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Nested Sequent Calculi for Some Modal Logics with Non-Standard Modalities*

Abstract. This paper introduces nested sequent calculi for modal logics that include non-standard modalities as primitive operators in their languages. By non-standard modalities, we mean non-contingency, contingency, essence, accident, impossibility, and unnecessity. We consider basic normal modal logic **K** and its serial, reflexive, transitive, and symmetric extensions. Our research begins by using Poggiolesi's nested sequent calculi as a foundation. These calculi are specifically designed for logics that are formulated in a language that includes the necessity operator. Next, we proceed to modify their rules to accommodate non-standard modalities. We then establish the soundness and completeness of the resulting calculi. As a consequence, we get that the nested sequent calculus for **K** is cut-free. Subsequently, we provide a constructive cut admissibility proof for **K**. Finally, we discuss the issues pertaining to the cut admissibility for the extensions of **K** and their relationships with the so-called special structural rules as well as the potential for considering other forms of non-standard modalities.

Keywords: nested sequent calculus; cut elimination; modal logic; contingency logic; essence logic; accident logic; paraconsistent logic; paracomplete logic

1. Introduction

Alethic modal logic is commonly interpreted as a logic that deals with concepts of necessity and possibility. These operators, either alone or together, are typically included in the language of a modal logic. Modal

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logic, specifically proof theory for modal logic, is a highly advanced and extensively studied field of research. However, it is worth considering alternative forms of alethic modalities. The most evident and philosophically significant choices are negative modalities (unnecessity and impossibility) along with contingency and non-contingency. Negative modalities can be considered as a specific form of negation. Unnecessity can be seen as a paraconsistent negation, influenced by Jaśkowski's findings [16] and further developed by Béziau [2]. Impossibility, on the other hand, can be viewed as a paracomplete negation, inspired by Béziau [2] and expanded upon by Marcos [22]. Contingency and non-contingency are understood as follows: a proposition A is contingent, if it is possible and its negation $\neg A$ is also possible ($\blacklozenge A := \lozenge A \wedge \lozenge\neg A$); conversely, A is non-contingent, if it is either necessary or its negation $\neg A$ is necessary ($\blacksquare A := \Box A \vee \Box\neg A$).[2]

Alternatively, there is a slightly distinct approach that employs the concepts of essence and accident. A proposition A is essentially true, if its truth implies that it is necessarily true ($\circ A := A \to \Box A$). On the other hand, a proposition A is accidentally true, if it is true, but its negation $\neg A$ is possibly true ($\bullet A := A \wedge \lozenge\neg A$). These two modalities were proposed by Marcos [23] (see also Gilbert and Venturi [14] for further analysis). As Marcos notes in a further paper, 'one could read $\bullet A$ as saying that 'A is the case, but could have been otherwise': It works as a kind of (local) connective for 'accidental truth'. Similarly, $\circ$ could be read as expressing a (local) notion of 'essential truth" [22, pp. 202–203, notation adjusted]. The falsehood counterparts for these modalities are as follows. A statement A is considered to be essentially false, if its falsity implies that it is necessarily false ($\widetilde{\circ} A := \neg A \to \Box\neg A$). On the other hand, A is accidentally false, if it is false, but there is a possibility that it could be true ($\widetilde{\bullet} A := \neg A \wedge \lozenge A$). These two modalities are also due to Marcos [22], who interpreted $\widetilde{\bullet}$ 'as a kind of (local) connective for counterfactual truth', a statement A 'is not the case, but it could have been' [22, p. 191]. Their interpretation as 'essentially false' and 'accidentally false' modalities is presented in [32].

Other forms of essence and accident are also conceivable. Pan and Yang [31] introduced the modalities of *weak* essence and *strong* accident:

[2] The operator $\blacksquare$ can also be understood as representing concepts such as 'ignorance' [42], 'knowing whether' [9], 'no belief' or 'undecided' [20], '(moral) indifference' [44], 'topological border' [39], or 'undecidability in Peano Arithmetic' [45].

A is weakly essentially true, if the fact that A is true implies that A is *possibly* true ($\circ_\flat A := A \to \Diamond A$); A is strongly accidentally true, if A is true, but $\neg A$ is *necessarily* true ($\bullet_\sharp A := A \wedge \Box\neg A$). In [32], their falsity counterparts are given: A is weakly essentially false, if the fact that A is false implies that A is possibly false ($\widetilde{\circ}_\flat A := \neg A \to \Diamond\neg A$); A is strongly accidentally false, if A is false, but A is necessarily true ($\widetilde{\bullet}_\sharp A := \neg A \wedge \Box A$).

Let P be the set $\{p_0, p_1, \ldots\}$ of propositional variables and $\circledcirc$ be one of the following modal operators: $\Box$ (necessity), $\Diamond$ (possibility), $\blacksquare$ (non-contingency), $\blacklozenge$ (contingency), $\circ$ (true essence), $\widetilde{\circ}$ (false essence), $\bullet$ (true accident), $\widetilde{\bullet}$ (false accident), $\circ_\flat$ (weak true essence), $\bullet_\sharp$ (strong true accident), $\widetilde{\circ}_\flat$ (weak false essence), $\widetilde{\bullet}_\sharp$ (strong false accident), $\sim$ (unnecessity/paraconsistent negation; $\sim A := \Diamond\neg A$), and $\dot\sim$ (impossibility/paracomplete negation; $\dot\sim A := \Box\neg A$). Let $\mathrm{For}_\circledcirc$ be the set of formulas built in the standard inductive way from the alphabet $\langle P, \circledcirc, \neg, \wedge, \vee, \to, \leftrightarrow, (,)\rangle$. We write $\boldsymbol{L}^\circledcirc$ to emphasize that a logic $\boldsymbol{L}$ is built in $\mathrm{For}_\circledcirc$. We also consider extensions of $\mathrm{For}_\circledcirc$ that include multiple modalities simultaneously.

Subsequently, we will refer to these alternatives to the conventional selection of necessity and possibility as non-standard modalities. They capture different aspects of concepts of being (non)contingent, essential, or accidental. They might be helpful for philosophical reasoning and clarification of various nuances of modal notions.

Although non-standard modalities are expressed via standard ones, their proof-theoretic investigation is not always straightforward. For example, Zolin [45] developed an uniform method of constructing Hilbert-style axiomatic calculi for reflexive non-contingent logics, due to the equation $\Box A = A \wedge \blacksquare A$. However, this method is not applicable to non-reflexive non-contingent logics.[3] Zolin developed also sequent calculi for reflexive [45] and non-reflexive [46, 47] non-contingent logics. However, it is important to note that none of these calculi are cut-free. Furthermore, all of their modal rules diverge from the conventional classification of the sequent rules as right and left rules.

Hilbert-style calculi for the essence and accident logics were developed by Marcos [23], Steinsvold [38], and Fan [10]. Fan also developed

[3] The first works about Hilbert-style calculi for (non)contingent logics were conducted in [26, 27, 28] ($\mathbf{T}^\blacksquare$, $\mathbf{T}^\blacklozenge$, $\mathbf{S4}^\blacksquare$, $\mathbf{S4}^\blacklozenge$, $\mathbf{S5}^\blacksquare$, $\mathbf{S5}^\blacklozenge$). Additionally, it is worth noting the contributions of Humberstone [15] ($\mathbf{K}^\blacksquare$), Kuhn [18] ($\mathbf{K4}^\blacksquare$), Fan, Wang, and Ditmarsch [8] ($\mathbf{KB}^\blacksquare$), and Zolin [46, 47] ($\mathbf{K}^\blacksquare$, $\mathbf{K4}^\blacksquare$, $\mathbf{K5}^\blacksquare$, $\mathbf{K45}^\blacksquare$).

a combination of accident and contingent logics [11, 12]. Venturi and Yago developed labelled analytic tableaux for these logics [43].

The Hilbert-style calculus for the $\neg$-free fragment of $\mathbf{S5}^{\sim}$ (also known as Béziau's [2] logic $\mathbf{Z}$) was introduced by Béziau [2] (and simplified by Omori and Waragai [30]). The Hilbert-style calculus for $\mathbf{S4}^{\sim}$ (also referred to as Coniglio and Prieto-Sanabria's [6] paraconsistent logic with a topological semantics **LTop**) was presented by Coniglio and Prieto-Sanabria's [6]. Hilbert-style calculi for other normal modal logics with $\sim$ or $\dot{\sim}$ in their languages were built by Marcos [22] as well as Mruczek-Nasieniewska and Nasieniewski [25]. Sequent calculi for the logics with unnecessity and impossibility operators were developed by Dodó and Marcos [7] as well as Lahav, Marcos, and Zohar [19]. Both negative modalities must be present in a language for their method, and some of their calculi are cut-free.

Let $\circledcirc \in \{\blacksquare, \blacklozenge, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \circ_\flat, \bullet_\sharp, \widetilde{\circ}_\flat, \widetilde{\bullet}_\sharp, \sim, \dot{\sim}\}$, There are cut-free, sound, and complete hypersequent calculi for $\mathbf{S5}^{\circledcirc}$ [32] and Béziau's [2] logic $\mathbf{Z}$ [1] based on Restall's cut-free hypersequent calculus for $\mathbf{S5}^{\Box}$ [36]. In this paper, we present a cut-free, sound, and complete nested sequent calculus for $\mathbf{K}^{\circledcirc}$ (with admissible structural rules, except contraction) and sound and complete nested sequent calculi for $\mathbf{KX}^{\circledcirc}$ (with primitive structural rules), where $\mathbf{X} \subseteq \{\mathbf{T}, \mathbf{D}, \mathbf{4}, \mathbf{B}\}$, based on Poggiolesi's [34] cut-free nested sequent calculi for $\mathbf{K}^{\Box}$ and $\mathbf{KX}^{\Box}$.[4]

Hypersequent and nested sequent calculi extend the scope of an ordinary sequent calculus framework. Sequent calculi are less convenient and useful for researching non-standard modalities. If one wishes to achieve the cut admissibility result, it is more preferable to use their generalisations. In the case of standard modalities, also there are examples of logics which fail to have an ordinary cut-free sequent calculus (**KB**, **DB**, **TB**, **KB4** (these four logics, nevertheless, have the subformula property); **K5** and **D5** [41]). However, in many cases, an ordinary sequent calculus is sufficient to prove the cut elimination theorem for logics that include $\Box$ or $\Diamond$ in their languages. Therefore, the analysis of non-standard modalities appears to be more intricate when considering proof theory. Regarding the logics examined in this study, it appears that hypersequent calculi

[4] However, in certain specific instances, some logics may fall out of this list: as follows from Zolin's research [48], serial and non-serial non-contingency (and, hence, contingency) logics have the same sets of tautologies; as follows from Fan's studies [10], reflexive and non-reflexive essence and accidence logics have the same sets of tautologies.

are insufficient for achieving cut elimination, notwithstanding their effectiveness for **S5**-based logics [32]. Therefore, we must employ a more comprehensive framework: nested sequent calculi.

The structure of the paper is as follows. In Section 2, we describe semantics of the logics in question. In Section 3, we provide an explanation of nested calculi and introduce these calculi for the logics being discussed. In Section 4, we prove soundness and completeness for the considered calculi. In Section 5, we give a constructive cut elimination proof for $\mathbf{K}^{\circledcirc}$. In Section 6, we address potential areas for further research, issues related to cut admissibility, and examine additional non-standard modalities inspired by provability logic.

2. Semantics

Let $\circledcirc \in \{\Box, \Diamond, \blacksquare, \blacklozenge, \circ, \bullet, \tilde{\circ}, \tilde{\bullet}, \circ_\flat, \bullet_\sharp, \tilde{\circ}_\flat, \tilde{\bullet}_\sharp, \sim, \dot{\sim}\}$. A triple $\mathcal{M} = \langle W, R, \vartheta\rangle$ is said to be an $\mathbf{K}^{\circledcirc}$-model iff W is a non-empty set, $R \subseteq W \times W$, and ϑ is a mapping from $W \times \mathrm{For}_{\circledcirc}$ to $\{1, 0\}$ such that it preserves classical conditions for truth-value connectives and for all $A \in \mathrm{For}_{\circledcirc}$ and $x \in W$ we have the following clauses for a given $\circledcirc$, where $R[x] := \{y \in W : xRy\}$:

- $\vartheta(\Box A, x) = 1$ iff $\forall_{y\in R[x]}\ \vartheta(A, y) = 1$,
- $\vartheta(\Diamond A, x) = 1$ iff $\exists_{y\in R[x]}\ \vartheta(A, y) = 1$,
- $\vartheta(\blacksquare A, x) = 1$ iff $\forall_{y\in R[x]}\ \vartheta(A, y) = 1$ or $\forall_{y\in R[x]}\ \vartheta(A, y) = 0$,
- $\vartheta(\blacklozenge A, x) = 1$ iff $\exists_{y\in R[x]}\ \vartheta(A, y) = 1$ and $\exists_{yR[x]}\ \vartheta(A, y) = 0$,
- $\vartheta(\circ A, x) = 1$ iff $\vartheta(A, x) = 0$ or $\forall_{y\in R[x]}\ \vartheta(A, y) = 1$,
- $\vartheta(\bullet A, x) = 1$ iff $\vartheta(A, x) = 1$ and $\exists_{y\in R[x]}\ \vartheta(A, y) = 0$,
- $\vartheta(\tilde{\circ} A, x) = 1$ iff $\vartheta(A, x) = 1$ or $\forall_{y\in R[x]}\ \vartheta(A, y) = 0$,
- $\vartheta(\tilde{\bullet} A, x) = 1$ iff $\vartheta(A, x) = 0$ and $\exists_{y\in R[x]}\ \vartheta(A, y) = 1$,
- $\vartheta(\circ_\flat A, x) = 1$ iff $\vartheta(A, x) = 1$ implies $\exists_{y\in R[x]}\ \vartheta(A, y) = 1$,
- $\vartheta(\bullet_\sharp A, x) = 1$ iff $\vartheta(A, x) = 1$ and $\forall_{y\in R[x]}\ \vartheta(A, y) = 0$,
- $\vartheta(\tilde{\circ}_\flat A, x) = 1$ iff $\vartheta(A, x) = 0$ implies $\exists_{y\in R[x]}\ \vartheta(A, y) = 0$,
- $\vartheta(\tilde{\bullet}_\sharp A, x) = 1$ iff $\vartheta(A, x) = 0$ and $\forall_{\in R[x]}\ \vartheta(A, y) = 1$,
- $\vartheta({\sim} A, x) = 1$ iff $\exists_{y\in R[x]}\ \vartheta(A, y) = 0$,
- $\vartheta(\dot{\sim} A, x) = 1$ iff $\forall_{y\in R[x]}\ \vartheta(A, y) = 0$.

A formula A is true at a world $x \in W$ in a model $\mathcal{M} = \langle W, R, \vartheta\rangle$ (in symbols, $\mathcal{M} \models_x A$) iff $\vartheta(A, x) = 1$. We say that A is true in $\mathcal{M}$ (in symbols, $\mathcal{M} \models A$) iff $\mathcal{M} \models_x A$ for each $x \in W$.

We say that a model $\mathcal{M} = \langle W, R, \vartheta\rangle$ is based in the frame $\mathcal{M} = \langle W, R\rangle$. A formula A is valid in a frame $\mathcal{F}$ (in symbols, $\mathcal{F} \models A$) iff

$\mathcal{M} \models A$ for each model $\mathcal{M}$ based on $\mathcal{F}$. For any class $\mathscr{F}$ of frames, we say that A is valid in $\mathscr{F}$ (in symbols, $\mathscr{F} \models A$) iff $\mathcal{F} \models A$ for each $\mathcal{F} \in \mathscr{F}$.

A formula A follows from a set Γ of formulas (in symbols, $\Gamma \models_{\mathbf{K}^{\circledcirc}} A$) iff for every $\mathbf{K}^{\circledcirc}$-model $\mathcal{M} = \langle W, R, \vartheta \rangle$ and every $x \in W$, if each $\mathcal{M} \models_x B$ for each $B \in \Gamma$, then $\mathcal{M} \models_x A$. A formula A is $\mathbf{K}^{\circledcirc}$-valid iff $\emptyset \models_{\mathbf{K}^{\circledcirc}} A$, i.e., $\mathcal{M} \models A$ for each $\mathbf{K}^{\circledcirc}$-model $\mathcal{M}$.

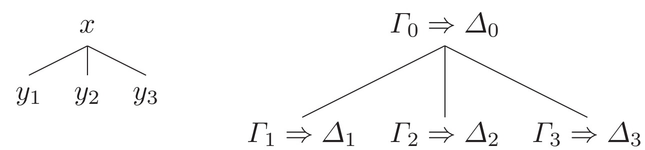
We consider the following restrictions on R (by adding them on an $\mathbf{K}^{\circledcirc}$-model one gets an $\mathbf{KX}^{\circledcirc}$-model, where $\mathbf{X} \subseteq \{\mathbf{T}, \mathbf{D}, \mathbf{B}, \mathbf{4}, \mathbf{5}\}$):

- reflexivity: $\forall_{x \in W}\ xRx$ (**T**-logics);
- seriality: $\forall_{x \in W} \exists_{y \in W}\ xRy$ (**D**-logics);
- symmetry: $\forall_{x,y \in W}(xRy \text{ implies } yRx)$ (**B**-logics);
- transitivity: $\forall_{x,y,z \in W}\big((xRy \text{ and } yRz) \text{ implies } xRz\big)$ (**4**-logics),
- the Euclideanness: $\forall_{x,y,z \in W}\big((xRy \text{ and } xRz) \text{ implies } yRz\big)$ (**5**-logics).

3. Nested sequents

Definition 3.1 (Sequent). By a sequent we mean an ordered pair written as $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas.

A nested sequent calculus is a generalisation of a hypersequent calculus proposed independently by numerous authors under different names: nested sequents (Kashima [17]), deep sequents (Brünnler [4]), and tree-hypersequents (Poggiolesi [34, 35]). Fitting [13] showed that "modal nested sequents and prefixed modal tableaus are notational variants of each other, roughly in the same way that Gentzen sequent calculi and tableaus are notational variants" [13, p. 291]. We will follow Poggiolesi's explication of this method, but using Kashima's name for it. Initially, we will provide an informal explanation of the concept of a nested sequent. Let us examine the Kripke tree depicted on the left:



To present this structure proof-theoretically we replace worlds with sequents and get the tree presented on the right. We use a more compact notation to shorten the formulation of the rules:

$$\Gamma_0 \Rightarrow \Delta_0 / \Gamma_1 \Rightarrow \Delta_1; \Gamma_2 \Rightarrow \Delta_2; \Gamma_3 \Rightarrow \Delta_3$$

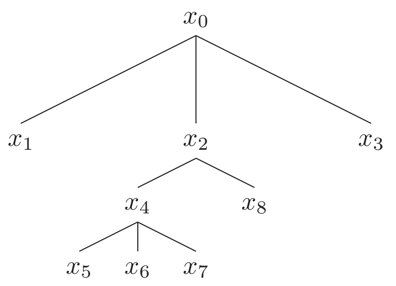


Figure 1. An example of a Kripke tree later transformed into a nested sequent

A forward slash plays here the role of the accessibility relation, the semicolon represent the fact that sequents have the same height in the tree. This nested sequent can be translated into the following modal formula, where $\tau(\Gamma_i \Rightarrow \Delta_i) = \bigwedge \Gamma_i \to \bigvee \Delta_i$, $0 \leqslant i \leqslant 3$:

$$\tau(\Gamma_0 \Rightarrow \Delta_0) \vee \Box\tau(\Gamma_1 \Rightarrow \Delta_1) \vee \Box\tau(\Gamma_2 \Rightarrow \Delta_2) \vee \Box\tau(\Gamma_3 \Rightarrow \Delta_3)$$

Let us consider a more complicated example presented on Figure 1. We put $\mathfrak{S}_i = \Gamma_i \Rightarrow \Delta_i$ (where $0 \leqslant i \leqslant 8$) and replace x_i with $\mathfrak{S}_i$:

$$\mathfrak{S}_0/\mathfrak{S}_1; (\mathfrak{S}_2/(\mathfrak{S}_4/\mathfrak{S}_5; \mathfrak{S}_6; \mathfrak{S}_7); \mathfrak{S}_8); \mathfrak{S}_3$$

It can be translated into the following modal formula, where τ is defined as in the previous example:

$$\tau(\mathfrak{S}_0) \vee \Box\tau(\mathfrak{S}_1) \vee \Box\big(\tau(\mathfrak{S}_2) \vee \Box(\tau(\mathfrak{S}_4) \vee \Box\tau(\mathfrak{S}_5) \vee \Box\tau(\mathfrak{S}_6) \vee \Box\tau(\mathfrak{S}_7)) \vee \Box\tau(\mathfrak{S}_8)\big) \vee \Box\tau(\mathfrak{S}_3)$$

A nested sequent encodes Kripke trees through the use of sequents. We will now provide its formal definition, in accordance with [34, 35].

Definition 3.2 (Nested sequent; [34, Definition 6.1]).

- every sequent is a nested sequent;
- if $\mathfrak{S}$ is a sequent as well as $\mathfrak{N}_1, \ldots, \mathfrak{N}_l$ are nested sequents, then $\mathfrak{S}/\mathfrak{N}_1; \ldots; \mathfrak{N}_l$ is a nested sequent.

Let $\mathfrak{N}$ be a nested sequent, and let $\mathfrak{S}$ represent a sequent that is a component of $\mathfrak{N}$. We write $\mathfrak{N}[\mathfrak{S}]$ when we intend to make a statement regarding $\mathfrak{S}$. A formal analysis of the expression $\mathfrak{N}[\mathfrak{S}]$ is presented in [34], which introduces the concept of a zoom tree-hypersequent.

Definition 3.3. [34, Definition 6.3; slightly modified] The notion of a zoom nested sequent is inductively defined as follows:

- $[*]$ is a zoom nested sequent,
- if $\mathfrak{N}_1, \ldots, \mathfrak{N}_l$ are nested sequents, then $[*]/\mathfrak{N}_1; \ldots; \mathfrak{N}_l$ is a zoom nested sequent,
- if $\mathfrak{N}_i[*]$ is a zoom nested sequent and $\mathfrak{N}_1, \ldots, \mathfrak{N}_{i-1}, \mathfrak{N}_{i+1}, \ldots, \mathfrak{N}_l$ are nested sequents, then $[*]/\mathfrak{N}_1, \ldots, \mathfrak{N}_{i-1}, \mathfrak{N}_i[*], \mathfrak{N}_{i+1}, \ldots, \mathfrak{N}_l$ is a zoom nested sequent,
- if $\mathfrak{S}$ is a sequent, $\mathfrak{N}_i[*]$ is a zoom nested sequent, and $\mathfrak{N}_1, \ldots, \mathfrak{N}_{i-1}$, $\mathfrak{N}_{i+1}, \ldots, \mathfrak{N}_l$ are nested sequents, then $\mathfrak{S}/\mathfrak{N}_1, \ldots, \mathfrak{N}_{i-1}, \mathfrak{N}_i[*], \mathfrak{N}_{i+1}$, $\ldots, \mathfrak{N}_l$ is a zoom nested sequent,
- if $\mathfrak{S}$ is a sequent, $\mathfrak{N}_i[*][*]$ is a zoom nested sequent, and $\mathfrak{N}_1, \ldots, \mathfrak{N}_{i-1}$, $\mathfrak{N}_{i+1}, \ldots, \mathfrak{N}_l$ are nested sequents, then $\mathfrak{S}/\mathfrak{N}_1, \ldots, \mathfrak{N}_{i-1}, \mathfrak{N}_i[*][*]$, $\mathfrak{N}_{i+1}, \ldots, \mathfrak{N}_l$ is a zoom nested sequent.

DEFINITION 3.4. [34, Definition 6.4; slightly modified] For all zoom nested sequents $\mathfrak{N}[*]$, or $\mathfrak{N}[*][*]$, and nested sequents $\mathfrak{K}$ and $\mathfrak{L}$, we define $\mathfrak{N}[\mathfrak{K}]$ and $\mathfrak{N}[\mathfrak{K}][\mathfrak{L}]$, the result of substituting $\mathfrak{K}$ into $\mathfrak{N}[*]$, and the result of substituting $\mathfrak{K}$ and $\mathfrak{L}$ in $\mathfrak{N}[*][*]$, respectively, as follows, where $\mathfrak{S}$ is a sequent:

- if $\mathfrak{N}[*] = [*]$, then $\mathfrak{N}[\mathfrak{K}] = \mathfrak{K}$,
- if $\mathfrak{N}[*] = [*]/\mathfrak{N}_1; \ldots; \mathfrak{N}_l$ and $\mathfrak{K} = \mathfrak{S}/\mathfrak{M}_1; \ldots; \mathfrak{M}_n$, then $\mathfrak{N}[\mathfrak{K}] = \mathfrak{S}/\mathfrak{N}_1; \ldots; \mathfrak{N}_l; \mathfrak{M}_1; \ldots; \mathfrak{M}_n$,
- if $\mathfrak{N}[*][*] = [*]/\mathfrak{N}_1, \ldots, \mathfrak{N}_{i-1}, \mathfrak{N}_i[*], \mathfrak{N}_{i+1}, \ldots, \mathfrak{N}_l$ and $\mathfrak{K} = \mathfrak{S}/\mathfrak{M}_1; \ldots;$ $\mathfrak{M}_n$, then $\mathfrak{N}[\mathfrak{K}][\mathfrak{L}] = \mathfrak{S}/\mathfrak{N}_1; ...; \mathfrak{N}_{i-1}; \mathfrak{N}_i[\mathfrak{L}]; \mathfrak{N}_{i+1}; ...; \mathfrak{N}_l; \mathfrak{M}_1; ...; \mathfrak{M}_n$,
- if $\mathfrak{N}[*] = \mathfrak{S}/\mathfrak{N}_1, \ldots, \mathfrak{N}_{i-1}, \mathfrak{N}_i[*], \mathfrak{N}_{i+1}, \ldots, \mathfrak{N}_l$, then $\mathfrak{N}[\mathfrak{K}] = \mathfrak{S}/\mathfrak{N}_1$, $\ldots, \mathfrak{N}_{i-1}, \mathfrak{N}_i[\mathfrak{L}], \mathfrak{N}_{i+1}, \ldots, \mathfrak{N}_l$,
- if $\mathfrak{N}[*][*] = \mathfrak{S}/\mathfrak{N}_1, \ldots, \mathfrak{N}_{i-1}, \mathfrak{N}_i[*][*], \mathfrak{N}_{i+1}, \ldots, \mathfrak{N}_l$, then $\mathfrak{N}[\mathfrak{K}][\mathfrak{L}] =$ $\mathfrak{S}/\mathfrak{N}_1, \ldots, \mathfrak{N}_{i-1}, \mathfrak{N}_i[\mathfrak{K}][\mathfrak{L}], \mathfrak{N}_{i+1}, \ldots, \mathfrak{N}_l$.

Let us describe Poggiolesi's nested sequent (tree-hypersequent) calculi for modal logics [34, pp. 126–127]. The axiom (which is applied for each propositional variable p and can be generalized for each formula A) and propositional logical rules are as follows:

$$\mathfrak{N}[p, \Gamma \Rightarrow \Delta, p]$$

$$[\neg \Rightarrow]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A]}{\mathfrak{N}[\neg A, \Gamma \Rightarrow \Delta]} \qquad [\Rightarrow \neg]\ \frac{\mathfrak{N}[A, \Gamma \Rightarrow \Delta]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg A]}$$

$$[\wedge \Rightarrow]\ \frac{\mathfrak{N}[A, B, \Gamma \Rightarrow \Delta]}{\mathfrak{N}[A \wedge B, \Gamma \Rightarrow \Delta]} \qquad [\Rightarrow \wedge]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A] \quad \mathfrak{N}[\Gamma \Rightarrow \Delta, B]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, A \wedge B]}$$

$$[\vee\Rightarrow]\ \frac{\mathfrak{N}[A,\Gamma\Rightarrow\Delta]\quad\mathfrak{N}[B,\Gamma\Rightarrow\Delta]}{\mathfrak{N}[A\vee B,\Gamma\Rightarrow\Delta]}\qquad[\Rightarrow\vee]\ \frac{\mathfrak{N}[\Gamma\Rightarrow\Delta,A,B]}{\mathfrak{N}[\Gamma\Rightarrow\Delta,A\vee B]}$$

$$[\rightarrow\Rightarrow]\ \frac{\mathfrak{N}[\Gamma\Rightarrow\Delta,A]\quad\mathfrak{N}[B,\Gamma\Rightarrow\Delta]}{\mathfrak{N}[A\rightarrow B,\Gamma\Rightarrow\Delta]}\qquad[\Rightarrow\rightarrow]\ \frac{\mathfrak{N}[A,\Gamma\Rightarrow\Delta,B]}{\mathfrak{N}[\Gamma\Rightarrow\Delta,A\rightarrow B]}$$

$$[\leftrightarrow\Rightarrow]\ \frac{\mathfrak{N}[B,\Gamma\Rightarrow\Delta,A]\quad\mathfrak{N}[A,\Gamma\Rightarrow\Delta,B]}{\mathfrak{N}[\Gamma\Rightarrow\Delta,A\leftrightarrow B]}$$

$$[\Rightarrow\leftrightarrow]\ \frac{\mathfrak{N}[A,B,\Gamma\Rightarrow\Delta]\quad\mathfrak{N}[\Gamma\Rightarrow\Delta,A,B]}{\mathfrak{N}[A\leftrightarrow B,\Gamma\Rightarrow\Delta]}$$

Modal rules for the logic $\mathbf{K}^{\Box}$, where X is a multiset of nested sequents, are given below:

$$[\Box\Rightarrow]\ \frac{\mathfrak{N}[\Box A,\Gamma\Rightarrow\Delta/(A,\Theta\Rightarrow\Lambda/X)]}{\mathfrak{N}[\Box A,\Gamma\Rightarrow\Delta/(\Theta\Rightarrow\Lambda/X)]}\qquad[\Rightarrow\Box]\ \frac{\mathfrak{N}[\Gamma\Rightarrow\Delta/\Rightarrow A]}{\mathfrak{N}[\Gamma\Rightarrow\Delta,\Box A]}$$

Informally, the relationship between these rules and the semantics can be expressed as follows: the right rules, if read top-down, express the right-to-left truth clause and the left rules, if read bottom-up, express the left-to-right truth clause (see [29] for more details).

Poggiolesi [34] does not give the rules for $\Diamond$, but it is quite easy to formulate them, employing the equality $\Diamond A = \neg\Box\neg A$:

$$[\Diamond\Rightarrow]\ \frac{\mathfrak{N}[\Gamma\Rightarrow\Delta/A\Rightarrow]}{\mathfrak{N}[\Diamond A,\Gamma\Rightarrow\Delta]}\qquad[\Rightarrow\Diamond]\ \frac{\mathfrak{N}[\Gamma\Rightarrow\Delta,\Diamond A/(\Theta\Rightarrow\Lambda,A/X)]}{\mathfrak{N}[\Gamma\Rightarrow\Delta,\Diamond A/(\Theta\Rightarrow\Lambda/X)]}$$

Definition 3.5 (Proof). By a proof in a nested sequent calculus we mean a tree which nodes are nested sequents such that leaves are axioms and other nodes are obtained from the upper ones by applications of the rules of the calculus.

Poggiolesi presents specific logical rules for extensions of $\mathbf{K}^{\Box}$ (in two versions, pp. 125 and 127 in [34]; we describe the second version) together with specific structural rules [34, p. 125]. Every pair of these rules corresponds to the axioms/properties of the accessibility relation R. Both special logical and structural rules are sound with respect to frames that possess the corresponding properties of their accessibility relations. **T**-, **D**-, **B**-, **4**-, and **5**-axioms[5] can be proved employing the corresponding special logical rules as well as special structural rules. Both types of the rules are helpful for the cut elimination: in order to eliminate cuts

[5] The formulas we refer to are as follows, respectively: $\Box A\rightarrow A$, $\Box A\rightarrow\Diamond A$, $A\rightarrow\Box\Diamond A$, $\Box A\rightarrow\Box\Box A$, and $\Diamond A\rightarrow\Box\Diamond A$.

generated by special logical rules, one requires special structural rules. Poggiolesi proved that special structural rules are height-preserving admissible in the calculi with special logical rules [34, Lemmas 6.13–6.17]. She observes that the rules associated with the **5**-axiom "do not reflect the strength and power" of this axiom [34, p. 126], because to issues with cut elimination, the calculus for **K5** obtained in this manner is not cut-free and also lacks completeness. Consequently, in our examination of non-standard modalities, we exclude Euclidean logics to circumvent such issues.

Let us present the aforementioned special logical and structural rules:

$$[\mathbf{D}]\ \frac{\mathfrak{N}[\Box A, \Gamma \Rightarrow \Delta / A \Rightarrow]}{\mathfrak{N}[\Box A, \Gamma \Rightarrow \Delta]} \qquad [\widetilde{\mathbf{D}}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / \Rightarrow]}{\mathfrak{N}[\Gamma \Rightarrow \Delta]}$$

$$[\mathbf{T}]\ \frac{\mathfrak{N}[\Box A, A, \Gamma \Rightarrow \Delta]}{\mathfrak{N}[\Box A, \Gamma \Rightarrow \Delta]} \qquad [\widetilde{\mathbf{T}}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda / X)]}{\mathfrak{N}[\Gamma, \Theta \Rightarrow \Delta, \Lambda / X]}$$

$$[\mathbf{4}]\ \frac{\mathfrak{N}[\Box A, \Gamma \Rightarrow \Delta / (\Box A, \Theta \Rightarrow \Lambda / X)]}{\mathfrak{N}[\Box A, \Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda / X)]} \qquad [\widetilde{\mathbf{4}}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda / X)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Rightarrow / \Theta \Rightarrow \Lambda / X)]}$$

$$[\mathbf{B}]\ \frac{\mathfrak{N}[A, \Gamma \Rightarrow \Delta / (\Box A, \Theta \Rightarrow \Lambda / X)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda / X)]} \qquad [\widetilde{\mathbf{B}}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda / (\Xi \Rightarrow \Pi / X); Y)]}{\mathfrak{N}[\Gamma, \Xi \Rightarrow \Delta, \Pi / (\Theta \Rightarrow \Lambda / X; Y)]}$$

$$[\mathbf{5}]\ \frac{\mathfrak{N}[\Box A, \Gamma \Rightarrow \Delta / (\Box A, \Theta \Rightarrow \Lambda / X)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Box A, \Theta \Rightarrow \Lambda / X)]}$$

$$[\widetilde{\mathbf{5}}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda / (\Xi \Rightarrow \Pi / X); Y)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Xi \Rightarrow \Pi / X); (\Theta \Rightarrow \Lambda / Y)]}$$

As follows from [34, Lemma 6.18], all the propositional rules, the modal rules and the special logical rules are height-preserving invertible. Poggiolesi showed that the following structural rules are height-preserving admissible [34, Lemmas 6.10–6.12, 6.19]:

$$[\mathrm{EW}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta]}{\mathfrak{N}[\Gamma \Rightarrow \Delta / \Pi \Rightarrow \Sigma]} \qquad [\mathrm{IW}\Rightarrow]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta]}{\mathfrak{N}[A, \Gamma \Rightarrow \Delta]}$$

$$[\Rightarrow\mathrm{IW}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, A]}$$

$$[\mathrm{Merge}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Pi \Rightarrow \Sigma / X); (\Theta \Rightarrow \Lambda / Y)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Pi, \Theta \Rightarrow \Sigma, \Lambda / X; Y)]}$$

$$[\mathrm{rn}]\ \frac{\mathfrak{N}}{\Rightarrow / \mathfrak{N}} \qquad [\mathrm{C}\Rightarrow]\ \frac{\mathfrak{N}[A, A, \Gamma \Rightarrow \Delta]}{\mathfrak{N}[A, \Gamma \Rightarrow \Delta]} \qquad [\Rightarrow\mathrm{C}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A, A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, A]}$$

To formulate the rule of cut in the nested sequent framework, the following auxiliary definitions are required.

DEFINITION 3.6. [34, Definition 6.5] Given two nested sequents, $\mathfrak{N}[\Gamma \Rightarrow \Delta]$ and $\mathfrak{M}[\Theta \Rightarrow \Lambda]$ together with an occurrence of a sequent in each, the relation of an equivalent position between two of their sequents, in this case $\Gamma \Rightarrow \Delta$ and $\Theta \Rightarrow \Lambda$, $\mathfrak{N}[\Gamma \Rightarrow \Delta] \approx \mathfrak{M}[\Theta \Rightarrow \Lambda]$, is defined inductively in the following way:

- $\Gamma \Rightarrow \Delta \approx \Theta \Rightarrow \Lambda$,
- $\Gamma \Rightarrow \Delta/X \approx \Theta \Rightarrow \Lambda/Y$,
- if $\mathfrak{K}[\Gamma \Rightarrow \Delta] \approx \mathfrak{L}[\Theta \Rightarrow \Lambda]$, then
  $\Phi \Rightarrow \Pi/\mathfrak{K}[\Gamma \Rightarrow \Delta]; X \approx \Sigma \Rightarrow \Upsilon/\mathfrak{L}[\Theta \Rightarrow \Lambda]; Y$.

> Intuitively, given two nested sequents, $\mathfrak{N}[\Gamma \Rightarrow \Delta]$ and $\mathfrak{M}[\Theta \Rightarrow \Lambda]$ together with an occurrence of a sequent in each, the relation of equivalent position between two of their sequents holds when, by considering $\mathfrak{N}[\Gamma \Rightarrow \Delta]$ and $\mathfrak{M}[\Theta \Rightarrow \Lambda]$ as trees, and $\Gamma \Rightarrow \Delta$ and $\Theta \Rightarrow \Lambda$ as nodes of the trees, the two nodes have the same height in their respective trees. [35, p. 36, the notation and terminology adjusted]

DEFINITION 3.7. [34, Definition 6.6] Given two nested sequents $\mathfrak{N}[\Gamma \Rightarrow \Delta]$ and $\mathfrak{M}[\Theta \Rightarrow \Lambda]$ together with an occurrence of a sequent in each, such that $\mathfrak{N}[\Gamma \Rightarrow \Delta] \approx \mathfrak{M}[\Theta \Rightarrow \Lambda]$, the operation of product, $\mathfrak{N}[\Gamma \Rightarrow \Delta] \otimes \mathfrak{M}[\Theta \Rightarrow \Lambda]$, is defined inductively in the following way:

- $\Gamma \Rightarrow \Delta \otimes \Theta \Rightarrow \Lambda = \Gamma, \Theta \Rightarrow \Delta, \Lambda$
- $(\Gamma \Rightarrow \Delta/X) \otimes (\Theta \Rightarrow \Lambda/Y) = \Gamma, \Theta \Rightarrow \Delta, \Lambda/X; Y$
- $(\Phi \Rightarrow \Pi/\mathfrak{K}[\Gamma \Rightarrow \Delta]; X) \otimes (\Psi \Rightarrow \Upsilon/\mathfrak{L}[\Theta \Rightarrow \Lambda]; Y) =$
  $\Phi, \Psi \Rightarrow \Pi, \Upsilon/(\mathfrak{K}[\Gamma \Rightarrow \Delta] \otimes \mathfrak{L}[\Theta \Rightarrow \Lambda]); X; Y$.

Given two tree-hypersequents $\mathfrak{N}[\Gamma \Rightarrow \Delta, A]$ and $\mathfrak{M}[A, \Theta \Rightarrow \Lambda]$ together with an occurrence of a sequent in each, such that $\mathfrak{N}[\Gamma \Rightarrow \Delta, A] \approx \mathfrak{M}[A, \Theta \Rightarrow \Lambda]$, the cut rule is as follows:

$$[\mathrm{Cut}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A] \quad \mathfrak{M}[A, \Theta \Rightarrow \Lambda]}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Theta \Rightarrow \Delta, \Lambda]}$$

Let $\circledcirc \in \{\blacksquare, \blacklozenge, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \circ_\flat, \bullet_\sharp, \widetilde{\circ}_\flat, \widetilde{\bullet}_\sharp, \sim, \dot{\sim}\}$. Let us formulate a nested sequent calculus $\mathbb{NSK}^{\circledcirc}$ for the logic $\mathbf{K}^{\circledcirc}$. It has the above mentioned axiom and propositional rules, the contraction rules [C⇒] and [⇒C] as well as the following modal rules:

$$[\blacksquare \Rightarrow]\ \frac{\mathfrak{N}[\blacksquare A, \Gamma \Rightarrow \Delta/(A, \Theta \Rightarrow \Lambda/X)] \quad \mathfrak{N}[\blacksquare A, \Gamma \Rightarrow \Delta/(\Xi \Rightarrow \Pi, A/Y)]}{\mathfrak{N}[\blacksquare A, \Gamma \Rightarrow \Delta/(\Theta \Rightarrow \Lambda/X); (\Xi \Rightarrow \Pi/Y)]}$$

$$[\Rightarrow \blacksquare_L]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / A \Rightarrow]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \blacksquare A]} \qquad [\Rightarrow \blacksquare_R]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / \Rightarrow A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \blacksquare A]}$$

$$[\blacklozenge \Rightarrow_L]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / A \Rightarrow]}{\mathfrak{N}[\blacklozenge A, \Gamma \Rightarrow \Delta]} \qquad [\blacklozenge \Rightarrow_R]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / \Rightarrow A]}{\mathfrak{N}[\blacklozenge A, \Gamma \Rightarrow \Delta]}$$

$$[\Rightarrow \blacklozenge]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \blacklozenge A / (A, \Theta \Rightarrow \Lambda / X)] \quad \mathfrak{N}[\Gamma \Rightarrow \Delta, \blacklozenge A / (\Xi \Rightarrow \Pi, A / Y)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \blacklozenge A / (\Theta \Rightarrow \Lambda / X); (\Xi \Rightarrow \Pi / Y)]}$$

$$[\circ \Rightarrow]\ \frac{\mathfrak{N}[\circ A, \Gamma \Rightarrow \Delta / (A, \Theta \Rightarrow \Lambda / X)] \quad \mathfrak{N}[\circ A, \Gamma \Rightarrow \Delta, A / Y]}{\mathfrak{N}[\circ A, \Gamma \Rightarrow \Delta / Y; (\Theta \Rightarrow \Lambda / X)]}$$

$$[\Rightarrow \circ_L]\ \frac{\mathfrak{N}[A, \Gamma \Rightarrow \Delta]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \circ A]} \qquad [\Rightarrow \circ_R]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / \Rightarrow A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \circ A]}$$

$$[\bullet \Rightarrow_L]\ \frac{\mathfrak{N}[A, \Gamma \Rightarrow \Delta]}{\mathfrak{N}[\bullet A, \Gamma \Rightarrow \Delta]} \qquad [\bullet \Rightarrow_R]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / \Rightarrow A]}{\mathfrak{N}[\bullet A, \Gamma \Rightarrow \Delta]}$$

$$[\Rightarrow \bullet]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \bullet A / (A, \Theta \Rightarrow \Lambda / X)] \quad \mathfrak{N}[\Gamma \Rightarrow \Delta, A, \bullet A / Y]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \bullet A / Y; (\Theta \Rightarrow \Lambda / X)]}$$

$$[\widetilde{\circ} \Rightarrow]\ \frac{\mathfrak{N}[\widetilde{\circ} A, A, \Gamma \Rightarrow \Delta / X] \quad \mathfrak{N}[\widetilde{\circ} A, \Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda, A / Y)]}{\mathfrak{N}[\widetilde{\circ} A, \Gamma \Rightarrow \Delta / X; (\Theta \Rightarrow \Lambda / Y)]}$$

$$[\Rightarrow \widetilde{\circ}_L]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / A \Rightarrow]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \widetilde{\circ} A]} \qquad [\Rightarrow \widetilde{\circ}_R]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \widetilde{\circ} A]}$$

$$[\widetilde{\bullet} \Rightarrow_L]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / A \Rightarrow]}{\mathfrak{N}[\widetilde{\bullet} A, \Gamma \Rightarrow \Delta]} \qquad [\widetilde{\bullet} \Rightarrow_R]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A]}{\mathfrak{N}[\widetilde{\bullet} A, \Gamma \Rightarrow \Delta]}$$

$$[\Rightarrow \widetilde{\bullet}]\ \frac{\mathfrak{N}[A, \Gamma \Rightarrow \Delta, \widetilde{\bullet} A / X] \quad \mathfrak{N}[\Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda, A, \widetilde{\bullet} A / Y)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \widetilde{\bullet} A / X; (\Theta \Rightarrow \Lambda / Y)]}$$

$$[\circ_\flat \Rightarrow]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A] \quad \mathfrak{N}[\Gamma \Rightarrow \Delta / A \Rightarrow]}{\mathfrak{N}[\circ_\flat A, \Gamma \Rightarrow \Delta]}$$

$$[\Rightarrow \circ_{\flat L}]\ \frac{\mathfrak{N}[A, \Gamma \Rightarrow \Delta]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \circ_\flat A]} \qquad [\Rightarrow \circ_{\flat R}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \circ_\flat A / \Theta \Rightarrow \Lambda, A / X]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \circ_\flat A / \Theta \Rightarrow \Lambda / X]}$$

$$[\bullet_{\sharp L} \Rightarrow]\ \frac{\mathfrak{N}[A, \Gamma \Rightarrow \Delta]}{\mathfrak{N}[\bullet_\sharp A, \Gamma \Rightarrow \Delta]} \qquad [\bullet_{\sharp R} \Rightarrow]\ \frac{\mathfrak{N}[\bullet_\sharp A, \Gamma \Rightarrow \Delta / \Theta \Rightarrow \Lambda, A / X]}{\mathfrak{N}[\bullet_\sharp A, \Gamma \Rightarrow \Delta / \Theta \Rightarrow \Lambda / X]}$$

$$[\Rightarrow \bullet_\sharp]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A] \quad \mathfrak{N}[\Gamma \Rightarrow \Delta / A \Rightarrow]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \bullet_\sharp A]}$$

$$[\widetilde{\circ}_\flat \Rightarrow]\ \frac{\mathfrak{N}[A, \Gamma \Rightarrow \Delta] \quad \mathfrak{N}[\Gamma \Rightarrow \Delta / \Rightarrow A]}{\mathfrak{N}[\widetilde{\circ}_\flat A, \Gamma \Rightarrow \Delta]}$$

$$
\cfrac{
[\cdot]\,\cfrac{\cfrac{\mathfrak{X}/A \Rightarrow A \qquad \mathfrak{X}/B \Rightarrow B}{\mathfrak{X}/A, A \to B \Rightarrow B} \qquad \mathfrak{X}/A \Rightarrow A}{\mathfrak{X}/A \to B \Rightarrow B; A \Rightarrow}
\qquad
[\cdot]\,\cfrac{\cfrac{\mathfrak{X}/A, A, B \Rightarrow B}{\mathfrak{X}/A, B \Rightarrow A \to B} \qquad \mathfrak{X}/A \Rightarrow A}{\mathfrak{X}/B \Rightarrow A \to B; A \Rightarrow}
}{
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\blacksquare(A \to B), \blacksquare A \Rightarrow /A \Rightarrow; A \Rightarrow; \Rightarrow B; B \Rightarrow}{A, \blacksquare(A \to B), \blacksquare A \Rightarrow /A \Rightarrow; \Rightarrow B; B \Rightarrow}[\widetilde{\mathbf{T}}]}{A, A, \blacksquare(A \to B), \blacksquare A \Rightarrow / \Rightarrow B; B \Rightarrow}[\widetilde{\mathbf{T}}]}{A, \blacksquare(A \to B), \blacksquare A \Rightarrow / \Rightarrow B; B \Rightarrow}[\mathrm{C} \Rightarrow]}{A, \blacksquare(A \to B), \blacksquare A \Rightarrow \blacksquare B / B \Rightarrow}[\Rightarrow \blacksquare_R]}{A, \blacksquare(A \to B), \blacksquare A \Rightarrow \blacksquare B, \blacksquare B}[\Rightarrow \blacksquare_L]}{A, \blacksquare(A \to B), \blacksquare A \Rightarrow \blacksquare B}[\Rightarrow \mathrm{C}]}{\Rightarrow A \to \big(\blacksquare(A \to B) \to (\blacksquare A \to \blacksquare B)\big)}[\Rightarrow\to], 3x}
}[\cdot]
$$

Figure 2. An example of a proof in $\mathbb{N}\$\mathbf{KT}^{\blacksquare}$, where $\mathfrak{X}$ is $\blacksquare A, \blacksquare(A \to B) \Rightarrow$ and $[\cdot]$ stands for $[\blacksquare \Rightarrow]$

$$
[\Rightarrow \widetilde{\circ}_{\flat L}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \widetilde{\circ}_{\flat} A]}
\qquad
[\Rightarrow \widetilde{\circ}_{\flat R}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \widetilde{\circ}_{\flat} A / A, \Theta \Rightarrow \Lambda / X]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \widetilde{\circ}_{\flat} A / \Theta \Rightarrow \Lambda / X]}
$$

$$
[\widetilde{\bullet}_{\sharp L} \Rightarrow]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A]}{\mathfrak{N}[\widetilde{\bullet}_{\sharp} A, \Gamma \Rightarrow \Delta]}
\qquad
[\widetilde{\bullet}_{\sharp R} \Rightarrow]\ \frac{\mathfrak{N}[\widetilde{\bullet}_{\sharp} A, \Gamma \Rightarrow \Delta / A, \Theta \Rightarrow \Lambda / X]}{\mathfrak{N}[\widetilde{\bullet}_{\sharp} A, \Gamma \Rightarrow \Delta / \Theta \Rightarrow \Lambda / X]}
$$

$$
[\Rightarrow \widetilde{\bullet}_{\sharp}]\ \frac{\mathfrak{N}[A, \Gamma \Rightarrow \Delta] \quad \mathfrak{N}[\Gamma \Rightarrow \Delta / \Rightarrow A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \widetilde{\bullet}_{\sharp} A]}
$$

$$
[\sim \Rightarrow]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / \Rightarrow A]}{\mathfrak{N}[\sim A, \Gamma \Rightarrow \Delta]}
\qquad
[\Rightarrow \sim]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \sim A / (A, \Theta \Rightarrow \Lambda / X)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \sim A / (\Theta \Rightarrow \Lambda / X)]}
$$

$$
[\dot{\sim} \Rightarrow]\ \frac{\mathfrak{N}[\dot{\sim} A, \Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda, A / X)]}{\mathfrak{N}[\dot{\sim} A, \Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda / X)]}
\qquad
[\Rightarrow \dot{\sim}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / A \Rightarrow]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \dot{\sim} A]}
$$

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n \in \{\mathbf{T}, \mathbf{D}, \mathbf{4}, \mathbf{B}\}$. A nested sequent calculus $\mathbb{N}\$\mathbf{K}\mathbf{X}_1 \ldots \mathbf{X}_n^{\obar}$ for $\mathbf{K}\mathbf{X}_1 \ldots \mathbf{X}_n^{\obar}$ is an extension of $\mathbb{N}\$\mathbf{K}^{\obar}$ by the above mentioned special structural rules $[\widetilde{\mathbf{X}_1}], \ldots, [\widetilde{\mathbf{X}_n}]$ as well as the rules [EW], [IW⇒], [⇒IW], [Merge], and [rn].

On Figures 2–5, we give some examples of proofs in nested sequent calculi for several non-contingency logics. In these examples, we use formulas that are axioms in Hilbert-style calculi presented in [45, 46] for the logics in question. Similarly to [34, Lemma 6.18], we get:

Proposition 3.1. *All the propositional rules and the modal rules of nested sequent calculi* $\mathbb{N}\$\mathbf{K}^{\obar}$ *and* $\mathbb{N}\$\mathbf{K}\mathbf{X}_1 \ldots \mathbf{X}_n^{\obar}$ *are height-preserving invertible.*

$$
\begin{array}{cl}
\blacksquare A \Rightarrow /A \Rightarrow A \qquad \blacksquare A \Rightarrow /A \Rightarrow A & \\ \hline
\blacksquare A \Rightarrow / \Rightarrow A; A \Rightarrow & [\blacksquare \Rightarrow] \\ \hline
\blacksquare A \Rightarrow / \Rightarrow A; \Rightarrow \neg A & [\Rightarrow \neg] \\ \hline
\blacksquare A \Rightarrow /\neg A \Rightarrow; \Rightarrow \neg A & [\neg \Rightarrow] \\ \hline
\blacksquare A \Rightarrow \blacksquare\neg A/ \Rightarrow \neg A & [\Rightarrow \blacksquare_L] \\ \hline
\blacksquare A \Rightarrow \blacksquare\neg A, \blacksquare\neg A & [\Rightarrow \blacksquare_R] \\ \hline
\blacksquare A \Rightarrow \blacksquare\neg A & [\Rightarrow \mathrm{C}] \\ \hline
\Rightarrow \blacksquare A \rightarrow \blacksquare\neg A & [\Rightarrow\rightarrow]
\end{array}
$$

Figure 3. An example of a proof in $\mathbb{N}\$\mathbf{K}^{\blacksquare}$.

$$
\begin{array}{cl}
\Rightarrow /\blacksquare A \Rightarrow /A \Rightarrow A \qquad \Rightarrow /\blacksquare A \Rightarrow /A \Rightarrow A & \\ \hline
\Rightarrow /\blacksquare A \Rightarrow / \Rightarrow A; A \Rightarrow & [\blacksquare \Rightarrow] \\ \hline
A \Rightarrow /\blacksquare A \Rightarrow / \Rightarrow A & [\widetilde{\mathbf{B}}] \\ \hline
A \Rightarrow /\blacksquare A \Rightarrow A & [\widetilde{\mathbf{T}}] \\ \hline
A \Rightarrow / \Rightarrow \blacksquare A \rightarrow A & [\Rightarrow\rightarrow] \\ \hline
A \Rightarrow \blacksquare(\blacksquare A \rightarrow A) & [\Rightarrow \blacksquare_R] \\ \hline
\Rightarrow A \rightarrow \blacksquare(\blacksquare A \rightarrow A) & [\Rightarrow\rightarrow]
\end{array}
$$

Figure 4. An example of a proof in $\mathbb{N}\$\mathbf{KTB}^{\blacksquare}$

$$
\begin{array}{cl}
\blacksquare A \Rightarrow /A \Rightarrow A \qquad \blacksquare A \Rightarrow /A \Rightarrow A & \\ \hline
\blacksquare A \Rightarrow / \Rightarrow A; A \Rightarrow & [\blacksquare \Rightarrow] \\ \hline
\blacksquare A \Rightarrow / \Rightarrow / \Rightarrow A; A \Rightarrow & [\widetilde{\mathbf{4}}] \\ \hline
\blacksquare A \Rightarrow / \Rightarrow \blacksquare A/A \Rightarrow & [\Rightarrow \blacksquare_R] \\ \hline
\blacksquare A \Rightarrow / \Rightarrow \blacksquare A, \blacksquare A & [\Rightarrow \blacksquare_L] \\ \hline
\blacksquare A \Rightarrow / \Rightarrow \blacksquare A & [\Rightarrow \mathrm{C}] \\ \hline
\blacksquare A \Rightarrow \blacksquare\blacksquare A & [\Rightarrow \blacksquare_R] \\ \hline
\Rightarrow \blacksquare A \rightarrow \blacksquare\blacksquare A & [\Rightarrow\rightarrow]
\end{array}
$$

Figure 5. An example of a proof in $\mathbb{N}\$\mathbf{K4}^{\blacksquare}$

Moreover, similarly to Lemmas 6.10–6.12 from [34], we get:

PROPOSITION 3.2. *The rules* [EW], [IW⇒], [⇒IW], [Merge], *and* [rn], *are height-preserving admissible in* $\mathbb{N}\$\mathbf{K}^{\copyright}$.

The admissibility of [EW], [IW⇒], [⇒IW], [Merge], [C⇒], [⇒C], and [rn] in $\mathbb{N}\$\mathbf{KX}^{\copyright}$ remains an open issue and requires further investigation. Unlike Lemmas 6.10–6.12 and 6.19 in [34], our approach requires treating special structural rules as primitive, leading to certain complications. In particular, we have not succeeded in proving the admissibility of the contraction rules in $\mathbb{N}\$\mathbf{KX}^{\circ}$. A similar problem has been discussed in [24]: as an anonymous reviewer pointed out, "for standard modalities we can have a cut-free nested calculus using the special structural rules instead of the special logical ones only if we have a primitive rule of contraction, otherwise there is a flaw in the proof of the admissibility of cut". In our case, the problem looks as follows. For example, it is not clear how the admissibility of contraction rules can be shown in the case of $[\widetilde{\mathbf{T}}]$ (a similar problem occurs with the rule $[\widetilde{\mathbf{B}}]$; double line signifies multiple applications of the rules):

$$\dfrac{\dfrac{\mathfrak{N}[\Gamma \Rightarrow \Delta/(\Gamma \Rightarrow \Delta/X)]}{\mathfrak{N}[\Gamma, \Gamma \Rightarrow \Delta, \Delta/X]}\,[\widetilde{\mathbf{T}}]}{\mathfrak{N}[\Gamma \Rightarrow \Delta/X]}\,[\mathrm{C}\Rightarrow], [\Rightarrow\mathrm{C}]$$

In the case of the logic $\mathbf{K}^{\blacksquare}$, the proof of the admissibility of contraction in certain instances resembles the argument given in [34, Lemma 6.19]. For example, by induction on the derivation of the premises of the contraction rules, we obtain the following transformation of the deduction (cf. the case of $\Box K$ (in our notation, $[\Rightarrow \Box]$) in [34, p. 137]):

$$\dfrac{\dfrac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \blacksquare A/ \Rightarrow A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \blacksquare A, \blacksquare A]}\,[\Rightarrow \blacksquare_R]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \blacksquare A]}\,[\Rightarrow\mathrm{C}] \quad \dashrightarrow \quad \dfrac{\dfrac{\dfrac{\mathfrak{N}[\Gamma \Rightarrow \Delta/ \Rightarrow A; \Rightarrow A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta/ \Rightarrow A, A]}\,[\mathrm{Merge}]}{\mathfrak{N}[\Gamma \Rightarrow \Delta/ \Rightarrow A]}\,[\mathrm{ind.h.}]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \blacksquare A]}\,[\Rightarrow \blacksquare_R]$$

However, since there are two right rules for $\blacksquare$, unlike in the proof of [34, Lemma 6.19], the following case is also possible. The induction hypothesis cannot be applied to the premise $\mathfrak{N}[\Gamma \Rightarrow \Delta/A \Rightarrow; \Rightarrow A]$. See also Figure 3 for an example of a deduction in which contraction is required.

$$\dfrac{\dfrac{\dfrac{\mathfrak{N}[\Gamma \Rightarrow \Delta/A \Rightarrow; \Rightarrow A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \blacksquare A; \Rightarrow A]}\,[\Rightarrow \blacksquare_L]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \blacksquare A, \blacksquare A]}\,[\Rightarrow \blacksquare_R]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \blacksquare A]}\,[\Rightarrow\mathrm{C}]$$

4. Soundness and completeness

Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be a model and $x \in W$. For any multiset Γ of formulas, $\mathcal{M} \models_x \Gamma$ iff $\exists_{A \in \Gamma}\ \mathcal{M} \models_x A$. Moreover, for any sequent

$\Gamma \Rightarrow \Delta$, $\mathcal{M} \models_x \Gamma \Rightarrow \Delta$ iff $\exists_{B \in \Gamma}\, \mathcal{M} \not\models_x B$ or $\mathcal{M} \models_x \Delta$. Finally, for any nested sequent $\mathfrak{N}$, $\mathcal{M} \models_x \mathfrak{N}$ is inductively defined as follows:

- if $\mathfrak{N}$ is a sequent $\mathfrak{S}$, then $\mathcal{M} \models_x \mathfrak{N}$ iff $\mathcal{M} \models_x \mathfrak{S}$,
- if $\mathfrak{N} = \mathfrak{S}/X$, where X is a multiset of nested sequents, then $\mathcal{M} \models_x \mathfrak{N}$ iff $\mathcal{M} \models_x \mathfrak{S}$ or $\exists_{\mathfrak{M} \in X} \forall_{y \in R[x]}\, \mathcal{M} \models_y \mathfrak{M}$.

Let $\mathfrak{X}$ be a formula, or a multiset of formulas, or a multiset of sequents, or a nested sequent, or a multiset of a nested sequent. We write $\mathcal{M} \models_x^* \mathfrak{X}$ iff $\forall_{y \in R[x]}\, \mathcal{M} \models_y \mathfrak{X}$. Hence, we can write $\mathcal{M} \models_x \mathfrak{S}/X$ iff $\mathcal{M} \models_x \mathfrak{S}$ or $\exists_{\mathfrak{M} \in X}\, \mathcal{M} \models_x^* \mathfrak{M}$.

Let $\mathcal{F}$ be a frames and $\mathfrak{N}$ be a nested sequent. We say that $\mathfrak{N}$ is valid in $\mathcal{F}$ (we write $\mathcal{F} \models \mathfrak{N}$) iff $\mathcal{M} \models \mathfrak{N}$ for each model $\mathcal{M}$ based on a frame of $\mathscr{F}$. Let $\mathscr{F}$ be a class of frames. We say that $\mathfrak{N}$ is valid in $\mathscr{F}$ (we write $\mathscr{F} \models \mathfrak{N}$) iff $\mathcal{F} \models \mathfrak{N}$ for each $\mathcal{F} \in \mathscr{F}$. If $\mathscr{F}$ is the class of all $\mathbf{K}^{\circledcirc}$-frames, then we write $\mathbf{K}^{\circledcirc} \models \mathfrak{N}$.

LEMMA 4.1. [34, Lemma 8.2] *Let $\mathscr{F}$ be a class of frames, $\Gamma \Rightarrow \Delta$ and $\Theta \Rightarrow \Lambda$ be sequents, and $\mathfrak{K}$, $\mathfrak{L}$ and $\mathfrak{N}$ be nested sequents. If Then*:

1. *if $\mathscr{F} \models \Gamma \Rightarrow \Delta$ implies $\mathscr{F} \models \Theta \Rightarrow \Lambda$, then $\mathscr{F} \models \mathfrak{N}[\Gamma \Rightarrow \Delta]$ implies $\mathscr{F} \models \mathfrak{N}[\Theta \Rightarrow \Lambda]$.*
2. *If $\mathscr{F} \models \mathfrak{K}$ implies $\mathscr{F} \models \mathfrak{L}$, then $\mathscr{F} \models \mathfrak{N}[\mathfrak{K}]$ implies $\mathscr{F} \models \mathfrak{N}[\mathfrak{L}]$.*

Let $\circledcirc \in \{\blacksquare, \blacklozenge, \circ, \bullet, \tilde{\circ}, \tilde{\bullet}, \circ_\flat, \bullet_\sharp, \tilde{\circ}_\flat, \tilde{\bullet}_\sharp, \sim, \dot{\sim}\}$. As Theorem 8.3 from [34], we obtain:

LEMMA 4.2. *All the rules of $\mathbb{N}\$\mathbf{K}^{\circledcirc}$ are sound w.r.t. $\mathscr{F}$.*

PROOF. As an example, we consider the rule $[\Rightarrow\blacksquare_L]$. Suppose that $\mathscr{F} \models \Gamma \Rightarrow \Delta/A \Rightarrow$ and $\mathcal{M} = \langle W, R, \vartheta\rangle$ be any model based on a frame of $\mathscr{F}$, and $x \in W$. Then $\mathcal{M} \models_x \Gamma \Rightarrow \Delta/A \Rightarrow$, i.e., $\mathcal{M} \models_x \Gamma \Rightarrow \Delta$ or $\mathcal{M} \not\models_x^* A$, i.e., $\mathcal{M} \models_x \Gamma \Rightarrow \Delta$ or $\mathcal{M} \models_x \blacksquare A$. Hence $\mathcal{M} \models_x \Gamma \Rightarrow \Delta, \blacksquare A$. So, $\mathscr{F} \models \Gamma \Rightarrow \Delta, \blacksquare A$. Thus, by Lemma 4.1(2): $\mathscr{F} \models \mathfrak{N}[\Gamma \Rightarrow \Delta/A \Rightarrow$ implies $\mathscr{F} \models \mathfrak{N}[\Gamma \Rightarrow \Delta, \blacksquare A]$. ⊣

By induction on the height of the derivation, by Lemma 4.2, we get:

THEOREM 4.1. *For each nested sequent $\mathfrak{N}$, if $\mathbb{N}\$\mathbf{K}^{\circledcirc} \vdash \mathfrak{N}$, then $\mathbf{K}^{\circledcirc} \models \mathfrak{N}$.*

From Theorem 4.1 and soundness of the special structural rules established in [34], we obtain:

THEOREM 4.2. *For $\mathbf{X} \subseteq \{\mathbf{T}, \mathbf{D}, \mathbf{4}, \mathbf{B}\}$ and each nested sequent $\mathfrak{N}$, if $\mathbb{N}\$\mathbf{KX}^{\circledcirc} \vdash \mathfrak{N}$, then $\mathbf{KX}^{\circledcirc} \models \mathfrak{N}$.*

We provide a semantic completeness proof for a nested sequent calculus for $\mathbf{K}^{\circ}$ in accordance with Poggiolesi [34], who in turn follows Brünnler [4]. The completeness for the extensions of $\mathbf{K}^{\circ}$ follows from the established in [34] correspondence of the structural rules $[\widetilde{\mathbf{D}}]$, $[\widetilde{\mathbf{T}}]$, $[\widetilde{\mathbf{4}}]$, $[\widetilde{\mathbf{B}}]$ for the properties of the accessibility relation. The calculus in question requires minor reformulation, which will be designated as $\mathbb{NS}\mathbf{K}^{\circ+}$. For each rule $\Re$, we define a rule $\Re^{+}$ that incorporates the principal formula from the conclusion within its premises. At that point, we have $\Re = \Re^{+}$ for the following rules: $[\blacksquare \Rightarrow]$, $[\Rightarrow \blacklozenge]$, $[\circ \Rightarrow]$, $[\Rightarrow \bullet]$, $[\tilde{\circ} \Rightarrow]$, $[\Rightarrow \tilde{\bullet}]$, $[\circ_\flat \Rightarrow]$, $[\Rightarrow \bullet_\sharp]$, $[\tilde{\circ}_\flat \Rightarrow]$, $[\Rightarrow \tilde{\bullet}_\sharp]$, $[\Rightarrow \sim]$, $[\dot{\sim} \Rightarrow]$. Regarding the remaining rules, $\Re^{+}$ is as follows:

$$[\neg \Rightarrow]^{+}\ \frac{\mathfrak{N}[\neg A, \varGamma \Rightarrow \varDelta, A]}{\mathfrak{N}[\neg A, \varGamma \Rightarrow \varDelta]} \qquad [\Rightarrow \neg]^{+}\ \frac{\mathfrak{N}[A, \varGamma \Rightarrow \varDelta, \neg A]}{\mathfrak{N}[\varGamma \Rightarrow \varDelta, \neg A]}$$

$$[\wedge \Rightarrow]^{+}\ \frac{\mathfrak{N}[A, B, A \wedge B, \varGamma \Rightarrow \varDelta]}{\mathfrak{N}[A \wedge B, \varGamma \Rightarrow \varDelta]}$$

$$[\Rightarrow \wedge]^{+}\ \frac{\mathfrak{N}[\varGamma \Rightarrow \varDelta, A, A \wedge B] \quad \mathfrak{N}[\varGamma \Rightarrow \varDelta, B, A \wedge B]}{\mathfrak{N}[\varGamma \Rightarrow \varDelta, A \wedge B]}$$

$$[\vee \Rightarrow]^{+}\ \frac{\mathfrak{N}[A \vee B, A, \varGamma \Rightarrow \varDelta] \quad \mathfrak{N}[A \vee B, B, \varGamma \Rightarrow \varDelta]}{\mathfrak{N}[A \vee B, \varGamma \Rightarrow \varDelta]}$$

$$[\Rightarrow \vee]^{+}\ \frac{\mathfrak{N}[\varGamma \Rightarrow \varDelta, A, B, A \vee B]}{\mathfrak{N}[\varGamma \Rightarrow \varDelta, A \vee B]}$$

$$[\to\Rightarrow]^{+}\ \frac{\mathfrak{N}[A \to B, \varGamma \Rightarrow \varDelta, A] \quad \mathfrak{N}[A \to B, B, \varGamma \Rightarrow \varDelta]}{\mathfrak{N}[A \to B, \varGamma \Rightarrow \varDelta]}$$

$$[\Rightarrow\to]^{+}\ \frac{\mathfrak{N}[A, \varGamma \Rightarrow \varDelta, B, A \to B]}{\mathfrak{N}[\varGamma \Rightarrow \varDelta, A \to B]}$$

$$[\leftrightarrow\Rightarrow]^{+}\ \frac{\mathfrak{N}[B, \varGamma \Rightarrow \varDelta, A, A \leftrightarrow B] \quad \mathfrak{N}[A, \varGamma \Rightarrow \varDelta, B, A \leftrightarrow B]}{\mathfrak{N}[\varGamma \Rightarrow \varDelta, A \leftrightarrow B]}$$

$$[\Rightarrow\leftrightarrow]^{+}\ \frac{\mathfrak{N}[A \leftrightarrow B, A, B, \varGamma \Rightarrow \varDelta] \quad \mathfrak{N}[A \leftrightarrow B, \varGamma \Rightarrow \varDelta, A, B]}{\mathfrak{N}[A \leftrightarrow B, \varGamma \Rightarrow \varDelta]}$$

$$[\Rightarrow \blacksquare_L]^{+}\ \frac{\mathfrak{N}[\varGamma \Rightarrow \varDelta, \blacksquare A / A \Rightarrow]^{\dagger}}{\mathfrak{N}[\varGamma \Rightarrow \varDelta, \blacksquare A]} \qquad [\Rightarrow \blacksquare_R]^{+}\ \frac{\mathfrak{N}[\varGamma \Rightarrow \varDelta, \blacksquare A / \Rightarrow A]^{\ddagger}}{\mathfrak{N}[\varGamma \Rightarrow \varDelta, \blacksquare A]}$$

† $\varGamma \Rightarrow \varDelta, \blacksquare A$ does not have any immediate successive sequent (or more succinctly, childsequent) that contains A on the right side.
‡ $\varGamma \Rightarrow \varDelta, \blacksquare A$ does not have any immediate successive sequent (or more succinctly, childsequent) that contains A on the left side.

$$[\blacklozenge \Rightarrow_L]^+ \ \frac{\mathfrak{N}[\blacklozenge A, \Gamma \Rightarrow \Delta / A \Rightarrow]^{\dagger}}{\mathfrak{N}[\blacklozenge A, \Gamma \Rightarrow \Delta]} \qquad [\blacklozenge \Rightarrow_R]^+ \ \frac{\mathfrak{N}[\blacklozenge A, \Gamma \Rightarrow \Delta / \Rightarrow A]^{\ddagger}}{\mathfrak{N}[\blacklozenge A, \Gamma \Rightarrow \Delta]}$$

† $\blacklozenge A, \Gamma \Rightarrow \Delta$ does not have any immediate successive sequent that contains A on the right side. ‡ $\blacklozenge A, \Gamma \Rightarrow \Delta$ does not have any immediate successive sequent that contains A on the left side.

$$[\sim \Rightarrow]^+ \ \frac{\mathfrak{N}[\sim A, \Gamma \Rightarrow \Delta / \Rightarrow A]^{\dagger}}{\mathfrak{N}[\sim A, \Gamma \Rightarrow \Delta]} \qquad [\Rightarrow \dot{\sim}]^+ \ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \dot{\sim} A / A \Rightarrow]^{\ddagger}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \dot{\sim} A]}$$

† $\sim A, \Gamma \Rightarrow \Delta$ does not have any immediate successive sequent that contains A on the left side. ‡ $\Gamma \Rightarrow \Delta, \dot{\sim} A$ does not have any immediate successive sequent that contains A on the right side.

The rules $[\Rightarrow\circ_L]^+$, $[\Rightarrow\circ_R]^+$, $[\bullet\Rightarrow_L]^+$, $[\bullet\Rightarrow_R]^+$, $[\Rightarrow\tilde{\circ}_L]^+$, $[\Rightarrow\tilde{\circ}_R]^+$, $[\tilde{\bullet}\Rightarrow_L]^+$, $[\tilde{\bullet}\Rightarrow_R]^+$, $[\Rightarrow\circ_{\flat L}]^+$, $[\Rightarrow\circ_{\flat R}]^+$, $[\bullet_{\sharp L}\Rightarrow]^+$, $[\bullet_{\sharp R}\Rightarrow]^+$, $[\Rightarrow\tilde{\circ}_{\flat L}]^+$, $[\Rightarrow\tilde{\circ}_{\flat R}]^+$, $[\tilde{\bullet}_{\sharp L}\Rightarrow]^+$, $[\tilde{\bullet}_{\sharp R}\Rightarrow]^+$ are formulated in a similar way.

DEFINITION 4.1. [34, Definition 8.4] A *set nested sequent* of a nested sequent $\Gamma \Rightarrow \Delta / \mathfrak{M}_1; \ldots; \mathfrak{M}_m$ is an underlying set of $\Theta \Rightarrow \Lambda / \mathfrak{N}_1; \ldots; \mathfrak{N}_n$, where $\mathfrak{N}_1; \ldots; \mathfrak{N}_n$ are sets nested sequents of $\mathfrak{M}_1; \ldots; \mathfrak{M}_m$. Clearly, a set nested sequent of a nested sequent is still a nested sequent since a set is a multiset.

For each rule $\mathfrak{R}^+$ it is stipulated that for all of its premises, the set nested sequent is different from the set nested sequent of the conclusion.

By induction on the height of derivations in $\mathbb{N}\$\mathbf{K}^{\circ+}$, employing contraction and weakening, we obtain:

LEMMA 4.3. *For any nested sequent $\mathfrak{N}$, if $\mathbb{N}\$\mathbf{K}^{\circ+} \vdash \mathfrak{N}$, then $\mathbb{N}\$\mathbf{K}^{\circ} \vdash \mathfrak{N}$.*

DEFINITION 4.2. [34, Definitions 8.10–8.13] 1. A leaf of a nested sequent is *cyclic* iff in its branch there exists a sequent that contains the same set of formulas.

2. A sequent of a nested sequent is *finished* for a nested sequent calculus $\mathbb{N}\$\mathbf{K}^{\circ+}$ if no rule of that calculus applies to one of its formulas. A nested sequent is finished for a nested sequent calculus $\mathbb{N}\$\mathbf{K}^{\circ+}$ iff all sequents that compose it are finished or cyclic.

3. A procedure prove($\mathfrak{N}, \mathbb{N}\$\mathbf{K}^{\circ+}$) is defined as follows. It takes a nested sequent $\mathfrak{N}$ and a calculus $\mathbb{N}\$\mathbf{K}^{\circ+}$, and builds a tree for $\mathfrak{N}$ by applying rules from that calculus to non-initial and unfinished derivation leaves in the bottom-up fashion, as follows:

(i) keep applying all the rules of $\mathbb{N}\$\mathbf{K}^{\circ+}$ which are not the rules with the provisos above indicated as † and ‡ long as possible;

(ii) wherever possible, apply these rules with the † and ‡ provisos once.

Repeat this operation until each non-initial derivation leaf of the nested sequent $\mathfrak{N}$ is finished. If prove($\mathfrak{N}, \mathbb{N}\$\mathbf{K}^{\circ+}$) terminates and all derivation leaves are initial, then it succeeds; otherwise, i.e., if it terminates and there is a non-initial derivation leaf, it fails.

4. The *size* of a nested sequent $\mathfrak{N}$, s($\mathfrak{N}$), is the number of sequents that compose it. The *set of subformulas* of a nested sequent $\mathfrak{N}$, a nested sequent sf($\mathfrak{N}$), is the set of all subformulas of all formulas that compose all sequents that belong to the nested sequent.

Definition 4.3. [34, Definition 8.15, slightly modified] A nested sequent $\mathfrak{N}_i$ ($1 \leqslant i \leqslant l$) is an *immediate subtree* of a nested sequent $\mathfrak{M}$, if $\mathfrak{M} = \Gamma \Rightarrow \Delta/\mathfrak{N}_1; ...; \mathfrak{N}_l$. It is a proper subtree if it is an immediate subtree either of $\mathfrak{M}$ or of a proper subtree of $\mathfrak{M}$, and it is a subtree if it is either a proper subtree of $\mathfrak{M}$ or $\mathfrak{M} = \mathfrak{N}_i$. The set of all subtrees of $\mathfrak{M}$ is denoted by st($\mathfrak{M}$).

Similarly to [34, Lemma 8.14], we get:

Lemma 4.4. *The procedure* prove($\mathfrak{N}, \mathbb{N}\$\mathbf{K}^{\circ+}$) *terminates after at most* $2^{|\mathrm{sf}(\mathfrak{N})|}$ *iterations, for each nested sequent* $\mathfrak{N}$.

Similarly to Theorem 8.16 from [34], we obtain:

Theorem 4.3. *For each nested sequent* $\mathfrak{N}$:

1. *If* $\mathbf{K}^{\circ} \models \mathfrak{N}$, *then* $\mathbb{N}\$\mathbf{K}^{\circ} \vdash \mathfrak{N}$.
2. *If* prove($\mathfrak{N}, \mathbb{N}\$\mathbf{K}^{\circ+}$) *fails, then* $\mathbf{K}^{\circ+} \not\models \mathfrak{N}$.

Proof. The contraposition of 1 follows from 2. Suppose that $\mathbb{N}\$\mathbf{K}^{\circ} \nvdash \mathfrak{N}$. By Lemma 4.3, $\mathbb{N}\$\mathbf{K}^{\circ+} \nvdash \mathfrak{N}$. By Lemma 4.4, prove($\mathfrak{N}, \mathbb{N}\$\mathbf{K}^{\circ+}$) has to fail. Let us define a countermodel for $\mathfrak{N}$.

Let $\mathfrak{N}^*$ be the set nested sequent obtained from a non-axiomatic nested sequent. Let $\mathbb{Y}$ be the set of all cyclic leaves in $\mathfrak{N}^*$. Let $W_0 := \mathrm{st}(\mathfrak{N}^*) \setminus \mathbb{Y}$ and let $f_0 \colon \mathbb{Y} \to W$ be a function which maps a cyclic leaf to a nested sequent in W_0 whose root carries the same set of formulas, and extend f_0 to st($\mathfrak{N}^*$) by the identity on W. We define the binary relation R_0 on W such that $\mathfrak{K}R\mathfrak{L}$ iff either (i) $\mathfrak{L}$ is an immediate subtree of $\mathfrak{K}$, or (ii) $\mathfrak{K}$ has an immediate subtree $\mathfrak{M} \in \mathbb{Y}$ and $f(\mathfrak{M}) = \mathfrak{L}$. Let $\vartheta_0(\mathfrak{N}, p)$ be such that $\vartheta_0(\mathfrak{N}, p) = 1$, if p is positioned on the left side of a sequent $\Gamma \Rightarrow \Delta \in \mathfrak{N}$, and $\vartheta_0(\mathfrak{N}, p) = 0$ otherwise. We put $\mathcal{M}_0 := \langle W_0, R_0, \vartheta_0 \rangle$.

Claim 1. For all $\mathfrak{K}, \mathfrak{L} \in W_0$ such that $\mathfrak{K} R_0 \mathfrak{L}$, for each A occurring on the left side of a sequent that belongs to the nested sequent $\mathfrak{N}$, we have the following: if $\blacksquare A \in \mathfrak{K}$, then $A \in \mathfrak{L}$ or $\neg A \in \mathfrak{L}$. By the definition of R_0 and the rules $[\blacksquare\Rightarrow]$ and $[\neg\Rightarrow]$, we get A in (the root sequent of) all immediate subtrees of $\mathfrak{K}$. The cases with the other modalities are considered similarly.

Claim 2. For each $\mathfrak{K} \in W_0$, we have:

1. for each $A \in \mathfrak{K}$ such that they are on the left side of the sequent, $\mathcal{M}_0 \models_{\mathfrak{K}} A$,
2. for each $A \in \mathfrak{K}$ such that they are on the right side of the sequent, $\mathcal{M}_0 \not\models_{\mathfrak{K}} A$.

By induction on the complexity of A. The basic case follows from the definition of the valuation. The propositional cases are quite evident. Let $A = \blacksquare B$. Suppose that it occurs on the right side of the sequent, then by the rules $[\Rightarrow \blacksquare_L]^+$ and $[\Rightarrow \blacksquare_R]^+$ as well as the rules for negations, we have at least one $\mathfrak{M} \in \mathfrak{K}$ with $B \in \mathfrak{M}$ and at least one $\mathfrak{M}' \in \mathfrak{K}$ with $\neg B \in \mathfrak{M}'$. By the inductive hypothesis, $\mathcal{M}_0 \not\models_{\mathfrak{M}} B$ and $\mathcal{M}_0 \not\models_{\mathfrak{M}'} \neg B$ (that is $\mathcal{M}_0 \models_{\mathfrak{M}'} B$). Thus, $\mathcal{M}_0 \not\models_{\mathfrak{K}} \blacksquare B$. Suppose that $\blacksquare B$ occurs on the left side of the sequent. By Claim 1, $B \in \mathfrak{M}$, for all $\mathfrak{M}$ such that $\mathfrak{K} R_0 \mathfrak{M}$, or $\neg B \in \mathfrak{M}'$, for all $\mathfrak{M}'$ such that $\mathfrak{K} R_0 \mathfrak{M}'$. Using the inductive hypothesis, $\mathcal{M}_0 \models_{\mathfrak{K}} \blacksquare B$.

The cases with the other modalities are considered similarly.

Claim 3. For each $\mathfrak{K} \in \mathrm{st}(\mathfrak{N}^*)$, $\mathcal{M}_0 \not\models_{f_0(\mathfrak{K})} \mathfrak{K}$.

By induction on the complexity of the nested sequent $\mathfrak{K}$, by Claim 2 [see 34, Theorem 8.16].

Since all rules seen top-down preserve countermodels, Claim 3 implies that $\mathcal{M}_0 \not\models \mathfrak{N}$. ⊣

From Theorem 4.3 and the fact that in its proof, the rule of cut has not been used, we get:

Theorem 4.4 (Cut admissibility). *For any nested sequent $\mathfrak{N}$, $\mathbb{NS}\mathbf{K}^{\circ} \vdash \mathfrak{N}$ implies that there is a cut-free proof of $\mathfrak{N}$ in $\mathbb{NS}\mathbf{K}^{\circ}$.*

Moreover, from Theorem 4.3 and the fact (which follows from the results of [34]) that special structural rules correspond to the properties of the accessibility relation, we obtain:

Theorem 4.5 (Completeness). *For $\mathbf{X} \subseteq \{\mathbf{T}, \mathbf{D}, \mathbf{4}, \mathbf{B}\}$ and each nested sequent $\mathfrak{N}$, if $\mathbf{KX}^{\circ} \models \mathfrak{N}$, then $\mathbb{NS}\mathbf{KX}^{\circ} \vdash \mathfrak{N}$.*

5. Constructive cut admissibility

Let $\circledcirc \in \{\blacksquare, \blacklozenge, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \circ_\flat, \bullet_\sharp, \widetilde{\circ}_\flat, \widetilde{\bullet}_\sharp, \sim, \dot{\sim}\}$. In this section, we consider a modification of Definition 3.1: multisets of formulas are replaced with sets of formulas. This change allows us to avoid treating cases related to contraction rules. Throughout this section, we assume that $\mathbb{N}\$\mathbf{K}^{\circledcirc}$ is defined accordingly. We proceed to demonstrate a constructive proof of cut admissibility for $\mathbb{N}\$\mathbf{K}^{\circledcirc}$, employing the techniques outlined in [34], where such a proof is provided for logics formulated in the language $\mathrm{For}_{\Box}$. It is worth noting that in [34], the definition of a sequent remains unchanged, as the admissibility of contraction rules is established there.

LEMMA 5.1. *Given three zoom nested sequents* $\mathfrak{K}[*]$, $\mathfrak{L}[*]$, *and* $\mathfrak{M}[*]$ *such that* $\mathfrak{K}[*] \sim \mathfrak{L}[*] \sim \mathfrak{M}[*]$, *if there is a rule* $\mathfrak{R}$ *of* $\mathbb{N}\$\mathbf{L}$ *and a sequent* Γ *such that*

$$\mathfrak{R}\,\frac{\mathfrak{L}[\Gamma]}{\mathfrak{K}[\Gamma]}$$

then, for each Δ, *we have that*

$$\mathfrak{R}\,\frac{\mathfrak{L}\otimes\mathfrak{M}[\Delta]}{\mathfrak{K}\otimes\mathfrak{M}[\Delta]}$$

PROOF. We follow the method from [34, p. 143, Lemma 7.1]. By induction on the form of nested sequents $\mathfrak{K}[*]$, $\mathfrak{L}[*]$, and $\mathfrak{M}[*]$. The proof consists of the following parts:

(a) $\mathfrak{K}[*]$, $\mathfrak{L}[*]$, and $\mathfrak{M}[*] \equiv *$.

(b) $\mathfrak{K}[*] \equiv */X$, $\mathfrak{L}[*] \equiv */Y$, and $\mathfrak{M}[*] \equiv */Z$.

(c) $\mathfrak{K}[*] \equiv \Gamma_1 \Rightarrow \Delta_1/\mathfrak{K}'[*]; X$, $\quad \mathfrak{L}[*] \equiv \Gamma_2 \Rightarrow \Delta_2/\mathfrak{L}'[*]; Y$, and $\mathfrak{M}[*] \equiv \Gamma_3 \Rightarrow \Delta_3/\mathfrak{M}'[*]; Z$.

Let
$$\mathfrak{R}\,\frac{\Gamma_1 \Rightarrow \Delta_1/\mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1]; X}{\Gamma_2 \Rightarrow \Delta_2/\mathfrak{L}'[\Theta_2 \Rightarrow \Lambda_2]; Y}$$

For each $\Pi \Rightarrow \Sigma$, we have that

$$\mathfrak{R}\,\frac{\Gamma_1, \Gamma_3 \Rightarrow \Delta_1, \Delta_3/\mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1] \otimes \mathfrak{M}'[\Pi \Rightarrow \Sigma]; X; Z}{\Gamma_2, \Gamma_3 \Rightarrow \Delta_2, \Delta_3/\mathfrak{L}'[\Theta_2 \Rightarrow \Lambda_2] \otimes \mathfrak{M}'[\Pi \Rightarrow \Sigma]; Y; Z}$$

We following subcases are distinguished:

(c1) The rule $\mathfrak{R}$ operates on $\Gamma_1 \Rightarrow \Delta_1$:

(c1.1) The rule $\mathfrak{R}$ operates on $\Gamma_1 \Rightarrow \Delta_1$ only, (c1.2) The rule $\mathfrak{R}$ operates between $\Gamma_1 \Rightarrow \Delta_1$ and $\mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1]; X$.

(c2) The rule $\mathfrak{R}$ operates on X.

(c3) The rule $\mathfrak{R}$ operates on $\mathfrak{K}'$.

Cases (a), (b), (c1.1), (c2), and (c3) are proven in [34]. Case (c1.2) deals with modal rules. Let us consider it on the example of the rules for $\blacksquare$.

Let $\mathfrak{R}$ be the rule $[\Rightarrow \blacksquare_L]$. There are two subcases: (i) $\mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1]; X$ is of the form $B \Rightarrow ; \mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1]; X'$, (ii) $\mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1]; X$ is of the form $B \Rightarrow ; \mathfrak{K}''[\Theta_1 \Rightarrow \Lambda_1]; X$. As an example, consider the case (i):

$$\frac{\Gamma_1 \Rightarrow \Delta_1 / B \Rightarrow ; \mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1]; X'}{\Gamma_1 \Rightarrow \Delta_1, \blacksquare B \Rightarrow ; \mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1]; X'}$$

For each $\Pi \Rightarrow \Sigma$, we have that

$$\frac{\Gamma_1, \Gamma_3 \Rightarrow \Delta_1, \Delta_3 / B \Rightarrow ; \mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1] \otimes \mathfrak{M}'[\Pi \Rightarrow \Sigma]; X'; Z}{\Gamma_1, \Gamma_3 \Rightarrow \Delta_1, \Delta_3, \blacksquare B / \mathfrak{L}'[\Theta_2 \Rightarrow \Lambda_2] \otimes \mathfrak{M}'[\Pi \Rightarrow \Sigma]; X'; Z}$$

The cases of the rules $[\Rightarrow \blacksquare_R]$ and $[\blacksquare \Rightarrow]$ are treated similarly. ⊣

Theorem 5.1. *Let* $\mathfrak{N}[\Gamma \Rightarrow \Delta, A]$ *and* $\mathfrak{M}[A, \Pi \Rightarrow \Sigma]$ *be such that* $\mathfrak{N}[\Gamma \Rightarrow \Delta, A] \sim \mathfrak{M}[A, \Pi \Rightarrow \Sigma]$. *If*

$$[\text{Cut}]\ \frac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2 \\ \mathfrak{N}[\Gamma \Rightarrow \Delta, A] & \mathfrak{M}[A, \Pi \Rightarrow \Sigma]\end{matrix}}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma]}$$

and $\mathfrak{D}_1$ *and* $\mathfrak{D}_2$ *do not contain any other application of the cut-rule, then we can construct a derivation of* $\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma]$ *with no application of the cut-rule.*

Proof. Similarly to the proof of Lemma 7.2 from [34]. By a double induction on the complexity of the cut-formula $\mathfrak{c}(A)$[6] and on the sum of the heights of the derivations of the premises of the cut-rule. The cases are separated according to the last rule applied to the left premise.

Case 1. $\mathfrak{N}[\Gamma \Rightarrow \Delta, A]$ is an axiom. This case is considered in the proof of Lemma 7.2 from [34]: either the conclusion is also an axiom or it can be inferred from $\mathfrak{M}[A, \Pi \Rightarrow \Sigma]$ by internal and external weakening rules.

[6] The definition is standard: $\mathfrak{c}(p) = 1$, $\mathfrak{c}(*A) = \mathfrak{c}(A) + 1$, where $*$ is an unary connective, $\mathfrak{c}(A \star B) = \max(\mathfrak{c}(A), \mathfrak{c}(B)) + 1$, where $\star$ is a binary connective.

Case 2. $\mathfrak{N}[\Gamma \Rightarrow \Delta, A]$ is derived by a rule $\mathfrak{R}$ such that A is not principal. The case is resolved by induction on the sum of the heights of the derivations of the premises of the cut-rule, employing Lemma 5.1.

Subcase 2.1. $\mathfrak{N}[\Gamma \Rightarrow \Delta, A]$ was obtained by the rule $[\Rightarrow \blacksquare_L]$. The following application of cut

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1'\\ \mathfrak{N}[\Gamma \Rightarrow \Delta, A/B \Rightarrow]\end{array}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, A, \blacksquare B]}[\Rightarrow \blacksquare_L] \qquad \begin{array}{c}\mathfrak{D}_2\\ \mathfrak{M}[A, \Pi \Rightarrow \Sigma]\end{array}}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma, \blacksquare B]}[\text{Cut}]$$

is reduced to the subsequent deduction, where we have a lesser sum of the heights of the derivations:

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1'\\ \mathfrak{N}[\Gamma \Rightarrow \Delta, A/B \Rightarrow]\end{array} \qquad \begin{array}{c}\mathfrak{D}_2\\ \mathfrak{M}[A, \Pi \Rightarrow \Sigma]\end{array}}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma/B \Rightarrow]}[\text{Cut}]}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma, \blacksquare B]}[\Rightarrow \blacksquare_L]$$

Subcase 2.2. $\mathfrak{N}[\Gamma \Rightarrow \Delta, A]$ was obtained by the rule $[\Rightarrow \blacksquare_R]$. Similarly to the previous subcase.

Subcase 2.3. $\mathfrak{N}[\Gamma \Rightarrow \Delta, A]$ was obtained by the rule $[\blacksquare \Rightarrow]$. Let $S_1 = \Gamma \Rightarrow \Delta$, $S_2 = \Theta \Rightarrow \Lambda$, $S_3 = \Xi \Rightarrow \Upsilon$, and $S_4 = \Pi \Rightarrow \Sigma$. The following application of cut

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1'\\ \mathfrak{N}[\blacksquare B, S_1, A/(B, S_2/X)]\end{array} \qquad \begin{array}{c}\mathfrak{D}_1''\\ \mathfrak{N}[\blacksquare B, S_1, A/(S_3, B/Y)]\end{array}}{\mathfrak{N}[\blacksquare B, S_1, A/(S_2/X); (S_3/Y)]}[\blacksquare \Rightarrow] \qquad \begin{array}{c}\mathfrak{D}_2\\ \mathfrak{M}[A, S_4]\end{array}}{\mathfrak{N} \otimes \mathfrak{M}[\blacksquare B, \Gamma, \Pi \Rightarrow \Delta, \Sigma/(\Theta \Rightarrow \Lambda/X); (\Xi \Rightarrow \Upsilon/Y)]}$$

is reduced to the subsequent deduction, where we have a lesser sum of the heights of the derivations:

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1'\\ \mathfrak{N}[\blacksquare B, S_1, A/(B, S_2/X)]\end{array} \quad \begin{array}{c}\mathfrak{D}_2\\ \mathfrak{M}[A, S_4]\end{array}}{\mathfrak{N} \otimes \mathfrak{M}[\blacksquare B, \Gamma, \Pi \Rightarrow \Delta, \Sigma/(B, S_2/X)]} \qquad \dfrac{\begin{array}{c}\mathfrak{D}_1''\\ \mathfrak{N}[\blacksquare B, S_1, A/(S_3, B/Y)]\end{array} \quad \begin{array}{c}\mathfrak{D}_2\\ \mathfrak{M}[A, S_4]\end{array}}{\mathfrak{N} \otimes \mathfrak{M}[\blacksquare B, \Gamma, \Pi \Rightarrow \Delta, \Sigma/(S_3, B/Y)]}}{\mathfrak{N} \otimes \mathfrak{M}[\blacksquare B, \Gamma, \Pi \Rightarrow \Delta, \Sigma/(\Theta \Rightarrow \Lambda/X); (\Xi \Rightarrow \Upsilon/Y)]}$$

The subcases produced by other connectives are treated similarly.

Case 3. $\mathfrak{N}[\Gamma \Rightarrow \Delta, A]$ is derived by a rule $\mathfrak{R}$ such that A is principal. The cases where $\mathfrak{R}$ is a propositional rule (or a rule for $\Box$) are covered in [34]. As an example, we consider the case when $\mathfrak{R}$ is $[\Rightarrow \blacksquare_L]$.

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}'_1\\ \mathfrak{N}[\Gamma \Rightarrow \Delta / B \Rightarrow]\end{matrix}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \blacksquare B]}\,[\Rightarrow \blacksquare_L] \qquad \begin{matrix}\mathfrak{D}_2\\ \mathfrak{M}[\blacksquare B, \Pi \Rightarrow \Sigma]\end{matrix}}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma]}\,[\text{Cut}]$$

We need to consider the ways $\mathfrak{M}[\blacksquare B, \Pi \Rightarrow \Sigma]$ could be derived. If it is an axiom, then we go to the case 1. If $\blacksquare B$ is not the principal formula in $\mathfrak{D}_2$, then we go to the case 2. The last option is as follows: $\mathfrak{M}[\blacksquare B, \Pi \Rightarrow \Sigma]$ was obtained by $[\blacksquare \Rightarrow]$. Then the following deduction holds, where $S_1 = \Gamma \Rightarrow \Delta$, $S_2 = \Theta \Rightarrow \Lambda$, $S_3 = \Xi \Rightarrow \Upsilon$ and $S_4 = \Pi \Rightarrow \Sigma$:

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}'_1\\ \mathfrak{N}[S_1 / B \Rightarrow]\end{matrix}}{\mathfrak{N}[S_1, \blacksquare B]}\,[\Rightarrow \blacksquare_L] \qquad \dfrac{\begin{matrix}\mathfrak{D}'_2\\ \mathfrak{M}[\blacksquare B, S_4/(B, S_2/X)]\end{matrix} \qquad \begin{matrix}\mathfrak{D}''_2\\ \mathfrak{M}[\blacksquare B, S_4/(S_3, B/Y)]\end{matrix}}{\mathfrak{M}[\blacksquare B, S_4/(S_2/X);(S_3/Y)]}}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma/(\Theta \Rightarrow \Lambda/X);(\Xi \Rightarrow \Upsilon/Y)]}$$

We perform the transformation as follows.

$$\dfrac{\dfrac{[\text{Cut}]\ \dfrac{[\text{Cut}]\ \dfrac{[\Rightarrow \blacksquare_L]\ \dfrac{\begin{matrix}\mathfrak{D}'_1\\ \mathfrak{N}[S_1 / B \Rightarrow]\end{matrix}}{\mathfrak{N}[S_1, \blacksquare B]} \qquad \begin{matrix}\mathfrak{D}''_2\\ \mathfrak{M}[\blacksquare B, S_4/(S_3, B/Y)]\end{matrix}}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi, \Rightarrow \Delta, \Sigma/(\Xi \Rightarrow \Upsilon, B/Y)]} \qquad \begin{matrix}\mathfrak{D}'_1\\ \mathfrak{N}[S_1 / B \Rightarrow]\end{matrix}}{\mathfrak{N} \otimes \mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Gamma, \Pi, \Rightarrow \Delta, \Delta, \Sigma/(\Xi \Rightarrow \Upsilon/Y)]}}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi, \Rightarrow \Delta, \Sigma/(\Xi \Rightarrow \Upsilon/Y)]}}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi, \Rightarrow \Delta, \Sigma/(\Theta \Rightarrow \Lambda/X);(\Xi \Rightarrow \Upsilon/Y)]}\,[\text{EW}]$$

Double lines indicate multiple applications of the merge rule. (One might refer to contraction as well, but since we have modified the notion of a sequent to use sets of formulas instead of multisets, contraction is no longer required.) The initial application of [Cut] is eliminable by the induction hypothesis on the sum of the heights of the derivations of the premises. The subsequent application of [Cut] is eliminable by the induction hypothesis on the complexity of the cut formula. ⊣

By induction on the number of cuts, using Theorem 5.1, we get:

THEOREM 5.2 (Constructive cut admissibility). *All derivations* $\mathfrak{D}$ *in* $\mathbb{NSK}^{\circ}$ *can be effectively transformed into derivations, where there is no application of the rule of cut.*

As for the extensions of $\mathbb{N}\$\mathbf{K}^{\circledcirc}$, it seems to be rather problematic to prove the cut admissibility theorem constructively for them. For example, in $\mathbb{N}\$\mathbf{KT}^{\circledcirc}$, Case 3 has an additional option: $\mathfrak{M}[\blacksquare B, \Pi \Rightarrow \Sigma]$ was obtained by $[\widetilde{\mathbf{T}}]$.

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}'_1\\ \mathfrak{N}[\Gamma \Rightarrow \Delta / B \Rightarrow]\end{array}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \blacksquare B]}\,[\Rightarrow \blacksquare_L] \qquad \dfrac{\begin{array}{c}\mathfrak{D}'_2\\ \mathfrak{M}[\Pi' \Rightarrow \Sigma' / \blacksquare B, \Pi'' \Rightarrow \Sigma'']\end{array}}{\mathfrak{M}[\blacksquare B, \Pi \Rightarrow \Sigma]}\,[\widetilde{\mathbf{T}}]}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma]}\,[\mathrm{Cut}]$$

The transformation of the deduction could be as follows, where the cut could be eliminated based on the induction hypothesis concerning the sum of the heights of the derivation of the premises of the cut rules:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}'_1\\ \mathfrak{N}[\Gamma \Rightarrow \Delta / B \Rightarrow]\end{array}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \blacksquare B]}\,[\Rightarrow \blacksquare_L] \qquad \begin{array}{c}\mathfrak{D}'_2\\ \mathfrak{M}[\Pi' \Rightarrow \Sigma' / \blacksquare B, \Pi'' \Rightarrow \Sigma'']\end{array}}{\mathfrak{N} \otimes \mathfrak{M}[\Pi' \Rightarrow \Sigma' / \Gamma, \Pi'' \Rightarrow \Sigma'', \Delta]}\,[\mathrm{Cut}]}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma]}\,[\widetilde{\mathbf{T}}]$$

The problem here is that the application of the rule of cut here is not correct. As Poggiolesi notes [34, p. 125], as follows from the definition of the rule of cut, “given two tree-hypersequents, we can cut on any two sequents belonging to them provided that they are in equivalent position.” How to eliminate this cut application remains unclear. Similar problems arise with the other special structural rules. A potential solution could be to develop special logical rules for the modalities in question; then the cut could be eliminated, as in the cases covered by Poggiolesi. With the number of modalities we deal with, this solution seems impractical; too many rules are needed.

6. Conclusion

This paper introduces nested sequent calculi inspired by Poggiolesi [34] for modal logics **K**, **D**, **T**, **K4**, **KB**, **D4**, **S4**, **KB4**, **DB**, and **B** formulated not through necessity or possibility operators, but utilizing various non-standard modalities: non-contingency, contingency, essence, accident, impossibility, and unnecessity. Unfortunately, we have managed to prove the cut admissibility theorem for **K**-based logics only, utilizing two approaches: semantically, as a consequence of Hintikka-style completeness proof, and syntactically, in a constructive manner. It is not clear if the cut admissibility theorem applies to the rest of logics. Moreover, the situation remains ambiguous even with standard modalities, if we consider nested sequent calculi with special structural rules, but without special logical rules. Poggiolesi references the work [5] and states that "tree-hypersequent calculi composed by generalised initial tree-hypersequents, propositional rules, modal rules, special structural rules and contraction rules are sound and complete with respect to their corresponding Hilbert systems. Moreover they are cut-free and modular" [34, p. 140]. However, as follows from [24, p. 6], there is a mistake in the cut elimination proof of [5]. The paper [24] further examines nested sequent calculi with both special logical and structural rules, demonstrating that they are cut-free. Poggiolesi [34] provides semantic and syntactic proofs of cut admissibility for nested sequent calculi with both special logical rules and height-preserving admissible special structural rules. Consequently, it is uncertain, if a nested sequent calculus with special structural rules only possesses cut admissibility. This subject necessitates additional investigation.

Euclidean logics are absent in our current investigation. In our prior publication [32], we developed cut-free hypersequent calculi for **S5** formulated in the language incorporating those modalities. We assign the creation of cut-free Gentzen calculi for the remaining Euclidean logics from the modal cube, namely **K5**, **D5**, **K45**, **D45**, as a target for future research. One may also contemplate the exploration of alternative modal logics beyond the modal cube. Zolin presented non-cut-free sequent calculi for $\mathbf{GL}^{\blacksquare}$ [46] and $\mathbf{Grz}^{\blacksquare}$ [45]: one might attempt to develop a cut-free calculus for these systems and their variants incorporating other non-standard modalities.

Another subject for further investigation is Craig interpolation property. Some results are already known from the literature: e.g., Zolin showed [45] that $\mathbf{T}^{\blacksquare}$, $\mathbf{S4}^{\blacksquare}$, $\mathbf{D}^{\blacksquare}$, and $\mathbf{S5}^{\blacksquare}$ have Craig interpolation property. Nevertheless, it seems that we still lack the full picture for all the logics in question. One more topic for future research is the consideration of quantifiers for the calculi in question (for the case of nested sequent calculi for the logic $\mathbf{K}^{\square}$ and its extensions, this issue has recently been addressed by Lyon and Orlandelli [21]).

Finally, one may search for other non-standard modalities. For example, let us consider Boolos' [3] 'boxdot' modality introduced in the context of provability logic: $\boxdot A = \Box A \wedge A$ ('provable and true'[7]; for its application in context of essence and accident logics see [40]). The semantic condition for $\boxdot A$ is as follows:

- $\vartheta(\boxdot A, x) = 1$ iff $\vartheta(A, x) = 1$ and $\forall_{y \in R[x]}\, \vartheta(A, y) = 1$.

The appropriate nested sequent rules for $\boxdot$ are given below:

$$[\boxdot \Rightarrow_1]\ \frac{\mathfrak{N}[A, \varGamma \Rightarrow \varDelta]}{\mathfrak{N}[\boxdot A, \varGamma \Rightarrow \varDelta]} \qquad [\boxdot \Rightarrow_2]\ \frac{\mathfrak{N}[\boxdot A, \varGamma \Rightarrow \varDelta/(A, \varTheta \Rightarrow \varLambda/X)]}{\mathfrak{N}[\boxdot A, \varGamma \Rightarrow \varDelta/(\varTheta \Rightarrow \varLambda/X)]}$$

$$[\Rightarrow \boxdot]\ \frac{\mathfrak{N}[\varGamma \Rightarrow \varDelta, A] \quad \mathfrak{N}[\varGamma \Rightarrow \varDelta/ \Rightarrow A]}{\mathfrak{N}[\varGamma \Rightarrow \varDelta, \boxdot A]}$$

One may investigate other modal operators of a similar nature. For example, one might define a 'diamonddot' operator as follows: $\dot{\Diamond} A = \neg\boxdot\neg A = \Diamond A \vee A$. It might be interpreted as 'non-contradictory or true'. The semantic condition for it is as follows:

- $\vartheta(\dot{\Diamond} A, x) = 1$ iff $\vartheta(A, x) = 1$ or $\exists_{y \in R[x]}\, \vartheta(A, y) = 1$.

The appropriate rules are as follows:

$$[\dot{\Diamond} \Rightarrow]\ \frac{\mathfrak{N}[A, \varGamma \Rightarrow \varDelta] \quad \mathfrak{N}[\varGamma \Rightarrow \varDelta/A \Rightarrow]}{\mathfrak{N}[\dot{\Diamond} A, \varGamma \Rightarrow \varDelta]}$$

$$[\Rightarrow \dot{\Diamond}_1]\ \frac{\mathfrak{N}[\varGamma \Rightarrow \varDelta, A]}{\mathfrak{N}[\varGamma \Rightarrow \varDelta, \dot{\Diamond} A]} \qquad [\Rightarrow \dot{\Diamond}_2]\ \frac{\mathfrak{N}[\varGamma \Rightarrow \varDelta, \dot{\Diamond} A/(\varTheta \Rightarrow \varLambda, A/X)]}{\mathfrak{N}[\varGamma \Rightarrow \varDelta, \dot{\Diamond} A/(\varTheta \Rightarrow \varLambda/X)]}$$

One might continue this analogy and introduce the negated counterparts of 'boxdot' and 'diamonddot' operators: $\widetilde{\boxdot} A = \neg\Box A \wedge \neg A$ ('non-provable and false') and $\widetilde{\dot{\Diamond}} A = \neg\Diamond A \vee \neg A$ ('contradictory or false'). The semantic conditions are as follows:

- $\vartheta(\widetilde{\boxdot} A, x) = 1$ iff $\vartheta(A, x) = 0$ and $\exists_{y \in R[x]}\, \vartheta(A, y) = 0$,
- $\vartheta(\widetilde{\dot{\Diamond}} A, x) = 1$ iff $\vartheta(A, x) = 0$ or $\forall_{y \in R[x]}\, \vartheta(A, y) = 0$.

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[7] This interpretation might remind the reader Grzegorczyk's modal logic **Grz**, "which can be characterized by reflexive partially ordered Kripke frames without infinite ascending chains. This logic is complete w.r.t. the arithmetical semantics, where the modal connective $\Box$ corresponds to the strong provability operator "*... is true and provable in Peano arithmetic*". There is a translation from **Grz** into the Gödel-Löb provability logic **GL** such that **Grz** $\vdash A \Longleftrightarrow$ **GL** $\vdash A^*$, where A^* is obtained from A by replacing all subformulas of the form $\Box B$ by $B \wedge \Box B$" [37, p. 23, boldface is ours]. Since we are studying the $\boxdot$-modality not in the context of **GL** (in contrast to Boolos [3]), it means that, in our case, it is not a **Grz**-modality and its interpretation as "is true and provable" should not be straightforwardly associated with Peano arithmetics.

The appropriate rules are given below:

$$[\widetilde{\boxdot}\Rightarrow_1]\ \frac{\mathfrak{N}[\Gamma\Rightarrow\Delta,A]}{\mathfrak{N}[\widetilde{\boxdot}A,\Gamma\Rightarrow\Delta]}\qquad[\widetilde{\boxdot}\Rightarrow_2]\ \frac{\mathfrak{N}[\Gamma\Rightarrow\Delta/\Rightarrow A]}{\mathfrak{N}[\widetilde{\boxdot}A,\Gamma\Rightarrow\Delta]}$$

$$[\Rightarrow\widetilde{\boxdot}]\ \frac{\mathfrak{N}[A,\Gamma\Rightarrow\Delta]\quad\mathfrak{N}[\Gamma\Rightarrow\Delta,\widetilde{\boxdot}A/(A,\Theta\Rightarrow\Lambda/X)]}{\mathfrak{N}[\Gamma\Rightarrow\Delta,\widetilde{\boxdot}A/(\Theta\Rightarrow\Lambda/X)]}$$

$$[\widetilde{\diamonddot}\Rightarrow]\ \frac{\mathfrak{N}[\Gamma\Rightarrow\Delta,A]\quad\mathfrak{N}[\widetilde{\diamonddot}A,\Gamma\Rightarrow\Delta/(\Theta\Rightarrow\Lambda,A/X)]}{\mathfrak{N}[\widetilde{\diamonddot}A,\Gamma\Rightarrow\Delta/(\Theta\Rightarrow\Lambda/X)]}$$

$$[\Rightarrow\widetilde{\diamonddot}_1]\ \frac{\mathfrak{N}[A,\Gamma\Rightarrow\Delta]}{\mathfrak{N}[\Gamma\Rightarrow\Delta,\widetilde{\diamonddot}A]}\qquad[\Rightarrow\widetilde{\diamonddot}_2]\ \frac{\mathfrak{N}[\Gamma\Rightarrow\Delta/A\Rightarrow]}{\mathfrak{N}[\Gamma\Rightarrow\Delta,\widetilde{\diamonddot}A]}$$

THEOREM 6.1. *Let* $\circledcirc\in\{\boxdot,\widetilde{\boxdot},\diamonddot,\widetilde{\diamonddot}\}$ *and* $\mathbf{X}\subseteq\{\mathbf{T},\mathbf{D},\mathbf{4},\mathbf{B}\}$. *Then for any nested sequent* $\mathfrak{N}$:

1. $\mathbf{K}^{\circledcirc}\models\mathfrak{N}$ *iff* $\mathbb{N}\$\mathbf{K}^{\circledcirc}\vdash\mathfrak{N}$.
2. *If* $\mathbb{N}\$\mathbf{K}^{\circledcirc}\vdash\mathfrak{N}$, *then there is a cut-free proof of* $\mathfrak{N}$ *in* $\mathbb{N}\$\mathbf{K}^{\circledcirc}$.
3. $\mathbf{KX}^{\circledcirc}\models\mathfrak{N}$ *iff* $\mathbb{N}\$\mathbf{KX}^{\circledcirc}\vdash\mathfrak{N}$.

PROOF. 1. Similarly to Theorems 4.1 and 4.3. 2. Similarly to Theorems 4.4 and 5.2. 3. Similarly to Theorems 4.2 and 4.5. ⊣

We suppose that other similar non-standard modalities can be proposed, motivated, and explored in the future investigations.

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