever (of the appropriate grammatical type) without change of truth-value." The weakness of this way of putting things is illustrated in Appendix B with the example *Neptune is less than 1000 light years away from a star.*

Since Prior has just been cited in connection with the more straightforward simple notion of essential occurrence, we pause to note how widespread the appeal to this notion became after its enunciation by

[93] Further discussion on aspects of essentiality as applied to (the occurrence of) compound formulas can be found in several of Schurz's publications; for example, in [154, p. 92], the (in)essentiality of Bridge Principles in alethic–deontic logic is discussed.

Quine in [144] (originally 1936). Within a few years, Morton White [175] was deploying it the course of a discussion of historical explanation,[94] with Pap [125] soon afterwards on the analytic/synthetic and *a priori*/*a posteriori* distinctions (and again on many occasions in [126]). In the next two decades, it popped up from time to time in analytic philosophy of science, as one sees from the collection of Hempel's papers [58], in which it emerges on pp. 37, 208, and 294. Of course there are numerous further examples in the philosophical literature, several of them (Goodman, Nakhnikian, ...) mentioned in Appendices A, D, and elsewhere in the present discussion.

D. 'Contradicting with a contrary' issue. We return, as announced in note 37 to the case sketched in the paragraph to which that note is appended, retaining the informal presentation the example there — taking a single interpretation to be under discussion and use-mention confusing names and their denotations, if you and I, both intuitionists, are in conversation. You say A ($= (Fa \vee Fb) \wedge \neg\neg Fa$) and thereby say something in part about about b, but when I respond with $\neg A$ I am trying to contradict you but apparently I have not said anything about b at all. That was the scenario. First we should notice that the traditional contrast between contradictories and contraries — the latter not being required to be incapable of joint falsity, but both required to be incapable of joint truth — there is no corresponding difference available for *verbs*: if I say something which cannot be true alongside something you have just said, especially when this is immediately evident, then I have contradicted you, whether or not our two claims are such that of necessity one or other must be true. For example, if you say that Jane visited in India in 1990 and I then say that Jane has never been to India, I have contradicted you, even though the two claims are not subcontraries (and would both be false if Jane visited India but only once, that being in 1986). With this in mind, we might say that whether or not Example 10 raises a problem for IL, given the e.o.-based account of aboutness, this is no more serious than a problem we can extract from Example 6. And there, intuitionistic negation is not in play at all. This is because a statement and the result of prefixing it by "It is necessarily not the case that" will be contraries, even if not contradictories, by the

[94] And in particular White argues that history has no distinctive technical vocabulary of its own, distinguishing itself in that respect from (applied) scientific work. (Paluch [124] takes up this issue.)

lights of $\mathsf{S4}$, in which the negation of their conjunction is provable (even though in general their disjunction is not).

We can adapt Example 6 to check that the context $\Box\neg(\cdot)$ is not e.o.-preserving in $\mathsf{S4}$. Let A be the formula so labelled in Example 6. We saw there that q occurs $\mathsf{S4}$-essentially in A, so by Prop. 5(iv), q occurs $\mathsf{S4}$-essentially in $\neg A$. Now embed $\neg A$ in the context of current concern, giving the formula $\Box\neg\neg((\Box p \vee q) \wedge p)$ which is $\mathsf{S4}$-equivalent to the formula $\Box((\Box p \vee q) \wedge q)$, in which q was seen in Example 6 not to occur $\mathsf{S4}$-essentially, so q does not occur in the longer $\neg\neg$-featuring version, by Prop. 5(i).

One might accordingly think to reply to the suggestion that the fact that $\neg$ is not e.o.-preserving in IL is somehow problematic: not at all, the modal case of impossibility (of $\Box\neg$, that is) just worked through shows that contrary-forming operators which do not at the same time form subcontraries and so do not contradictory-forming operators are no more problematic than operators (such as $\Box$ itself) which do not raise such qualms.[95] What this response overlooks is that intuitionistic negation doesn't just convert a statement into a contrary — something or other incompatible with the original statement — but specifically into the *deductively weakest* contrary of the given statement, implied by any other contrary.[96] You can't say anything less than (IL's) $\neg A$ and still be disagreeing with your interlocutor's assertion that A.

E. A Subtlety Emerging from Example 17. The issue was raised at the end of Example 17 by the suggestion that there is no need to fuss about the details of the logic under consideration: we "can always define such projection connectives without worrying about the available primitive connectives are and how they behave accord to the logic under consideration, *simply by stipulating* that for any A, B, the formula $A \#^2_1 B$ is to be the formula A, and $A \#^2_2 B$, the formula B."

[95] Here for reasons of space we skip over the terminological issues involved — principally that of the choice of taking contradictories to be both contraries and subcontraries, rather than taking these three relations as mutually exclusive — as well as the question of what subcontrariety amounts to precisely in an intuitionistic setting. Further discussion of such matters can be found in Appendices 1 and 2 in Humberstone [78], and references there cited.

[96] See (if necessary) Humberstone [74, p. 1169*f.*]; it is best not to get waylaid by toying with *dual* intuitionistic negation, which instead forms weakest contraries, but goes against the currently relevant grain by violating the Disjunction Property, as recalled in note 33; see also Appendix 1 of [78].

A subtlety arises here, though, and one somewhat independent of the distinction between the object-linguistic and metalinguistic treatments of defined expressions mentioned in note 45 and the text to which it is appended, though most readily illustrated with the object-linguistic approach in mind. As an unavoidable preamble, we rehearse a background point familiar from any good introduction to definitional hygiene (such as Chapter 8 of Suppes [164]). The usual constraints on defining a new functor expression (in the sense explained in Remark 7, such as a predicate symbol, function symbol, or connective, is that the n-ary expression being defined should have n distinct variables of the appropriate type as it appears on the left (as *definiendum*) and no additional (free) variables on the right (in the *definiens*), on pain of risking that the definitionally extended logic or theory may extend the original non-conservatively ('creatively', as it is usually put in this connection).[97] Thus if we are working in the language of arithmetic (with addition and multiplication) and someone proposes the introduce a new two-place function symbol $*$ with the stipulation that $x \circledast y$ is to be $(x + y) + z$, we object that instantiating the variables x, y, z respectively to 1, 2, and 4, say, tells us that $1 \circledast 2 = (1 + 2) + 4$, and instantiating them to 1, 2, and 8 tells us that $1 \circledast 2 = (1 + 2) + 8$ from which we conclude that $(1 + 2) + 4 = (1 + 2) + 8$ — a conclusion we not only want to avoid but also, more to the present point, one not available for us to draw before the would-be definition was employed, despite being formulated without the aid of the new symbol $*$ being introduced.

What is not usually given much attention, on the other hand, is the reverse imbalance in free variables: having a variable free on the left that is not free on the right. Continuing to illustrate with the arithmetical case, suppose the proposed introduction of the new — and now ternary (so we'll use prefix notation for it) function symbol $\circledast$ goes, instead, like this: $\circledast(x, y, z) = (x + x) + (y + y)$. We can calculate away consistently using this, immediately noting that though it is a ternary function symbol, and indeed picks out, given the interpretation of $+$ as addition, a perfectly good ternary function (or operation), though one that is *essentially binary* (as in note 1): arguably pointless, but certainly harmless. As the point was made at by the imagined interlocutor at the end of in

[97] Here for connectives and function symbols we have in mind specifically equational definitions, the subject of §8.4 in [164] — not that Suppes is considering these, since he is explicitly discussing definition non-logical first-order theories.

Example 17, we can simply define (for any A, B), the formula $A \#_1^2 B$ to be the formula A itself — and hey presto! — the first-coordinate binary projection connective is a defined connective. The need to make space for a rejection of this description of the situation is pressing. Recall the relevant equivalence property from [82], taken up after Examples 3: any two equivalent formulas have the same sentence letters occurring in them (not necessarily the same number of times), with the upshot that any such letter occurs L-essentially in any formula in which it occurs at all, for L with the relevant equivalence property. But if $\#_1^2$ is a binary connective, then certainly q occurs in $p \#_1^2 q$, regardless of whether it is introduced into our chosen L or into some L' without the relevant equivalence property, in which it is now declared to occur in this $\#_1^2$-compound, despite q's non-occurrence in a formula (namely the formula p itself), in apparent contradiction with the fact that we are dealing with an L for which all occurrence is L-essential occurrence.

A resolution of this apparent contradiction will need to be tailored to one's preferred treatment of defining a connective, object-linguistic or metalinguistic. In either case we can most conveniently take languages in the traditional manner of the Polish logic school, i.e., as absolutely free algebras (of their similarity types/signatures, as given taking by the primitive connectives as fundamental operations). On the object-linguistic view we are adding a new symbol to the language which represents a new fundamental operation — a new way of combining formulas to make a formula — giving a new free algebra, and the definition tells us what the definitionally extended logic should be: the least extension of the original logic in which the *definiendum* and *definiens* are (not just equivalent but) synonymous.[98] On this approach, $p \#_1^2 q$ is a formula constructed from two formulas — in this case, as sentence letters, two of the language's free generators — in which both p and q occur, the latter inessentially, relative to the definitionally extended logic (rather than the original logic). On the metalinguistic view of definition, the defined vocabulary is a way of talking about the formulas of a single language (as free algebra, again) which sometimes mirrors the effect of a language-expanding object-linguistically viewed definition, as when we announce an intention to refer to any formula of the form $\neg A \vee B$ more concisely as

[98] This formulation is taken from p. 59 of [65], formulated specifically for the case of formulas/sentences but readily adapted to arbitrary expressions on the basis of the synonymy of formulas containing them.

$A \to B$, so "$\to$" is genuinely a binary connective — albeit not a primitive such a connective — taking two formulas to a formula constructed from them with the single application of an item of the object language's own primitive vocabulary. But not always. For example, we may find it convenient, especially if we are talking about pure implicational formulas to introduce the notation $(\cdot)^{-1}$ according to the stipulation that for C of the form $A \to B$, C^{-1} is the formula $B \to A$, while for any other C (which in the pure implicational setting amounts to: when C is some p_i), C^{-1} is C itself. Thus $(p \to q)^{-1}$ is the formula $q \to p$, not some new formula with $p \to q$ as a subformula with a 1-ary 'conversifying connective' as its main connective. In [74] functions such as $(\cdot)^{-1}$ here are referred to as non-connectival operations on formulas, terminology already employed to this effect in Remark 19(iii) in our discussion here.[99] On this approach $\#^2_1$ is a non-connectival operation and $p \#^2_1 q$ is just the formula p, so "q" occurs in this particular designation of that formula but not in the formula designated (as in the case of "$\mathsf{del}_p(p \wedge q)$" of 18, with p appearing in this label but not in the formula for which it serves as a label). Taking either the object-linguistic or the metalinguistic approach to defined connectives without illicitly combining the two approaches in one breath thus dissipates the apparent contradiction in contemplating projection connectives and what does or doesn't occur in $p \#^2_1 q$.

A closely related issue arises with such metalinguistic references as occur with "let $C(p, q)$ be a formula in which at most the sentence letters p, q occur" such as arise (e.g., in our discussion of contexts) when we want $C(A, B)$ to be the result of substituting formulas A, B uniformly for any occurrences of p, q respectively in $C(p, q)$. We may want to prove something about all formulas $C(A, B)$ by induction on the complexity of $C(p, q)$, and we come to the inductive step for the case in which $C(p, q)$ is, say, a disjunction $C_1(p, q) \vee C_2(p, q)$. Even if each of p, q occurs in $C(p, q)$, either or both of them may be absent from one of the disjuncts, for example because the first disjunct is $\Box p$ and the second is $p \wedge \neg q$. But the inductive hypothesis will still be able to handle the first disjunct since the notation "$C(p, q)$" was set up only to require that no variable other than those exhibited occurred, rather than that each variable exhibited

[99] See the index entry "non-connectival operations on formulas" in [74] for numerous further examples. The example just given may be more naturally presented as having $(\cdot)^A$ undefined for the case in which A is not an implicational formula, but we are simply avoiding the extraneous novelty of accommodating *partial* non-connectival operations.

had to occur in the formula.[100] The latter condition would render the inductive step excessively complicated as all possible cases of C_1 and C_2 are separately considered depending as to exactly which of p, q occurred in which disjunct, so that C_1, for example, had featured (1) neither of them,[101] (2) p but not q, (3) q but not p, or (4) both p and q. Thus we again have a case — say, with (2), in which the notation "$C_1(p, q)$" contains "q" while standing for a formula in which "q" does not occur. In particular, it does not stand for a formula in which q occurs inessentially. It might even be desirable to revive the old term "fictitious variable" (formerly used for *inessential variable* in the sense of note 1) for this kind of occurrence of a metalinguistic occurrence.

We close with something of an aside prompted by aspects of the foregoing discussion. The usual notion of *derived operation* to contrast with the fundamental operations of an algebra is that the derived operations comprise those in the *clone* of operations generated by the fundamental operations — the familiar *term functions* — of universal algebra, and this automatically builds in the projection functions as derived operations. If one wants to attend to the case in which the projection functions arise as more genuinely *compositionally* derived operations, as with when the algebra is a lattice, then a more restrictive conception of derived operation is required.[102] Instead of looking at the operations in the clone generated by the fundamental operations, one could at those in the *composition-closed class* generated by them, in the sense described in Waldhauser [170] or [171], where several clone-like collections of operations are considered.

F. Goodman and Others. Goodman [50, p. 7] writes:

> Let us say that a statement T follows from S differentially with respect to k if T contains an expression designating k and follows logically

[100] Likewise in the case of Williamson's notation (note 5), generalized in the obvious way: $\langle C, p, q\rangle$ might have $C = C_1 \vee C_2$ and we can regard the context $\langle C, p, q\rangle$ as the disjunction of $\langle C_1, p, q\langle$ with $\langle C_2, p, q\rangle$, with putting A, B in that order into the context $\langle C_1, p, q\langle$, say, where only q occurs in C_1, as the result of as substituting A and B respectively for any occurrences of p, q as there may be in C_1 (which is a vacuously satisfied condition in the case of p).

[101] E.g., because C_1 was $\Box\bot$.

[102] We cite this example because of the stress placed in our discussion on the absorption laws for 'dummying in'. For lattices, the fundamental operations are meet and join operations $\wedge$ and $\vee$ and the n-ary projection to the k^{th} coordinate is given by $x_k \vee (x_1 \wedge \cdots \wedge x_k \wedge \cdots \wedge x_n)$ or its dual.

> from S, while no generalization of T with respect to any part of that expression also follows logically from S. Then our final definition of absolute aboutness runs: S is *absolutely about* k if and only if some statement T follows from S differentially with respect to k.

He also defines [50, p. 12] S to be *immediately about* k when S follows from itself differentially w.r.t. k. (So here we are presuming a specific intended model — or at least a contextually salient model — to relative to which some expression denotes k, that model's domain.) Various aspects of these notions of Goodman's were the subject of subsequent discussion and development in such papers as Goodman [51], Ullian [166], Putnam and Ullian [142], and numerous others detailed in Hart [53] and Osorio-Kupferblum [122]. By-passing the perhaps overly complicated aspects of some of those proposals,[103] here we prefer to identify the key concept of Goodman's account with the distillation of it credited to H. E. Hendry at p. 28 of Hart [53], in this extended form of the quotation from that source given in Section 3 above (with the same ellipse as there — see note 42):

> A designator δ *occurs essentially* in a sentence ψ if and only if ψ does not imply its own generalization with respect [...] δ. In terms of this notion, we define a notion of "absolute aboutness":
>
> φ is *absolutely about* k if and only if ϕ is equivalent to some sentence ψ containing an essential occurrence of a designator δ that designates k.

The formulation of this second definition is somewhat peculiar, since if φ is equivalent to a sentence in which δ occurs essentially, then ϕ is already itself such a sentence.[104] That is clear, for the definition of essential occurrence given informally in the early part of Section 1 and then more formally (and for the special case of sentence letters) in Definition 2: Suppose δ occurs essentially in φ and ψ is equivalent to φ: then δ occurs in ψ (since otherwise its occurrence in φ would not be 'essential'), so it remains to check that δ occurs *essentially* in ψ. If it doesn't, there is a δ-free χ with ψ equivalent to χ. But in that case χ is a δ-free formula equivalent to φ, contradicting the supposition that δ occurs essentially in φ.

[103] The distinction between absolute and immediate aboutness remains obscure to this reader, for instance.

[104] Here I am re-wording Hart's "containing an essential occurrence of a designator δ" as "in which δ occurs essentially" to avoid any possible confusion with notions of essentiality for individual token-occurrences (as described at the end of Section 1), which are presumably not meant to be in play.

(1)	$\vdash_{\mathsf{QCL}} \psi(a) \to \forall x \centerdot \psi(x)$	(2)	$\vdash_{\mathsf{QCL}} \psi(a) \leftrightarrow \forall x \centerdot \psi(x)$
(3)	$\vdash_{\mathsf{QCL}} \psi(a) \to \psi(b)$	(4)	$\vdash_{\mathsf{QCL}} \psi(a) \leftrightarrow \psi(b)$
(5)	$\vdash_{\mathsf{QCL}} \forall x \forall y(\psi(x) \to \psi(y))$	(6)	$\vdash_{\mathsf{QCL}} \forall x \forall y(\psi(x) \leftrightarrow \psi(y))$
(7)	$\vdash_{\mathsf{QCL}} \exists x \centerdot \psi(x) \to \psi(a)$	(8)	$\vdash_{\mathsf{QCL}} \exists x \centerdot \psi(x) \leftrightarrow \psi(a)$
(9)	$\vdash_{\mathsf{QCL}} \exists x \centerdot \psi(x) \to \forall x \centerdot \psi(x)$	(10)	$\vdash_{\mathsf{QCL}} \exists x \centerdot \psi(x) \leftrightarrow \forall x \centerdot \psi(x)$

Table 2. Equivalent Claims (for any given $\psi(\cdot)$))

How do things stand, though, with Hendry's own suggested definition of occurring essentially, based on Goodman's 'generality' approach? Let us write the ψ of the definition as $\psi(\delta)$ with $\psi(x)$ indicating the replacement of all occurrences of δ in $\psi(\delta)$ by the variable x. The proposal is then that $\psi(\delta) \to \forall x \centerdot \psi(x)$ should be valid/provable in classical first-order predicate logic. The latter we denote by QCL, for (standardly) Quantified CL. Recalling the developments of Section 3, where propositional quantifiers took more of the limelight than standard first-order quantifiers. To make things look more familiar from the point of view of current logical work, rather than reproducing Goodman's δ, in Table 2 we replace it with a, and take b to be a second individual constant[105] not occurring in $\psi(a)$ but occurring exactly where a does there, in $\psi(b)$. All of the claims in the table are equivalent — all correct or none of them correct, that is; those in the second column feature biconditional versions of the one-way implications in the first column, and of course there are numerous further possibilities that could have been included (such as, in the same vein as (9) and (10): $\forall x \centerdot \psi(x) \vee \forall x \centerdot \neg\psi(x)$).

Note that the description of these claims are equivalent for an arbitrarily selected $\psi(x)$ to which $\varphi(a)$, $\varphi(b)$, are related as described is not to be confused with the claim that for any such selection the formulas involved are themselves QCL-equivalent. The fact that (3) implies (4) does not mean, for example, that for all such $\psi(x)$ we have:

$$\psi(a) \to \psi(b) \vdash_{\mathsf{QCL}} \psi(a) \leftrightarrow \psi(b),$$

and the fact that (3) implies (1) does not mean that

$$\psi(a) \to \psi(b) \vdash_{\mathsf{QCL}} \psi(a) \to \forall x \centerdot \psi(x);$$

[105] Or *parameter*, the point simply being to avoid having free variables in $\vdash$ formulations, and more generally to avoid open formulas as anything other than subformulas of closed sentences. In this respect we avoid also the formulations of Goodman and Ullian as well as those of Patton [130], on whose (R), at p. 313, Table 2 here can be regarded as a convenient elaboration.

in either case, taking $\psi(x)$ as Fx (F a monadic predicate letter) gives a counterexample. This is the familiar distinction between rules of proof and rules of inference, stressed for modal logic in Smiley [160, p. 115] (footnote included). Indeed the transition, in either direction, between the modal analogues of (1) and (7) — $\varphi \to \Box\varphi$ and $\Diamond\varphi \to \varphi$ — is a rule of proof studied under the name *the MacIntosh rule* in the lattice of normal modal logics in Chellas and Segerberg [20] and Williamson [176].[106] Of course in that setting the analogues of rule versions of (1)–(10) are not in general interderivable, and indeed in some cases do not exist (namely, cases (3) and (4)).

G. 'Disjunctive' Variations on Essential Occurrence. Might it be that although no variable occurs L-essentially in A, it is "essential" that this or that variable — p_1 or else p_2, say, from our official enumeration in the opening paragraph of Section 1, occurs in any given L-equivalent of A, making the formula in question count as constant-like by the definition, though intuitively not deserving any such categorization. This will be the first of the two disjunction-involving characterizations, in this appendix, of a relation in the 'essential occurrence' family. Note the partial; similarity to 'existential' formulations such as those in encountered in Appendix A, such as the Donnellan-related case of every equivalent (for a given sentence) containing some definite description without there being some description which every equivalent sentence contains. (The second will involve taking the disjunction down into the object language.) More precisely, the possibility under consideration is that no variable occurs L-essentially in A but every formula L-equivalent to A contains at least one of the variables $q_1, \ldots, q_n$ in some finite list.[107]

To check out the credentials of the possibility here mooted, we consider a simple representative case. Suppose that we have a formula $A(p, q, r)$ in which, whether or not p occurs L-essentially, for some L

[106] Unusually, the working notion of normality (or perhaps more generally, of being a modal logic at all) in the first of these publications does not build in the condition of closure of uniform substitution, as is noted in the latter, p. 88; see note 113 in Humberstone [81] for a slightly fuller discussion.

[107] Of course this can happen if we drop the "finite" since the official enumeration just alluded to insisted on a countable supply of sentence letters, so any case in which the language of L has no nullary connectives/sentential constants presents a case in which every formula, and so every formula equivalent to a given formula, has *some* variable(s) occurring in it.

satisfying the Default Background Conditions,

(1) q does not occur L-essentially;
(2) r does not occur L-essentially; but
(3) every L-equivalent of $A(p,q,r)$ contains q or r.

(1) and (2) imply that there are formulas B, C, L-equivalent to A, in which, respectively, q does not occur, and r does not occur, as a reminder of which, we write them as $B_{\mathsf{no}\text{-}q}(p,r)$ and $C_{\mathsf{no}\text{-}r}(p,q)$. More explicitly, then, we have

(4) $\vdash_{\mathsf{L}} A(p,q,r) \leftrightarrow B_{\mathsf{no}\text{-}q}(p,r)$ and
(5) $\vdash_{\mathsf{L}} A(p,q,r) \leftrightarrow C_{\mathsf{no}\text{-}r}(p,q)$.

In view of (3), r does indeed occur in on the right of the biconditional in (4), since q does not and q occurs in the formula on the right of the biconditional in (5), since r does not. From (4) and (5) we conclude that

$$\vdash_{\mathsf{L}} B_{\mathsf{no}\text{-}q}(p,r) \leftrightarrow C_{\mathsf{no}\text{-}r}(p,q).$$

Since r does not occur in the formula on the right here, uniformly substituting some completely new variable, s, say, for r in this biconditional affects only its left-hand side, giving:

$$\vdash_{\mathsf{L}} B_{\mathsf{no}\text{-}q}(p,s) \leftrightarrow C_{\mathsf{no}\text{-}r}(p,q).$$

In view of (5), we can replace the r.h.s. here with $A(p,q,r)$, revealing this formula to be L-equivalent to a formula (namely $B_{\mathsf{no}\text{-}q}(p,s)$) in which neither q nor r appears, contradicting (3). Of course, we could have used $\top$ or $\bot$ or p itself to substitute for r in $B_{\mathsf{no}\text{-}q}(p,r)$ to provide a q,r-free L-equivalent $A(p,q,r)$. There is, then, after all, no 'disjunctive' essentiality along the lines envisaged: if every equivalent to A contains either q or r, then either every equivalent to A contains q or every equivalent contains r, so at least one of q,r occurs in A essentially *tout court*. Although we have been working through the special case of $n = 2$, it is clear that a more general conclusion is available, which is left to the interested reader to formulate and prove.[108]

We turn now to the second of our disjunction-involving candidate variations on the theme of essential occurrence. In an experimental

[108] Similar considerations show that for any pair — where again the $n = 2$ case illustrative rather than special — of atomic expressions, of every sentence in which equivalent to a given sentence contains at least one of those atomic expressions, then at least one of them occurs essentially in its own right in the given sentence.

spirit, let us explore the possibility of a disjunctive variant of the notion of a variable's occurring in A, in which the disjunctive element lies in the object language itself. Since this will involve us in disjoining object-linguistic conditionals in the scope of disjunction, to replace logical equivalence — the outright provability of a biconditional — with what we may loosely call the active occurrence of several such disjoined biconditionals. This calls for a strengthening of the conditions on $\leftrightarrow$ given in the Default Background Conditions of Section 1 in order to turn $\leftrightarrow$ into an appropriately equivalential connective — or more accurately, to make $\{p \leftrightarrow q\}$ (or $\{p \rightarrow q, q \rightarrow p\}$) into a *set of equivalence formulas* for $\vdash_{\mathsf{L}}$ — in the sense of abstract algebraic logic,[109] by imposing the following 'replacement'-style *extensionality* condition (a strengthening of the $\leftrightarrow$-involving congruentiality condition on other primitive connectives), for every n-ary connective # in the language of L:

$$A_1 \leftrightarrow B_1, \ldots, A_n \leftrightarrow B_n \vdash_{\mathsf{L}} \#(A_1, \ldots, A_n) \leftrightarrow \#(A_1, \ldots, A_n) \quad (\#\ \text{Ext})$$

We also need a familiar condition on disjunction to be satisfied:

$$\Gamma, A \vee B \vdash_{\mathsf{L}} C \text{ if and only if } \Gamma, A \vdash_{\mathsf{L}} C \text{ and } B \vdash_{\mathsf{L}} C \qquad (\vee)$$

Finally, we need the concept of a minimally provable disjunction, though the terminology has no connection with (Johansson's) minimal logic, with the aid of which we can define our object-language 'disjunctive' version of essential occurrence — which we offer in a weaker and a stronger form, labelled with the aid of a prefixed 'alter' to recall the traditional terminology of *alternation* for disjunction:

Definitions 29. Suppose that $\vdash_{\mathsf{L}}$ satisfies the Default Background Conditions as well as ($\vee$) and (# Ext) for each primitive connective # of the language of L. Then

(i) A disjunction $C = C_1 \vee \ldots \vee C_m$ is *minimally provable* in L notated as — $\vdash^{\mathsf{min}}_{\mathsf{L}} C$ — iff C is L-provable but for no i ($1 \leq i \leq m$) is $C_1 \vee \ldots C_{i-1}, \vee C_{i+1} \vee \ldots \vee C_m$ is L-provable.

(ii) q occurs (at most) *alter-L-inessentially* in A iff and only if there are formulas $B_1, \ldots, B_m$ with $\vdash^{\mathsf{min}}_{\mathsf{L}} (A \leftrightarrow B_1) \vee \ldots \vee (A \leftrightarrow B_m)$, and q not occurring in some B_i ($i = 1, \ldots, m$).

[109] The other conditions, listed on p. 424 of Hermann [59] are that $\vdash_{\mathsf{L}} A \leftrightarrow A$, and $A, A \leftrightarrow B \vdash_{\mathsf{L}} B$; Hermann also mentions some additional conditions given in the original definition from the 1970s, which follow from these and (# *Ext*) below. Further discussion and references can be found in Czelakowski and Pigozzi [21].

Thus, we have in alter-inessential occurrence, a weaker sense of being equivalent to a formula not containing which the variable whose essentiality is at issue: being equivalent in one or more of the irredundantly arising possibilities (represented by the various disjuncts). It would altogether too weak if we dropped the "irredundantly," captured in Def. 29(i) by the notion of minimally provable disjunctions deployed in (ii): otherwise we could always add a redundant q-free B_{m+1}; for example q would occur alter-L-inessentially in, say, $\neg q$, in view of the fact that $\vdash_{\mathsf{L}} (\neg q \leftrightarrow \neg q) \vee (\neg q \leftrightarrow r)$, for any L with $\leftrightarrow$, $\vee$, satisfying the conditions above.

Indeed, the logics satisfying the Default Background Conditions in play in the main body of the present paper, CL, IL, do not provide a suitable habitat for illustrating alter-essential occurrence. Taking the case of CL first, q does indeed occur alter-CL-inessentially in any formula A in which it occurs at all. If A or its negation is CL provable then we have the $m = 1$ case from Def. 29(ii) since we have $\vdash_{\mathsf{CL}} A \leftrightarrow \top$ or $\vdash_{\mathsf{CL}} A \leftrightarrow \bot$, where the superscripted min goes without saying since there is only one disjunct, and if A is neither classically provable or refutable, when we have the Law of excluded middle in the form:

$$\vdash^{\mathsf{min}}_{\mathsf{CL}} (A \leftrightarrow \top) \vee (A \leftrightarrow \bot). \qquad (\mathrm{LEM}_{\top\bot})$$

On the other hand IL fares no better as a source of examples, since the Disjunction Property prevents it from ever being the case that a disjunction with more than one disjunct is never minimally provable — so alter-IL-inessentiality is just the original IL-inessentiality.

Thus we are led to consider the strictly intermediate logics (between IL and CL, that is) lacking the disjunction property. One that jumps out as particularly promising is Dummett's LC, with consequence relation $\vdash_{\mathsf{LC}}$ as the axiomatic extension of $\vdash_{\mathsf{IL}}$ by all instances of the schema $(A \to B) \vee (B \to A)$. The reason is that this is the weakest intermediate logic in which certain connectives — in particular $\wedge$ and $\vee$ — share with their classical analogues the property of being *componential*, as it is called in [76, p. 695], which is roughly speaking the claim, for an n-ary connective # that it forms compounds equivalent to one of their components, and which more precisely can be take for present purposes to the provability of all formulas of the form

$$(\#(A_1, \ldots, A_n) \leftrightarrow A_1) \vee \cdots \vee (\#(A_1, \ldots, A_n) \leftrightarrow A_n) \qquad (\#\ \mathrm{Comp})$$

with $\leftrightarrow$ and $\vee$ satisfying the conditions imposed on them before Definitions 29 above.[110] This does indeed provide us with the desired illustration:

Example 30. The componentiality of $\wedge$ in LC thus amounts to the fact that

$$\vdash_{\mathsf{LC}} \big((p \wedge q) \leftrightarrow p\big) \vee \big((p \wedge q) \leftrightarrow q\big),$$

where the formula involved is a minor rewriting of Dummett's $(p \to q) \vee (q \to p)$ axiom, as indeed does the analogous formula for # as $\vee$ itself, though we stick to the $\wedge$ case to minimize distraction from the other role played by $\vee$ here (as main connective, doing its "alter-" work). Further, since neither of the disjuncts of this formula is LC provable, we are dealing with a minimally provable disjunction, and so we indeed in the territory of alter-essential occurrence: p occurs on the right of one of the disjoined biconditionals but not the other, and the same goes for q. So each of these variables occurs alter-LC-inessentially in $p \wedge q$, or as we may equally well put it, each of them occurs alter-LC-essentially in that conjunction. $\lhd$

Picking p to concentrate on for illustrative purposes, we can ask about whether this sentence letter occurs strongly or weakly alter-essentially in $p \wedge q$ with respect to LC.

In fact we could also illustrate a non-trivial case of alter-inessentiality (one in which the "alter" cannot be dropped, that is) without taking the above componentiality-of-$\wedge$ route, in the setting of the other famous Dummett–Lemmon intermediate logic, LC's younger brother KC, which lacks the componentiality features for conjunction and disjunction. In its best known axiomatization, KC extends IL with all instances of the schema $\neg A \vee \neg\neg A$.[111] In the style of ($\mathrm{LEM}_{\top\bot}$) we can rewrite the nega-

[110] Definition 3.95 on p. 695 of [76], gives the more general formulation in which $\leftrightarrow$ is replaced by suitable references to a 'set of equivalence formulas'. The important point to note is that in the rough formulation above "equivalent to one of their components" does not involve logical (i.e., outright provable) equivalence — we are not requiring # to be one of the *projection* connectives — and that replacements licensed by the extensionality conditions can be performed 'disjunct by disjunct', as one might put it. Further details on componentiality and on the significance of LC in this connection can be found in [76]. (*Correction*: the reference to $\vdash_{\mathsf{CL}}$ in the third line of Example 4.2 there should be to $\vdash_{\mathsf{LC}}$.)

[111] Dummett and Lemmon [31, esp. p. 252]; the "W" in the label below is for "Weak," as in the title of Jankov [86].

tions from the representative instance, $\neg p \vee \neg\neg p$, of the above schema thus:

$$\vdash^{\mathsf{min}}_{\mathsf{LC}} (p \leftrightarrow \bot) \vee ((p \leftrightarrow \bot) \leftrightarrow \bot) \qquad (\text{WLEM}_\bot)$$

Now, $\text{WLEM}_\bot$ does not, as it stands, have the form dictated by Definitions 29(ii) and further evident in (# Comp), for which we need all the disjuncts involved to have the form $A \leftrightarrow \_)$ for different fillings of the blank. For that, our second disjunct in the case of $\text{WLEM}_\bot$ needs to have the form $p \leftrightarrow B$ for some formula B. We cannot just regroup the parentheses in the second disjunct, since, as is well known, the only intermediate logic in which $\leftrightarrow$ is associative is CL.[112] So the question becomes: can we rewrite (to within CL-synonymy) $\neg\neg p$ or, to use the above formulation, $(p \leftrightarrow \bot) \leftrightarrow \bot$, in the format $p \leftrightarrow B$?

And the answer is: yes, already in IL, in fact, by choosing B as $p \leftrightarrow \sigma p$, where "$\sigma p$" abbreviates "$\sigma(p)$" and $\sigma(A)$ is $A \leftrightarrow \neg\neg A$.[113] We can state the relevant observation, whose proof is left to the reader, as follows, before making the use it as foreshadowed:

LEMMA 31. *$\neg\neg p \dashv\vdash_{\mathsf{IL}} p \leftrightarrow \sigma p$.*

Example 32. In view of Lemma 31, we can rewrite $(\text{WLEM}_\bot)$, thus:

$$\vdash^{\mathsf{min}}_{\mathsf{KC}} (p \leftrightarrow \bot) \vee (p \leftrightarrow \sigma p),$$

from which we conclude that p occurs alter-L-inessentially in p itself, since p does not occur on the right in the first disjunct. $\lhd$

Definition 29(ii) used the condition that $\vdash_{\mathsf{L}} \bigvee_{i=1}^{m} A \leftrightarrow B_i$ for some given $B_1, \ldots, B_m$ not all of which B_i featured q to say what q's alter-inessential occurrence (relative to L) in A came to. In the course of our development of the theme we have encountered cases falling under this description in Examples 30 and 32, but also, for CL a case — taking A as q, for example, in $(\text{LEM}_{\top\bot})$ in which a much stronger disjunctive inessentiality condition was satisfied with "not all of the B_i feature q"

[112] This is illustrated in the present case by the fact that the mooted re-association produces a minor variation on $\text{LEM}_{\top\bot}$ itself.

[113] "σ" to suggest *stable*, a term applied to formulas A for which $\sigma(A)$ is IL-provable. We could of course equivalently regard $\sigma(A)$ as just being the formula $\neg\neg A \to A$, since we are here considering only superintuitionistic logics. It is for this same reason that we treat $\neg A$ indifferently as synonymous with $A \to \bot$ or $A \leftrightarrow \bot$. The latter pair give different negation-like connectives in the more general setting of arbitrary extensions of Johansson's minimal logic [74, p. 1271].

replaced by "none of the B_i feature q". This stronger variation on the alter-inessentiality and alter-essentiality theme might well be worth devoting some attention to, though we have perhaps seen enough of the these disjunctivizing moves for the moment.

H. A Little More Early (and not so early) Carnap. Here we continue note 57, which ended by remarking that though Demolombe and Fariñas del Cerro cite Carnap [15] when discussing aboutness, they do not cite anything more specific than the whole book. Perhaps the reference is there simply because it appears in note 1 of Goodman [50], though more explicitly there, with the passage pp. 284–292 singled out.

On p. 285 there, Carnap (as translated) tells us that the sentence "Yesterday's lecture was about Babylon" *appears* to assert something about Babylon but is in fact only a "pseudo-object-sentence," saying nothing about Babylon, "but merely something about yesterday's lecture and the word 'Babylon'." It is not unclear why, in a case in which the sentence is true, it doesn't tell us that Babylon, even if not as famous as it might have been in its heyday, was still famous enough to attract a lecture audience two and half thousand years later. Of course, being famous (at a given time) is not an *intrinsic* property, but does not seem to bear on the aboutness issue; see further the paragraph below. Yet when he returns to the example on p. 308 of [15], Carnap seems to have something in this general area in mind, writing: "For the qualities (in the ordinary sense) of the city in question, it is not of the least importance whether it has the property of having been treated of in yesterday's lecture or not." (Indeed, never mind the sentence mentioning yesterday's lecture, what about the lecture itself, supposing there to have been one? Did even the lecture itself manage to be about Babylon in Carnap's view, if it consisted mostly of one visiting Assyriologist contesting the evidence on which colleagues had based their views of certain events in Babylon's history? More on Carnap, Babylon, and 'Babylon' can be found in Osorio-Kupferblum [122, pp. 6–7].)

To return briefly to the issue of intrinsic properties: many suggestions have been made as to how to spell out the intrinsic/extrinsic distinction, and it is far from evident, even, that there is a single notion of intrinsicality to spell out. Francescotti [42] collects a number of pertinent discussions of such matters. There is a heavy emphasis on intrinsic — indeed *fundamental* intrinsic properties — combined with mereological essentialism in the theory of various notions of (objectual *de re*) aboutness in Marshall [103], which relates these notions to each other without attempting to define or explicate them in other terms, presenting instead what one diagram caption in [103, p. 1669] characterizes as 'non-reductive analyses of *de re* aboutness'. (In fact, Marshall offers instead: a definition of intrinsicness *in terms of* aboutness: p. 1645, top line.) A somewhat similar line, methodologically, is taken in Bar-Elli [4, p. 161], which begins with the point, mentioned in connection with Strawson and Partee in note 75, about the extreme fluidity of what "about" comes down to in this or that setting:

> Aboutness is a notoriously vague notion. It is vague in the sense that in many cases we may be perplexed by the request to judge what a proposition is about. Its application, particularly in compound statements, is notoriously indeterminate (and it has, of course, a great variety of other uses, not necessarily connected with propositions). However, for our purposes here we should not be too troubled by that. I do not assume that aboutness is a sharply bounded or precise notion. I do not propose to use it in defining the formal structure of the reference function, but rather in explaining its non-formal, substantive, realistic aspects (and this is how the "definitions" offered in the sequel should be read).
>
> The vagueness of the notion of aboutness notwithstanding, it is important to realize that it is a basic notion: It is fundamental to the way we understand our thought and language. With all its vagueness, the idea that in uttering a meaningful statement we talk about something, and that, in general, a meaningful statement is about something, is both a common and an elementary notion. In particular, its application to simple atomic statements is fairly clear, or clear enough for my purposes here. Everybody understands that in stating that Jane is pretty, I am talking about her, stating something about her, and that one has to look at her, or find out about her in some other way, in order to know whether what I said was true.

Bar-Elli emphasizes Frege's use of the idea of what is being talked or written about in getting across his notion of reference (*Bedeutung*), and with Carnap's following suit in [16, Chapter 3]. A conflicting opinion as to the tightness of the reference/aboutness connection is evident in Osorio-Kupferblum [123], with such remarks (quoted already in note 31) as "(A)boutness is a relation internal to language," which, if taken at face value, would suggest that there was no call, after all, for the fuss made in Section 4 between *about* and *about**, or Goodman's attention to this or that expressions's *denoting k*. Partially suggestive of similar sentiments voiced like this on the opening page of Van der Auwera [167] — noting in passing the resemblance of the book's title to that of Tichý's paper [165]:

> Perhaps somebody might object to the preceding considerations with the claim that what language is (all) about, is neither reality nor certain fractions of it, but only somebody's mental picture of reality or of its fractions. I think that this objection is really a correction. Language is about the world as well as about the mind.

By the end here we seem to have drifted into an "about" in the style of *What's it all about (Alfie)?* and a sentiment with which it is hard to imagine disagreement — though this is of course only a preliminary overview of Van der Auwera's subsequent discussion.

I. Spelling out some proofs for Section 4. Here we take up, first, the point mentioned in note 63 — that the double occurrence of "iff" in a definition by Demolombe and Fariñas del Cerro may seem puzzling. The version given initially, as (⊛), and repeated here:

$$NAbout(F, c) \text{ iff } \forall M \forall M'\big((M \approx_{\|c\|^M} M' \;\&\; M \models F) \Rightarrow M' \models F\big). \quad (\circledast)$$

can have the right-hand side of the (single) "iff" appearing, given in 'exported' form, after commuting the conjuncts in (⊛):

$$\forall M \forall M'\big(M \models F \;\Rightarrow\; (M \approx_{\|c\|^M} M' \Rightarrow M' \models F)\big),$$

so that, now the "$\forall M'$" no longer binds a variable in the antecedent of the main ⇒, we can move to the consequent:

$$\forall M\big(M \models F \;\Rightarrow\; \forall M'(M \approx_{\|c\|^M} M' \Rightarrow M' \models F)\big).$$

One more step is needed to get this to match the authors' formulation with the two "iff"s appearing in the paragraph to which note 63 was appended, which after the "iff" linking the *definiendum* and *definiens* was this:

> for every interpretation M, we have $M \models F$ iff for every interpretation M' in M^c we have $M' \models F$,

namely boosting the initial "⇒" to a "⇔":

$$\forall M\big(M \models F \;\Leftrightarrow\; \forall M'(M \approx_{\|c\|^M} M' \Rightarrow M' \models F)\big).$$

Secondly, near the end of Section 4 it is mentioned that (⊛), spelled out as below, admits of an equivalent supervenience-style formulation (⊛′)

$$NAbout(F,c) \Longleftrightarrow \forall M\forall M'\big((M \approx_{\|c\|^M_{\mathsf{prd}}} M' \;\&\; M \models F) \;\Rightarrow\; M' \models F\big) \tag{⊛}$$

$$NAbout(F,c) \Longleftrightarrow \forall M\forall M'\big(M \approx_{\|c\|^M_{\mathsf{prd}}} M' \Rightarrow (M \models F \Leftrightarrow M' \models F)\big) \tag{⊛′}$$

To check that these two biconditionals are indeed equivalent, we show this for their right-hand sides (resp. (1) and (5) below), since their left-hand sides match. It will suffice to start from the right-hand side of (⊛) and derive the r.h.s. of (⊛′), since each of the steps is in an obvious way reversible:

$$\forall M\forall M'\big((M \approx_{\|c\|^M_{\mathsf{prd}}} M' \;\&\; M \models F) \;\Rightarrow\; M' \models F\big) \tag{1}$$

'Export' the second conjunct of the antecedent:

$$\forall M\forall M'\big(M \approx_{\|c\|^M_{\mathsf{prd}}} M' \Rightarrow (M \models F \;\Rightarrow\; M' \models F)\big) \tag{2}$$

Now reletter M as M' and conversely:

$$\forall M'\forall M\big(M' \approx_{\|c\|^{M'}_{\mathsf{prd}}} M \Rightarrow (M' \models F \;\Rightarrow\; M \models F)\big) \tag{3}$$

Next, commuting like quantifiers at the front and using the fact that $M' \approx_{\|c\|^{M'}_{\mathsf{prd}}} M$ and $M \approx_{\|c\|^M_{\mathsf{prd}}} M'$ are themselves equivalent, we infer:

$$\forall M\forall M'\big(M \approx_{\|c\|^M_{\mathsf{prd}}} M' \Rightarrow (M' \models F \;\Rightarrow\; M \models F)\big) \tag{4}$$

Now (2) and (4) are, within the scope of the two initial universal quantifiers, alike except in having the two conditionals in their consequents being each other's converses. So, purely propositional reasoning allows us to combine them into a biconditional, giving

$$\forall M\forall M'\big(M \approx_{\|c\|^M_{\mathsf{prd}}} M' \Rightarrow (M \models F \Leftrightarrow M' \models F)\big) \tag{5}$$

and (5) is the r.h.s. of (⊛′). Similar reasoning shows the equivalence of the right-hand sides of (⊛⊛) and its reformulation as (⊛⊛′), and hence of (⊛⊛) and (⊛⊛′) themselves, from Section 4.

J. Demolombe and Fariñas del Cerro's account with worlds replacing models. Abrusán [1, §2] adapts the account of aboutness in Demolombe and Fariñas del Cerro [26] so as to have possible worlds playing the role of models (interpretations, structures, . . .) in [26]. Osorio-Kupferblum [121] discerns a similarity between [26]'s account and that of David Lewis (in [99], [100] and elsewhere), an assimilation which also involves having the role of models (in the former account) played by worlds (in the latter). Aspects of the two proposals will be discussed here, in the order just presented.

For Abrusán, we continue the discussion of note 55, which recalled the difference between *model-fixed* and *model-changing* semantic accounts of variable-binding devices in first order languages. The model-fixed (and no doubt philosophically preferable) treatment evaluates open formulas relative to a model and variable-assignment, while the model-changing treatment takes such an assignment to be part and parcel of the model itself, so that the interpretation of quantifiers requires us to 'change models' in the course of evaluating a formula relative to a given model. Now, that if *possible worlds* are used to play the role played by models in Demolombe and Fariñas del Cerro's account of aboutness, as in Abrusán [1, §2)], then, *pace* Abrusán (p. 686), the analogous 'world-changing' treatment would have no plausibility, whether possible worlds are thought of abstractly, as ways things might have been, or concretely, as mereologically maximal realities like the actual world we find ourselves immersed in.[114] This feature of Abrusán's presentation could be adjusted, though, by doing the evaluations relative to worlds paired with variable-assignments, though talk of worlds rather than models does suggest that we are on specifically the representational side of Etchemendy's distinction mentioned in note 54 and we should remember that worlds themselves need to be collected and appropriately interrelated if we happened to want a purely formal account of the notions in play. Alternatively, if we are prepared to go along with the idea of an intended such model,[115] not and think of proper names along the lines suggested in Kripke [88], as

[114] In a paper that finds its way into our bibliography for other reasons, Tichý [165, §6] thinks that this is all part of a confused way of thinking: "Unfortunately, the misconstrual of the term 'the actual world' as a name of a world is fostered by many possible-world semanticists" (p. 91).

[115] This is presumably Abrusán's intention, since the discussion begins [1, p. 686] "Let M be a model. . . " where the model is a Kripke model, equipped with a set of worlds and a domain of individuals, for a first-order language; aboutness is then defined in terms of this apparatus and an adaptation of the Demolombe and Fariñas del Cerro notion of variants (over the denotation of their privileged constant c) related worlds rather than models, but nowhere is the potential sensitivity of aboutness statements to the initial choice of a Kripke model acknowledged. So one assumes

rigid designators, this would remove the difficulty faced by Demolombe and Fariñas del Cerro's discussion of thinking of individual constants as having a model-invariant or model-independent denotation, since if c is taken rigidly it does indeed denote the same object in any worlds in which it denotes anything.[116] There is also a downside, though, since if the denotation of, say, the proper name or individual constant c is the wooden table ostended by Kripke in those lectures (p. 113) in the course of urging persuasively that *it* ("that very table") could not have been made of ice from the River Thames — then neither *c is made of ice* nor *c is made of wood* is a statement about the table in question, since the properties of the table that bear on the truth in a world of either statement are not susceptible to the variation required by Abrusán's version of the Demolombe and Fariñas del Cerro account of aboutness. One may think this is no disadvantage at all, and the case is just like that of *If c has a vase sitting on it, then c has a vase sitting on it* a formal rendering of which would be ruled similarly not to be about the table in question by Demolombe and Fariñas del Cerro's model-theoretic account, as well as by Goodman-inspired suggestion — which otherwise fares somewhat differently from theirs (Proposition 22) — that aboutness requires the essential occurrence of a denoting expression. In that connection, it is perhaps mildly amusing to see that with Kripke's table example and the like, we have the *essential properties* of the referent providing precisely the reason for a term rigidly designating it to occur *inessentially* in sentences ascribing those properties (or their complements) to the object.

We turn to Osorio-Kupferblum and the assimilation of Lewis's account to that of Demolombe and Fariñas del Cerro, which needs to begin with her recapitulation of Lewis [121, p. 531], in which footnote positions are cited for reference, below:

> Lewis treats statements as the class of worlds in which they are true, and subject matters M as parts of worlds. Thus, the subject matter 'the seventeenth century' picks out the seventeenth centuries in all worlds that have one. Worlds that are alike with respect to these parts form a

such statements are taken relative to an intended model (the *real* model, for the modal realist).

[116] Here we are in close proximity to an objection to the idea — Russell [151, mid-p. 229] — that model-theoretic semantics for natural languages can be done on either the interpretational or the representational side of Etchemendy's distinction (in Russell's preferred formulation: either the semantic or the metaphysical conception of the semantics).

> cell (there are usually many cells for each subject matter). The *content* of a statement is then given by the class E of worlds the statement excludes. [Footnote numbered 15.] With logical space thus partitioned, any two worlds that are equivalent with respect to M are either both in or both outside E. These worlds, therefore, stand to each other in an equivalence relation;[footnote numbered 16] Lewis defines subject matter as just that equivalence relation, on which the *truth-value* of the statement *supervenes.*

The first sentence of this passage, included for the continuity of what follows it, is highly misleading in a way that is incidental to the reason the passage is being quoted here, since it makes it sound as though a subject matters are generally parts of worlds, rather than equivalence relations (or the partitions induced thereby), with the case of parts of worlds being a special case in virtue of the equivalence relation of having the corresponding parts — their seventeenth centuries, in the illustration — being duplicates. In Lewis [99, p. 11*f.*], being (entirely) about a subject matter is introduced like this:

> I suggest that this is a matter of supervenience: a statement is entirely about some subject matter iff its truth value supervenes on that subject matter. Two possible worlds which are exactly alike so far as that subject matter is concerned must both make the statement true, or else both make it false. So a statement is entirely about the 17th Century iff, whenever two worlds have duplicate 17th Centuries (or both lack 17th Centuries), then both worlds give the statement the same truth value. [. . .] It is otherwise for other subject matters. For instance, consider the subject matter: how many stars there are. Two possible worlds are exactly alike with respect to this subject matter iff they have equally many stars. A statement is entirely about how many stars there are iff, whenever two worlds have equally many stars, the statement has the same truth value at both. Maybe an ingenious ontologist could devise a theory saying that each world has its *nos-part*, as we may call it, such that the nos-parts of two worlds are exact duplicates iff those two worlds have equally many stars. Maybe — and maybe not. We shouldn't rely on it. Rather, we should say that being exactly alike with respect to a subject matter may or may not be a matter of duplication between the parts of worlds which that subject matter picks out.

At this point it would be nice to observe that there could not be a one-to-one correspondence between the parts and the partitions because of differences in the structure of the associated lattices (with meets and joins given in the former case given by mereological overlap/intersection and fusion, and in the latter by greatest common refinement and least common conflation): in the mereological case we have complements, for instance, and in the partition case we do not — consider, for example, the (five-element) lattice of partitions of a three-element set.[117] But this would not quite fit Lewis's approach since he does not acknowledge that every partition of the set of worlds deserves to count as a subject matter: here we see the intrusion of his (one might well suspect) subjective/anthropocentric tendency to take seriously issues of (objective) relative naturalness.

Whatever one makes of that, it is clear from Lewis's discussion, quoted above, that the passage from Osorio-Kupferblum [121] quoted before it gets off on the wrong foot: not every subject-matter partition is a part-based partition.[118] The locations of two footnote flags in the passage from Osorio-Kupferblum are indicated. That numbered 15 refers to an aspect of circularity in Lewis's treatment (acknowledged by him). It is the footnote numbered 16 that accounts for the appearance of [121] in this appendix. The main part of the footnote in question reads: "This account is used in information science by Demolombe and Fariñas del Cerro, for example their (2000)," which is our [26]. As note 69 recalled, quite apart from the modulation from the environment of (essentially, first-order) models to that of possible worlds, Lewis gives a supervenience account of *being about* a subject matter — agreement in truth-value implies by standing in the (subject matter *qua*) corresponding equivalence relation — whereas Demolombe and Fariñas del Cerro use their equivalence relation (notated in Section 4 by "$\approx_d$"[119]) to define *not being about* ('NAboutness'): it was *not* being about (the referent of) c that was to consist in any models standing in this relation agreeing on the truth-value of the sentence in question.

[117] I am thinking of the mereological lattices as essentially Boolean algebras, though there is a historical tendency to throw out the bottom element on nominalistic grounds — the dreaded *null individual* (Martin [104]) — rendering intersection only a partial operation.

[118] Humberstone [68] hoped to exploit special features of the part-based subject matters to fix an elusive notion of independence that would resolve some issues concerning logical subtraction (as in Remark 19(iii), and the text on p. 41).

[119] Or "$\overset{\mathsf{prd}}{\approx}_d$," in the more explicit notation given after Definitions 21 to stress that it is the interpretation of *predicate* letters that is affected by variation among $\approx_d$-equivalent models, not of individual constants, despite d's being, as an element of the domain, a candidate denotation for such a constant.

One may think that this apparent anomaly is not as serious as it may seem. After all, more precisely, Lewis's account as summarised above is an account of being *entirely about* a given subject matter, and one may be thinking of this as a matter of *not* being about anything else. That wouldn't be quite right, as it stands, since if one subject matter, conceived as an equivalence relation, is included in (i.e., stands in the relation $\subseteq$ to) another, any statement entirely about the first is automatically entirely about the second.[120] And Lewis [99, pp. 16–18] devotes extensive discussion to how to obtain distinct notions of being *partly about* a subject matter on the basis of his conception of being *entirely about* a subject matter among which one might search for a suitable *rapprochement* with Demolombe and Fariñas del Cerro, if one were so minded. There are several discouraging signs, though. One is that Lewis is not concerned with being about an individual except to the extent that this can be subsumed under the partiform subject matter we get by thinking of an individual as a part of the world — not a very promising subsumption perhaps, calling at least for replacing the role played by the duplicate-of relation by the counterpart-of relation, as Lewis sees things.[121] Even with such an identification of objects/individuals as parts of the world, as well as any steps taken (as in Abrusán [1]) to get the role of models to be played by possible worlds, Lewis-style aboutness verdicts do not align with those of Demolombe and Fariñas del Cerro. To take the simplest case, those authors, we recall, were happy to have their account rule that $\exists x(Px)$, $\forall x(Qx)$ were both about (the referent of) c — though it is hard to see Lewis's account stretching to find "Something is pink" being (entirely or otherwise) about, say, the Eiffel Tower.

With this effort at discouraging any assimilation of Lewis's account to Demolombe and Fariñas del Cerro (or vice versa) behind us, a few further words are in order on this business of individuals/objects vs. sub-

[120] Note that if these are partiform subject matters, the mereological inclusion relation runs in the opposite direction from the set-theoretic $\subseteq$ relation: how-things-are-with-France (as a subject matter) is included in how-things-are-with-Paris because duplicating the whole requires duplicating the parts, so having duplicate *France* parts implies having duplicate *Paris* parts.

[121] We recall that, for Lewis, the intrinsic properties of an individual are those share by duplicates, and the essential properties of an individual are those shared by counterparts — with a rider in the latter case to the effect that the relevant counterparts may depend on how the individual is described. (See the editor's introduction to, and the paper by Langton and Lewis in, Francescotti [42] on the former topic, and Lewis [98] on the latter.)

ject matters in the more abstract and general sense. This is intended as supplementing the discussion in the paragraph toward the end of which note 36 appeared. In that discussion, the significance of this objects-vs.-subject-matters contrast, stressed also by Demolombe and Fariñas del Cerro in contrasting their papers' concerns with those of Demolombe and Jones [29], was taken for granted (note 49). Indeed, at least initially, these appear to be different aboutness relations, as the authors note, and as our earlier discussion observed had been stressed also in Yablo [178], which picks up and develops roughly Lewisian themes from where Lewis left them in [99], with an admixture of (what from the perspective of classical logic would be considered) of hyperintensionality — a rather un-Lewisian element. But consider this. One of two people is talking to the other about a third, who then appears, and the speaker may naturally say: "Oh, what a surprise! You know, we were just talking about you." Now of course, it's not every day that you see a Lewisian subject-matter — whether conceived as a partition of the set of worlds, or as the associated equivalence relation — come into a room. But: two aboutness relations? The speaker might just as naturally have said: We were just talking about you and where you were born; so we appear to have 'conjoinable types' here, though "where you were born" is not here a term referring to the country the third party was in fact born in.[122] I detect no zeugmatic jarring in such a case. So: more by way of unification may be required than is achievable by thinking of individuals as parts inducing abstract subject matters via duplication relations, or even vaguely delimited 'how things stand with so-&-so' partitions.

K. Wallace's Sortally Restricted Predicate Logic. By way of elaboration on note 70, recalling Hempel's 'ravens paradox' (the paradox of confirmation), and the possibility of a 'predicate' version of the usual

[122] The original interlocutors may have been in disagreement as to whether the birthplace was Guinea or, instead, the neighbouring country, Guinea-Bissau, so they were certainly talking about both those places. But suppose that both were wrong and the actual birthplace was Guyana, an ocean away and not plausibly describable as something they were talking about. (The references in note 66 are relevant here, as are those in the 'Intermission on Copular Sentence Types' in Appendix above. Consider the ambiguity of "They were discussing what was in the box on the table," as between "What they were discussing was in the box," on the one hand, and "what they were discussing was what was in the box," on the other: discussing the thing — a trapped koala, for instance, perhaps unbeknownst to them — as contrasted to discussing an issue/topic/question).

restricted quantifiers of Demolombe and Fariñas del Cerrro. Some suggestion, though not under that description, of course, is made in, for example, Wallace [172, p. 153] in connection with his own 'Aristotelian quantification theory' developed in part through aboutness-related motivations in order "to respect the fact that the sortal predicates [used as the quantifier-restrictors] in a sentence determine what entities enter into its truth conditions" (p. 136), by contrast with standard predicate logic, for which "[t]he truth conditions of any sentence in a language call into play every assignment and, thus, involve everything that can be talked about in the language." Wallace describes his favoured remedial action [172, p. 140]:

> Consider the classical quantification theory version of "All men are mortal", viz., "(x)(is a man $\supset$ x is mortal)". When we spell out its truth conditions by applying to it the definition of truth we see that the assignments used are extravagant in two ways. The only variable that occurs in the sentence is 'x', thus the only feature of an assignment that can matter to the truth value of the sentence is its value at 'x'. Yet we use arbitrary assignments. And we use assignments that assign arbitrary members of the domain of discourse to 'x'; it would be enough to consider only assignments that assign men to 'x'.

The usual ground given for treating "All/some men are mortal" by restricted 'sortal' quantification is the fact that the unrestricted quantifier + appropriately mated connective (conditional for the "all" case, conjunctive for the "some" case) is that the analogous "most" case can't be handled analogously. But Wallace's objections are based on a felt inadequacy of the standard quantifier cases, and here, in particular, that of "all". The first objection arises only for a certain formulation of the usual satisfaction conditions, which can be replaced by one in which the metalinguistic quantification is over objects: assignment $\mathbf{a}$ satisfies $\forall x\,.\,\varphi(x)$ iff for every domain element d, $\mathbf{a}^{x\mapsto d}$ satisfies $\varphi(x)$, where $\mathbf{a}^{x\mapsto d}$ is the unique assignment differing from $\mathbf{a}$ at most in mapping x to d. The second objection seems more interesting, and somewhat reminiscent of Ladd-Franklin's comment (quoted in Appendix A) on the difference between "no men are mortal," and "there are no mortal men." Though Wallace writes that it suffices to consider *only* "assignments that assign men to 'x' " remember that we do need to consider *all* such assignments, raising an issue about how this metalinguistic quantification is itself to be taken — whether in the Fregean way, *if the assignment assigns a man to* x,... or, instead, along the lines of *such that the assignment* *assigns a man to x*, where "such that" indicates restricted quantifier in the metalanguage. While "(is a) man" might be a sortal predicate in roughly the sense popularized by Strawson, this looks rather less plausible for "(is a) variable-assignment meeting such-&-such conditions," and even less so for "(is a) non-black thing" in "All non-black things are non-ravens," supposedly differing in aboutness from the pre-contraposed version sufficiently for the two to differ in confirmation conditions apropos of Hempel's problem. (Further references to Wallace, Strawson and Quine are supplied in Davies [23, pp 123–131 and p. 148] along with a more thoroughgoing discussion of the issue of homophonic semantics in this connection.)

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References

[1] Abrusán, Márta, "Triggering Verbal Presuppositions", *Semantics and Linguistic Theory* (SALT) 20 (2011), 684–701. DOI: 10.3765/salt.v20i0.2579

[2] Anand, Pranav, and Andrew Nevins, "Shifty Operators in Changing Contexts", *Semantics and Linguistic Theory* (SALT) 14 (2004), 20–37. DOI: 10.3765/salt.v0i0.2913

[3] Bacon, Andrew, *A Philosophical Introduction to Higher-Order Logics*, Taylor and Francis, New York 2024. DOI: 10.4324/9781003039181

[4] Bar-Elli, Gilead, *The Sense of Reference*, de Gruyter, Berlin 1996. DOI: 10.1515/9783110813036

[5] Bar-Hillel, Y., "Bolzano's Definition of Analytic Propositions", *Theoria* 16 (1950), 91–117. DOI: 10.1111/j.1755-2567.1950.tb00899.x

[6] Berman, Joel, and Ralph McKenzie, "Clones Satisfying the Term Condition", *Discrete Mathematics* 52 (1984), 7–29. DOI: 10.1016/0012-365X(84)90100-6

[7] Berto, Francesco, "Aboutness in Imagination", *Philosophical Studies* 175 (2018), 1871–1886. DOI: 10.1007/s11098-017-0937-y

[8] Berto, Francesco, *Topics of Thought*, Oxford University Press, Oxford 2022. DOI: 10.1093/oso/9780192857491.001.0001

[9] Blackburn, Patrick, Maarten de Rijke, and Yde Venema, *Modal Logic*, Cambridge University Press, Cambridge 2001. DOI: 10.1017/CBO9781107050884

[10] Brast-McKie, Benjamin, "Identity and Aboutness", *Journal of Philosophical Logic* 50 (2021), 1471–1503. DOI: 10.1007/s10992-021-09612-w

[11] Briciu, Adrian , "On Context Shifters and Compositionality in Natural Languages", *Organon F* 25 (2018), 2–20.

[12] Brożek, Anna, "On Essential and Inessential Occurrence of Expressions in Another Expression[s]", *Filozofia Nauki* 13 (2005), 5–25. (Polish article title: "O istotnym i nieistotnym występowaniu jakiegoś wyrażenia w innym wyrażeniu". Publishers' abstract in English at https://fn.uw.edu.pl/index.php/fn/article/view/425

[13] Brożek, Bartosz, Adam Olszewski and Mateusz Hohol (eds.), *Logic in Theology*, Copernicus Center Press, Kraków 2013.

[14] Butchart, Sam, and Susan Rogerson, "On the Algebraizability of the Implicational Fragment of Abelian Logic", *Studia Logica* 102 (2014), 981–1001. DOI: 10.1007/s11225-013-9515-2

[15] Carnap, Rudolf, *The Logical Syntax of Language* (transl. Amethe Smeaton), Kegan Paul, Trench, Trubner & Co Ltd., London 1937. Reprinted: Routledge, London 2000. DOI: 10.4324/9781315823010

[16] Carnap, Rudolf, *Meaning and Necessity*, Chicago University Press, Chicago 1956 (enlarged edition; first edn. 1947).

[17] Carlson, Lauri, *Dialogue Games: An Approach to Discourse Analysis*, Reidel, Dordrecht 1983. DOI: 10.1007/978-94-015-3963-0

[18] Castañeda, Hector-Neri, *Thinking and Doing*, Reidel, Dordrecht 1975. DOI: 10.1007/978-94-010-9888-5

[19] Chellas, Brian, *Modal Logic: An Introduction*, Cambridge University Press, Cambridge 1980. Reprinted (with corrections) 1988 and subsequently. DOI: 10.1017/CBO9780511621192

[20] Chellas, Brian, and Krister Segerberg, "Modal Logics with the MacIntosh Rule", *Journal of Philosophical Logic* 23 (1994), 67–86. DOI: 10.1007/BF01417958

[21] Czelakowski, Janusz, and Don Pigozzi, "Fregean Logics", *Annals of Pure and Applied Logic* 127 (2004), 17–76. DOI: 10.1016/j.apal.2003.11.008

[22] Darwiche, Adnan, and Pierre Marquis, "On Quantifying Literals in Boolean Logic and its Applications to Explainable AI", *Journal of Artificial Intelligence Research* 72 (2021) 285–328. DOI: 10.1613/jair.1.12756

[23] Davies, Martin, *Meaning, Quantification, Necessity: Themes in Philosophical Logic*, Second Edn., Routledge, Oxford 2020. (First edn.: Routledge, London 1981.) DOI: 10.4324/9780367853952

[24] Davies, Martin, "Cognitive Science", pp. 358–394 in Frank Jackson and Michael Smith (eds.), *The Oxford Handbook of Contemporary Philosophy*, Oxford University Press, Oxford 2005. DOI: 10.1093/oxfordhb/9780199234769.003.0014

[25] Deal, Amy Rose, "Indexiphors: Notes on Embedded Indexicals, Shifty Agreement, and Logophoricity", pp. 59–86 in R. Ivan (ed.), *The Leader of the Pack: A Festschrift in Honor of Peggy Speas*, University of Massachusetts Occasional Papers in Linguistics #40, GLSA Amherst 2018.

[26] Demolombe, Robert, and Luis Fariñas del Cerro, "Towards a Logical Characterization of Sentences of the Kind 'Sentence p is about object c' ", pp. 85–99 in S. Hölldobler (ed.), *Intellectics and Computational Logic: Papers in Honor of Wolfgang Bibel*, Kluwer, Dordrecht 2000. DOI: 10.1007/978-94-015-9383-0_6

[27] Demolombe, Robert, and Luis Fariñas del Cerro, "How to Produce Information About a Given Entity Using Automated Deduction Methods", *Electronic Notes in Theoretical Computer Science* 256 (2009), 19–31. DOI: 10.1016/j.entcs.2009.11.003

[28] Demolombe, Robert, and Luis Fariñas del Cerro, "Information About a Given Entity: From Semantics Towards Automated Deduction", *Logic Journal of the IGPAL* 20 (2010), 1231–1250. DOI: 10.1093/logcom/exp083

[29] Demolombe, Robert, and Andrew J. I. Jones, "On Sentences of the Kind 'Sentence p is about topic t': Some steps toward a formal-logical analysis", pp. 115–133 in H. J. Ohlbach and U. Reyle (eds.), *Logic, Language and Reasoning: Essays in Honor of Dov Gabbay*, Kluwer, Dordrecht 1999. DOI: 10.1007/978-94-011-4574-9_7

[30] Donnellan, Keith, "Reference and Definite Descriptions", *Philosophical Review* 75 (1966), 281–304. DOI: 10.1093/acprof:osobl/9780199857999.003.0001

[31] Dummett, M. A., and E. J. Lemmon, "Modal Logics Between S4 and S5", *Zeitschr. für math. Logik und Grundlagen der Math.* 5 (1959), 250–264. DOI: 10.1002/malq.19590051405

[32] Dunn, J. M., "Relevant Predication 1: The Formal Theory", *Journal of Philosophical Logic* 16 (1987), 347–381. DOI: 10.1007/BF00431183

[33] Endriss, Cornelia, *Quantificational Topics: A Scopal Treatment of Exceptional Wide Scope Phenomena*, Springer, Dordrecht 2009. DOI: 10.1007/978-90-481-2303-2

[34] Etchemendy, John, *The Concept of Logical Consequence*, CSLI Publications, Stanford 1999.

[35] Ferguson, Thomas Macaulay, *Meaning and Proscription in Formal Logic*, Springer International, Cham (Switzerland) 2017. DOI: 10.1007/978-3-319-70821-8

[36] Ferguson, Thomas Macaulay, "Topic-Theoretic Extensions of Analytic Implication", *Notre Dame Journal of Formal Logic* 64 (2023), 471–493. DOI: 10.1215/00294527-2023-0019

[37] Fine, Kit, "Angellic Content", *Journal of Philosophical Logic* 45 (2016) 199–226. DOI: 10.1007/s10992-015-9371-9

[38] Fine, Kit, "Truthmaker Semantics", pp. 556–577 in Bob Hale, C. Wright and A. Miller (eds.), *A Companion to the Philosophy of Language*, Second edn., Volume 2, Wiley-Blackwell, Oxford 2017. DOI: 10.1002/9781118972090.ch22

[39] Fine, Kit, "Yablo on Subject-matter", *Philosophical Studies* 177 (2020), 129–171.

[40] Frana, Ilaria, "Copular Questions and Concealed Questions", pp. 134–150 in M. Prinzhorn, V. Schmitt and S. Zobel (eds.), *Proceedings of SuB14* (Sinn und Bedeutung, Vol 14), Vienna 2010.

[41] Frana, Ilaria, *Concealed Questions*, Oxford University Press, Oxford 2017. DOI: 10.1093/acprof:oso/9780199670925.001.0001

[42] Francescotti, Robert M. (ed.), *Companion to Intrinsic Properties*, De Gruyter, Berlin 2014. DOI: 10.1515/9783110292596

[43] French, Rohan , "An Argument against General Validity?", *Thought* 1 (2012), 4–9. DOI: 10.1002/tht3.1

[44] French, Rohan, "A Simple Sequent Calculus for Angell's Logic of Analytic Containment", *Studia Logica* 105 (2017), 971–994. DOI: 10.1007/s11225-017-9719-y

[45] Fritz, Peter, *Propositional Quantifiers*, Cambridge University Press, Cambridge 2024. DOI: 10.1017/9781009177740

[46] Gabbay, Dov M., “On Some New Intuitionistic Propositional Connectives. I”, *Studia Logica* 36 (1977), 127–141. DOI: 10.1007/BF02121119

[47] Gildea, Spike, “The Referential Hierarchy and Attention”, *Faits de Langues* 39 2012, 33–47. DOI: 10.1163/19589514-039-01-900000004

[48] Giordani, Alessandro, “On the Notion of Aboutness in Logical Semantics”, pp. 407–467 in F. L. G. Faroldi and F. Van De Putte (eds.), *Kit Fine on Truthmakers, Relevance, and Non-classical Logic*, Springer, Cham (Switzerland) 2023. DOI: 10.1007/s10992-018-9488-8

[49] Goodman, Jeremy, “Agglomerative Algebras”, *Journal of Philosophical Logic* 48 (2019), 631–648; correction: *ibid.* 49 (2020), p. 1257. DOI: 10.1007/s10992-018-9488-8

[50] Goodman, Nelson, “About”, *Mind* 70 (1961), 1–24.

[51] Goodman, Nelson “‘About’ Mistaken (a reply to Rescher)”, *Mind* 74 (1965), 248.

[52] Hale, Bob, and Crispin Wright, “Bolzano’s Definition of Analytic Propositions”, *Gratzer Philosophische Studien* 91 (2015), 325–364.

[53] Hart, Allan M., *Aboutness: A Logico-Philosophical Analysis*, Ph.D. thesis, Michigan State University 1981.

[54] Hawke, Peter, “Theories of Aboutness”, *Australasian Journal of Philosophy* 96 (2018), 697–723. DOI: 10.1080/00048402.2017.1388826

[55] Hawke, Peter, Levin Hornischer, and Francesco Berto, “Truth, Topicality, and Transparency: One-component versus two-component semantics”, *Linguistics and Philosophy* 47 (2024), 481–503. DOI: 10.1007/s10988-023-09408-y

[56] Hazen, Allen, “The Eliminability of the Actuality Operator in Propositional Modal Logic”, *Notre Dame Journal of Formal Logic* 19 (1978), 617–622. DOI: 10.1305/ndjfl/1093888509

[57] Heller, Daphna, and Lynsey Wolter, “That is Rosa: Identificational Sentences as Intensional Predication”, pp. 226–240 in Atle Grønn (ed.), *Proceedings of SuB12* (Sinn und Bedeutung, Vol 12), Oslo 2008.

[58] Hempel, Carl C., *Aspects of Scientific Explanation – and other Essays in the Philosophy of Science*, The Free Press, New York 1965.

[59] Herrmann, Burghard, “Equivalential and Algebraizable Logics”, *Studia Logica* 57 (1996), 419–436. DOI: 10.1007/BF00370843

[60] Higgins, F. R., *The Pseudo-Cleft Construction in English*, Garland Publishing, New York 1979. (Originally a 1973 MIT doctoral dissertation; later re-issued by Routledge, New York 2015.) DOI: 10.4324/9781315693545

[61] Hinterwimmer, Stefan, "Information structure and truth-conditional semantics", pp. 1875–1908 in K. von Heusinger, C. Maienborn and P. Portner (eds.), *Semantics: An International Handbook of Natural Language Meaning, Vol. 2*, Mouton de Gruyter, Berlin 2011. DOI: 10.1515/9783110255072.1875

[62] Hodges, Wilfrid, "Tarski's Theory of Definition", pp. 94–132 in D. Patterson (ed.), *New Essays on Tarski and Philosophy*, Oxford University Press, New York 2008. DOI: 10.1093/acprof:oso/9780199296309.003.0005

[63] Hösli, Brigitte, "Stable Logic", pp. 201–202 in Egon Börger, Gerhard Jäger, Hans Kleine Büning, Michael M. Richter (eds.), *CSL '91: Proceedings of the 5th Workshop on Computer Science Logic* (Berne, Switzerland, October 7–11, 1991), Springer, Berlin 1991. DOI: 10.1007/BFb0023768

[64] Humberstone, Lloyd, Critical Notice of E. L. Keenan (ed.), *Formal Semantics of Natural Language* (Cambridge: 1975), *Australasian Journal of Philosophy* 57 (1979), 171–182.

[65] Humberstone, Lloyd, "Zero-Place Operations and Functional Completeness, and the Definition of New Connectives", *History and Philosophy of Logic* 14 (1993). 39–66. DOI: 10.1080/01445349308837209

[66] Humberstone, Lloyd, "A Study in Philosophical Taxonomy", *Philosophical Studies* 83 (1996), 121–169. DOI: 10.1007/BF00354286

[67] Humberstone, Lloyd, "Equivalential Interpolation", pp. 36–53 in *The Goldblatt Variations (Eight Papers in Honour of Rob)*, anonymously edited by Krister Segerberg, Uppsala Prints and Preprints in Philosophy, 1999.

[68] Humberstone, Lloyd, "Parts and Partitions", *Theoria* 66 (2000), 41–82. DOI: 10.1111/j.1755-2567.2000.tb01144.x

[69] Humberstone, Lloyd "What *Fa* Says About *a*", *Dialectica* 54 (2000), 1–28.

[70] Humberstone, Lloyd, "Contra-Classical Logics", *Australasian Journal of Philosophy* 78 (2000), 438–474. 10.1080/00048400012349741

[71] Humberstone, Lloyd, "Archetypal Forms of Inference", *Synthese* 141 (2004), 45–76. DOI: 10.1023/B:SYNT.0000035850.89516.e1

[72] Humberstone, Lloyd, "Two-Dimensional Adventures", *Philosophical Studies* 118 (2004), 17–65. DOI: 10.1023/B:PHIL.0000019542.43440.d1

[73] Humberstone, Lloyd, "Can Every Modifier be Treated as a Sentence Modifier?", *Philosophical Perspectives* 22 (2008), 241–275. DOI: 10.1111/j.1520-8583.2008.00148.x

[74] Humberstone, Lloyd, *The Connectives*, MIT Press, Cambridge, MA 2011. DOI: 10.7551/mitpress/9055.001.0001

[75] Humberstone, Lloyd, "Inverse Images of Box Formulas in Modal Logic", *Studia Logica* 101 (2013), 1031–1060. DOI: 10.1007/s11225-012-9419-6

[76] Humberstone, Lloyd, "Aggregation and Idempotence", *Review of Symbolic Logic* 6 (2013), 680–708. DOI: 10.1017/S175502031300021X

[77] Humberstone, Lloyd, *Philosophical Applications of Modal Logic*, College Publications, London 2016.

[78] Humberstone, Lloyd, "Priest on Negation", pp. 285–341 in C. Başkent and T. M. Ferguson (eds.), *Graham Priest on Dialetheism and Paraconsistency*, Springer, Cham (Switzerland) 2019. DOI: 10.1007/978-3-030-25365-3_15

[79] Humberstone, Lloyd, "Recent Thought on *Is* and *Ought*: Connections, Confluences and Rediscoveries", *Journal of Applied Logics (IfCoLog Journal of Logics and their Applications)* 6, 7 (2019), 1373–1446. https://www.collegepublications.co.uk/downloads/ifcolog00036.pdf#page=196

[80] Humberstone, Lloyd, "Propositional Variables Occurring Exactly Once in Candidate Modal Axioms", *Filosofiska Notiser* 8 (2021), 27–73.

[81] Humberstone, Lloyd, "A Farewell to Fragility' (Critical Notice of Russell [151]), *Australasian Journal of Logic* 21 (2024) 210–282. DOI: 10.26686/ajl.v21i5.9597

[82] Humberstone, Lloyd, and Robert K. Meyer, "The Relevant Equivalence Property", *Logic Journal of the IGPL* 15 (2007), 165–181. DOI: 10.1093/jigpal/jzm005

[83] Humberstone, Lloyd, and Timothy Williamson, "Inverses for Normal Modal Operators", *Studia Logica* 59 (1997), 33–64. DOI: 10.1023/A:1004995316790

[84] Iemhoff, Barbara, "Uniform Interpolation and the Existence of Sequent Calculi", *Annals of Pure and Applied Logic* 170 (2011), Issue 11, Article #102711, 33pp. DOI: 10.1016/j.apal.2019.05.008

[85] Jago, Mark, *The Impossible: An Essay on Hyperintensionality*, Oxford University Press, Oxford 2014. DOI: 10.1093/acprof:oso/9780198709008.001.0001

[86] Jankov, V. A., "The Calculus of the Weak 'Law of Excluded Middle' ", *Math. USSR (Izvestija)* 2 (1968), 997–1004. DOI: 10.1070/IM1968v002n05ABEH000690

[87] Körner, Stephan, *Conceptual Thinking*, Dover Publications, New York 1959 (first publ. 1955).

[88] Kripke, Saul A., *Naming and Necessity*, Blackwell, Oxford 1980. DOI: 10.1007/978-94-010-2557-7_9 (only for the original 1972 paper).

[89] Kreisel, G., and J. L. Krivine, *Elements of Mathematical Logic*, North-Holland, Amsterdam 1967.

[90] Ladd, Christine, "On the Algebra of Logic", pp. 17–71 in C. S. Peirce (ed.), *"Studies in Logic' by Members of the Johns Hopkins University (1883)*, John Benjamins, Amsterdam 1983. DOI: 10.1037/12811-004

[91] Lamarque, Peter, "About", *Journal of Literary Theory* 8 (2014), 257–269. DOI: 10.1515/jlt-2014-0013

[92] Lambrecht, Knud, *Information Structure and Sentence Form: Topic, focus, and the mental representations of discourse referents*, Cambridge University Press, Cambridge 1994. DOI: 10.1017/CBO9780511620607

[93] Lang, Jérôme, Paolo Liberatore, and Pierre Marquis, "Propositional independence: formula-variable independence and forgetting", *Journal of Artificial Intelligence Research* 18 (2003), 391–443. DOI: 10.1613/jair.1113

[94] Lapointe, Sandra, *Bolzano's Theoretical Philosophy: an introduction*, Palgrave Macmillan, Basingstoke 2011. DOI: 10.1057/9780230308640

[95] Lapointe, Sandra, "Bolzano, Quine, and Logical Truth", pp. 296–312 in G. Harman and E. Lepore (eds.), *A Companion to W. V. O. Quine*, Wiley Blackwell, Oxford 2014. DOI: 10.1002/9781118607992.ch15

[96] Leblanc, Hugues, "Syntactically Free, Semantically Bound (A Note on Variables)", *Notre Dame Journal of Formal Logic* 9 (1968), 167–170. DOI: 10.1305/ndjfl/1093893413

[97] Lemmon, E. J., "Sentences, Statements and Propositions", pp. 87–107 in Bernard Williams and Alan Montefiore (eds.) *British Analytical Philosophy*, Routledge & Kegan Paul, London 1966.

[98] Lewis, David, "Counterparts of Persons and Their Bodies", *Journal of Philosophy* 68 (1971), 203–211. DOI: 10.2307/2024902

[99] Lewis, David, "Statements Partly about Observation", *Philosophical Papers* 17 (1988), 1–31. DOI: 10.1080/05568648809506282 (journal version), DOI: 10.1017/CBO9780511625237.011 (subsequent reprint)

[100] Lewis, David, "Relevant Implication", *Theoria* 54 (1988), 162–174. DOI: 10.1111/j.1755-2567.1988.tb00716.x (journal version), DOI: 10.1017/CBO9780511625237.010 (subsequent reprint)

[101] Lin, Fangzhen, and Ray Reiter, "Forget It!", pp. 141–146 in *AAAI Technical Report FS-94-02*, 1994.

[102] Major, Travis, "Revisiting the Syntax of Monsters in Uyghur", *Linguistic Inquiry* 53 (2022), 780–807. DOI: 10.1162/ling_a_00424

[103] Marshall, Dan, "Objectual Aboutness", *Inquiry* 67 (2024), 1640–1715. DOI: 10.1080/0020174X.2021.1954992

[104] Martin, R. M., "Of Time and the Null Individual", *Journal of Philosophy* 62 (1965), 723–736. DOI: 10.2307/2023874

[105] Martinich, A. P., "The Attributive Use of Proper Names' *Analysis* 37 (1977), 159–163. DOI: 10.1093/analys/37.4.159

[106] McKinsey, J. C. C., "A New Definition of Truth", *Synthese* 7 (1948/1949), 428–433.

[107] McNally, Louise, "On Recent Formal Analyses of Topic", pp. 147–160 in J. Ginzburg, Z. Khasidashvili, C. Vogel, J.-J. Lévy and Enric Vallduví (eds.), *The Tbilisi Symposium on Logic, Language and Computation: Selected Papers*, CSLI Publications, Stanford, California 1998.

[108] Meyer, Robert K., and John K. Slaney, "Abelian Logic (From A to Z)", pp. 245–288 in G. Priest, R. Routley and J. Norman (eds.), *Paraconsistent Logic: Essays on the Inconsistent*, Philosophia Verlag, Munich 1989. DOI: 10.2307/j.ctv2x8v8c7.12

[109] Meyer, Robert K., and John K. Slaney, "A, Still Adorable", pp. 241–260 in in W. A. Carnielli, M.E. Coniglio and I. M. L. D'Ottaviano (eds.), *Paraconsistency: The Logical Way to Inconsistency*, Marcel Dekker, New York 2002. DOI: 10.1201/9780203910139.pt2

[110] Mikkelsen, Line, *Copular Clauses: Specification, predication and equation*, John Benjamins, Amsterdam 2005. DOI: 10.1075/la.85

[111] Mikkelsen, Line, "Copular Clauses", pp. 1805–1829 in K. von Heusinger, C. Maienborn, and P. Portner (eds.), *Semantics, Volume 2*, De Gruyter Mouton, Berlin 2011. DOI: 10.1515/9783110589863-007

[112] Miller, David, "Verisimilitude Redeflated", *British Journal for the Philosophy of Science* 27 (1976), 363–402. DOI: 10.1093/bjps/27.4.363

[113] Miller, David, "Not Much has Changed: A rejoinder to Raclavský", *Organon F* 15 (2008), 178–190.

[114] Miller, David, and Miloš Taliga, "Through a Glass Darkly: A final rejoinder to Raclavský, *Organon F* 15 (2008), 473–476.

[115] Morton, Adam, "Denying the Doctrine and Changing the Subject", *Journal of Philosophy* 70 (1973), 503–510. DOI: 10.2307/2025282

[116] Motohashi, Nobuyoshi, "Equality and Lyndon's Interpolation Theorem", *Journal of Symbolic Logic* 49 (1984), 123–128. DOI: 10.2307/2274095

[117] Nakhnikian, George, *Introduction to Philosophy*, Alfred A. Knopf, New York 1967. DOI: 10.2307/2217696

[118] Niebergall, Karl-Georg, "On 'About': Definitions and Principles", pp. 137–169 in G. Ernst, J. Steinbrenner, and O. R. Scholz (eds.), *From Logic to Art: Themes from Nelson Goodman*, Ontos Verlag, Heusenstamm 2009. DOI: 10.1515/9783110327199.137

[119] Nozick, Robert, *Anarchy, State and Utopia*, Basic Books, New York 1974. DOI: 10.1515/9781400848393-024

[120] Olson, J. S., J. G. Raftery, and C. J. van Alten, "Structural Completeness in Substructural Logics", *Logic Journal of the IGPL* 16 (2008) 454-495. DOI: 10.1093/jigpal/jzn014

[121] Osorio-Kupferblum, Naomi, "Aboutness' (Critical Notice of Yablo [178]) *Analysis* 76 (2016), 528–546. DOI: 10.1093/analys/anw027

[122] Osorio-Kupferblum, Naomi, "Goodman's 'About': the Ryle factor", *Journal for the History of Analytical Philosophy* 12, 5 (2024), 1–27. DOI: 10.15173/jhap.v12i5.5463

[123] Osorio-Kupferblum, Naomi, "As you embed, so Ködel must lie. . . ", to appear in *Inquiry*. DOI: 10.1080/0020174X.2024.2338792

[124] Paluch, Stanley, "The Specificity of Historical Language", *History and Theory* 7 (1968), 76–82.

[125] Pap, Arthur, "Logic and the Synthetic *A Priori*", *Philosophy and Phenomenological Research* 10 (1950), 500–514. Reprinted as Chapter 3 in A. Keupink and S. Shieh (eds.), *The Limits of Logical Empiricism: Selected Papers of Arthur Pap*, Springer, Dordrecht 2006. DOI: 10.1007/1-4020-4299-X

[126] Pap, Arthur, *Semantics and Necessary Truth: An Inquiry into the Foundations of Analytic Philosophy*, Yale University Press, New Haven 1959.

[127] Partee, Barbara H., "Speaker's Reply (to comments by Perry and Rieser)", pp. 152–161 in S. Allén (ed.), *Possible Worlds in Humanities, Arts and Sciences*, De Gruyter, Berlin 1989. The comments in question are those on her paper "Possible Worlds in Model-Theoretic Semantics: A Linguistic Perspective", *ibid.*, pp. 93–123. DOI: 10.1515/9783110866858

[128] Partee, Barbara H., "Implicit Arguments and Invisible Variables", §7.4 in Partee, *Compositionality in Formal Semantics: Selected Papers*, Blackwell, Oxford 2004. DOI: 10.1002/9780470751305.ch7

[129] Partee, Barbara H., "Ambiguous Pseudoclefts with Unambiguous *Be*", Chapter 9 in Partee, *Compositionality in Formal Semantics: Selected Papers*, Blackwell, Oxford 2004. DOI: 10.1002/9780470751305.ch9

[130] Patton, Thomas E., "Some Comments on 'About' ", *Journal of Philosophy* 62 (1965), 311–325. DOI: 10.2307/2023638

[131] Pietruszczak, Andrzej, "The Consequence Relation Preserving Logical Information", *Logic and Logical Philosophy* 13 (2004), 89–120. DOI: 10.12775/LLP.2004.006

[132] Pietruszczak, Andrzej, "The Axiomatization of Horst Wessel's Strict Logical Consequence Relation", *Logic and Logical Philosophy* 13 (2004), 121–138. DOI: 10.12775/LLP.2004.007

[133] Pigden, Charles R., "Logic and the Autonomy of Ethics", *Australasian Journal of Philosophy* 67 (1989), 127–151. DOI: 10.1080/00048408912343731

[134] Pigden, Charles R. (ed.), *Hume on "Is' and "Ought'*, Palgrave Macmillan, NY 2010. DOI: 10.1007/978-0-230-29688-6

[135] Pigden, Charles R., "On the Triviality of Hume's Law: A Reply to Gerhard Schurz", pp. 217–238 in [134]. DOI: 10.1007/978-0-230-29688-6_19

[136] Plebani, Matteo, and Giuseppe Spolaore, "Predicative Subject Matter", *Philosophical Studies* 181 (2024), 247–265. DOI: 10.1007/s11098-023-02081-8

[137] Połacik, Tomasz, "Pitts' Quantifiers Are Not Topological Quantification", *Notre Dame Journal of Formal Logic* 39 (1998), 531–544. DOI: 10.1305/ndjfl/1039118868

[138] Potts, Timothy C., "Fregean Categorial Grammar", pp. 245–284 in R. J. Bogdan and I. Niiniluoto (eds.), *Logic, Language and Probability*, Reidel, Dordrecht 1973. DOI: 10.1007/978-94-010-2568-3_25

[139] Potts, Timothy C., *Structures and Categories for the Representation of Meaning*, Cambridge University Press, Cambridge 1994. DOI: 10.1017/cbo9780511554629

[140] Prior, A. N., "The Autonomy of Ethics", *Australasian Journal of Philosophy*, 38 (1960), 199–206. (Reprinted as Chapter 10 in P. T. Geach and A. J. Kenny (eds.), *A. N. Prior: Papers on Logic and Ethics*, Duckworth, London 1976, and as Chapter 1.1 in Pigden [134].) DOI: 10.1080/00048406085200221 (journal version), DOI: http://dx.doi.org/10.1007/978-0-230-29688-6_2 (Pigden collection version)

[141] Putnam, Hilary, "Formalization of the Concept 'About' ", *Philosophy of Science* 25 (1958), 125–130. DOI: 10.1086/287584

[142] Putnam, Hilary, and J. S. Ullian, "More about 'About' ", *Journal of Philosophy* 62 (1965), 305–310. DOI: 10.2307/2023637

[143] Quine, W. V. O., *Mathematical Logic*, Harvard University Press, Cambridge Mass. 1940, Revised edition 1961. DOI: 10.4159/9780674042469

[144] Quine, W. V. O., "Truth by Convention", pp. 70–99 in Quine, *The Ways of Paradox and Other Essays*, Random House, New York 1966; also pp. 3–30 in R. F. Gibson (ed.), *Quintessence: Basic Readings from the Philosophy of W. V. Quine*, Harvard University Press, Cambridge, Mass. 2004. (Original publ. date 1936.) DOI: 10.1017/CBO9781139171519.018or10.2307/j.ctv1pncnhw.4

[145] Randriamahazaka, Thomas, "A note on Signed Occurrences of Propositional Variables", *Australasian Journal of Logic* 19 2022), 59–72. DOI: 10.26686/ajl.v19i1.7432

[146] Roberts, Craige, "Topics", pp. 1908–1934 in K. von Heusinger, C. Maienborn and P. Portner (eds.), *Semantics: An International Handbook of Natural Language Meaning, Vol. 2*, Mouton de Gruyter, Berlin 2011.

[147] Roelofsen, Floris, "A Bare Bone Attentive Semantics for *Might*", pp. 190–125 in M. Aloni, M. Franke and F. Roelofsen (eds.), *The dynamic, inquisitive, and visionary life of* φ, $?\varphi$ *and* $\Diamond\varphi$*: a Festschrift for Jeroen Groenendijk, Martin Stokhof, and Frank Veltman*, ILLC Publications, Amsterdam 2013.

[148] Routley, Richard, and Valerie Routley, "Semantics of First Degree Entailment", *Noûs* 6 (1972), 335–359. DOI: 10.2307/2214309

[149] Rowe, William L., *The Cosmological Argument*, Princeton University Press, Princeton 1975. DOI: 10.1515/9780823296774 (1998 edition)

[150] Rubin, Mariela, and Damian Szmuc, "Pure Refined Variable Inclusion Logics", *Australasian Journal of Logic* 19 (2022), 147–166. DOI: 10.26686/ajl.v19i5.7991

[151] Russell, Gillian K., *Barriers to Entailment: Hume's Law and other limits on logical consequence*, Oxford University Press, Oxford 2023. DOI: 10.1093/oso/9780192874733.001.0001

[152] Sidorenko, E. A., "Semantic Truth Conditionals and Relevant Calculi", *Synthese* 66 (1986), 55–62. DOI: 10.1007/BF00413578

[153] Schlenker, Philippe, "A Plea for Monsters", *Linguistics and Philosophy* 26 (2003), 29–120. DOI: 10.1023/A:1022225203544

[154] Schurz, Gerhard, *The Is–Ought Problem: An Investigation in Philosophical Logic*, Kluwer, Dordrecht 1997. DOI: 10.1007/978-94-017-3375-5

[155] Schurz, Gerhard, "Non-trivial Versions of Hume's Is-Ought Thesis", pp. 198–216 in Pigden [134]. DOI: 10.1007/978-0-230-29688-6_18

[156] Schurz, Gerhard, "Relevance as Difference-making: a generalized theory of relevance and its applications", *Philosophical Studies* 181 (2024), 2279–2316. DOI: 10.1007/s11098-023-01988-6

[157] Shtrakov, Slavcho, "Essential Variables and Psitions in Terms", *Algebra Universalis* 61 (2009) 381–397. DOI: 10.1007/s00012-009-0023-1

[158] Shtrakov, S., and D. Kenecke, "Essential Variables and Separable Sets in Universal Algebra", *Multiple-Valued Logic* 8 (2002), 165–181.

[159] Simchen, Ori, *Semantics, Metasemantics, Aboutness*, Oxford University Press, Oxford 2017. DOI: 10.1093/acprof:oso/9780198792147

[160] Smiley, T. J., "Relative Necessity", *Journal of Symbolic Logic* 28 (1963), 113–134. DOI: 10.2307/2271593

[161] Stalnaker, Robert C., "Pragmatics", pp. 380–397 in D. Davidson and G. Harman (eds.), *Semantics of Natural Language*, Reidel Publ. Company, Dordrecht 1972. DOI: 10.1007/BF00413603

[162] Standefer, Shawn, "What is a Relevant Connective?", *Journal of Philosophical Logic* 51 (2022), 919–950. DOI: 10.1007/s10992-022-09655-7

[163] Sudo, Yasutada, "Person Indexicals in Uyghur Indexical Shifting", pp. 441–56 in N. Rolle, J. Steffman and John Sylak-Glassman (eds.), *BLS 36: General Session and Special and Parasessions* (Annual Meeting of the Berkeley Linguistics Society) 2010.

[164] Suppes, Patrick, *Introduction to Logic*, Van Nostrand, New York 1957.

[165] Tichý, Pavel, "What Do We Talk About?", *Philosophy of Science* 42 (1975), 80–93. DOI: 10.1086/288622

[166] Ullian, J. S., "Corollary to Goodman's Explication of 'About' ", *Mind* 71 (1962), 345.

[167] Van der Auwera, Johan, *What Do We Talk About When We Talk?*, John Benjamins, Amsterdam 1981.

[168] van Fraassen, Bas, "Facts and Tautological Entailments", *Journal of Philosophy* 66 (1969), 477–487.

[169] Vranas, Peter B. M., "Informative Aboutness", *Australasian Journal of Philosophy* 95 (2017), 354–364. DOI: 10.1080/00048402.2016.1157196

[170] Waldhauser, Tamás, "On Composition-closed Classes of Boolean Functions", pp. 313–317 in *41st IEEE International Symposium on Multiple-Valued Logic* 2011. (Shortened version of [171].) DOI: 10.1109/ISMVL.2011.35

[171] Waldhauser, Tamás, "On Composition-closed Classes of Boolean Functions", *Multiple-Valued Logic and Soft Computing* 19 (2012), 493–518.

[172] Wallace, John R., *Philosophical Grammar: A Study of Classical Quantification Theory, the Theory of Sense and Reference, and the Logic of Sortal Predicates*, PhD Dissertation, Stanford University 1964. (Xerox facsimile by University Microfilms, Ann Arbor 1971.)

[173] Weber, Andreas, "Algorithms for Propositional Updates", pp. 335–346 in E. Börger, H. Kleine Büning, M. M. Richter (eds.), *CSL '87* (Procs. 1st Workshop on Computer Science Logic, Karlsruhe, FRG, October 12–16, 1987), Springer, Berlin 1988. DOI: 10.1007/3-540-50241-6_48

[174] Wehmeier, Kai F., and Helge Rückert, "Still in the Mood: The Versatility of Subjunctive Markers in Modal Logic", *Topoi* 38 (2019), 361–377. DOI: 10.1007/s11245-016-9426-8

[175] White, Morton G., "Historical Explanation", *Mind* 52 (1943), 212–229.

[176] Williamson, Timothy, "Non-genuine MacIntosh Logics", *Journal of Philosophical Logic* 23 (1994), 87–101. 10.1007/BF01417959

[177] Williamson, Timothy, "Indicative versus Subjunctive Conditionals, Congruential versus Nonhyperintensional Contexts", *Philosophical Issues* 16 (2006), 310–333. DOI: 10.1111/j.1533-6077.2006.00116.x

[178] Yablo, Stephen, *Aboutness*, Princeton University Press, Princeton, NJ 2014. DOI: 10.1515/9781400845989

[179] Zakharova, E. Y., and S. V. Yablonskii, "Certain properties of nondegenerate superpositions in P_k", *Mathematical Notes of the Academy of Sciences of the USSR* 12 (1972), 435–440. DOI: 10.1007/BF01094386

[180] Zdanowski, Konrad, "On Second Order Intuitionistic Propositional Logic Without a Universal Quantifier", *Journal of Symbolic Logic* 74 (2009), 157–167. DOI: 10.2178/jsl/1231082306

[181] Zimmerman, David, "Coercive Wage Offers", *Philosophy and Public Affairs* 10 (1981), 121–145.

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