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Simplified Semantics for Further Relevant Logics II: Propositional Constants

Abstract. It is shown how to model propositional constants within the simplified Routley-Meyer semantics. Various axioms and rules allowing the definition of modal operators, implicative negations, enthymematical conditionals, and propositions expressing various infinite conjunctions and disjunctions are set forth and shown to correspond to specific frame conditions. Two propositional constants which are both often designated as "the Ackermann constant" are shown to capture two such "infinite" propositions: The conjunction of every logical law and the conjunction of every truth — what Anderson and Belnap called the "world" constant.

Keywords: relevant logics; simplified Routley-Meyer semantics; propositional constants

1. Introduction

Anderson and Belnap initiated the research program of relevant/relevance logics in the late 1950s. Two of its initial sources of inspiration were (Ackermann, 1956) and (Church, 1951), both of which cite the Norwegian logician Ingebrigt Johansson in introducing a propositional constant so as to define negation (Church) and modalities (Ackermann). Church defined $\neg A$ as $A \to f$, with no extra logical principles governing f. This makes for an intuitionistic-type negation in logics like $\mathbf{R}$ with $\neg A$ equivalent to the standard De Morgan negation $\sim$ of $\mathbf{R}$ if $((A \to f) \to f) \to A$ is added.[1] Defining negation that way would, however, fail to capture

[1] See (Meyer, 1966, pp. 179f) and (Slaney, 1989) for two examples in which negation is introduced this way.

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the intended negation operator of most relevant logics: $\sim A$ is simply not equivalent to any $\to$-formula in many such logics. This paper shows the precise conditions under which the standard De Morgan negation is implicational — the conditions, semantically speaking, a propositional constant $\mathbf{f}$ needs to satisfy for $\sim A \leftrightarrow (A \to \mathbf{f})$ to hold.

Church introduced f with the intended meaning of "denoting falsehood." In contrast to the propositional constant $\perp$, which, then, misappropriately is often called a *Church constant*, Church's constant fails to be *trivially* false even in $\mathbf{R}$, in the sense that $f \to A$ fails to be a logical theorem for some A, even though it is false in the sense that $f \to f$ is a logical theorem. In this paper it will be shown how to model such a triviality constant within the so-called *simplified* Routley-Meyer semantics. This type of semantics was first set out in (Priest and Sylvan, 1992) for the weak relevant logics **BM** and **B** and then extended so as to cover a range of relevant logics (as well as some related non-relevant ones) in (Restall, 1993), and so as to cover the characteristic axioms of Anderson and Belnap's favorite logic **E** and the rule of Ackermann's logic $\mathbf{\Pi}'$, as well as intensional conjunction (fusion) and the converse conditional in (Øgaard, 2024).

One of Anderson and Belnap's first formal results was to show that Ackermann's propositional constant is in fact contextually definable in the sense that if A is a logical theorem of Ackermann's constant-enhanced logic $\mathbf{\Pi}''$, then the formula A' obtained by replacing his propositional constant '⅄' with $\sim\bigwedge_{i\in I_A}(p_i \to p_i)$, where I_A is the index-set for the set of propositional variables occurring in A, is a logical theorem of the constant-free original logic $\mathbf{\Pi}'$ (cf. Anderson and Belnap, 1959). Although, then, one may not *need* such constants for the applications Church and Ackermann envisaged, many relevant logicians have found reasons for retaining them. One early such view was voiced by Meyer who defended such a constant in the name of elegance: "My view is that the sentential constant is well-motivated and ought to be introduced in the name of elegance; theirs (read Anderson and Belnap) seems to be that it is superfluous and ought to be thrown out in the name of Ockham" (Meyer, 1970, fn. 4). Other logicians have found use for propositional constants in applications such as naïve set theory, or to investigate whether relevance really does force paraconsistency.[2] This paper is writ-

[2] See for instance (Routley, 1980, appx.), (Priest, 2006, ch. 18), and (Weber, 2010) for examples of the former, and (Øgaard, 2021b) for the latter.

ten under the conviction that such constants are both technically useful and philosophically illuminating.

There are four lines of inquiry which have been pursued in this paper: First of all the Johansson-line: propositional constants may be used to define implicational negation operators. Secondly, as Ackermann realized, propositional constants may be used to define modal operators. Thirdly, any such constant $\mathbf{c}$ may be used to define *enthymematical conditionals* on the form $A \overset{c}{\mapsto} B := A \wedge \mathbf{c} \to B$.[3] Lastly, propositional constants can be used to express operations on the totality of propositions. Although they preferred $\mathbf{E}$ without propositional constants, Anderson and Belnap (1975, § 27.1.2) suggested that $\mathbf{E}$ could be augmented with no less than six such with the following intuitive interpretations:

- $\boldsymbol{t}$: conjunction of all logical truths,
- $\boldsymbol{f}$: disjunction of all logical falsehoods,
- $\boldsymbol{w}$: conjunction of all truths (“the world”),
- $\boldsymbol{w}'$: disjunction of all falsehoods,
- $\boldsymbol{T}$: disjunction of all propositions,
- $\boldsymbol{F}$: conjunction of all propositions.

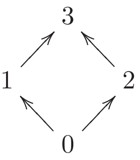
Anderson and Belnap’s suggested axiomatization of the latter two constants use one axiom for each — $A \to \boldsymbol{T}$ and $\boldsymbol{F} \to A$. This can be shown to be conservative,[4] but within the Routley-Meyer semantics this is insufficient since it fails to yield $A \to (\boldsymbol{T} \to \boldsymbol{T})$ and $A \to (\boldsymbol{F} \to \boldsymbol{F})$ as logical theorems even when added to $\mathbf{\Pi}'$ (a counter-model is displayed in figure 1). One can simply add these axioms, but to stay in keep with the “one axiom, one frame condition” policy, I’ve rather chosen the equivalent $A_1 \to (\ldots \to (A_n \to (B \to \boldsymbol{T}))\ldots)$ and $A_1 \to (\ldots \to (A_n \to (\boldsymbol{F} \to B))\ldots)$, where $n \geqslant 0$.

The success criteria for an axiomatization of any of the other propositions, however, is far more contentious. Anderson and Belnap’s axiomatization of $\boldsymbol{t}$ can be argued to capture the conjunction of all logical truths when added to $\mathbf{E}$. The heart of the justification for reading $\boldsymbol{t}$ as expressing

[3] See (Øgaard, 2021b) for a discussion of such in the context of Anderson and Belnap’s philosophy of entailment. There are other enthymematic-type conditionals such as the modal-type enthymematical conditional $(\mathbf{c} \to A) \to B$. Investigating such is regrettably beyond the scope of this paper.

[4] The standard proof is algebraic: any algebra fit for $\mathbf{E}$ can be “completed” so as to include a top and bottom element. See (Restall, 2000, chs. 8–9) for more on the algebraic semantics for relevant logics.

$\mathcal{T} = \{x \mid 1 \preceq x\} = \{1, 3\}$
$[\![\boldsymbol{T}]\!] = 3 \qquad [\![\boldsymbol{F}]\!] = 0$
$3 \to (3 \to 3) = 3 \to (0 \to 0) = 0$



$\to$	0	1	2	3	$\sim$
0	1	1	1	1	3
1	0	1	0	1	2
2	0	0	1	1	1
3	0	0	0	1	0

Figure 1. A $\mathbf{\Pi}'$-model

this conjunction is that the necessitation-like claim holds: A is a logical theorem of **E** with $\boldsymbol{t}$ just in case $\boldsymbol{t} \to A$ is. To ensure such a reading in other logics, it is in some cases necessary to ensure that the "necessitated" version of any of the rules of the logic are derivable. Finding the correct frame condition for the necessitated version of disjunctive syllogism in particular, have proven difficult, and so the path chosen in this paper is to rather focus on the enthymematical-axiomatic versions of the primitive rules. Not all such strengthenings are conservative, however. It is, therefore, far from evident that a propositional constant expressing the conjunction of all logical theorems can be added to all logics.

The axiomatization argued to be correct in the case of the world constant has long been known to yield a conservative extension when added to relevant logics. When added to **E** augmented by $\boldsymbol{t}$, however, it becomes impossible to uphold the reading of $\boldsymbol{t}$ as expressing the conjunction of every logical theorem. It is suggested, however, that the more fitting intuitive reading of it is as expressing the conjunction of every logical *law*—logical theorems which are $\to$-formulas. Although Anderson and Belnap's theorem-criteria for $\boldsymbol{w}$ can be met, they asked for an axiomatization of it which does not make use of additional primitive rules. In short, they asked of for a purely axiomatic world-constant. It is shown that such an axiomatic constant satisfying all of Anderson and Belnap's desiderata, can indeed be added to **E**. However, the question as to whether this can be done *conservatively* is regrettably left open, leaving it an open question whether a purely axiomatic world can truly be added to **E**.

It was shown in (Øgaard, 2021b) that if **E** and **R** are augmented with Anderson and Belnap's $\mathbf{t}$, but augmented by the "explosive" axiom $\mathbf{t} \wedge {\sim}\mathbf{t} \to A$, then the resultant logics, **Æ** and **M**, retain the selection criteria for a logic of entailment (**E**&**Æ**) and relevant implication (**R**&**M**). Along with Anderson and Belnap's axioms for $\mathbf{t}$, it will be shown how to model this explosive axiom within the simplified semantics. For better or worse,

Æ turned out to be a non-conservative extension of **E**. The question whether **E** augmented by $\mathbf{w} \wedge \sim \mathbf{w} \rightarrow A$ makes for a conservative extension, however, is yet another question raised in this paper, but left unanswered.

Only a selection of possible postulates for implicative negations, modal operators, and enthymematical implications will be considered, but sufficiently many, it is hoped, so as to convey the expressive richness afforded by propositional constants.[5] The method used will be general so as to allow for any number of propositional constants. A list of possible axioms and rules for any such constant will be given and each such will be shown to correspond to a particular frame condition of the simplified semantics.

This paper is a continuation of (Øgaard, 2024) and so definitions and results from that paper will in general not be repeated herein. The plan for this paper is as follows: The next section will give some initial definitions and state a result regarding the notion of a *regular* modal logic intended to motivate the study of enthymematically strenthened rules. Section 3 will then show how to interpret propositional constants within the simplified semantics. It will also be shown how to define accessibility relations for any definable modal and negation operator and how the truth conditions for these relate to these relations. The frame conditions for axioms and rules involving propositional constants will then be presented. The task of section 4 and section 5 is to show that these frame conditions suffice for a strong soundness and completeness result. Section 6 states some results regarding conservative extensions before section 7 discusses Anderson and Belnap's notion of a world constant. Lastly, section 8 gives a brief summary.

2. Initial definitions

Defined connectives:

$$\begin{aligned} A \leftrightarrow B &:= (A \rightarrow B) \wedge (B \rightarrow A) \\ A \supset B &:= \sim A \vee B \\ A \equiv B &:= (A \supset B) \wedge (B \supset A) \\ \Box A &:= (A \rightarrow A) \rightarrow A \\ \overset{B}{\blacksquare} A &:= B \rightarrow A \end{aligned}$$

[5] I refer the interested reader to look up (Restall, 2000) and (Standefer, n.d.) for a more extensive list of modal and negation principles.

$$\overset{B}{\blacklozenge} A := {\sim}\overset{B}{\blacksquare}{\sim}A$$

$$A \overset{C}{\hookrightarrow} B := A \wedge C \to B$$

$$\overset{B}{\neg} A := A \to B$$

A logic may have any finite number of propositional constants. I will use '$\mathbf{t}$' and '$\mathbf{w}$' when the above mentioned meanings of Anderson and Belnap (or something closely related) are intended, but will otherwise use the more generic '$\mathbf{c}$' or '$\mathbf{c}_i$' for $i \in \omega$. The language of a logic is to be identified as its set of connectives, which, then, will be those explicitly occurring in its axioms and rules.[6] As in (Øgaard, 2024), the logics that will be studied in this paper are the disjunctive logics which extend $\mathbf{B}^{\mathrm{d}}$ and are obtainable from the axioms and rules (along with any disjunctive version of such rules) found in (Øgaard, 2024), or those occurring in table 1. Since a logic may be equipped with any number of propositional constants, the intent, then, is that the axiomatization of any such is obtained by selecting any number of axioms and rules from the list in table 1.

The $\mathbf{c}$-principles R$\mathbf{c}$, A$\mathbf{c}$1, $\mathrm{N}_1\overset{\mathbf{c}}{\blacksquare}$, $\mathrm{N}_2\overset{\mathbf{c}}{\blacksquare}$, $\mathrm{T}\overset{\mathbf{c}}{\blacksquare}$, and A$\mathbf{c}$3 as well as the enthymematical strengthenings of the rules R2 and R4 — that is $\mathrm{E}^{\mathrm{R2}}_{\mathbf{c}}$ and $\mathrm{E}^{\mathrm{R4}}_{\mathbf{c}}$ — were considered in (Routley et al., 1982, ch. 5.1), in which also frame postulates for these principles were set forth. Anderson and Belnap's axiomatization of $\boldsymbol{t}$ uses $\mathrm{N}_1\overset{\mathbf{c}}{\blacksquare}$ and $\mathbf{c}\mathrm{T}\overset{\mathbf{c}}{\blacksquare}$, the latter, then, suffices for deriving the generally stronger principle $\mathrm{T}\overset{\mathbf{c}}{\blacksquare}$ within **E**. A$\mathbf{c}$2, along with $\mathrm{N}_1\overset{\mathbf{c}}{\blacksquare}$ and $\mathrm{T}\overset{\mathbf{c}}{\blacksquare}$, were added to **E** and **R** in (Øgaard, 2021b) yielding the logics **Æ** and **M**.

Some might be tempted to view R$\mathbf{c}$ as a "necessity rule." I should stress, however, that if such a rule is derivable for a propositional constant $\mathbf{c}$, then $\overset{\mathbf{c}}{\blacksquare}$ can hardly be read as expressing anything reminiscent of necessity seeing as any assumption would be deemed necessarily true. Nor is it in fact evident that such a necessity rule ought to be even admissible in the current context. Anderson and Belnap's "world"-constant is a case in point: it should, according to their criteria, be true, but not necessarily so. Thus $\boldsymbol{w}$ should be a logical theorem, but $\overset{\boldsymbol{t}}{\blacksquare}\boldsymbol{w}$, that is $\boldsymbol{t} \to \boldsymbol{w}$, should not.

[6] The Church-type case wherein no additional axioms/rules are set forth for a propositional constant $\mathbf{c}$ can be handled using this convention by simply adding, say, $\mathbf{c} \to \mathbf{c}$ as an axiom of the logic.

(R$\mathbf{c}$)	$\{A\} \Vdash \mathbf{c} \to A$	
(A$\mathbf{c}$1)	$\mathbf{c}$	
(A$\mathbf{c}$2)	$\mathbf{c} \wedge {\sim}\mathbf{c} \to A$	
(A$\mathbf{c}$3)	$A \to (\mathbf{c} \to A)$	
(A$\mathbf{c}$4)	$A_1 \to (\ldots \to (A_n \to (B \to \mathbf{c}))\ldots)$	$(0 \leqslant n)$
(A$\mathbf{c}$5)	$A_1 \to (\ldots \to (A_n \to (\mathbf{c} \to B))\ldots)$	$(0 \leqslant n)$
($\overset{\mathbf{c}}{\neg}$1)	$A \to \overset{\mathbf{c}}{\neg}\overset{\mathbf{c}}{\neg}A$	
($\overset{\mathbf{c}}{\neg}$2)	$\overset{\mathbf{c}}{\neg}\overset{\mathbf{c}}{\neg}A \to A$	
($\overset{\mathbf{c}}{\neg}$3)	$\overset{\mathbf{c}}{\neg}A \to {\sim}A$	
($\overset{\mathbf{c}}{\neg}$4)	${\sim}A \to \overset{\mathbf{c}}{\neg}A$	
($\overset{\mathbf{c}}{\neg}$5)	$A \wedge \overset{\mathbf{c}}{\neg}A \to B$	
($\overset{\mathbf{c}}{\neg}$6)	$B \to A \vee \overset{\mathbf{c}}{\neg}A$	
($N_1\overset{\mathbf{c}}{\blacksquare}$)	$\overset{\mathbf{c}}{\blacksquare}(A \to A)$	
($N_2\overset{\mathbf{c}}{\blacksquare}$)	$\overset{\mathbf{c}}{\blacksquare}(A \vee {\sim}A)$	
($\mathbf{c}$T$\overset{\mathbf{c}}{\blacksquare}$)	$\overset{\mathbf{c}}{\blacksquare}\mathbf{c} \to \mathbf{c}$	
(T$\overset{\mathbf{c}}{\blacksquare}$)	$\overset{\mathbf{c}}{\blacksquare}A \to A$	
(4$\overset{\mathbf{c}}{\blacksquare}$)	$\overset{\mathbf{c}}{\blacksquare}A \to \overset{\mathbf{c}}{\blacksquare}\overset{\mathbf{c}}{\blacksquare}A$	
($E_{\mathbf{c}}^{R2}$)	$A \wedge (A \to B) \overset{\mathbf{c}}{\mapsto} B$	
($E_{\mathbf{c}}^{R3}$)	$(A \to B) \overset{\mathbf{c}}{\mapsto} ((C \to A) \to (C \to B))$	
($E_{\mathbf{c}}^{R4}$)	$(A \to B) \overset{\mathbf{c}}{\mapsto} ((B \to C) \to (A \to C))$	
($E_{\mathbf{c}}^{R5}$)	$(A \to B) \overset{\mathbf{c}}{\mapsto} ({\sim}B \to {\sim}A)$	
($E_{\mathbf{c}}^{R6}$)	$A \wedge ({\sim}A \vee B) \overset{\mathbf{c}}{\mapsto} B$	
($E_{\mathbf{c}}^{R7}$)	$A \overset{\mathbf{c}}{\mapsto} ((A \to B) \to B)$	
($E_{\mathbf{c}}^{R\mathbf{c}}$)	$A \overset{\mathbf{c}}{\mapsto} (\mathbf{c} \to A)$	

Table 1. List of possible axioms and rules for any propositional constant $\mathbf{c}$

The enthymematical axioms $E_{\mathbf{c}}^{Ri}$, for $i \leqslant 7$, and $E_{\mathbf{c}}^{R\mathbf{c}}$ are interesting as they reflect in axiomatic form the primitive rules considered in this paper. Note, then, that adjunction is trivially reflected since $A \wedge B \overset{\mathbf{c}}{\mapsto} A \wedge B$ is an instance of the axiom A3. There are two interesting features of logics which validate their primitive rules in form of enthymemes:

THEOREM 2.1. *Let* $\boldsymbol{L}$ *be a logic extending* $\mathbf{B}^{\mathrm{d}}$ *using any axioms and rules displayed above such that*

- $\varnothing \vdash_{\boldsymbol{L}} \mathbf{c}$, *and*
- $E_{\mathbf{c}}^{Ri}$ *is a logical theorem for every of its primitive rules.*

Then $\boldsymbol{L}$ *is disjunctive.*

PROOF. It is evident that $\boldsymbol{L}$ can be reaxiomatized using only adjunction and modus ponens. The claim follows from (Øgaard, 2024, lem. 2.1). ⊣

DEFN 2.1. A logic is called REGULAR relative to a modal operator $\overset{c}{\blacksquare}$, just in case for every set of formulas $\Delta \cup \{B\}$,

$$\Delta \vdash_{\boldsymbol{L}} B \Longrightarrow \overset{c}{\blacksquare}\Delta \vdash_{\boldsymbol{L}} \overset{c}{\blacksquare}B,$$

where $\overset{c}{\blacksquare}\Delta := \{\overset{c}{\blacksquare}A \mid A \in \Delta\}$.

THEOREM 2.2. *Let* $\boldsymbol{L}$ *be any logic extending* $\mathbf{B}^{\mathrm{d}}$ *using any axioms and rules displayed above such that*

- $\varnothing \vdash_{\boldsymbol{L}} A \Rightarrow \varnothing \vdash_{\boldsymbol{L}} \overset{c}{\blacksquare}A$, *and*
- $\mathrm{E}_{\mathbf{c}}^{\mathrm{R}i}$ *is a logical theorem for every of its primitive rules.*

Then $\boldsymbol{L}$ *is regular.*

PROOF. Assume that $\Delta \vdash_{\boldsymbol{L}} B$. Then for some formula $\delta = \delta_1 \wedge \ldots \wedge \delta_n$, where $\delta_{i \leqslant n} \in \Delta$, $\{\delta\} \vdash_{\boldsymbol{L}} B$. Since $\overset{c}{\blacksquare}\Delta \vdash_{\boldsymbol{L}} \overset{c}{\blacksquare}\delta$ (left for the reader to verify), it suffices to show that $\{\overset{c}{\blacksquare}\delta\} \vdash_{\boldsymbol{L}} \overset{c}{\blacksquare}B$

Let $A_1, \ldots, A_m = B$ be a derivation of B from δ. The rest of the proof is a simple inductive proof to the effect that for $i \leqslant m$, $\{\overset{c}{\blacksquare}\delta\} \vdash_{\boldsymbol{L}} \overset{c}{\blacksquare}A_i$. If A_i is the formula δ, the claim is trivial. If, on the other hand, A_i is a logical axiom, then by assumption $\varnothing \vdash_{\boldsymbol{L}} \overset{c}{\blacksquare}A_i$. Assume that A_i was obtained by some rule from some finite subset $\Theta \subseteq \{A_j \mid j < i\}$. We may assume for inductive hypothesis that $\{\overset{c}{\blacksquare}\delta\} \vdash_{\boldsymbol{L}} \overset{c}{\blacksquare}A_j$ for every $j < i$. Since all of the primitive rules are reflected as enthymematical theorems, we have that $\varnothing \vdash_{\boldsymbol{L}} \bigwedge\Theta \overset{c}{\mapsto} A_i$, that is $\varnothing \vdash_{\boldsymbol{L}} \bigwedge\Theta \wedge \mathbf{c} \rightarrow A_i$. By the prefixing rule (R3) it follows that $\varnothing \vdash_{\boldsymbol{L}} \overset{c}{\blacksquare}(\bigwedge\Theta \wedge \mathbf{c}) \rightarrow \overset{c}{\blacksquare}A_i$. Since $\overset{c}{\blacksquare}\Theta \vdash_{\boldsymbol{L}} \overset{c}{\blacksquare}(\bigwedge\Theta \wedge \mathbf{c})$ (left for the reader to verify) it therefore follows that $\{\overset{c}{\blacksquare}\delta\} \vdash_{\boldsymbol{L}} \overset{c}{\blacksquare}A_i$ which ends the proof. ⊣

That $\mathbf{E}$ is regular relative to $\Box$ follows from the known property that if $\Delta \vdash_{\mathbf{E}} B$, then for some finite subset $\Delta' \subseteq \Delta$ and some conjunction Th of logical $\mathbf{E}$-theorems, $\varnothing \vdash_{\mathbf{E}} \bigwedge\Delta' \wedge Th \rightarrow B$ (for a proof see Øgaard, 2021a, p. 7009). The result then follows by the fact that $\Box$ has familiar $\mathbf{S4}$-properties in $\mathbf{E}$. Note, then, that Ackermann's logic $\mathbf{\Pi}'$ fails to be regular relative to $\Box$: $\{A, A \supset B\} \vdash_{\mathbf{\Pi}'} B$ obviously holds, but $\{\Box A, \Box(A \supset$

$B)\} \nvdash_{\mathbf{\Pi}'} \Box B$.[7] In fact, any disjunctive logic **L** extending $\mathbf{\Pi}'$ using any axioms and rules displayed above which is regular relative to $\Box$, is a non-conservative extension of **E** in that the non-**E**-theorem $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is a theorem of **L** (for details, see Øgaard, 2020). The notion of regularity makes for an arguably more well-motivated property of logic choice which tells in favor of **E** over $\mathbf{\Pi}'$, than that of *rule normality* appealed to by Anderson and Belnap.[8]

The above regularity theorem assumes that the necessity rule with regard to $\overset{c}{\blacksquare}$ is admissible. One familiar trick to avoid having to deal with primitive, yet merely admissible rules, is to augment the set of logical axioms. Note, then, that if $A \to B$ is a logical axiom, then $(A \to A) \to (A \to B)$ is a logical theorem (use the prefixing rule). To ensure that $\overset{c}{\blacksquare}(A \to B)$ is a logical theorem, it therefore suffices to add the axiom $\mathrm{N}_1\overset{c}{\blacksquare}$. For non-conditional axioms, however, one will in general need to add the necessitated version of the axiom. This, then, is the reason behind $\mathrm{N}_2\overset{c}{\blacksquare}$ — if the logic has excluded middle as a logical theorem, and the necessity rule is to be admissible for $\overset{c}{\blacksquare}$, then add $\mathrm{N}_2\overset{c}{\blacksquare}$ as a primitive axiom. It is not, however, only the stock of axioms that might need to be necessitated in order for the necessity rule to be admissible. The necessitated version of any rule which is itself not superfluous with regards to logical theoremhood must also be derivable.[9] One way to ensure this is to add as primitive axioms (if not already a theorem), the enthymematical version of any primitive rule.

Similar to theorem 2.2 we obtain:

LEMMA 2.1. *Let $\boldsymbol{L}$ be any logic extending $\mathbf{B}^{\mathrm{d}}$ using any axioms and rules displayed above and $\Delta \Vdash A$ be a primitive rule of $\boldsymbol{L}$. If $\varnothing \vdash_{\boldsymbol{L}} \bigwedge \Delta \overset{c}{\mapsto} A$ is a logical theorem of $\boldsymbol{L}$ for a propositional constant $\mathbf{c}$ which is such that $\varnothing \vdash_{\boldsymbol{L}} \mathbf{c}$, then the rule $\overset{c}{\blacksquare}\Delta \Vdash \overset{c}{\blacksquare}A$ is a derivable rule of $\boldsymbol{L}$.*

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[7] A counter model: take the algebraic model set forth in (Ackermann, 1956, p. 126), but assign $a \to b$ to the value 4 where Ackermann assigned 3. The model still validates all of $\mathbf{\Pi}'$ as the reader can verify. Now let A and B be propositional variables and evaluate these to, respectively, 4 and 3. Since $\Box 3$ evaluates to the undesignated element 0, whereas $\Box 4$ and $\Box(4 \supset 3)$ to the designated 4, this is indeed a counter model.

[8] For a systematic account of the philosophy of relevance adhered to by Anderson-Belnap (see Øgaard, 2021b).

[9] R7 is famously admissible in **E** (cf. Meyer and Dunn, 1969), and so superfluous wrt. logical theoremhood in $\mathbf{\Pi}'$.

3. Interpretations

We extend the notion a frame from (Øgaard, 2024) so as to be able to interpret the propositional constants.

DEFN 3.1. A FRAME for a logic with propositional constants $\mathbf{c}_1, \ldots, \mathbf{c}_n$, is a sextuple $\mathcal{F} = \langle g, W, R, *, \sqsubseteq, T\rangle$ such that $\langle g, W, R, *, \sqsubseteq\rangle$ is a frame as defined in (Øgaard, 2024, df. 4.1), and for which for all $a, b \in W$ and $i \leqslant n$

- $T = \{T_1, \ldots, T_n\}$ with $\bigcup T_i \subseteq W$
- $a \sqsubseteq b \Rightarrow (T_i a \Rightarrow T_i b)$

The notion of a model defined in (Øgaard, 2024, df. 4.3) is then augmented by demanding that any propositional constant $\mathbf{c}_i$ be evaluated using its truth set T_i:[10]

(viii) $a \vDash \mathbf{c}_i \Leftrightarrow T_i a$.

LEMMA 3.1. *For any model $\mathfrak{M}$, with $a, b \in W$ and any formula A,*

$$a \sqsubseteq b \;\&\; a \vDash A \Longrightarrow b \vDash A.$$

PROOF. For propositional constants and variables this follows by definition of a frame/model. See (Øgaard, 2024, lem. 4.3) for the inductive part of the proof. ⊣

DEFN 3.2. For any frame $\langle g, W, R, *, \sqsubseteq, T\rangle$ with $a, b, c, d, \in W$, and $T_i \in T$,

1. $S_i := \{\langle a, b\rangle \mid \exists x(Raxb \;\&\; T_i x)\}$
2. $C_i := \{\langle a, b\rangle \mid \exists x(Rabx \;\&\; \neg T_i x)\}$

[10] Chapter 5.1 of (Routley et al., 1982) investigates "the logic of t," where t is a propositional constant. The truth condition for t is determined by the clause

$$\exists x(Px \;\&\; Rxab) \Rightarrow a \sqsubseteq b,$$

where P is a property on the set of point W of the frame in question (cf. Routley et al., 1982, p. 351). This is the standard way to state the truth condition for a propositional constant within the Routley-Meyer semantics. Note, then, that Routley et al. (1982, p. 352) mentions the notion of a "varied models structure" where a set S is used to interpret t rather than the set P with $a \vDash t \Leftrightarrow a \in S$. This, then, is the backdrop of the current use of "truth sets" T_i to interpret propositional constants. I should also like to note that the current truth condition using truth sets is also used in (Restall, 2000, p. 241). However, any (left) truth set T must therein, in addition to be closed upwards by $\sqsubseteq$, satisfy the condition that

$$\forall x \forall y(x \sqsubseteq y \Leftrightarrow \exists z \in T : Rzxy)$$

something which forces $t \to (A \to A)$ to be valid in every frame, where t, then, is the propositional constant which is evaluated using T.

To ease the notion, let for any propositional constant $\mathbf{c}_i$,

$$C_{\mathbf{c}_i} := C_i \mid S_{\mathbf{c}_i} := S_i \mid T_{\mathbf{c}_i} := T_i$$

Each propositional constant $\mathbf{c}_i$ can be used to define a necessity operator and a negation operator:

$$\overset{i}{\blacksquare} A := \overset{\mathbf{c}_i}{\blacksquare} A := \mathbf{c}_i \to A$$
$$\overset{i}{\neg} A := \overset{\mathbf{c}_i}{\neg} A := A \to \mathbf{c}_i$$

Any propositional constant $\mathbf{c}$ holds true at a point a just in case a is in its truth set $T_{\mathbf{c}}$. A similar result holds for any necessity and negation operator: The truth condition for any necessity operator can be stated in terms the binary accessibility relation S which works, then, in the same way as is familiar from modal logics: $\overset{i}{\blacksquare} A$ is true at a point just in case A is true at all of its S_i-accessible points. The truth condition for any negation operator, on the other hand, can be stated in terms of the *compatibility* relation C_i: $\overset{i}{\neg} A$ holds true at a point just in case A fails to hold at all of its C_i-accessible (compatible) points.[11]

Lemma 3.2. *For any model $\mathfrak{M}$, point a, and formula A,*

$$a \models \overset{i}{\blacksquare} A \Leftrightarrow \forall y(aS_i y \Rightarrow y \models A)$$

Proof. $a \models \overset{i}{\blacksquare} A$ iff $\forall x \forall y(Raxy \ \& \ x \models \mathbf{c}_i \Rightarrow y \models A)$ iff $\forall x \forall y(Raxy \ \& \ T_i x \Rightarrow y \models A)$ iff $\forall y(\exists x(Raxy \ \& \ T_i x) \Rightarrow y \models A)$ iff $\forall y(aS_i y \Rightarrow y \models A)$. ⊣

Lemma 3.3. *For any model $\mathfrak{M}$, any point a, and formula A,*

$$a \models \overset{i}{\blacklozenge} A \Leftrightarrow \exists y(a^* S_i y \ \& \ y^* \models A)$$

Proof. $a \models \overset{i}{\blacklozenge} A$ iff $a^* \not\models \overset{i}{\blacksquare} {\sim} A$ iff $\exists y(a^* S_i y \ \& \ y \not\models {\sim} A)$ iff $\exists y(a^* S_i y \ \& \ y^* \models A)$. ⊣

The above conditions fall short of the standard truth condition for possibility due to the non-Boolean negation. However, it is possible to uphold the classical truth conditions for possibility without forcing negation to be Boolean: Let $a\bar{S}_i b := aS_i b \ \& \ a^* S_i b^*$. If the frame in question satisfies the frame condition corresponding to the axiom called *modal*

[11] For more on the compatibility interpretation of negation, see (Berto, 2015) and (Berto and Restall, 2019).

	Frame condition	Axiom/rule
$\mathcal{F}(\mathrm{R}\mathbf{c})$	$T_{\mathbf{c}}a \Rightarrow g \sqsubseteq a$	$\{A\} \Vdash \mathbf{c} \to A$
$\mathcal{F}(\mathrm{A}\mathbf{c}1)$	$T_{\mathbf{c}}g$	$\mathbf{c}$
$\mathcal{F}(\mathrm{A}\mathbf{c}2)$	$T_{\mathbf{c}}a \Rightarrow T_{\mathbf{c}}a^*$	$\mathbf{c} \wedge {\sim}\mathbf{c} \to A$
$\mathcal{F}(\mathrm{A}\mathbf{c}3)$	$T_{\mathbf{c}}b \Rightarrow (Rabc \Rightarrow a \sqsubseteq c)$	$A \to (\mathbf{c} \to A)$
$\mathcal{F}(\mathrm{A}\mathbf{c}4)$	$T_{\mathbf{c}}a$	$A_1 \to (\ldots \to (A_n \to (B \to \mathbf{c}))\ldots)$
$\mathcal{F}(\mathrm{A}\mathbf{c}5)$	$\neg T_{\mathbf{c}}a$	$A_1 \to (\ldots \to (A_n \to (\mathbf{c} \to B))\ldots)$
$\mathcal{F}(\overset{\mathbf{c}}{\neg}1)$	$aC_{\mathbf{c}}b \Rightarrow bC_{\mathbf{c}}a$	$A \to \overset{\mathbf{c}}{\neg}\overset{\mathbf{c}}{\neg}A$
$\mathcal{F}(\overset{\mathbf{c}}{\neg}2)$	$\exists y(aC_{\mathbf{c}}y \ \&\ \forall z(yC_{\mathbf{c}}z \Rightarrow z \sqsubseteq a))$	$\overset{\mathbf{c}}{\neg}\overset{\mathbf{c}}{\neg}A \to A$
$\mathcal{F}(\overset{\mathbf{c}}{\neg}3)$	$aC_{\mathbf{c}}a^*$	$\overset{\mathbf{c}}{\neg}A \to {\sim}A$
$\mathcal{F}(\overset{\mathbf{c}}{\neg}4)$	$aC_{\mathbf{c}}b \to b \sqsubseteq a^*$	${\sim}A \to \overset{\mathbf{c}}{\neg}A$
$\mathcal{F}(\overset{\mathbf{c}}{\neg}5)$	$aC_{\mathbf{c}}a$	$A \wedge \overset{\mathbf{c}}{\neg}A \to B$
$\mathcal{F}(\overset{\mathbf{c}}{\neg}6)$	$aC_{\mathbf{c}}b \Rightarrow b \sqsubseteq a$	$B \to A \vee \overset{\mathbf{c}}{\neg}A$
$\mathcal{F}(\mathrm{N}_1\overset{\mathbf{c}}{\blacksquare})$	$T_{\mathbf{c}}a \Rightarrow (Rabc \Rightarrow b \sqsubseteq c)$	$\overset{\mathbf{c}}{\blacksquare}(A \to A)$
$\mathcal{F}(\mathrm{N}_2\overset{\mathbf{c}}{\blacksquare})$	$T_{\mathbf{c}}a \Rightarrow a^* \sqsubseteq a$	$\overset{\mathbf{c}}{\blacksquare}(A \vee {\sim}A)$
$\mathcal{F}(\mathbf{c}\mathrm{T}\overset{\mathbf{c}}{\blacksquare})$	$\neg T_{\mathbf{c}}a \Rightarrow \exists y(aS_{\mathbf{c}}y \ \&\ \neg T_{\mathbf{c}}y)$	$\overset{\mathbf{c}}{\blacksquare}\mathbf{c} \to \mathbf{c}$
$\mathcal{F}(\mathrm{T}\overset{\mathbf{c}}{\blacksquare})$	$aS_{\mathbf{c}}a$	$\overset{\mathbf{c}}{\blacksquare}A \to A$
$\mathcal{F}(4\overset{\mathbf{c}}{\blacksquare})$	$aS_{\mathbf{c}}b \ \&\ bS_{\mathbf{c}}c \Rightarrow aS_{\mathbf{c}}c$	$\overset{\mathbf{c}}{\blacksquare}A \to \overset{\mathbf{c}}{\blacksquare}\overset{\mathbf{c}}{\blacksquare}A$
$\mathcal{F}(\mathrm{E}^{\mathrm{R}2}_{\mathbf{c}})$	$T_{\mathbf{c}}a \Rightarrow Raaa$	$A \wedge (A \to B) \overset{\mathbf{c}}{\mapsto} B$
$\mathcal{F}(\mathrm{E}^{\mathrm{R}3}_{\mathbf{c}})$	$T_{\mathbf{c}}a \Rightarrow (R^2abcd \Rightarrow R^2a(bc)d)$	$(A \to B) \overset{\mathbf{c}}{\mapsto} ((C \to A) \to (C \to B))$
$\mathcal{F}(\mathrm{E}^{\mathrm{R}4}_{\mathbf{c}})$	$T_{\mathbf{c}}a \Rightarrow (R^2abcd \Rightarrow R^2b(ac)d)$	$(A \to B) \overset{\mathbf{c}}{\mapsto} ((B \to C) \to (A \to C))$
$\mathcal{F}(\mathrm{E}^{\mathrm{R}5}_{\mathbf{c}})$	$T_{\mathbf{c}}a \Rightarrow (Rabc \Rightarrow Rac^*b^*)$	$(A \to B) \overset{\mathbf{c}}{\mapsto} ({\sim}B \to {\sim}A)$
$\mathcal{F}(\mathrm{E}^{\mathrm{R}6}_{\mathbf{c}})$	$T_{\mathbf{c}}a \Rightarrow a \sqsubseteq a^*$	$A \wedge ({\sim}A \vee B) \overset{\mathbf{c}}{\mapsto} B$
$\mathcal{F}(\mathrm{E}^{\mathrm{R}7}_{\mathbf{c}})$	$T_{\mathbf{c}}a \Rightarrow (Rabc \Rightarrow \exists x(a \sqsubseteq x \ \&\ Rbxc))$	$A \overset{\mathbf{c}}{\mapsto} ((A \to B) \to B)$
$\mathcal{F}(E^{\mathrm{R}\mathbf{c}}_{\mathbf{c}})$	$T_{\mathbf{c}}a \Rightarrow (T_{\mathbf{c}}b \Rightarrow (Rabc \Rightarrow a \sqsubseteq c))$	$A \overset{\mathbf{c}}{\mapsto} (\mathbf{c} \to A)$

Table 2. Frame conditions for **c**-axioms and -rules

confinement, $\overset{i}{\blacksquare}(A \vee B) \to \overset{i}{\blacklozenge}A \vee \overset{i}{\blacksquare}B$, then $\bar{S}_i$ is a standard accessibility relation which allows for the standard truth conditions for both $\overset{i}{\blacksquare}$ and $\overset{i}{\blacklozenge}$ (for more on this, see Standefer, n.d.).

The following lemma shows that an implicationally negated proposition is true at a point a just in case the unnegated formula fails to be true at every point compatible with a:

LEMMA 3.4. *For any model $\mathfrak{M}$, any point a, and formula A,*

$$a \vDash \overset{i}{\neg}A \Leftrightarrow \forall x(aC_ix \Rightarrow x \nvDash A)$$

PROOF. $a \vDash \overset{i}{\neg}A$ iff $a \vDash A \to \mathbf{c}_i$ iff $\forall x\forall y(Raxy \ \&\ x \vDash A \Rightarrow y \vDash \mathbf{c}_i)$ iff $\forall x\forall y(Raxy \ \&\ y \nvDash \mathbf{c}_i \Rightarrow x \nvDash A)$ iff $\forall x\forall y(Raxy \ \&\ \neg T_iy \Rightarrow x \nvDash A)$ iff $\forall x(\exists y(Raxy \ \&\ \neg T_iy) \Rightarrow x \nvDash A)$ iff $\forall x(aC_ix \Rightarrow x \nvDash A)$. ⊣

The frame conditions for the axioms and rules involving propositional constants listed above are stated in table 2, where a, b, c, d are arbitrary point. The goal of the following two sections is to show that these frame conditions are indeed correct with regards to their respective axiom/rule. As an easy corollary, then, it will follow that any logic with any number of propositional constants axiomatized using a collection of these logical principles is strongly sound and complete with regards to the simplified semantics.

4. Soundness

The following lemma — see (Øgaard, 2024, lem. 5.1) for a proof — allows for slightly shorter proofs. I will in the following make use of it, as well as lemma 3.2 and lemma 3.4, without reference.

LEMMA 4.1 (Semantic entailment in a model).

$$g \vDash A \rightarrow B \iff \forall x \in W(x \vDash A \Rightarrow x \vDash B)$$

The following lemmas will all be on the form "logical principle P holds true in any model $\mathfrak{M}$ which satisfies $\mathcal{F}(P)$" which I'll symbolize using "$\mathcal{F}(P) \rightsquigarrow P$."

4.1. Truth principles

LEMMA 4.2. $\mathcal{F}(\mathrm{R}\mathbf{c}) \rightsquigarrow \mathrm{R}\mathbf{c}\&R^{\mathrm{d}}\mathbf{c}$

PROOF. In order to show that $\{A\} \vDash_{\mathfrak{M}} \mathbf{c} \rightarrow A$, assume that $g \vDash A$ and let a be any point such that $a \vDash \mathbf{c}$. Then $T_{\mathbf{c}}a$ and so $g \sqsubseteq a$ by $\mathcal{F}(\mathrm{R}\mathbf{c})$. It now follows from lemma 3.1 that $a \vDash A$ and so $g \vDash \mathbf{c} \rightarrow A$.

That $R^{\mathrm{d}}\mathbf{c}$ holds given $\mathcal{F}(\mathrm{R}\mathbf{c})$ is left for the reader. ⊣

LEMMA 4.3. $\mathcal{F}(\mathrm{A}\mathbf{c}1) \rightsquigarrow \mathrm{A}\mathbf{c}1$

LEMMA 4.4. $\mathcal{F}(\mathrm{A}\mathbf{c}2) \rightsquigarrow \mathrm{A}\mathbf{c}2$

PROOF. In order to show that $g \vDash \mathbf{c} \wedge {\sim}\mathbf{c} \rightarrow A$, assume that a is any point such that $a \vDash \mathbf{c} \wedge {\sim}\mathbf{c}$. Then $T_{\mathbf{c}}a$ and $a^* \notin T_{\mathbf{c}}$ which contradicts $\mathcal{F}(\mathrm{A}\mathbf{c}2)$. Thus *no* such a can exists and so it is trivially true that if $a \vDash \mathbf{c} \wedge {\sim}\mathbf{c}$, then $a \vDash A$ for every formula A. ⊣

LEMMA 4.5. $\mathcal{F}(\mathrm{A}\mathbf{c}3) \rightsquigarrow \mathrm{A}\mathbf{c}3$

PROOF. In order to show that $g \vDash A \rightarrow (\mathbf{c} \rightarrow A)$, assume that $a \vDash A$ and let b, c be such that $Rabc$ and $b \vDash \mathbf{c}$. Then $T_{\mathbf{c}}b$ and so it follows by $\mathcal{F}(\mathrm{A}\mathbf{c}3)$ that $a \sqsubseteq c$. From lemma 3.1 it therefore follows that $c \vDash A$, and so $a \vDash \mathbf{c} \rightarrow A$. ⊣

LEMMA 4.6. $\mathcal{F}(\mathrm{A}\mathbf{c}4) \rightsquigarrow \mathrm{A}\mathbf{c}4$

PROOF. The proof that $g \vDash A_1 \to (\ldots \to (A_n \to (B \to \mathbf{c}))\ldots)$ is split into two for easier reading. First, then, that $g \vDash B \to \mathbf{c}$: This easily follows from the fact that $T_{\mathbf{c}} = W$. Next, assume that $n \geqslant 1$ and let $A_1, \ldots, A_n$ be any list of formulas. Assume that $a_1 \vDash A_1$, that a_2, b_2 are any points such that $Ra_1a_2b_2$ and $a_2 \vDash A_2$, that a_3, b_3 are any points such that $Rb_2a_3b_3$ and $a_3 \vDash A_3, \ldots,$ that a_n, b_n are any points such that $Rb_{n-1}a_nb_n$ and $a_n \vDash A_n$. To show that $b_n \vDash B \to \mathbf{c}$ it now suffices to not that this holds trivially since $T_{\mathbf{c}} = W$. $\dashv$

LEMMA 4.7. $\mathcal{F}(\mathrm{A}\mathbf{c}5) \rightsquigarrow \mathrm{A}\mathbf{c}5$

PROOF. The proof that $g \vDash A_1 \to (\ldots \to (A_n \to (\mathbf{c} \to B))\ldots)$ is similar to lemma 4.6 and is therefore left for the reader. $\dashv$

4.2. Negation principles

LEMMA 4.8. $\mathcal{F}(\overset{c}{\neg}1) \rightsquigarrow \overset{c}{\neg}1$

PROOF. In order to show that $g \vDash A \to \overset{c}{\neg}\overset{c}{\neg}A$ provided $aC_{\mathbf{c}}b \Rightarrow bC_{\mathbf{c}}a$ for any a, b, assume that $a \vDash A$. $a \vDash \overset{c}{\neg}\overset{c}{\neg}A$ if and only if $\forall x(aC_{\mathbf{c}}x \Rightarrow x \nvDash \overset{c}{\neg}A)$. Let x, therefore, be any point such that $aC_{\mathbf{c}}x$. The frame assumption yields that $xC_{\mathbf{c}}a$. Since x is compatible with a and $a \vDash A$, if follows that $x \nvDash \overset{c}{\neg}A$. $\dashv$

For the proof of the following lemma see (Restall, 2000, p. 261).

LEMMA 4.9. $\mathcal{F}(\overset{c}{\neg}i) \rightsquigarrow \overset{c}{\neg}i$ *for* $i \in \{2, 5, 6\}$.

LEMMA 4.10. $\mathcal{F}(\overset{c}{\neg}3) \rightsquigarrow \overset{c}{\neg}3$

PROOF. In order to show that $g \vDash \overset{c}{\neg}A \to {\sim}A$ provided $aC_{\mathbf{c}}a^*$ for any a, let a be any point such that $a \vDash \overset{c}{\neg}A$. Using $\mathcal{F}(\overset{c}{\neg}3)$, it follows that $a^* \nvDash A$, and therefore that $a \vDash {\sim}A$. $\dashv$

LEMMA 4.11. $\mathcal{F}(\overset{c}{\neg}4) \rightsquigarrow \overset{c}{\neg}4$

PROOF. In order to show that $g \vDash {\sim}A \to \overset{c}{\neg}A$ provided $aC_{\mathbf{c}}b \to b \sqsubseteq a^*$ for any a, b, let a be any point such that $a \vDash {\sim}A$. It follows that $a^* \nvDash A$. Let b be any point such that $aC_{\mathbf{c}}b$. Using $\mathcal{F}(\overset{c}{\neg}4)$ it follows that $b \sqsubseteq a^*$. From lemma 3.1 it therefore follows that $b \nvDash A$, and therefore that $a \vDash \overset{c}{\neg}A$. $\dashv$

4.3. Modal principles

LEMMA 4.12. $\mathcal{F}(\mathrm{N}_1\overset{\mathbf{c}}{\blacksquare}) \rightsquigarrow \mathrm{N}_1\overset{\mathbf{c}}{\blacksquare}$

PROOF. In order to show that $g \vDash \mathbf{c} \to (A \to A)$, assume that a is any point such that $a \vDash \mathbf{c}$, and let b, c be any points such that $Rabc$ and $b \vDash A$. Then $T_{\mathbf{c}}a$ and so using $\mathcal{F}(\mathrm{N}_1\overset{\mathbf{c}}{\blacksquare})$ it follows that $b \sqsubseteq c$ and so by lemma 3.1 that $c \vDash A$. Thus $a \vDash A \to A$. ⊣

LEMMA 4.13. $\mathcal{F}(\mathrm{N}_2\overset{\mathbf{c}}{\blacksquare}) \rightsquigarrow \mathrm{N}_2\overset{\mathbf{c}}{\blacksquare}$

PROOF. In order to show that $g \vDash \mathbf{c} \to A \vee {\sim}A$, assume that a is any point such that $a \vDash \mathbf{c}$. Then $T_{\mathbf{c}}a$ and so using $\mathcal{F}(\mathrm{N}_2\overset{\mathbf{c}}{\blacksquare})$ it follows that $a^* \sqsubseteq a$. Assume that $a \nvDash A \vee {\sim}A$. Then $a \vDash A$ and $a \nvDash {\sim}A$. That $a \nvDash {\sim}A$ yields that $a^* \vDash A$. So, by lemma 3.1, $a \nvDash A$. Contradiction. ⊣

LEMMA 4.14. $\mathcal{F}(\mathbf{c}\mathrm{T}\overset{\mathbf{c}}{\blacksquare}) \rightsquigarrow \mathbf{c}\mathrm{T}\overset{\mathbf{c}}{\blacksquare}$

PROOF. In order to show that $g \vDash \overset{\mathbf{c}}{\blacksquare}\mathbf{c} \to \mathbf{c}$, provided the frame in question satisfies $\neg T_{\mathbf{c}}a \Rightarrow \exists y(aS_{\mathbf{c}}y \;\&\; \neg T_{\mathbf{c}}y)$ for every point a, let a be any point such that $a \vDash \overset{\mathbf{c}}{\blacksquare}\mathbf{c}$. Assume for contradiction that $a \nvDash \mathbf{c}$. Then $\neg T_{\mathbf{c}}a$, and so by $\mathcal{F}(\mathbf{c}\mathrm{T}\overset{\mathbf{c}}{\blacksquare})$ let b be a point such that $aS_{\mathbf{c}}b \;\&\; \neg T_{\mathbf{c}}b$. Since $a \vDash \overset{\mathbf{c}}{\blacksquare}\mathbf{c}$ and $aS_{\mathbf{c}}b$, it follows that $b \vDash \mathbf{c}$, and therefore that $T_{\mathbf{c}}b$. Contradiction. ⊣

LEMMA 4.15. $\mathcal{F}(\mathrm{T}\overset{\mathbf{c}}{\blacksquare}) \rightsquigarrow \mathrm{T}\overset{\mathbf{c}}{\blacksquare}$

PROOF. In order to show that $g \vDash \overset{\mathbf{c}}{\blacksquare}A \to A$, assume that $a \vDash \overset{\mathbf{c}}{\blacksquare}A$. $aS_{\mathbf{c}}a$ by $\mathcal{F}(\mathrm{T}\overset{\mathbf{c}}{\blacksquare})$, and so there is a b such that $T_{\mathbf{c}}b$ and $Raba$. Then $b \vDash \mathbf{c}$ and so $a \vDash A$. ⊣

LEMMA 4.16. $\mathcal{F}(4\overset{\mathbf{c}}{\blacksquare}) \rightsquigarrow 4\overset{\mathbf{c}}{\blacksquare}$

PROOF. In order to show that $g \vDash \overset{\mathbf{c}}{\blacksquare}A \to \overset{\mathbf{c}}{\blacksquare}\overset{\mathbf{c}}{\blacksquare}A$ provided $aS_{\mathbf{c}}b \;\&\; bS_{\mathbf{c}}c \Rightarrow aS_{\mathbf{c}}c$ for every $a, b \in W$, assume that a is any point such that $a \vDash \overset{\mathbf{c}}{\blacksquare}A$. Let b, c be any points such that $aS_{\mathbf{c}}b$ and $bS_{\mathbf{c}}c$. Then $aS_{\mathbf{c}}c$, and so $c \vDash A$. Hence $b \vDash \overset{\mathbf{c}}{\blacksquare}A$ and $a \vDash \overset{\mathbf{c}}{\blacksquare}\overset{\mathbf{c}}{\blacksquare}A$. ⊣

4.4. Enthymematical principles

LEMMA 4.17. $\mathcal{F}(\mathrm{E}^{\mathrm{R2}}_{\mathbf{c}}) \rightsquigarrow \mathrm{E}^{\mathrm{R2}}_{\mathbf{c}}$

PROOF. In order to show that $g \vDash A \wedge (A \to B) \overset{\mathbf{c}}{\mapsto} B$, let $a \vDash A \wedge (A \to B) \wedge \mathbf{c}$. Then $T_{\mathbf{c}}a$ and so $Raaa$ by $\mathcal{F}(\mathrm{E}^{\mathrm{R2}}_{\mathbf{c}})$. Since $a \vDash A$ and $a \vDash A \to B$ it therefore follows that $a \vDash B$. ⊣

LEMMA 4.18. $\mathcal{F}(\mathrm{E}^{\mathrm{R3}}_{\mathbf{c}}) \rightsquigarrow \mathrm{E}^{\mathrm{R3}}_{\mathbf{c}}$

PROOF. In order to show that $g \vDash (A \to B) \overset{\mathbf{c}}{\mapsto} ((C \to A) \to (C \to B))$, let $a \vDash (A \to B) \wedge \mathbf{c}$ and let b, f be any points such that $Rabf$ with $b \vDash C \to A$. To show that $f \vDash C \to B$ let c, d be any points such that $Rfcd$ with $c \vDash C$. It suffices to show that $d \vDash B$. Since $Rabf$ and $Rfcd$, it follows that R^2abcd and so $R^2a(bc)d$ by $\mathcal{F}(\mathrm{E}^{\mathrm{R3}}_{\mathbf{c}})$. There is, then, some h such that $Rbch$ and $Rahd$. It follows that $h \vDash A$ and therefore that $d \vDash B$ which ends the proof. ⊣

LEMMA 4.19. $\mathcal{F}(\mathrm{E}^{\mathrm{R4}}_{\mathbf{c}}) \rightsquigarrow \mathrm{E}^{\mathrm{R4}}_{\mathbf{c}}$

PROOF. The proof that $g \vDash (A \to B) \overset{\mathbf{c}}{\mapsto} ((A \to C) \to (B \to C))$ is similar to lemma 4.18 and is therefore left for the reader. ⊣

LEMMA 4.20. $\mathcal{F}(\mathrm{E}^{\mathrm{R5}}_{\mathbf{c}}) \rightsquigarrow \mathrm{E}^{\mathrm{R5}}_{\mathbf{c}}$

PROOF. In order to show that $g \vDash (A \to B) \overset{\mathbf{c}}{\mapsto} ({\sim}B \to {\sim}A)$, let $a \vDash (A \to B) \wedge \mathbf{c}$ and let b, c be any points such that $Rabc$ with $b \vDash {\sim}B$. Since $a \vDash \mathbf{c}$ it follows that $T_{\mathbf{c}}a$ and so Rac^*b^* by $\mathcal{F}(\mathrm{E}^{\mathrm{R5}}_{\mathbf{c}})$. If $c \nvDash {\sim}A$, then $c^* \vDash A$, and so $b^* \vDash B$ which yields that $b \nvDash {\sim}B$. Contradiction. It follows, then, that $c \vDash {\sim}A$ and so $a \vDash {\sim}B \to {\sim}A$. ⊣

LEMMA 4.21. $\mathcal{F}(\mathrm{E}^{\mathrm{R6}}_{\mathbf{c}}) \rightsquigarrow \mathrm{E}^{\mathrm{R6}}_{\mathbf{c}}$

PROOF. In order to show that $g \vDash A \wedge ({\sim}A \vee B) \overset{\mathbf{c}}{\mapsto} B$, let $a \vDash A \wedge ({\sim}A \vee B) \wedge \mathbf{c}$. Then $a \vDash \mathbf{c}$, and so $T_{\mathbf{c}}a$. Furthermore, $a \vDash A$ and $a \vDash {\sim}A \vee B$ and so either $a \vDash {\sim}A$ or $a \vDash B$. It follows from $\mathcal{F}(\mathrm{E}^{\mathrm{R6}}_{\mathbf{c}})$ that $a \sqsubseteq a^*$ and so by lemma 3.1 that $a^* \vDash A$ and therefore that $a \nvDash {\sim}A$. It follows, then, that $a \vDash B$. ⊣

LEMMA 4.22. $\mathcal{F}(\mathrm{E}^{\mathrm{R7}}_{\mathbf{c}}) \rightsquigarrow \mathrm{E}^{\mathrm{R7}}_{\mathbf{c}}$

PROOF. In order to show that $g \vDash A \overset{\mathbf{c}}{\mapsto} ((A \to B) \to B)$, assume that $a \vDash A \wedge \mathbf{c}$ and let b, c be such that $Rabc$ and $b \vDash A \to B$. Since $a \vDash \mathbf{c}$, it follows that $T_{\mathbf{c}}a$, and so by $\mathcal{F}(\mathrm{E}^{\mathrm{R7}}_{\mathbf{c}})$, $Rbdc$ for some d such that $a \sqsubseteq d$. Since $a \vDash A$ it then follows by lemma 3.1 that $d \vDash A$, and therefore that $c \vDash B$ which suffices for establishing that $a \vDash (A \to B) \to B$. ⊣

LEMMA 4.23. $\mathcal{F}(E^{\mathrm{R}\mathbf{c}}_{\mathbf{c}}) \rightsquigarrow E^{\mathrm{R}\mathbf{c}}_{\mathbf{c}}$

PROOF. The proof that $g \vDash A \overset{\mathbf{c}}{\mapsto} (\mathbf{c} \to A)$ is similar to lemma 4.5 and is therefore left for the reader. ⊣

We have now seen that the axioms and rules all hold true provided the corresponding frame conditions are enforced. As an easy corollary, then, we have the following result:

THEOREM 4.1 (Strong soundness). *$\Theta \vdash_{\boldsymbol{L}} A \Longrightarrow \Theta \vDash_{\boldsymbol{L}} A$, where $\boldsymbol{L}$ is any disjunctive logic obtainable from* **B** *by adding any number of the axiom and rules mentioned in sect. 2 or the appendix of (Øgaard, 2024) or in table 2.*

5. Completeness

As in (Øgaard, 2024, sect. 6), it will be shown that any canonical frame for a logic with a logical axiom/rule θ is such as to validate θ's frame condition. That the canonical frame is indeed a model, follows from (Øgaard, 2024, thm. 6.3) together with the fact that T_i in the canonical frame, relative to a non-trivial, prime and Π-deductively closed Π-theory, is to be defined as the set of all Π-canonical theories Σ — Π-theories which are both non-trivial and prime — such that $\mathbf{c}_i \in \Sigma$. It is then evident that if $\Delta \subseteq \Gamma$, and $\Delta \in T_i$, then also $\Gamma \in T_i$. Thus the canonical frame is indeed a frame.

LEMMA 5.1. *For any Π-canonical theory Σ of any canonical model,*

$$A \in \Sigma \Leftrightarrow \Sigma \vDash A.$$

PROOF. This follows from (Øgaard, 2024, lem. 6.4) by the trivial addition that by definition of T_i in the canonical frame we have that $\mathbf{c}_i \in \Sigma \Leftrightarrow \Sigma \in T_i$, and that in any model, and hence in the canonical one, it is the case that $\Sigma \vDash \mathbf{c}_i \Leftrightarrow \Sigma \in T_i$. ⊣

We can now undertake the task of showing that the canonical model satisfies the frame conditions given that the logic in question validates the corresponding logical principle. In doing so it is important to have in mind that the ternary relation R is in the canonical model defined using a two-part definition, where for the base point $\underline{\Pi}$, $R\underline{\Pi}\Gamma\Delta \Leftrightarrow \Gamma = \Delta$, and for Π-canonical $\Sigma \neq \underline{\Pi}$, $R\Sigma\Gamma\Delta$ iff $\forall A \forall B(A \to B \in \Sigma \Rightarrow (A \in \Gamma \Rightarrow B \in \Delta))$ (for details, see Øgaard, 2024, def. 6.2). This implies, then, that the proofs needed to establish that R does obtain under given circumstances will typically require a two-part proof. Note, then, that Priest and Sylvan (1992) defines for any sets of formulas Σ, Γ, Δ, that

$R\Sigma\Gamma\Delta$ is true if and only if $\forall A \forall B(A \to B \in \Sigma \Rightarrow (A \in \Gamma \Rightarrow B \in \Delta))$. Their results carry over to the present context, but to avoid confusion and error, I'll use $\overline{R}$ as a relation over arbitrary sets of formulas, but will reserve the plain 'R' to designate the relation defined above holding only between Π-canonical theories — Π-theories which are prime and non-trivial.

5.1. Truth principles

LEMMA 5.2. R**c** $\rightsquigarrow \mathcal{F}$(R**c**)

PROOF. Assume that $\Sigma \in T_{\mathbf{c}}$. In order to show that $\underline{\Pi} \subseteq \Sigma$, let $A \in \underline{\Pi}$. Since R**c** is a rule of the logic, it follows that $\mathbf{c} \to A \in \underline{\Pi}$. Since $\Sigma \in T_{\mathbf{c}}$ it follows from lem. 5.1 that $\Sigma \vDash \mathbf{c}$. Since $R\underline{\Pi}\Sigma\Sigma$, it therefore follows that $A \in \Sigma$. Thus $\underline{\Pi} \subseteq \Sigma$ as required. ⊣

LEMMA 5.3. A**c**1 $\rightsquigarrow \mathcal{F}$(A**c**1)

PROOF. If A**c**1 is an axiom of the logic, then $\mathbf{c} \in \underline{\Pi}$, and so $\underline{\Pi} \in T_{\mathbf{c}}$ by the definition of $T_{\mathbf{c}}$ in the canonical frame. ⊣

LEMMA 5.4. A**c**2 $\rightsquigarrow \mathcal{F}$(A**c**2)

PROOF. Assume that $\mathbf{c} \wedge {\sim}\mathbf{c} \to A$ is an axiom of the logic, and let $\Sigma \in T_{\mathbf{c}}$. By the definition of the latter along with lem. 5.1, it follows that $\Sigma \vDash \mathbf{c}$. Since Σ is non-trivial, it follows that $\Sigma \nvDash {\sim}\mathbf{c}$, and so $\Sigma^* \vDash \mathbf{c}$ which by definition of $T_{\mathbf{c}}$ and lem. 5.1 yields that $\Sigma^* \in T_{\mathbf{c}}$. ⊣

LEMMA 5.5. A**c**3 $\rightsquigarrow \mathcal{F}$(A**c**3)

PROOF. Assume that $A \to (\mathbf{c} \to A)$ is an axiom of the logic, and let Δ, Σ, Γ be such that $R\Delta\Sigma\Gamma$ and $\Sigma \in T_{\mathbf{c}}$. We must show that $\Delta \subseteq \Gamma$, so assume that $A \in \Delta$. Since $\Sigma \in T_{\mathbf{c}}$, it follows that $\mathbf{c} \in \Sigma$. If $\Delta \neq \underline{\Pi}$, it follows by the definition of R in the canonical frame that $A \in \Gamma$. If $\Delta = \underline{\Pi}$, then $\Sigma = \Gamma$, and so it follows that $A \in \Gamma$ since Γ is a Π-theory. ⊣

LEMMA 5.6. A**c**4 $\rightsquigarrow \mathcal{F}$(A**c**4)

PROOF. Let Σ be any Π-canonical theory. Then $\Sigma \neq \varnothing$, and so assume that $B \in \Sigma$. $B \to \mathbf{c}$ is an instance of A**c**4, and so it follows that $\mathbf{c} \in \Sigma$ since Σ is a Π-theory. ⊣

LEMMA 5.7. A**c**5 $\rightsquigarrow \mathcal{F}$(A**c**5)

PROOF. Let Σ be any Π-canonical theory. Then $B \notin \Sigma$ for some formula B. $\mathbf{c} \to B$ is an instance of A**c**5, and so it follows that $\mathbf{c} \notin \Sigma$ since Σ is a Π-theory. ⊣

5.2. Negation principles

Lemma 5.8. $\overset{c}{\neg}1 \rightsquigarrow \mathcal{F}(\overset{c}{\neg}1)$

Proof. Suppose that $\Delta C_{\mathbf{c}} \Sigma$, that is, $R\Delta\Sigma\Gamma$ for some Γ such that $\Gamma \notin T_{\mathbf{c}}$. The proof splints in two:

$\Sigma \neq \underline{\Pi}$: Let $\Psi := \{B \mid \exists A(A \in \Delta \;\&\; A \to B \in \Sigma\}$. Then $\overline{R}\Sigma\Delta\Psi$. Assume for contradiction that $\mathbf{c} \in \Psi$. Then $\overset{c}{\neg}A \in \Sigma$ for some $A \in \Delta$. Since $\overset{c}{\neg}1$ is an axiom of the logic, it follows that $\overset{c}{\neg}\overset{c}{\neg}A \in \Delta$. From lem. 5.1 it then follows that both $\Sigma \vDash \overset{c}{\neg}A$ and $\Delta \vDash \overset{c}{\neg}\overset{c}{\neg}A$, and since $\Delta C_{\mathbf{c}} \Sigma$ that $\Sigma \nvDash \overset{c}{\neg}A$. Contradiction. By (Priest and Sylvan, 1992, lem. 5) there is a Π-canonical theory $\Theta \supseteq \Psi$ such that $\mathbf{c} \notin \Theta$ and $\overline{R}\Sigma\Delta\Theta$. Since $\mathbf{c} \notin \Theta$, it follows that $\Theta \notin T_i$. Since $\Sigma \neq \underline{\Pi}$, $\overline{R}\Sigma\Delta\Theta$ implies that $R\Sigma\Delta\Theta$, and therefore that $\Sigma C_{\mathbf{c}} \Delta$.

$\Sigma = \underline{\Pi}$: It will be shown that $\Delta \notin T_{\mathbf{c}}$, and therefore that $\Sigma C_{\mathbf{c}} \Delta$ since $R\underline{\Pi}\Delta\Delta$. Assume first that $\Delta = \underline{\Pi}$. It follows, then, that $\Gamma = \underline{\Pi}$, and therefore that $\Delta \notin T_{\mathbf{c}}$ and $\Sigma C_{\mathbf{c}} \Delta$. Lastly, assume that $\Delta \neq \underline{\Pi}$. For contradiction, assume that $\mathbf{c} \in \Delta$. By $\overset{c}{\neg}1$ it then follows that $\overset{c}{\neg}\overset{c}{\neg}\mathbf{c} \in \Delta$, that is $(\mathbf{c} \to \mathbf{c}) \to \mathbf{c} \in \Delta$. Since $R\Delta\Sigma\Gamma$ and $\Delta \neq \underline{\Pi}$, and $\mathbf{c} \to \mathbf{c} \in \underline{\Pi}$, it follows that $\mathbf{c} \in \Gamma$ and therefore that $\Gamma \in T_{\mathbf{c}}$. Contradiction. Since, then, $\mathbf{c} \notin \Delta$, it follows that $\Delta \notin T_{\mathbf{c}}$. Since $R\Sigma\Delta\Delta$ it therefore follows that $\Sigma C_{\mathbf{c}} \Delta$. ⊣

For the proof of the following, see (Restall, 2000, p. 262).

Lemma 5.9. $\overset{c}{\neg}i \rightsquigarrow \mathcal{F}(\overset{c}{\neg}i)$ *for* $i \in \{2, 5, 6\}$.

Lemma 5.10. $\overset{c}{\neg}3 \rightsquigarrow \mathcal{F}(\overset{c}{\neg}3)$

Proof. We must show that $\Sigma C_{\mathbf{c}} \Sigma^*$ for every Π-canonical theory Σ.

First, let $\Sigma = \underline{\Pi}$. Since $\mathbf{c} \to \mathbf{c} = \overset{c}{\neg}\mathbf{c} \in \Sigma$, it follows using $\overset{c}{\neg}3$ that ${\sim}\mathbf{c} \in \Sigma$ and therefore that $\mathbf{c} \notin \Sigma^*$. Since $R\Sigma\Sigma^*\Sigma^*$, it therefore follows that $\Sigma C_{\mathbf{c}} \Sigma^*$.

Let $\Sigma \neq \underline{\Pi}$. By setting $\Psi := \{B \mid \exists A(A \in \Sigma^* \;\&\; A \to B \in \Sigma\}$ it follows that $\overline{R}\Sigma\Sigma^*\Psi$. Assume for contradiction that $\mathbf{c} \in \Psi$. Then $\overset{c}{\neg}A \in \Sigma$ for some $A \in \Sigma^*$. Since $\overset{c}{\neg}3$ is an axiom of the logic, it follows that ${\sim}A \in \Delta$, and therefore that $A \notin \Sigma^*$. Contradiction. The rest of the proof is more or less identical to the middle part of lem. 5.8. ⊣

Lemma 5.11. $\overset{c}{\neg}4 \rightsquigarrow \mathcal{F}(\overset{c}{\neg}4)$

Proof. Assume that $\Delta C_{\mathbf{c}} \Sigma$. To show that $\Sigma \subseteq \Delta^*$, assume for contradiction that there is some A such that $A \in \Sigma$, but $A \notin \Delta^*$. It follows that $\sim A \in \Delta$, and since $\sim A \to \overset{c}{\neg} A$ is an axiom of the logic that $\overset{c}{\neg} A \in \Delta$. By lem. 5.1 it follows that both $\Delta \vDash \overset{c}{\neg} A$ and $\Sigma \vDash A$ which yields a contradiction since $\Delta C_{\mathbf{c}} \Sigma$. ⊣

5.3. Modal principles

LEMMA 5.12. $\mathrm{N}_1\overset{c}{\blacksquare} \rightsquigarrow \mathcal{F}(\mathrm{N}_1\overset{c}{\blacksquare})$

PROOF. Assume that $R\Delta\Sigma\Gamma$ with $\Delta \in T_{\mathbf{c}}$. We must show that $\Sigma \subseteq \Gamma$, so assume that $A \in \Sigma$. Since $\Delta \in T_{\mathbf{c}}$, it follows by definition that $\mathbf{c} \in \Delta$ and therefore by lem. 5.1 that $\Delta \vDash \mathbf{c}$. Since Δ is a Π-theory, it follows that $A \to A \in \Delta$ since $\mathrm{N}_1\overset{c}{\blacksquare}$ is the axiom $\mathbf{c} \to (A \to A)$. But then $\Delta \vDash A \to A$, and so $\Gamma \vDash A$, which by lem. 5.1 yields that $A \in \Gamma$. ⊣

LEMMA 5.13. $\mathrm{N}_2\overset{c}{\blacksquare} \rightsquigarrow \mathcal{F}(\mathrm{N}_2\overset{c}{\blacksquare})$

PROOF. Assume that $\Delta \in T_{\mathbf{c}}$. By definition of the truth set, $\mathbf{c} \in \Delta$. We must show that $\Delta^* \subseteq \Delta$, so assume that $A \in \Delta^*$. Since $\overset{c}{\blacksquare}(A \vee \sim A)$, that is $\mathbf{c} \to A \vee \sim A$, is an axiom of the logic, it follows that $A \vee \sim A \in \Delta$. Since $A \in \Delta^*$ it follows that $\sim A \notin \Delta$. Since $A \vee \sim A \in \Delta$ and Δ is prime it therefore follows that $A \in \Delta$, and therefore that $\Delta^* \subseteq \Delta$. ⊣

LEMMA 5.14. $\mathbf{c}\mathrm{T}\overset{c}{\blacksquare} \rightsquigarrow \mathcal{F}(\mathbf{c}\mathrm{T}\overset{c}{\blacksquare})$

PROOF. Assume that Σ is a Π-canonical theory such that $\Sigma \notin T_{\mathbf{c}}$. By definition it follows that $\mathbf{c} \notin \Sigma$. Since Σ is a Π-theory and $\mathbf{c} \to (\mathbf{c} \to \mathbf{c})$ is assumed to be an axiom, it follows that $\mathbf{c} \to \mathbf{c} \notin \Sigma$. Evidently, then, $\Sigma \neq \underline{\Pi}$. From (Priest and Sylvan, 1992, lem. 6) it then follows that there are Π-canonical theories Γ and Δ such that $\overline{R}\Sigma\Gamma\Delta$ with $\mathbf{c} \in \Gamma$ and $\mathbf{c} \notin \Delta$. Since $\Sigma \neq \underline{\Pi}$ it follows that $R\Sigma\Gamma\Delta$. From lemma 5.1 it then follows that $\Gamma \in T_{\mathbf{c}}$ and $\Delta \notin T_{\mathbf{c}}$, and therefore that $\Sigma S_{\mathbf{c}} \Delta$. ⊣

LEMMA 5.15. $\mathrm{T}\overset{c}{\blacksquare} \rightsquigarrow \mathcal{F}(\mathrm{T}\overset{c}{\blacksquare})$

PROOF. That $\underline{\Pi} S_{\mathbf{c}} \underline{\Pi}$ follows from the fact that $R\underline{\Pi\Pi\Pi}$ together with the fact that $\mathrm{T}\overset{c}{\blacksquare}$ yields that $\mathbf{c} \in \underline{\Pi}$ since $\mathbf{c} \to \mathbf{c} \in \underline{\Pi}$, and therefore that $\underline{\Pi} \in T_{\mathbf{c}}$. For the rest of the proof, see (Routley et al., 1982, p. 414). ⊣

LEMMA 5.16. $4\overset{c}{\blacksquare} \rightsquigarrow \mathcal{F}(4\overset{c}{\blacksquare})$

PROOF. Assume that $\Delta S_{\mathbf{c}} \Sigma$ and that $\Sigma S_{\mathbf{c}} \Gamma$. We must show that $\Delta S_{\mathbf{c}} \Gamma$, that is, that there is some Π-canonical theory Θ such that $\Theta \in T_{\mathbf{c}}$ and $R\Delta\Theta\Gamma$.

Assume first that $\Delta = \underline{\Pi}$. Since $\mathbf{c} \to \mathbf{c} = \overset{\mathbf{c}}{\blacksquare}\mathbf{c} \in \underline{\Pi}$, it follows using $4\overset{\mathbf{c}}{\blacksquare}$ that $(\mathbf{c} \to \mathbf{c}) \to \mathbf{c} = \overset{\mathbf{c}}{\blacksquare}\overset{\mathbf{c}}{\blacksquare}\mathbf{c} \in \underline{\Pi}$. Since $\Delta S_{\mathbf{c}} \Sigma$ it follows, therefore, that $\mathbf{c} \to \mathbf{c} \in \Sigma$. Since $\Sigma S_{\mathbf{c}} \Gamma$, it therefore follows that $\mathbf{c} \in \Gamma$. By lemma 5.1 it now follows that $\Gamma \in T_{\mathbf{c}}$. Since $R\Delta\Gamma\Gamma$, it therefore follows that $\Delta S_{\mathbf{c}} \Gamma$.

Assume that $\Delta \neq \underline{\Pi}$. Let $\Psi := \{A \mid \forall B(A \to B \in \Delta \Rightarrow B \in \Gamma\}$. Then $\overline{R}\Delta\Psi\Gamma$. Assume for contradiction that $\mathbf{c} \notin \Psi$. Then for some B, $\mathbf{c} \to B \in \Delta$, but $B \notin \Gamma$. Since $4\overset{\mathbf{c}}{\blacksquare}$ is an axiom it follows that $\overset{\mathbf{c}}{\blacksquare}\overset{\mathbf{c}}{\blacksquare}B \in \Delta$, and therefore that $\overset{\mathbf{c}}{\blacksquare}B \in \Sigma$, and therefore that $B \in \Gamma$. Contradiction. (Restall, 1993, lem. 4) then yields that there is a Π-canonical $\Theta \supseteq \Psi$ such that $\overline{R}\Delta\Theta\Gamma$ which yields that $R\Delta\Theta\Gamma$ since $\Delta \neq \underline{\Pi}$. Since $\mathbf{c} \in \Theta$, it follows that $\Theta \in T_{\mathbf{c}}$, and therefore that $\Delta S_{\mathbf{c}} \Gamma$. ⊣

5.4. Enthymematical principles

LEMMA 5.17. $\mathrm{E}^{\mathrm{R2}}_{\mathbf{c}} \rightsquigarrow \mathcal{F}(\mathrm{E}^{\mathrm{R2}}_{\mathbf{c}})$

PROOF. Suppose that $\Delta \in T_{\mathbf{c}}$. We must show that $R\Delta\Delta\Delta$ for every Π-canonical theory Δ. For $\Delta = \underline{\Pi}$, this follows by definition of R. Suppose, then, that $\Delta \neq \underline{\Pi}$ and letA, B be any formulas such that $A \to B \in \Delta$ and $A \in \Delta$. It follows that $A \wedge (A \to B) \wedge \mathbf{c} \in \Delta$, and therefore that $B \in \Delta$, since $A \wedge (A \to B) \overset{\mathbf{c}}{\mapsto} B$ is an axiom of the logic. ⊣

LEMMA 5.18. $\mathrm{E}^{\mathrm{R}i}_{\mathbf{c}} \rightsquigarrow \mathcal{F}(\mathrm{E}^{\mathrm{R}}_{\mathbf{c}})$ *for* $3 \leqslant i \leqslant 5$.

LEMMA 5.19. $\mathrm{E}^{\mathrm{R6}}_{\mathbf{c}} \rightsquigarrow \mathcal{F}(\mathrm{E}^{\mathrm{R6}}_{\mathbf{c}})$

PROOF. Suppose that $\Delta \in T_{\mathbf{c}}$. By definitition of $T_{\mathbf{c}}$ it follows that $\mathbf{c} \in \Delta$. To show that $\Delta \subseteq \Delta^*$, let $A \in \Delta$, and suppose for contradiction that $A \notin \Delta^*$. Then ${\sim}A \in \Delta$. Since Δ is non-trivial, $B \notin \Delta$ for some B. However, ${\sim}A \vee B \in \Delta$, and so $A \wedge ({\sim}A \vee B) \wedge \mathbf{c} \in \Delta$. Since Δ is a Π-theory it follows from the axiom $\mathrm{E}^{\mathrm{R6}}_{\mathbf{c}}$ that $B \in \Delta$. Contradiction. ⊣

LEMMA 5.20. $\mathrm{E}^{\mathrm{R7}}_{\mathbf{c}} \rightsquigarrow \mathcal{F}(\mathrm{E}^{\mathrm{R7}}_{\mathbf{c}})$

PROOF. Suppose that $\Delta \in T_{\mathbf{c}}$, and let Σ, Γ be such that $R\Delta\Sigma\Gamma$. We need to show that for some Ξ such that $\Delta \subseteq \Xi$, $R\Sigma\Xi\Gamma$. The proof is in three cases.

If $\Delta = \underline{\Pi} = \Sigma$: Since $R\Delta\Sigma\Gamma$, it follows by definition of R that $\Sigma = \Gamma$. Let $\Xi := \underline{\Pi}$. Since $R\underline{\Pi}\underline{\Pi}\underline{\Pi}$, we're done.

If $\Sigma \neq \underline{\Pi}$, let $\Xi := \Pi$.[12] Δ and Ξ are then coextensional qua sets of formulas, and so $\Delta \subseteq \Xi$. To show that $R\Sigma\Xi\Gamma$, note that since $\Sigma \neq \underline{\Pi}$, it suffices to show that for any formulas A and B, if $A \to B \in \Sigma$ and $A \in \Xi$, then $B \in \Gamma$. Let, therefore, $A \to B \in \Sigma$ and $A \in \Xi$. Since $\Delta \in T_{\mathbf{c}}$, $\mathbf{c} \in \Delta$, and so $\mathbf{c} \in \Xi$, and so by definition of $T_{\mathbf{c}}$ in the canonical model, $\Xi \in T_{\mathbf{c}}$. It follows that $A \wedge \mathbf{c} \in \Xi$. Since Ξ is a Π-theory and $\mathrm{E}_{\mathbf{c}}^{\mathrm{R7}}$ is a principle of the logic, it follows that $(A \to B) \to B \in \Xi$. But then $(A \to B) \to B \in \Delta$ from which it follows that $B \in \Gamma$ since $R\Delta\Sigma\Gamma$.

Lastly, assume that $\Delta \neq \underline{\Pi}$, but that $\Sigma = \underline{\Pi}$. In this case we let $\Xi := \Gamma$. It must, then, be shown that $\Delta \subseteq \Gamma$, so assume that $A \in \Delta$. It follows that $A \wedge \mathbf{c} \in \Delta$, and since $\mathrm{E}_{\mathbf{c}}^{\mathrm{R7}}$ is a principle of the logic that $(A \to A) \to A \in \Delta$. Since $R\Delta\Sigma\Gamma$ and $\Sigma = \underline{\Pi}$, it follows that $A \in \Gamma$ and therefore that $\Delta \subseteq \Gamma$. ⊣

The proof of the following is similar to the proof of lem. 5.5.

LEMMA 5.21. $E_{\mathbf{c}}^{\mathrm{R}\mathbf{c}} \rightsquigarrow \mathcal{F}(E_{\mathbf{c}}^{\mathrm{R}\mathbf{c}})$

We have now seen that the frame conditions hold in the canonical model provided the logic in question has the corresponding logical axiom/rule. As an easy corollary, then, we have the following result:

THEOREM 5.1 (Strong completeness). *$\Theta \vDash_{\boldsymbol{L}} A \Longrightarrow \Theta \vdash_{\boldsymbol{L}} A$, where $\boldsymbol{L}$ is any disjunctive logic obtainable from* **B** *by adding any number of the axiom and rules mentioned in sect. 2 or the appendix of (Øgaard, 2024) or in table 2.*

6. Conservative extension results

The notion of a strong conservative extension was set forth in (Øgaard, 2024) (cf. df. 6.3) in order to show that any logic fit for the simplified semantics can so be extended by fusion and the converse conditional. The following states similar results pertaining to propositional constants. The proof is in all cases the simple fact that a truth set for the constant in question is guaranteed to exists in every frame. The criterion for being a truth set is the property of persistence:

[12] The base point $\underline{\Pi}$ (as well as its starmate $\underline{\Pi}^*$) of the canonical model has a doppelgänger Π. This is achieved by letting the points of the canonical interpretation be pairs $\langle \Gamma, i \rangle$, where $0 \leqslant i \leqslant 1$, and Γ is a Π-theory. This detail, then, is glossed over when not needed. The base point $\underline{\Pi}$ is identified as $\langle \Pi, 1 \rangle$, whereas its doppelgänger Π is really the element $\langle \Pi, 0 \rangle$. Importantly, then, $\underline{\Pi}$ and Π are coextensional qua sets of formulas.

DEFN 6.1. A set $\varDelta \subseteq W$ of any frame $\mathcal{F} = \langle g, W, R, *, \sqsubseteq, T\rangle$ is called PERSISTENT if for all $a, b \in W$ it is the case that if $a \sqsubseteq b$ and $a \in \varDelta$, then also $b \in \varDelta$.

One such persistent set is the set

$$Z := \{a \mid \forall x \forall y(Raxy \Rightarrow x \sqsubseteq y\},$$

regarding which (Øgaard, 2024, lemma 4.4) states that for any frame and any points a, b, c that
(Z1): $g \in Z$
(Z2): $a \in Z \;\&\; Rabc \Rightarrow b \sqsubseteq c$
(Z3): $a \in Z \;\&\; a \sqsubseteq b \Rightarrow b \in Z$

Z3, then, is simply the claim that Z is a persistent set.

THEOREM 6.1. *Let $\boldsymbol{L}$ be any logic of the above completeness theorem any let $\boldsymbol{L}^{\dagger}$ be $\boldsymbol{L}$ augmented by any of the following four propositional constants $\{\top, \bot, \mathbf{w}, \mathbf{l}\}$ axiomatized using the following axioms and rules:*

(A⊤4)	$A_1 \to (\ldots \to (A_n \to (B \to \top))\ldots)$	$(n \geqslant 0)$
(A⊥5)	$A_1 \to (\ldots \to (A_n \to (\bot \to B))\ldots)$	$(n \geqslant 0)$
(R$\mathbf{w}$)	$\{A\} \Vdash \mathbf{w} \to A$	
(A$\mathbf{w}$1)	$\mathbf{w}$	
$(\mathrm{N}_1\overset{\mathbf{l}}{\blacksquare})$	$\mathbf{l} \to (A \to A)$	
(A$\mathbf{l}$1)	$\mathbf{l}$	

- *$\boldsymbol{L}^{\dagger}$ is a strong conservative extension of $\boldsymbol{L}$.*
- *If* A13 — $(((A \to A) \wedge (B \to B)) \to C) \to C$ — *is a logical theorem of $\boldsymbol{L}$, then $\boldsymbol{L}^{\ddagger}$ is a strong conservative extension of $\boldsymbol{L}$, where $\boldsymbol{L}^{\ddagger}$ is obtained by the addition of a propositional constant $\mathbf{t}$ axiomatized using the two axioms*

$(\mathrm{N}_1\overset{\mathbf{t}}{\blacksquare})$	$\mathbf{t} \to (A \to A)$
$(\mathrm{T}\overset{\mathbf{t}}{\blacksquare})$	$(\mathbf{t} \to A) \to A$

- *If* R6 — $\{A\} \Vdash (A \to B) \to B$ — *is a derivable rule of $\boldsymbol{L}$, then $\boldsymbol{L}^{\sharp}$ is a strong conservative extension of $\boldsymbol{L}$, where $\boldsymbol{L}^{\sharp}$ is obtained by the addition of a propositional constant $\mathbf{w}$ axiomatized using the two axioms*

(R$\mathbf{m}$)	$\{A\} \Vdash \mathbf{m} \to A$
$(\mathrm{T}\overset{\mathbf{m}}{\blacksquare})$	$(\mathbf{m} \to A) \to A$

PROOF. • It is evident that the truth sets for the first two constants — $\top \& \bot$ — must in any frame be, respectively, the entire set of points W, and the empty set of points. These sets are guaranteed to exist and are trivially persistent.

With regards to $\mathbf{w}$: The set $G := \{x \mid g \sqsubseteq x\}$ is persistent since $\sqsubseteq$ is transitive in any frame. Since it also exists in any frame and validates both R$\mathbf{w}$ and A$\mathbf{w}$1, any frame can interpret $\mathbf{w}$.[13]

That Z satisfies $\mathcal{F}(\mathrm{Al1})$ and $\mathcal{F}(\mathrm{N}_1\overset{\mathbf{l}}{\blacksquare})$ follows rather straight-forwardly from Z1 and Z2, respectively. Since it also exists in every frame, every such is ready-made to interpret $\mathbf{l}$.

• It was shown in (Øgaard, 2024) that Z satisfies $\mathcal{F}(\mathrm{T}\overset{\mathbf{t}}{\blacksquare})$ — $\forall x \exists y(y \in Z \;\&\; Rxyx)$ — given A13 is a logical theorem, and so a suitable truth set for $\mathbf{t}$ is guaranteed to exists in every frame fit for $\boldsymbol{L}^{\ddagger}$.

• It was shown in (Restall, 1993) that the correct frame condition for R6 is $Raga$ for all points a. Since $g \in G$ and this set always exists, is persistent and satisfies the frame condition $\mathcal{F}(\mathrm{T}\overset{\mathbf{m}}{\blacksquare})$ in any frame fit for a logic $\boldsymbol{L}^{\sharp}$, G can be the truth set of $\mathbf{m}$ axiomatized using R$\mathbf{m}$ and $\mathrm{T}\overset{\mathbf{m}}{\blacksquare}$. ⊣

COROLLARY 6.1. *The addition of any of the propositional constants* $\{\top, \bot, \mathbf{w}, \mathbf{t}\}$ *axiomatized as above to* $\mathbf{E}$ *and* $\mathbf{\Pi}'$ *makes for a strong conservative extension.*

There are other interesting persistent sets which are also readily available. Two such are as follows:

$$Z^{\sharp} := \{a \in Z \mid \forall y(a \sqsubseteq y \Rightarrow (y \notin G \;\&\; y^* \sqsubseteq y))\}$$
$$Z^{\flat} := \{a \in Z \mid \forall y(a \sqsubseteq y \Rightarrow y \sqsubseteq y^*)\}$$
$$G^{\flat} := \{a \in G \mid \forall x \forall y(Rxay \Rightarrow x \sqsubseteq y)\}$$

LEMMA 6.1. $Z^{\sharp}$, $Z^{\flat}$ *and* $G^{\flat}$ *are persistent sets.*

PROOF. That the first two sets are persistent follows trivially from the transitivity of $\sqsubseteq$. That $G^{\flat}$ is transitive: If $a \in G^{\flat}$ and $a \sqsubseteq b$, then $b \in G$ by the transitivity of $\sqsubseteq$. Assume that $Rxby$. If $x = g$, then $b = y$, and so it follows that $x \sqsubseteq y$. If $x \neq y$, then it follows from the definition of a frame (cf. Øgaard, 2024, df. 4.1) that $Rxay$, and so $x \sqsubseteq y$ since $a \in G^{\flat}$. ⊣

[13] I should like to note that Restall (1994) defines within a simplified semantics the truth condition for the propositional constant 't' using $a \vDash \mathrm{t} \Leftrightarrow g \sqsubseteq a$ with t axiomatized using two rules, namely $\{A\} \Vdash \mathrm{t} \to A$ and $\{\mathrm{t} \to A\} \Vdash A$. It is easy to show that the latter rule is interderivable with having t as a logical axiom, and so Restall's t amounts to the current propositional constant $\mathbf{w}$ with G as its truth set.

It is evident that $Z^{\flat}$ can be used as a truth set for a propositional constant $\mathbf{n}$ for which $A \wedge (\sim A \vee B) \overset{\mathbf{n}}{\mapsto} B$ and $\mathbf{n} \to (A \to A)$ are to be logical theorems, and that $Z^{\sharp}$ can be used if $\overset{\mathbf{n}}{\blacksquare}(A \to A)$ and $\overset{\mathbf{n}}{\blacksquare}(A \vee \sim A)$ are to be logical theorems, but $\mathbf{n}$ itself is not since $g \notin Z^{\sharp}$. Furthermore, using $Z^{\sharp}$ as a truth-set for $\mathbf{n}$ does not yield as truth-preserving the rule $\{A\} \Vdash \overset{\mathbf{n}}{\blacksquare}A$. If $Z^{\flat}$, on the other hand, is used, then it is easily verified that even $A \to (\mathbf{n} \to A)$ holds. However, since it is not in general the case that $g \in G^{\flat} \cup Z^{\flat}$, these truth-set cannot, unless other frame conditions are in place, be used to interpret a propositional constant which must hold true at the base point g in every interpretation.

7. The world of Anderson and Belnap

The defining axioms and rules of the propositional constants **w**, **l**, and **t** as axiomatized above have frequently been used in the literature on relevant logics. However, when occurring, they are designated in the singular as *the Ackermann constant*. The fact that one can add several such hasn't been systematically explored before. An early recognition of the fact that it may be possible to add several is witnessed in (Routley et al., 1982, p. 352) in which, using the original Routley-Meyer semantics, a "varied" model structure is presented. A propositional constant t is therein called *distinctive* if t is a logical theorem and $\{A\} \Vdash t \to A$ is derivable. Although they state that "[i]n cases of main interest, t is distinctive," but define, then, such "varied" models in which t need not be which, then, "illustrates the way in which constants other than t can be treated semantically" (ibid.). The possibility of adding not only **w**, but also propositional constants such as **t** and **l**, then, have already been recognized — both syntactically and semantically — within the literature on relevant logics. These constants are often glossed as expressing the conjunction of every truth or every logical truth. As we saw in the introduction, this view was voiced in (Anderson and Belnap, 1975, § 27.1.2), wherein **t** was given the latter interpretation, and **w** — "the world" — the former. Anderson and Belnap suggested three different axioms for **t** and one for **w**:

$$
\begin{array}{ll}
(\mathbf{t}1) & (\mathbf{t} \to \mathbf{t}) \to \mathbf{t} \\
(\mathbf{t}1') & \mathbf{t} \\
(\mathbf{t}2) & \mathbf{t} \to (A \to A) \\
(\mathbf{w}1) & A \equiv (\mathbf{w} \to A)
\end{array}
$$

I will discuss **w** first, before giving a short note on **t**.

Anderson and Belnap did not settle on an axiomatization of $\mathbf{w}$, but state the following criteria for how $\mathbf{w}$ ought turn out:

1. $A \to (\mathbf{w} \to A)$ should not be a logical theorem.
2. $(\mathbf{w} \to A) \to A$ should not be a logical theorem.
3. $(\mathbf{w} \to \mathbf{w}) \to \mathbf{w}$ should not be a logical theorem.
4. $\mathbf{w}$ should be a logical theorem.
5. $A \equiv (\mathbf{w} \to A)$ should be a logical theorem.
6. $(\mathbf{w} \to A \vee B) \equiv ((\mathbf{w} \to A) \vee (\mathbf{w} \to B))$ should be a logical theorem.

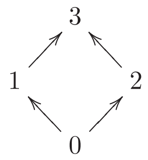
They then note that

> perhaps in the presence of (γ) $\mathbf{w}1$ suffices for *everything* one wants. But the question is: is there a formulation of $\mathbf{E}$ with $\mathbf{w}$ in which $\to$E and &I are the sole rules? We do not know.
>
> (Anderson and Belnap, 1975, p. 343)

Note, then, that γ, that is, R7, does suffice for both $\mathbf{w}$ as well as the derivability of the rule $\{A\} \Vdash \mathbf{w} \to A$, provided $\mathbf{w}1$. I want to suggest, rather, that these two logical principles are the basic properties of any world constant, regardless of which logic is at play. First of all, it seems evident that $\mathbf{w}$ ought to be true under any assumptions if it is to be read as the conjunction of every truth. $\mathbf{w}$, then, ought to come out a logical theorem. Furthermore, if A is true, then $\mathbf{w} \to A$ ought to be true, seeing as A, then, is to be found as one of $\mathbf{w}$'s conjuncts. $A \equiv (\mathbf{w} \to A)$, however, yields excluded middle (assuming $\mathbf{w}$ is a logical theorem), and so for logics such as $\mathbf{B}^{\mathrm{d}}$ in which excluded middle fails, it seems strange that a conjunction of every truth should force it to be valid. Furthermore, it seems hard to justify $(\mathbf{w} \to A \vee B) \equiv ((\mathbf{w} \to A) \vee (\mathbf{w} \to B))$ as a logical theorem without appealing to something like the counter-example rule — $\{A, \sim B\} \Vdash \sim(A \to B)$. For logics without this rule, however, it seems quite coherent that this formula should fail to be logically true. Note, then, that if $\mathbf{w}$ is axiomatized as suggested then all of Anderson and Belnap's wishlist items hold true — that is, if $\mathbf{w}$ is added to $\mathbf{E}$ axiomatized using $\mathbf{w}$ as an axiom along with the rule $\{A\} \Vdash \mathbf{w} \to A$, then 1–6 all obtain.[14] Such a propositional constant, then, can arguably be thought

[14] The claims of non-theoremhood can be verified by inspecting the algebraic model displayed in figure 2. $\mathbf{w}$ is a primitive axiom of $\mathbf{E}[\mathrm{A}\mathbf{w}1, \mathrm{A}\mathbf{w}3]$, and so also a theorem. That $A \equiv (\mathbf{w} \to A)$ and $(\mathbf{w} \to A \vee B) \equiv ((\mathbf{w} \to A) \vee (\mathbf{w} \to B))$ are logical theorems is easily shown using reasoning by cases (which holds in any disjunctive logic (Priest and Sylvan, 1992, p. 219)) on excluded middle, with the addition of the counter-example rule in the case of the latter formula — details are left for the reader.

$\mathcal{T} = \{x \mid 1 \preceq x\} = \{1, 3\}$
$[\![\mathbf{t}]\!] = 3 \qquad [\![\mathbf{w}]\!] = 1$
$(1 \to 1) \to 1 = 0$
$2 \to (1 \to 2) = 0$



$\to$	0	1	2	3	$\sim$
0	3	3	3	3	3
1	0	3	0	3	2
2	0	0	3	3	1
3	0	0	0	3	0

Figure 2. An **E**-model in which $A \overset{\mathbf{w}}{\mapsto} (\mathbf{w} \to A)$, $\mathbf{w}$, $\mathbf{t} \to (A \to A)$, $(\mathbf{t} \to A) \to A$ and $\mathbf{t} \wedge {\sim}\mathbf{t} \to A$ hold, but $(\mathbf{w} \to \mathbf{w}) \to \mathbf{w}$ and $A \to (\mathbf{w} \to A)$ do not.

of as expressing the conjunction of every truth, or maybe more precisely, the conjunction of the theory under investigation, seeing as if $\mathbf{w}$ is true under any set of assumptions, and A follows from a set of assumptions Γ, then so does $\mathbf{w} \to A$. The remaining question, however, is whether there is a way of adding it using only axioms. The minimal suggestion, then, seems to be to enthymematically strengthen the rule R$\mathbf{w}$ to $E^{\mathrm{R}\mathbf{w}}_{\mathbf{w}}$, that is to replace the rule $\{A\} \Vdash \mathbf{w} \to A$ with the axiom $A \wedge \mathbf{w} \to (\mathbf{w} \to A)$. One of the misgivings raised in Belnap and Dunn regarding Meyer's addition of Boolean negation to **R** was that although it is weakly conservative, it fails to be conservative in an "extended sense" since the explosion rule for the DeMorgan negation becomes derivable — $\{A, {\sim}A\} \Vdash B$ (cf. Belnap and Dunn, 1981, p. 341). In short, what they demanded was that any permissible extension be *strongly* conservative in the sense defined in (Øgaard, 2024). This, then, is one reason why one might be unhappy with a world constant if adding such requires also adding the γ-rule since this rule, then, is famously not derivable in **E**. The question, then, is whether the current suggestion fares better and does make for a strong conservative extension. Alas, I haven't been able to settle it, but post it here as an interesting question for further research:

Open question 7.1. Does adding a propositional constant $\mathbf{w}$ to **E** axiomatized using

$$\begin{array}{l|l} (\mathrm{A}\mathbf{w}1) & \mathbf{w} \\ (\mathrm{E}^{\mathrm{R}\mathbf{w}}_{\mathbf{w}}) & A \overset{\mathbf{w}}{\mapsto} (\mathbf{w} \to A) \end{array}$$

make for a strong conservative extension?And if so, is it also the case if the further axiom

$$\begin{array}{l|l} (\mathrm{A}\mathbf{w}2) & \mathbf{w} \wedge {\sim}\mathbf{w} \to A \end{array}$$

is added?

Notice, then, that the intersection of two persistent sets is always a persistent set, and so $G^\flat \cap Z^\flat$ is persistent. This set, if used as a truth-set for a propositional constant $\mathbf{s}$, validates what is in fact stronger than both of second and third $\mathbf{w}$-requirements above, namely

$$\begin{array}{l|l} (\mathrm{E}^{\mathrm{R6}}_{\mathbf{s}}) & A \wedge ({\sim}A \vee B) \overset{\mathbf{s}}{\mapsto} B \\ (\mathrm{As3}) & A \to (\mathbf{s} \to A) \end{array}$$

A truth-set for a world constant, however, must be such as to have g as one of its member, something which does not seem to be the case with $G^\flat \cap Z^\flat$.

Lastly, a note regarding $\mathbf{t}$. For adding $\mathbf{t}$ to $\mathbf{E}$, Anderson and Belnap set forth the axioms $\mathbf{t}1$ and $\mathbf{t}2$, and noted that $(\mathbf{t} \to A) \to A$ is a theorem of $\mathbf{E}$ thus extended. The perceived justification for reading $\mathbf{t}$ as expressing the conjunction of every logical truth in the context of $\mathbf{E}$ is that when added to $\mathbf{E}$, the resultant logic — $\mathbf{E}^{\mathbf{t}}$ — has the following property:

$$\varnothing \vdash_{\mathbf{E}^{\mathbf{t}}} A \Longleftrightarrow \varnothing \vdash_{\mathbf{E}^{\mathbf{t}}} \mathbf{t} \to A.$$

Note, then, that this property is lost if $\mathbf{w}$ is added, since $\mathbf{t} \to \mathbf{w}$ fails to be a logical theorem even though $\mathbf{w}$ is.

There is, however, a more stable meaning which can be attributed to $\mathbf{E}^{\mathbf{wt}}$'s $\mathbf{t}$, namely that of expressing the conjunction of every logical law, the latter term commonly being used to designate logical truths on the form $A \to B$. For $\mathbf{E}^{\mathbf{wt}}$, axiomatized as suggested above, it is easy to verify that

$$\varnothing \vdash_{\mathbf{E}^{\mathbf{wt}}} B \Longleftrightarrow \varnothing \vdash_{\mathbf{E}^{\mathbf{wt}}} \mathbf{t} \to B$$

for every $\to$-formula B. In fact, this interpretation is fitting not only in rather strong logics like $\mathbf{E}^{\mathbf{wt}}$, but also in $\mathbf{B}^{\mathrm{d}}$ with $\mathbf{t}$ axiomatized using $\mathbf{t} \to (A \to A)$ along with $\mathbf{t}$ as an axiom since this suffices for yielding that $A \to B$ is a logical theorem if and only if $\mathbf{t} \to (A \to B)$ is. Whether it is possible to add a propositional constant which for a given logic truly does express the conjunction of every logical theorem, however, is in general a more difficult question which requires attention to the particularity of the logic at hand. The notion of the minimal logical law, however, can, as we have seen, be added in a strong conservative fashion to all logics which the simplified semantics is fit for, provided the propositional constant is axiomatized using Anderson and Belnap's axioms $\mathbf{t}1'$ and $\mathbf{t}2$.

8. Summary

This paper has shown how to model proposition constants within the simplified Routley-Meyer semantics. It was shown that it is possible to add several such and that these can be used to define modal operators, implicative negations, enthymematical conditionals and to express propositions which intuitively require infinitely long formulas.

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