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Particular Reasoning Within Theories

Abstract. Particular reasoning enables the deductive proof of existential properties, such as the satisfiability/consistency of a set of formulas. In this work, we consider particular reasoning in the context of a theory of a given logic. The logic is presented by a semantic constraint specification. From this specification, we induce a particular calculus for the logic at hand. In this calculus we define what is a particular derivation in the context of a theory and show how to extract a model of the theory that satisfies the assertions within the derivation. We demonstrate that the induced particular calculus is both sound and complete with regard to the intended semantics. Our results are applicable to logics with a strong finite model property, including classical, intuitionistic, certain modal logics, and Nelson's N4 logic, among others.

Keywords: theories; particular reasoning; labelled deduction; knowledge representation

1. Introduction

A deductive calculus typically enables the proof of universal properties, such as determining whether a formula is a theorem or a consequence of a set of formulas. Semantically, this means we can conclude that a formula is valid, that is, it is true in all structures of the logic, or that a formula is a semantic consequence of a set of formulas, that is, every structure that satisfies the set also satisfies the formula.

Notions such as the satisfaction of a set of formulas or the existence of a model of a theory that also satisfies a given set of formulas are generally not addressed directly through deductive methods. Tableau systems [see

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9, 21] address this problem in a somewhat operational manner, but they do not provide a formal notion of particular derivation for that property. Furthermore, tableau systems address existential concepts and manage non-semantic entailment by incorporating a form of negation. Another approach is found in refutation systems, first described in [19]. Recent contributions to this field include [10, 22]. General requirements for refutation systems are detailed in [26].

In [24], we tackled the problem by defining a calculus for dealing with satisfiability of a sequence of formulas. We were able to cover a broad range of logics with relational semantics. To include logics lacking strong negation, we employed a labelled language that allows for positive assertions of the form $\omega : \varphi$ and negative assertions $\omega \not: \varphi$, indicating that the formula φ holds or does not hold in ω, respectively. The semantics of these logics are defined by a constraint specification. From this specification, we induce a deductive calculus and introduce the concept of particular derivation.

The main objective of this paper is to extend the approach in [24] to encompass theories. Herein we use the expression *particular reasoning* instead of *existential reasoning* in order to reach a broader audience. Indeed for researchers from philosophy the term *existential reasoning* has a different meaning. The problem of satisfiability of formulas in the presence of a theory plays an important role in applications, namely in artificial intelligence in general and in knowledge representation [4] in particular and also in formal systems specification and model checking [6]. In these cases we want to check if a set of assertions is compatible with a theory describing the universe of discourse. Other examples appear in argumentation theory [1], information security analysis [12], robot navigation [18]. The problem is also demanding from a theoretical point of view. Starting from a constraint specification for a given logic, we define a particular calculus for that logic. The concept of particular derivation must accommodate the presence of a theory. Specifically, the particular derivation should ensure that each formula in the theory is proven for every relevant world.

One of the advantages of our solution to the satisfiability of a sequence of labelled formulas in the presence of a theory is that we do so symbolically. Indeed we provide a particular calculus where we can construct a derivation proving that property.

The paper is organized as follows. In Section 2, we begin by defining the set of formulas and describing a running example involving the navigation of a robot over an area divided into squares. In Section 3, we explain how a particular calculus is generated from a constraint specification. After presenting several examples, we introduce the concept of particular derivation in the context of a theory. Section 4 focuses on proving that the induced calculus is sound for models that satisfy the constraint specification. Finally, in Section 5, we establish the completeness of the calculus when the constraint specification possesses a strong finite model property. This property is satisfied by classical, intuitionistic, some modal logics, as well as Nelson's N4 logic. We conclude the paper with an outlook on future research directions.

2. Constraint specification

We start by providing the linguistic setting. The main ingredient is a signature where we define the operators that we use for the particular logic at hand. We also consider a set of propositional symbols. Then we define the language or the set of formulas of the logic.

A *signature* C is a family $\{C_k\}_{k\in\mathbb{N}^+}$ where each C_k is the set of constructors of arity $k \in \mathbb{N}^+$. When defining a signature we only refer to the non-empty sets of constructors. We denote by $F(C, Q)$ the set of *formulas* inductively generated by C over a set Q of propositional symbols. When no confusion arises, we simply write F for $F(C, Q)$.

We provide an example in the context of a robot's navigation that will be used to illustrate the particular reasoning (see Example 3.5).

Example 2.1. Consider the navigation of a robot over a generic area divided in $n \times n$ squares denoted by the propositional symbols $s_{11}, \ldots, s_{nn}$ some of them with an obstacle. In each step the robot does not know in which square he is although he can learn. When in a square the robot has the capability of knowing if it is possible or not to move West, North, East and South denoted by the propositional symbols $poss_W, poss_N, poss_E, poss_S$. The actual moves of the robot are represented by the propositional symbols m_W, m_N, m_E, m_S. Hence the set of propositional symbols relevant for the navigation of the robot on the $n \times n$ area is

$$Q_{r,n} = \{s_{11}, \ldots, s_{nn}, poss_W, poss_N, poss_E, poss_S, m_W, m_N, m_E, m_S\}.$$

We adopt a modal logic for representing this *a priori* knowledge. The signature C_r is such that $(C_r)_1 = \{\neg, \Diamond\}$ and $(C_r)_2 = \{\supset\}$. The constructors $\wedge$, $\vee$ and ff are defined by abbreviation as usual. A possible generic theory $\Theta_{r,n}$ about general properties of the robot navigation is composed of the following formulas:

$$\bigwedge_{i=W,N,E,S} poss_i \supset \Diamond m_i, \qquad \bigwedge_{i=W,N,E,S} (\neg\, poss_i) \supset \neg \Diamond m_i,$$

$$\bigwedge_{\substack{i=2,\dots,n\\ j=1,\dots,n}} s_{ij} \supset \Diamond(m_W \supset s_{i-1j}), \qquad \bigwedge_{\substack{i=1,\dots,n\\ j=1,\dots,n-1}} s_{ij} \supset \Diamond(m_N \supset s_{ij+1}),$$

$$\bigwedge_{\substack{i=1,\dots,n-1\\ j=1,\dots,n}} s_{ij} \supset \Diamond(m_E \supset s_{i+1j}), \qquad \bigwedge_{\substack{i=1,\dots,n\\ j=2,\dots,n}} s_{ij} \supset \Diamond(m_S \supset s_{ij-1}),$$

$$\bigvee_{i,j=1,\dots,n} s_{ij}, \qquad \bigwedge_{\substack{i,i',j,j'=1,\dots,n\\ (i,j)\neq(i',j')}} (s_{ij} \supset \neg\, s_{i'j'}),$$

$$\bigwedge_{\substack{i,j=W,N,E,S\\ i\neq j}} (m_i \supset \neg\, m_j).$$

s_{13}	s_{23}	s_{33}
$\underline{s_{12}}$	s_{22}	s_{32}
s_{11}	s_{21}	$\underline{s_{31}}$

Figure 1. Description of an 3×3 area.

As an illustration we consider in Figure 1 a possible description d for an area a with $n = 3$. The underlined propositional symbols $\underline{s_{12}}$ and $\underline{s_{31}}$ indicate that there are obstacles in theses cells. The theory Θ_d over $Q_d = Q_{r,3}$ including $\Theta_{r,3}$ and

$$\neg\, s_{12} \wedge \neg\, s_{31},$$
$$s_{11} \supset (poss_E \wedge \neg\, poss_W \wedge \neg\, poss_N \wedge \neg\, poss_S),$$
$$s_{13} \supset (poss_E \wedge \neg\, poss_W \wedge \neg\, poss_N \wedge \neg\, poss_S),$$

$$s_{21} \supset (poss_W \wedge poss_N \wedge \neg\, poss_E \wedge \neg\, poss_S),$$

$$s_{22} \supset (poss_N \wedge poss_E \wedge poss_S \wedge \neg\, poss_W),$$

$$s_{23} \supset (poss_W \wedge poss_E \wedge poss_S \wedge \neg\, poss_N),$$

$$s_{32} \supset (poss_W \wedge poss_N \wedge \neg\, poss_E \wedge \neg\, poss_S),$$

$$s_{33} \supset (poss_W \wedge poss_S \wedge \neg\, poss_N \wedge \neg\, poss_E)$$

defines d. ⊣

Concerning the semantics we want to cope with several kinds of logics namely paraconsistent logics that seem to be very useful in Computer Science since inconsistencies may frequently occur in knowledge-based and intelligent information systems [2, 8, 15, 16]. We fulfil these objectives by defining the semantic properties of the constructors of a logic by a constraint specification. We start by identifying the envisaged (interpretation) structures. A *structure* over C is a triple (W, R, V) where W is a non-empty set, R is a binary relation over W, and $V : W \times F \to \{0, 1\}$ is a *valuation map*.

In the sequel, we use & and | for *and* and *or*, respectively. Moreover, we use $*$ to mean either & or | but in the same constraint we can only use one of the options. Furthermore, for simplifying the presentation, we may look at φ as $\mathsf{id}(\varphi)$ where id is the identity constructor.

To enhance readability, we no longer refer to the structure (W, R, V) or the signature C.

DEFINITION 2.1. A *local constraint* for $c \in C_n$ *depending on* $c_1, \dots, c_m \in C_1 \cup \{\mathsf{id}\}$ with c different from $c_1, \dots, c_m$ when $n = 1$ and *on* $i_1, \dots, i_m \in \{1, \dots, n\}$ is a constraint of the form

$$\text{if } V(w, c_1(\varphi_{i_1})) = b_1 * \cdots * V(w, c_m(\varphi_{i_m})) = b_m \text{ then} \qquad V(w, c(\varphi_1, \dots, \varphi_n)) = b$$

where $b_1, \dots, b_m, b \in \{0, 1\}$.

DEFINITION 2.2. A *non-local constraint* for $c \in C_n$ is a constraint of one of the forms

- *universal*: for all $w' \in W$ such that $(w, w') \in R$,

$$\text{if either } V(w', \varphi_1) = b_1 \mid \cdots \mid V(w', \varphi_n) = b_n \text{ then} \qquad V(w, c(\varphi_1, \dots, \varphi_n)) = b$$

- *existential*: for some $w' \in W$ such that $(w, w') \in R$,

$$\text{if } V(w', \varphi_1) = 1 - b_1 \& \ldots \& V(w', \varphi_n) = 1 - b_n \text{ then} \qquad V(w, c(\varphi_1, \ldots, \varphi_n)) = 1 - b$$

for unique $b, b_1, \ldots, b_n \in \{0, 1\}$.

In the sequel we do not strictly present the constraints in the form defined above but rather use a more readable presentation. Additionally, we will omit references to id on dependencies.

Example 2.2. We denote by $\mathcal{S}_{\mathsf{K}}$ the collection of local and non-local constraints for modal logic K [see 5] over a set of propositional symbols and a signature C_{K} with $(C_{\mathsf{K}})_1 = \{\neg, \Diamond\}$ and $(C_{\mathsf{K}})_2 = \{\supset\}$, composed of

- $V(w, \neg\,\varphi) = 1$ if and only if $V(w, \varphi) = 0$
- $V(w, \varphi_1 \supset \varphi_2) = 1$ if and only if $V(w, \varphi_1) = 0 \mid V(w, \varphi_2) = 1$
- $V(w, \Diamond\varphi) = 1$ if and only if there is $w' \in W$ such that $(w, w') \in R$ and $V(w', \varphi) = 1$.

Observe that we are simplifying the presentation of the specification since for example the first constraint should be seen as an abbreviation of

- if $V(w, \varphi) = 0$ then $V(w, \neg\,\varphi) = 1$,
- if $V(w, \varphi) = 1$ then $V(w, \neg\,\varphi) = 0$,

and similarly for the other constraints above. Observe that $\neg$ is a unary constructor with a local constraint depending on id and $\supset$ is a binary constructor with a local constraint depending twice on id. Note also that $\Diamond$ is a unary constructor with a non-local existential constraint. We will use this constraint specification for our running example of navigation of a robot (see Example 2.1). ⊣

DEFINITION 2.3. The *hereditary constraint* states that

$$\text{if } V(w, \varphi) = 1 \text{ and } (w, w') \in R \text{ then } V(w', \varphi) = 1.$$

This constraint means that if a formula is satisfied in world w then it will be satisfied in every world w' related with w. We are ready to define the last kind of constraints.

DEFINITION 2.4. A *relational constraint* is a constraint of one of the following forms

- *universal relational constraints*: either $\forall w\,(w,w) \in R$ or, for $k \in \mathbb{N}^+$, $i,j \in \{1,\ldots,k\}$,

$$\forall w_1,\ldots,w_k \text{ if } (w_1,w_2),\ldots,(w_{k-1},w_k) \in R \text{ then } (w_i,w_j) \in R$$

- *coherent relational constraints*: either $\forall w\,\exists w'\,(w,w') \in R$ or

$$\forall w_1,\ldots,w_k if (w_1,w_2),\ldots,(w_{k-1},w_k) \in R \text{ then } \exists w'\,(w_{i_1},w'),\ldots,(w_{i_n},w') \in R$$

  where $k \in \mathbb{N}^+$ and $i_1,\ldots,i_n \in \{1,\ldots,k\}$.

In the above definition we follow [20] for the language used for presenting the relational constraints.

Example 2.3. We denote by $\mathcal{S}_{\mathsf{J}}$ the set of constraints for intuitionistic logic J [see 25] over a set of propositional symbols and a signature C_{J} such that $(C_{\mathsf{J}})_1 = \{\neg\}$ and $(C_{\mathsf{J}})_2 = \{\wedge,\vee,\supset\}$, composed of

- $V(w,\varphi_1 \wedge \varphi_2) = 1$ if and only if $V(w,\varphi_1) = 1$ & $V(w,\varphi_2) = 1$
- $V(w,\varphi_1 \vee \varphi_2) = 1$ if and only if $V(w,\varphi_1) = 1 \mid V(w,\varphi_2) = 1$
- $V(w,\neg\varphi) = 1$ if and only if $V(w',\varphi) = 0$ for all $w' \in W$ such that $(w,w') \in R$
- $V(w,\varphi_1 \supset \varphi_2) = 1$ if and only if $V(w',\varphi_1) = 0 \mid V(w',\varphi_2) = 1$ for all $w' \in W$ st $(w,w') \in R$
- the hereditary constraint
- universal relational constraints: $\forall w\,(w,w) \in R$ and $\forall w_1,w_2,w_3$

$$\text{if } (w_1,w_2),(w_2,w_3) \in R \text{ then } (w_1,w_3) \in R.$$

Hence $\mathcal{S}_{\mathsf{J}}$ has two local constraints for binary constructors $\wedge$ and $\vee$ as well as universal constraints for unary constructor $\neg$ and for binary constructor $\supset$.

Moreover, we denote by $\mathcal{S}_{\mathsf{J}^+}$ the set of constraints for positive intuitionistic logic J^+ over a set of propositional symbols and a signature C_{J^+} with $(C_{\mathsf{J}^+})_2 = \{\wedge,\vee,\supset\}$, composed by the constraints in $\mathcal{S}_{\mathsf{J}}$ with the exception of the non-local constraint for $\neg$. ⊣

We will also consider the constraints for two paraconsistent logics.

Example 2.4. Given a set P, we denote by $\mathcal{S}_{\mathsf{N4}}$ the set of constraints for Nelson's logic $\mathsf{N4}$ [see 23] over $\{p, \sim p : p \in P\}$ and a signature C_{N4} with $(C_{\mathsf{N4}})_1 = \{\sim\sim\}$ and $(C_{\mathsf{N4}})_2 = \{\wedge,\vee,\supset,\sim\wedge,\sim\vee,\sim\supset\}$, composed of the constraints in $\mathcal{S}_{\mathsf{J}^+}$, defined in Example 2.3, as well as

- $V(w, \sim\!\sim \varphi) = 1$ if and only if $V(w, \varphi) = 1$,
- $V(w, \sim\!(\varphi_1 \supset \varphi_2)) = 1$ if and only if $V(w, \varphi_1) = 1$ & $V(w, \varphi_2) = 0$,
- $V(w, \sim\!(\varphi_1 \wedge \varphi_2)) = 1$ if and only if $V(w, \varphi_1) = 0 \mid V(w, \varphi_2) = 0$,
- $V(w, \sim\!(\varphi_1 \vee \varphi_2)) = 1$ if and only if $V(w, \varphi_1) = 0$ & $V(w, \varphi_2) = 0$.

This constraint specification involves $\wedge$, $\vee$ and $\supset$ that are shared with J^+ that we discussed in Example 2.3. The unary constructor $\sim\!\sim$ and binary constructors $\sim\!\supset$, $\sim\!\wedge$ and $\sim\!\vee$ are local. ⊣

Example 2.5. We denote by $\mathcal{S}_{\mathsf{imbC}}$ the collection of constraints for paraconsistent logic imbC [see 7]) over a set of propositional symbols and a signature C_{imbC} with $(C_{\mathsf{imbC}})_1 = \{\sim, \circ\}$ and $(C_{\mathsf{imbC}})_2 = \{\wedge, \vee, \supset\}$, composed of the constraints in $\mathcal{S}_{\mathsf{J}^+}$, defined in Example 2.3 plus

- if $V(w, \varphi) = 0$ then $V(w, \sim\!\varphi) = 1$,
- if $V(w, \varphi) = 1$ & $V(w, \sim\!\varphi) = 1$ then $V(w, \circ\varphi) = 0$.

For instance, in $\mathcal{S}_{\mathsf{imbC}}$, nothing is stated for $V(\omega, \sim\!\varphi) = 0$. Thus, both $V(\omega, \varphi) = 1$ and $V(\omega, \varphi) = 0$ are compatible with $V(\omega, \sim\!\varphi) = 0$.

The constraints for $\wedge$, $\vee$ and $\supset$ are shared with J^+ that we discussed in Example 2.3. The constructor $\sim$ has a local constraint depending on id. Finally, constructor $\circ$ has a local constraint that depends on id and $\sim$. ⊣

DEFINITION 2.5. A *constraint specification* $\mathcal{S}$ is a set of constraints containing either local or non-local constraints for the constructors in C, possibly the hereditary constraint and possibly universal or coherent constraints over relation R.

When $\mathcal{S}$ satisfies the constraint

$$\forall w \, (w, w) \in R,$$

we say that $\mathcal{S}$ is *reflexive*. Furthermore, when it satisfies the constraint

$$\forall w \, \exists w' \, (w, w') \in R,$$

we say that $\mathcal{S}$ is *serial*. An example of a coherent relational constraint is

$$\forall w, w_1, w_2 \text{ if } (w, w_1), (w, w_2) \in R \text{ then } \exists w' \, (w_1, w'), (w_2, w') \in R$$

in modal logic $\mathsf{K2}$. Additionally, we say that $\mathcal{S}$ is *hereditary* if it satisfies the hereditary constraint.

In the sequel, it becomes handy to unfold the constraint specifications by considering all compatible cases.

DEFINITION 2.6. Let $\mathcal{S}$ be a constraint specification, $c \in C_n$ and $b \in \{0,1\}$. Assume that c has local constraints in $\mathcal{S}$ depending on $c_1, \ldots, c_m$ and $i_1, \ldots, i_m$. We denote by B_b^c the set of all tuples $(b_1, \ldots, b_m)$ in $\{0,1\}^m$ such that

- either there is a local constraint of the form if $V(w, c_1(\varphi_{i_1})) = b_1$ & $\ldots$ & $V(w, c_m(\varphi_{i_m})) = b_m$ then $V(w, c(\varphi_1, \ldots, \varphi_n)) = b$
- or there is a local constraint in $\mathcal{S}$ of the form if $V(w, c_1(\varphi_{i_1})) = b'_1 \mid \cdots \mid V(w, c_m(\varphi_{i_m})) = b'_m$ then $V(w, c(\varphi_1, \ldots, \varphi_n)) = b$ such that b_i is b'_i for some $1 \leqslant i \leqslant m$.

Moreover, we denote by $\overline{B}_b^c$ the set $\{0,1\}^m \setminus B_{1-b}^c$. Observe that $\overline{B}_b^c$ extends B_b^c with all possibles alternatives that do not contradict b.

Example 2.6. Recall Example 2.5 where the set of constraints $\mathcal{S}_{\mathsf{imbC}}$ for paraconsistent logic imbC was introduced. In particular the constraints for $\vee$ in $\mathcal{S}_{\mathsf{imbC}}$ are if $V(w, \varphi_1) = 1 \mid V(w, \varphi_2) = 1$ then $V(w, \varphi_1 \vee \varphi_2) = 1$ and if $V(w, \varphi_1) = 0$ & $V(w, \varphi_2) = 0$ then $V(w, \varphi_1 \vee \varphi_2) = 0$. Thus

$$B_1^\vee = \{(1,0),(0,1),(1,1)\} \text{ and } B_0^\vee = \{(0,0)\}.$$

Moreover,

$$\overline{B}_1^\vee = B_1^\vee \text{ and } \overline{B}_0^\vee = B_0^\vee.$$

On the other hand, the constraint for $\sim$ in $\mathcal{S}_{\mathsf{imbC}}$ is if $V(w, \varphi) = 0$ then $V(w, {\sim}\varphi) = 1$. Thus, $B_1^\sim = \{0\}$ and $B_0^\sim = \emptyset$ and so $\overline{B}_1^\sim = \{0,1\}$ and $\overline{B}_0^\sim = \{1\}$. The only constraint for $\circ$ is if $V(w, \varphi) = 1$ & $V(w, {\sim}\varphi) = 1$ then $V(w, \circ\varphi) = 0$. Hence,

$$B_0^\circ = \{(1,1)\} \text{ and } B_1^\circ = \emptyset$$

and so $\overline{B}_1^\circ = \{(1,0),(0,1),(0,0)\}$, $\overline{B}_0^\circ = \{(1,1),(1,0),(0,1),(0,0)\}$. ⊣

We say that a structure M *meets* the constraint specification $\mathcal{S}$ whenever M fulfils all the constraints in $\mathcal{S}$. Observe that for every c with local constraints depending on $c_1, \ldots, c_m$ and $i_1, \ldots, i_m$ and $b \in \{0,1\}$ if $V(w, c(\varphi_1, \ldots, \varphi_n)) = b$ then there is $(b_1, \ldots, b_m) \in \overline{B}_b^c$ such that $V(w, c_j(\varphi_{i_j})) = b_j$ for $j = 1, \ldots, m$. We denote by $\mathcal{M}_{\mathcal{S}}$ the class of all structures that meet $\mathcal{S}$.

3. Particular reasoning in the context of a theory

In this section, we provide a symbolic method to draw particular inferences from a theory determining whether a formula is consistent with a theory, that is, if it holds in a model of the theory. For this purpose, we consider labelled formulas.

A *labelled formula* is an assertion of the form $\omega : \delta$ or $\omega \not: \delta$, where ω is a label and $\delta \in F$. We assume a fixed, countable set $\Omega = \{\omega_1, \dots\}$ of *labels* and denote by F^Ω the set of all labelled formulas. Additionally, we assume a fixed binary relational symbol S and denote by S^Ω the *set of relational assertions* $\{S\omega\omega' \mid \omega, \omega' \in \Omega\}$.

We begin by generating particular deductive calculus based on a given constraint specification for a logic system. The following notation is required. Let $b \in \{0,1\}$. Then

$$\omega \bowtie_b \varphi = \begin{cases} \omega : \varphi & \text{whenever } b = 1 \\ \omega \not: \varphi & \text{otherwise.} \end{cases}$$

Definition 3.1. A constraint specification $\mathcal{S}$ (over C and P) *induces a particular proof system* $\mathcal{D}_\mathcal{S} = (Ax, Rs)$ where Ax is the set of *particular axioms*

$$\text{Ax}_1 \ \frac{}{\omega : \alpha} \qquad \text{Ax}_2 \ \frac{}{\omega \not: \alpha} \qquad \text{Ax}_3 \ \frac{}{S\omega\omega'}$$

for $\alpha \in P$ and Rs includes the following rules:

- each $c \in C_n$ with local constraints in $\mathcal{S}$ depending on $c_1, \dots, c_m \in C_1 \cup \{\mathsf{id}\}$ and on $i_1, \dots, i_m$ and $b \in \{0,1\}$ induces the rule

$$c_b \ \frac{\bigoplus_{(b_1,\dots,b_m) \in \overline{B}^c_b} \quad \omega \bowtie_{b_1} c_1(\varphi_{i_1}) \ \dots \ \omega \bowtie_{b_m} c_m(\varphi_{i_m})}{\omega \bowtie_b c(\varphi_1, \dots, \varphi_n)}$$

- each universal non-local constraint of the form

$$\text{if } V(w', \varphi_1) = b_1 \mid \dots \mid V(w', \varphi_n) = b_n$$

  for all $w' \in W$ such that $(w, w') \in R$ then $V(w, c(\varphi_1, \dots, \varphi_n)) = b$, where $c \in C_n$ and $b \in \{0,1\}$ induces the *universal non-local rule*

$$c_b \ \frac{S\omega\omega' \ \lhd \ \omega' \bowtie_{b_1} \varphi_1 \ \oplus \ \cdots \ \oplus \ \omega' \bowtie_{b_n} \varphi_n}{\omega \bowtie_b c(\varphi_1, \dots, \varphi_n)} \ \omega'$$

- each existential non-local constraint of the form:
  if $V(w', \varphi_1) = b_1, \dots, V(w', \varphi_n) = b_n$ for some $w' \in W$ st $(w, w') \in R$, then $V(w, c(\varphi_1, \dots, \varphi_n)) = b$, where $c \in C_n$ and $b \in \{0,1\}$ induces the *existential non-local rule*

$$c_b \ \frac{S\omega\omega' \quad \omega' \bowtie_{b_1} \varphi_1 \quad \dots \quad \omega' \bowtie_{b_n} \varphi_n}{\omega \bowtie_b c(\varphi_1, \dots, \varphi_n)}$$

- the universal relational constraint $\forall w\,(w,w) \in R$ induces the *reflexivity closure axiom* $\mathrm{T}\ \dfrac{}{S\omega\omega}\ \forall$
- each universal relational constraint of the form $\forall w_1, \ldots, w_k$

$$\text{if } (w_1, w_2), \ldots, (w_{k-1}, w_k) \in R \text{ then } (w_i, w_j) \in R$$

  induces the *universal closure rule* $\dfrac{S\omega_1\omega_2 \quad \cdots \quad S\omega_{k-1}\omega_k}{S\omega_i\omega_j}\ \forall$
- the coherent relational constraint $\forall w\,\exists w'\,(w,w') \in R$ induces the *seriality closure axiom* $\mathrm{D}\ \dfrac{}{S\omega\omega'}\ \exists$
- each coherent relational constraint of the form $\forall w_1, \ldots, w_k$

$$\text{if } (w_1, w_2), \ldots, (w_{k-1}, w_k) \in \ \text{ then } \exists w'\,(w_{i_1}, w'), \ldots, (w_{i_n}, w') \in R$$

  induces the *coherent closure rule*

$$\frac{S\omega_1\omega_2 \quad \cdots \quad S\omega_{k-1}\omega_k}{S\omega_{i_1}\omega' \quad \ldots \quad S\omega_{i_n}\omega'}\ \exists$$

- the hereditary constraint induces the *hereditary closure rule*

$$\mathrm{H}\ \frac{S\omega\omega' \quad \omega : \varphi}{\omega' : \varphi}\ \forall$$

Observe that in a calculus for particular reasoning both $\omega : p$ and $\omega \not: p$ are axioms since each of them is satisfiable. Nevertheless both should not be present in the same derivation. The local rules follow straightforwardly the unfolding (see Definition 2.6) of the local constraints in $\mathcal{S}$. As a result of the unfolding each disjunctive local constraint originates several rules with conjunctive premises. For an illustration recall the disjunctive local constraint

$$\text{if } V(w, \varphi_1) = 1 \mid V(w, \varphi_2) = 1 \text{ then } V(w, \varphi_1 \vee \varphi_2) = 1$$

in Example 2.3. Its unfolding is $\overline{B}_1^{\vee} = \{(1,0), (0,1), (1,1)\}$ see Example 2.6. Hence in $\mathcal{D}_{\mathcal{S}_{\mathsf{imbC}}}$ we have the following rule

$$\vee_1 \quad \frac{\omega : \varphi_1 \quad \omega : \varphi_2 \quad \oplus \quad \omega : \varphi_1 \quad \omega \not: \varphi_2 \quad \oplus \quad \omega \not: \varphi_1 \quad \omega : \varphi_2}{\omega : \varphi_1 \vee \varphi_2}$$

Thus $\oplus$ means that there are three choices for $\vee$ to have value 1.

A universal non-local rule has the intuitive meaning that if

$$\text{for every } \omega' \text{ such that } S\omega\omega',\ \text{exists } i = 1, \ldots, n \text{ such that } \omega' \bowtie_{b_i} \varphi_i$$

then $\omega \bowtie_b c(\varphi_1, \dots, \varphi_n)$ also holds. So $\oplus$ means that for each ω' there is a choice.

An existential non-local rule has the intuitive meaning that if there is ω' such that $S\omega\omega'$ holds and $\omega' \bowtie_{b_i} \varphi_i$ also holds for each $i = 1, \dots, n$ then $\omega \bowtie_b c(\varphi_1, \dots, \varphi_n)$ also holds. The remaining axioms and rules are closure axioms and rules. The meaning of that will be made clear when defining the notion of derivation.

The reader may wonder why we may need the reflexivity and the serial closure axioms when we have Ax3. The difference between those closure axioms and Ax3 is that when the closure axioms are in the calculus the set of relational formulas in a derivation must comply with the respective properties.

Example 3.1. Recall Example 2.2. Let $\mathcal{D}_{\mathcal{S}_{\mathsf{K}}}$ be the labelled particular calculus for K with the following axioms and rules

$$\text{Ax}_1 \quad \frac{}{\omega : p} \qquad \text{Ax}_2 \quad \frac{}{\omega \not: p} \qquad \text{Ax}_3 \quad \frac{}{S\omega\omega'}$$

$$\supset_1 \quad \frac{\omega : \varphi_1 \quad \omega : \varphi_2 \quad \oplus \quad \omega \not: \varphi_1 \quad \omega : \varphi_2 \quad \oplus \quad \omega \not: \varphi_1 \quad \omega \not: \varphi_2}{\omega : \varphi_1 \supset \varphi_2}$$

$$\supset_0 \quad \frac{\omega : \varphi_1 \quad \omega \not: \varphi_2}{\omega \not: \varphi_1 \supset \varphi_2}$$

$$\neg_1 \quad \frac{\omega \not: \varphi}{\omega : \neg \varphi} \qquad \neg_0 \quad \frac{\omega : \varphi}{\omega \not: \neg \varphi}$$

$$\Diamond_1 \quad \frac{S\omega\omega' \qquad \omega' : \varphi}{\omega : \Diamond\varphi} \qquad \Diamond_0 \quad \frac{S\omega\omega' \ \lhd \ \omega' \not: \varphi}{\omega \not: \Diamond\varphi}\, \omega'$$

Furthermore, we define $\mathcal{D}_{\mathsf{K}2}$ as an extension of $\mathcal{D}_{\mathbf{K}}$ by adding the coherent closure rule

$$2 \quad \frac{S\omega\omega_1 \quad S\omega\omega_2}{S\omega_1\omega' \quad S\omega_2\omega'}\, \exists$$

For example rule $\supset_1$ indicates that there are three ways for having in $\varphi_1 \supset \varphi_2$ in ω: either we have both φ_1 and φ_2 in ω or we do not have φ_1 in ω but we have φ_2 in ω or we do not have both φ_1 and φ_2 in ω. On the other hand, rule $\Diamond_0$ states that $\Diamond\varphi$ does not hold in ω provided that for every ω' such that $S\omega\omega'$ we have that φ does not hold in ω'. $\dashv$

Example 3.2. The particular proof system $\mathcal{D}_{\mathcal{S}_\mathsf{J}}$ for intuitionistic logic J induced by constraint specification $\mathcal{S}_\mathsf{J}$ introduced in Example 2.3 is such that

$$\mathrm{Ax}_1\ \frac{}{\omega : p} \qquad \mathrm{Ax}_2\ \frac{}{\omega \not: p} \qquad \mathrm{Ax}_3\ \frac{}{S\omega\omega'}$$

$$\wedge_1\ \frac{\omega : \varphi_1 \quad \omega : \varphi_2}{\omega : \varphi_1 \wedge \varphi_2}$$

$$\wedge_0\ \frac{\omega \not: \varphi_1\ \omega : \varphi_2 \ \oplus\ \omega \not: \varphi_1\ \omega \not: \varphi_2 \ \oplus\ \omega : \varphi_1\ \omega \not: \varphi_2}{\omega \not: \varphi_1 \wedge \varphi_2}$$

$$\vee_1\ \frac{\omega : \varphi_1\ \omega : \varphi_2 \ \oplus\ \omega : \varphi_1\ \omega \not: \varphi_2 \ \oplus\ \omega \not: \varphi_1\ \omega : \varphi_2}{\omega : \varphi_1 \vee \varphi_2}$$

$$\vee_0\ \frac{\omega \not: \varphi_1 \quad \omega \not: \varphi_2}{\omega \not: \varphi_1 \vee \varphi_2}$$

$$\neg_1\ \frac{S\omega\omega' \ \lhd\ \omega' \not: \varphi}{\omega : \neg\varphi}\,\omega' \qquad \neg_0\ \frac{S\omega\omega' \quad \omega' : \varphi}{\omega \not: \neg\varphi}$$

$$\supset_1\ \frac{S\omega\omega' \ \lhd\ \omega' \not: \varphi_1 \ \oplus\ \omega' : \varphi_2}{\omega : \varphi_1 \supset \varphi_2}\,\omega'$$

$$\supset_0\ \frac{S\omega\omega' \quad \omega' : \varphi_1 \quad \omega' \not: \varphi_2}{\omega \not: \varphi_1 \supset \varphi_2}$$

$$\mathrm{H}\ \frac{S\omega\omega' \quad \omega : \varphi}{\omega' : \varphi}\,\forall \qquad \mathrm{T}\ \frac{}{S\omega\omega}\,\forall \qquad 4\ \frac{S\omega\omega' \quad S\omega'\omega''}{S\omega\omega''}\,\forall$$

Observe that $\mathcal{D}_{\mathcal{S}_{\mathsf{J}+}}$ is the particular calculus obtained from $\mathcal{D}_{\mathcal{S}_\mathsf{J}}$ by deleting rules $\neg_1$ and $\neg_0$ for negation. ⊣

Example 3.3. Recall Examples 2.4 and 3.2. Let $\mathcal{D}_{\mathcal{S}_{\mathsf{N4}}}$ be the particular calculus for $\mathsf{N4}$ extending $\mathcal{D}_{\mathcal{S}_{\mathsf{J}+}}$ with

$$\mathrm{Ax}_1\ \frac{}{\omega : \alpha} \qquad \mathrm{Ax}_2\ \frac{}{\omega \not: \alpha} \qquad \mathrm{Ax}_3\ \frac{}{S\omega\omega'}$$

where $\alpha \in \{p, {\sim}p : p \in P\}$, plus

$$\sim\sim_1\ \frac{\omega : \varphi}{\omega : {\sim}{\sim}\varphi} \qquad \sim\sim_0\ \frac{\omega \not: \varphi}{\omega \not: {\sim}{\sim}\varphi}$$

$${\sim}{\supset_1}\ \frac{\omega : \varphi_1 \quad \omega \not: \varphi_2}{\omega : {\sim}(\varphi_1 \supset \varphi_2)}$$

$$\sim\supset_0 \quad \frac{\omega \not: \varphi_1 \; \omega \not: \varphi_2 \;\oplus\; \omega \not: \varphi_1 \; \omega : \varphi_2 \;\oplus\; \omega : \varphi_1 \; \omega : \varphi_2}{\omega \not: \sim(\varphi_1 \supset \varphi_2)}$$

$$\sim\wedge_1 \quad \frac{\omega \not: \varphi_1 \; \omega \not: \varphi_2 \;\oplus\; \omega \not: \varphi_1 \; \omega : \varphi_2 \;\oplus\; \omega : \varphi_1 \; \omega \not: \varphi_2}{\omega : \sim(\varphi_1 \wedge \varphi_2)}$$

$$\sim\wedge_0 \quad \frac{\omega : \varphi_1 \quad \omega : \varphi_2}{\omega \not: \sim(\varphi_1 \wedge \varphi_2)}$$

$$\sim\vee_1 \quad \frac{\omega \not: \varphi_1 \quad \omega \not: \varphi_2}{\omega : \sim(\varphi_1 \vee \varphi_2)}$$

$$\sim\vee_0 \quad \frac{\omega \not: \varphi_1 \; \omega : \varphi_2 \;\oplus\; \omega : \varphi_1 \; \omega \not: \varphi_2 \;\oplus\; \omega : \varphi_1 \; \omega : \varphi_2}{\omega \not: \sim(\varphi_1 \vee \varphi_2)}$$

For example we do not have $\sim(\varphi_1 \wedge \varphi_2)$ provided that we have φ_1 and φ_2 in ω. ⊣

Example 3.4. Recall Examples 2.5 and 3.2. Let $\mathcal{D}_{\mathcal{S}_{\mathsf{imbC}}}$ be the labelled particular calculus for imbC extending $\mathcal{D}_{\mathcal{S}_{\mathsf{J}+}}$ with

$$\sim_1 \quad \frac{\omega : \varphi \;\oplus\; \omega \not: \varphi}{\omega : \sim\varphi} \qquad \sim_0 \quad \frac{\omega : \varphi}{\omega \not: \sim\varphi}$$

$$\circ_1 \quad \frac{\omega : \varphi \; \omega \not: \sim\varphi \;\oplus\; \omega \not: \varphi \; \omega : \sim\varphi \;\oplus\; \omega \not: \varphi \; \omega \not: \sim\varphi}{\omega : \circ\varphi}$$

$$\circ_0 \quad \frac{\omega : \varphi \; \omega : \sim\varphi \;\oplus\; \omega : \varphi \; \omega \not: \sim\varphi \;\oplus\; \omega \not: \varphi \; \omega : \sim\varphi \;\oplus\; \omega \not: \varphi \; \omega \not: \sim\varphi}{\omega \not: \circ\varphi}$$

To address assertions that apply universally, we use non-labelled formulas. The concept of a sequence of labelled formulas $\beta_1, \ldots, \beta_s$ to be particularly derived from a theory Θ (a set of formulas) is crucial.

Definition 3.2. We say that $\beta_1, \ldots, \beta_s$ in F^Ω is *particularly derived* from $\Theta \subseteq F$ in $\mathcal{D}_\mathcal{S}$, written

$$\Theta \vdash^{\exists}_{\mathcal{D}_\mathcal{S}} \beta_1, \ldots, \beta_s$$

whenever there is a sequence $\pi = \eta_1 \ldots \eta_t$ of assertions in $F^\Omega \cup S^\Omega$ with β_j occurring in π for each j and for every $i = 1, \ldots, t$

- either η_i is an axiom (justified by Ax);
- or there is a local rule c_b such that $\omega \bowtie_{b_1} c_1(\varphi_{i_1}), \ldots, \omega \bowtie_{b_m} c_m(\varphi_{i_m})$ occur in $\eta_1 \ldots \eta_{i-1}$ for some $(b_1, \ldots, b_m) \in \overline{B}^c_b$ and η_i is $\omega \bowtie_b c(\varphi_1, \ldots, \varphi_n)$ (justified by c_b);

- or there is a universal non-local rule c_b such that for every ω' in Ω, whenever $S\omega\omega'$ is in $\eta_1 \dots \eta_t$ then $\omega' \bowtie_{b_j} \varphi_j$ occurs in $\eta_1 \dots \eta_{i-1}$ for some $j = 1, \dots, n$ and η_i is $\omega \bowtie_b c(\varphi_1, \dots, \varphi_n)$ (justified by c_b);
- or there is an existential non-local rule c_b such that $S\omega\omega'$ and $\omega \bowtie_{b_1} \varphi_1, \dots, \omega \bowtie_{b_n} \varphi_n$ occur in $\eta_1 \dots \eta_{i-1}$ and η_i is $\omega \bowtie_b c(\varphi_1, \dots, \varphi_n)$ (justified by c_b);
- or there is an hereditary closure rule H with premises appearing in $\eta_1 \dots \eta_{i-1}$ and the conclusion is η_i (justified by H);

(such a sequence is called a *prederivation*) such that

- $\omega : \theta$ occurs in π for each ω in π and $\theta \in \Theta$
- if $\mathcal{D}_\mathcal{S}$ includes the reflexivity closure axiom or the seriality closure axiom then for each label ω in π, either $S\omega\omega$ is in π or there is ω' such that $S\omega\omega'$ is in π, respectively;
- for each coherent closure rule in $\mathcal{D}_\mathcal{S}$ if the premises occur in π then for some ω' the conclusions should also be in π;
- for each universal closure rule in $\mathcal{D}_\mathcal{S}$ if the premises occur in π then the conclusion should also be in π;
- the sequence is closed for the hereditary rule whenever it is in $\mathcal{D}_\mathcal{S}$. That is, if $S\omega\omega'$ and $\omega : \varphi$ appear in π then $\omega' : \varphi$ should also appears in π;
- for no $\varphi \in F$ both $w : \varphi$ and $w \not: \varphi$ occurs in π.

We say that the sequence $\eta_1 \dots \eta_t$ is an *particular derivation* for $\beta_1, \dots, \beta_s$ under the theory Θ. That is, a particular derivation for $\beta_1, \dots, \beta_s$ from Θ is a prederivation that satisfies the closure conditions above.

Moreover we say that $\beta_{1,1} \dots \beta_{1,s_1} \oplus \dots \oplus \beta_{\ell,1} \dots \beta_{\ell,s_\ell}$ is *particularly derived under* Θ, written

$$\Theta \vdash^{\exists}_{\mathcal{D}_\mathcal{S}} \beta_{1,1}, \dots, \beta_{1,s_1} \oplus \dots \oplus \beta_{\ell,1}, \dots, \beta_{\ell,s_\ell}$$

whenever there is $j \in \{1, \dots, \ell\}$ such that $\Theta \vdash^{\exists}_{\mathcal{D}_\mathcal{S}} \beta_{j,1}, \dots, \beta_{j,s_j}$.

Finally, given $\varphi_1, \dots, \varphi_s \in F$, we write

$$\Theta \vdash^{\exists}_{\mathcal{D}_\mathcal{S}} \varphi_1, \dots, \varphi_s$$

whenever $\Theta \vdash^{\exists}_{\mathcal{D}_\mathcal{S}} \omega : \varphi_1, \dots, \omega : \varphi_s$ for some $\omega \in \Omega$. In this case we say that $\eta_1 \dots \eta_t$ is a *particular derivation* for $\varphi_1, \dots, \varphi_s$ under the theory Θ.

Observe that Θ is necessarily finite whenever $\Theta \vdash^{\exists}_{\mathcal{D}_\mathcal{S}} \beta_1, \dots, \beta_s$. Moreover, Θ does not generate an infinite number of new labels.

Example 3.5. Consider Example 2.1 and the deductive system $\mathcal{D}_{\mathcal{S}_\mathsf{K}}$ in Example 3.1 over Q_d that we now use for reasoning about the navigation of the robot. We show that when the robot is in square s_{11} then, after moving one step, he cannot move East. That is,

$$\Theta_d \vdash^{\exists}_{\mathcal{D}_{\mathcal{S}_\mathsf{K}}} \omega : s_{11}, \omega \not: \Diamond poss_E$$

Indeed, consider the following particular derivation:

1.	$\omega : s_{11}$	Ax_1
2.	$\omega : poss_E$	Ax_1
3.	$S\omega\omega'$	Ax_3
4.	$\omega' \not: poss_E$	Ax_2
5.	$\omega \not: \Diamond poss_E$	$\Diamond_0$:3,4
6.	$\omega' : m_E$	Ax_1
7.	$\omega : \Diamond m_E$	$\Diamond_1$:3,6
8.	$\omega : poss_E \supset \Diamond m_E$	$\supset_1$:2,7
9.	$\omega \not: poss_W$	Ax_2
10.	$\omega' \not: m_W$	Ax_2
11.	$\omega \not: \Diamond m_W$	$\Diamond_0$:3,10
12.	$\omega : poss_W \supset \Diamond m_W$	$\supset_1$:9,11
	$\vdots$	

where we omit the proof of $\omega : \theta$ and $\omega' : \theta$ for each $\theta \in \Theta_d \setminus \{poss_E \supset \Diamond m_E, poss_W \supset \Diamond m_W\}$ since these cases follow in a similar way.

In steps 1 to 5 we prove that that there exists a prederivation for $\omega : s_{11}$ and $\omega \not: \Diamond poss_E$ where we use axioms and rule $\Diamond_0$.

From step 6 onwards, we must prove that all assertions in Θ_d can be inferred for ω and ω'. We start with the proof of $\omega : poss_E \supset \Diamond m_E$ in step 8 which comes from the application of $\supset_1$ over steps 2 and 7. Then we prove in step 12 that $\omega : poss_W \supset \Diamond m_W$. According to rule $\supset_1$ of $\mathcal{D}_{\mathcal{S}_\mathsf{K}}$ we have three possibilities for inferring the formula. We chose the third one in the rule. We omit the proof of the other steps.

In this case there are no relational constraints and no hereditary constraint. $\dashv$

Observe that we can start a derivation by introducing all the relational assertions needed. Thus we assume without loss of generality that a derivation is of the form $\tau\pi$ where τ is a sequence of relational assertions called the *relational subderivation* and π is a sequence of labelled formulas called the *formula subderivation*. Similarly for prederivations.

Moreover, we may write

$$\{\omega_1, \ldots, \omega_n\} \bowtie_b \psi$$

to denote the sequence $\omega_1 \bowtie_b \psi, \ldots, \omega_n \bowtie_b \psi$.

We end the section with a technical result that will be needed later on when dealing with completeness. It states that if we have a derivation of a sequence of labelled formulas then we also have a derivation of a sequence where we put all the labelled formulas that are not labelled with ω plus all the other labelled formulas where we replace ω by a sequence of labels not occurring in the first part of the sequence.

PROPOSITION 3.1. *Assume that* $\vdash^{\exists}_{\mathcal{D}_S} \delta_1, \ldots, \delta_k$ *and* $\delta'_1, \ldots, \delta'_{k'}$ *are the labelled formulas in* $\delta_1, \ldots, \delta_k$ *that are not labelled with* ω. *Moreover, let* $\delta''_1, \ldots, \delta''_{k''}$ *be the other formulas. Then*

$$\vdash^{\exists}_{\mathcal{D}_S} \delta'_1, \ldots, \delta'_{k'}, [\delta''_1]^{\omega}_{\{\omega_1,\ldots,\omega_n\}}, \ldots, [\delta''_{k''}]^{\omega}_{\{\omega_1,\ldots,\omega_n\}}$$

where $\omega_1, \ldots, \omega_n$ *do not occur in* $\delta'_1, \ldots, \delta'_{k'}$.

PROOF. Let $\tau\pi$ be a prederivation for $\delta_1, \ldots, \delta_k$ where τ is the relational subprederivation and $\pi = \eta_1 \ldots \eta_t$ is the formula subprederivation assuming that $\omega_1, \ldots, \omega_n$ do not occur in $\tau\pi$. Consider the sequence τ' obtained from τ by replacing each relational assertion of the form

- $S\omega\omega$ by $S\omega_i\omega_j$, for $i, j = 1, \ldots, n$
- $S\omega\omega'$ by $S\omega_1\omega', \ldots, S\omega_n\omega'$
- $S\omega'\omega$ by $S\omega'\omega_1, \ldots, S\omega'\omega_n$

and the sequence $\pi' = \eta'_1 \ldots \eta'_{t'}$ obtained from π by replacing each η_i of the form

$$\omega \bowtie_b \psi \quad \text{by} \quad \omega_1 \bowtie_b \psi, \ldots, \omega_n \bowtie_b \psi.$$

(1) We start by showing that $\tau'\pi'$ is a prederivation for

$$\delta'_1, \ldots, \delta'_{k'}, [\delta''_1]^{\omega}_{\{\omega_1,\ldots,\omega_n\}}, \ldots, [\delta''_{k''}]^{\omega}_{\{\omega_1,\ldots,\omega_n\}}$$

by induction on the length t of π.

(Base) $t = 1$. There are two cases to consider. (a) η_1 is not labelled by ω. Then π' is composed by η_1. (b) η_1 is labelled by ω. Then π' is $[\delta_1]^{\omega}_{\{\omega_1,\ldots,\omega_n\}}$.

(Step) There are several cases to consider. We denote by ω'^{-1} the label ω' if $\omega' \neq \omega_1, \dots, \omega_n$ otherwise is ω. Note that if $S\omega'\omega''$ occur in τ' then $S\omega'^{-1}\omega''^{-1}$ occur in τ. Moreover, if $S\omega'^{-1}\nu$ occur in τ then there is ν' such that $S\omega'\nu'$ occur in τ' and ν'^{-1} is ν. Furthermore, if $\omega' \bowtie_b \psi$ occur in π' then $\omega'^{-1} \bowtie_b \psi$ occur in π.

(a) η_t is $\delta''_1 = \omega \bowtie_b \psi$ where ψ is $c(\varphi_1, \varphi_2)$, c has a universal non-local constraint in $\mathcal{S}$ and $b = 1$, $b_1 = 0$ and $b_2 = 1$. Assume that $\{\omega' : S\omega_1\omega' \text{ occur in } \tau'\} = \{\omega'_1, \dots, \omega'_\ell\}$. Observe that

$$\{\omega' : S\omega_1\omega' \text{ occur in } \tau'\} = \cdots = \{\omega' : S\omega_n\omega' \text{ occur in } \tau'\}.$$

Note that $S\omega\omega_i'^{-1}$ occur in τ for every $i = 1, \dots, \ell$ and moreover no other relational assertions occurs in τ with ω as the first component. Thus, for each $i = 1, \dots, \ell$, either $\omega_i'^{-1} \not: \varphi_1$ or $\omega_i'^{-1} : \varphi_2$ occur in π. Therefore, by the induction hypothesis, for each $i = 1, \dots, \ell$, $\tau'\eta'_1 \dots \eta'_{t'-1}$ is a prederivation for $\delta'_1, \dots, \delta'_{k'}, [\delta''_2]^{\omega}_{\{\omega_1,\dots,\omega_n\}}, \dots, [\delta''_{k''}]^{\omega}_{\{\omega_1,\dots,\omega_n\}}$ and for each $i = 1, \dots, \ell$, if $\omega'_i \neq \omega_1, \dots, \omega_n$ then it is also a prederivation either for $\omega'_i \not: \varphi_1$ or for $\omega'_i : \varphi_2$ and otherwise a prederivation either for $\omega_1 \not: \varphi_1, \dots, \omega_n \not: \varphi_1$ or for $\omega_1 : \varphi_2, \dots, \omega_n : \varphi_2$. Then $\tau'\eta'_1 \dots \eta'_{t'-1}\omega_1 \bowtie_b \psi, \dots, \omega_n \bowtie_b \psi$ is a prederivation for $\delta'_1, \dots, \delta'_{k'}, [\delta''_1]^{\omega}_{\{\omega_1,\dots,\omega_n\}}, \dots, [\delta''_{k''}]^{\omega}_{\{\omega_1,\dots,\omega_n\}}$.

(b) η_t is $\nu \bowtie_b \psi$ justified by H. Then $\tau\eta_1 \dots \eta_{t-1}$ is a prederivation for $\delta_1, \dots, \delta_k$ not including η_t. Consider three cases: (i) η_t is not δ_i for each $i = 1, \dots . k$. Hence, $\tau'\eta'_1 \dots \eta'_{t'-1}$ is a prederivation for

$$\delta'_1, \dots, \delta'_{k'}, [\delta''_1]^{\omega}_{\{\omega_1,\dots,\omega_n\}}, \dots, [\delta''_{k''}]^{\omega}_{\{\omega_1,\dots,\omega_n\}},$$

by the induction hypothesis. (ii) η_t is $\delta'_{i'}$ for some $i' = 1, \dots, k'$. Then $\tau'\eta'_1 \dots \eta'_{t'-1}$ by the induction hypothesis is a prederivation for

$$\delta'_1, \dots \delta'_{i'-1}, \delta'_{i+1}, \dots, \delta'_{k'}, [\delta''_1]^{\omega}_{\{\omega_1,\dots,\omega_n\}}, \dots, [\delta''_{k''}]^{\omega}_{\{\omega_1,\dots,\omega_n\}}$$

So by H the sequence $\tau'\eta'_1 \dots \eta'_{t'}$ is a prederivation for

$$\delta'_1, \dots, \delta'_{k'}, [\delta''_1]^{\omega}_{\{\omega_1,\dots,\omega_n\}}, \dots, [\delta''_{k''}]^{\omega}_{\{\omega_1,\dots,\omega_n\}}$$

(iii) η_t is $\delta''_{i''}$ for some $i'' = 1, \dots, k''$. We omit the proof because it similar to (ii).

The other cases follow in a similar way.

(2) Let $\tau\pi$ be a derivation for $\vdash^{\exists}_{\mathcal{D}_{\mathcal{S}}} \delta_1, \dots . \delta_k$. Note that $\tau\pi$ is also a prederivation for $\delta_1, \dots . \delta_k$. Hence, by (1), $\tau'\pi'$ is also a prederivation for

$$\delta'_1, \dots, \delta'_{k'}, [\delta''_1]^{\omega}_{\{\omega_1,\dots,\omega_n\}}, \dots, [\delta''_{k''}]^{\omega}_{\{\omega_1,\dots,\omega_n\}}.$$

It remains to show that $\tau'\pi'$ is closed for the hereditary rule and for relational closure rules if present.

(a) $\mathcal{S}$ is hereditary. Assume that $S\omega'\omega'_1$ and $\omega' : \psi$ are in $\tau'\pi'$. Then, $S\omega'^{-1}\omega_1'^{-1}$ and $\omega'^{-1} : \psi$ are in $\tau\pi$. Since $\tau\pi$ is a derivation then $\omega_1'^{-1} : \psi$ occurs in $\tau\pi$. Thus, $\omega'_1 : \psi$ occurs in $\tau'\pi'$.

The other cases are proved in a similar way. ⊣

4. Soundness of particular reasoning over a theory

The main objective of soundness is to prove that a particular derivation of a sequence of labelled formulas from a theory can be interpreted semantically as entailment of the sequence from the theory. We start by introducing some preliminary semantic notions. Let $\mathcal{S}$ be a constraint specification. We consider a structure $M = (W, R, V) \in \mathcal{M}_{\mathcal{S}}$ and an assignment $\rho : \Omega \to W$ over M. The *satisfaction* by M and ρ is defined as follows:

- $M\rho \Vdash \omega \bowtie_b \varphi$ whenever $V(\rho(\omega), \varphi) = b$
- $M\rho \Vdash S\omega\omega'$ whenever $(\rho(\omega), \rho(\omega')) \in R$.

Moreover,

$$M \Vdash \varphi$$

whenever $M\rho \Vdash \omega : \varphi$ for every assignment ρ over M. We say that $\beta_1, \dots, \beta_s \in F^\Omega$ is *particularly entailed* from $\Theta \subseteq F$, denoted by

$$\Theta \vDash^{\exists}_{\mathcal{M}_{\mathcal{S}}} \beta_1, \dots, \beta_s$$

whenever there is $M \in \mathcal{M}_{\mathcal{S}}$ and ρ over M such that $M \Vdash \theta$ for every $\theta \in \Theta$ and $M\rho \Vdash \beta_j$ for each $j = 1, \dots, s$. We then say that M is a *witness* for $\Theta \vDash^{\exists}_{\mathcal{M}_{\mathcal{S}}} \beta_1, \dots, \beta_s$. In the sequel we need the following notation: given an assignment ρ, $\omega' \in \Omega$ and $w' \in W$, we denote by $\rho^{\omega'}_{w'}$ the assignment such that $\rho^{\omega'}_{w'}(\omega') = w'$ and otherwise $\rho^{\omega'}_{w'}(\omega) = \rho(\omega)$.

As usual we are going to prove soundness by starting to prove that rules are sound. We begin by defining the conditions under which the various types of rules in Definition 3.1 are sound.

A local rule

$$c_b \ \frac{\bigoplus_{(b_1,\dots,b_m)\in \overline{B}^c_b} \quad \omega \bowtie_{b_1} c_1(\varphi_{i_1}) \quad \dots \quad \omega \bowtie_{b_m} c_m(\varphi_{i_m})}{\omega \bowtie_b c(\varphi_1, \dots, \varphi_n)}$$

is *sound with respect to* $\mathcal{M}_\mathcal{S}$ providing that for every $M \in \mathcal{M}_\mathcal{S}$, assignment ρ and $(b_1, \dots, b_m) \in \overline{B}^c_b$, if $M\rho \Vdash \omega \ \bowtie_{b_j} \ c_j(\varphi_{i_j})$ for each $j = 1, \dots, m$ then $M\rho \Vdash \omega \bowtie_b c(\varphi_1, \dots, \varphi_n)$.

A universal non-local rule

$$c_b \ \frac{S\omega\omega' \ \lhd \ \omega' \bowtie_{b_1} \varphi_1 \oplus \cdots \oplus \omega' \bowtie_{b_n} \varphi_n}{\omega \bowtie_b c(\varphi_1, \dots, \varphi_n)} \ \omega'$$

is *sound with respect to* $\mathcal{M}_\mathcal{S}$ providing that, for each $M \in \mathcal{M}_\mathcal{S}$ and ρ, $M\rho \Vdash \omega \bowtie_b c(\varphi_1, \dots, \varphi_n)$ whenever for every $w' \in W$, if $M\rho^{\omega'}_{w'} \Vdash S\omega\omega'$ then $M\rho^{\omega'}_{w'} \Vdash \omega' \bowtie_{b_i} \varphi_i$ for some $i = 1, \dots, n$.

An existential non-local rule

$$c_b \ \frac{S\omega\omega' \quad \omega' \bowtie_{b_1} \varphi_1 \quad \dots \quad \omega' \bowtie_{b_n} \varphi_n}{\omega \bowtie_b c(\varphi_1, \dots, \varphi_n)}$$

is *sound with respect to* $\mathcal{M}_\mathcal{S}$ providing that for every $M \in \mathcal{M}_\mathcal{S}$ and ρ, if $M\rho \Vdash S\omega\omega'$ and $M\rho \Vdash \omega' \ \bowtie_{b_j} \ \varphi_j$ for each $j = 1, \dots, n$ then $M\rho \Vdash \omega \bowtie_b c(\varphi_1, \dots, \varphi_n)$.

The hereditary rule is *sound with respect to* $\mathcal{S}$ providing that for every $M \in \mathcal{M}_\mathcal{S}$ and ρ, $M\rho \Vdash \omega' : \varphi$ whenever $M\rho \Vdash S\omega\omega'$ and $M\rho \Vdash \omega : \varphi$.

The reflexivity closure axiom is *sound with respect to* $\mathcal{S}$ providing that for each $M \in \mathcal{M}_\mathcal{S}$ and ρ, $M\rho \Vdash S\omega\omega$.

A universal closure rule is *sound with respect to* $\mathcal{S}$ providing that for each $M \in \mathcal{M}_\mathcal{S}$ and ρ, if $M\rho \Vdash S\omega_\ell\omega_{\ell+1}$ for $\ell = 1, \dots, k-1$ then $M\rho \Vdash S\omega_i\omega_j$.

The seriality closure axiom is *sound with respect to* $\mathcal{S}$ if for every $M \in \mathcal{M}_\mathcal{S}$ and ρ there is $w' \in W$ such that $M\rho^{\omega'}_{w'} \Vdash S\omega\omega'$.

A geometric closure rule is *sound with respect to* $\mathcal{S}$ whenever for every $M \in \mathcal{M}_\mathcal{S}$ and ρ if $M\rho \Vdash S\omega_\ell\omega_{\ell+1}$ for $\ell = 1, \dots, k-1$ then there is $w' \in W$ such that $M\rho^{\omega'}_{w'} \Vdash S\omega_{i_j}\omega'$ for $j = 1, \dots, n$.

Proposition 4.1. *Every rule in $\mathcal{D}_\mathcal{S}$ is sound with respect to $\mathcal{S}$.*

We omit the proof of this result because it follows straightforwardly. We are ready to introduce the notion of soundness for a calculus induced by a constraint specification $\mathcal{S}$. A particular calculus $\mathcal{D}_\mathcal{S}$ is *sound with respect to* $\mathcal{M}_\mathcal{S}$ whenever

$$\Theta \vdash^{\exists}_{\mathcal{D}_\mathcal{S}} \omega_1 \bowtie_{b_1} \psi_1, \dots, \omega_s \bowtie_{b_s} \psi_s \Longrightarrow \Theta \vDash^{\exists}_{\mathcal{M}_\mathcal{S}} \omega_1 \bowtie_{b_1} \psi_1, \dots, \omega_s \bowtie_{b_s} \psi_s.$$

In order to construct a witness in $\mathcal{M}_\mathcal{S}$ for

$$\Theta \vDash^{\exists}_{\mathcal{M}_\mathcal{S}} \omega_1 \bowtie_{b_1} \psi_1, \dots, \omega_s \bowtie_{b_s} \psi_s$$

from a particular derivation for $\omega_1 \bowtie_{b_1} \psi_1, \ldots, \omega_s \bowtie_{b_s} \psi_s$ from Θ we need to work with marked labelled formulas: $F^{\Omega^{\checkmark}} = \{\gamma^{\checkmark} \mid \gamma \in F^{\Omega}\}$. Moreover, we use the notation $\bar{S}$ for the reflexive and transitive closure of S. We need also the (hereditary) map

$$H^T_{\mathcal{S}} \colon F^{\Omega} \cup F^{\Omega^{\checkmark}} \to \wp\,(F^{\Omega} \cup F^{\Omega^{\checkmark}})$$

for $T \subseteq S^{\Omega}$ (relevant when $\mathcal{S}$ includes the hereditary constraint) where

- $H^T_{\mathcal{S}}(\omega \not: \varphi) = \{\omega \not: \varphi\}$ and $H^T_{\mathcal{S}}(\omega \not: \varphi^{\checkmark}) = \{\omega \not: \varphi^{\checkmark}\}$
- $H^T_{\mathcal{S}}(\omega : \varphi) = \{\omega : \varphi\} \cup \{\omega' : \varphi \mid \bar{S}\omega\omega' \in T$ and $\mathcal{S}$ is hereditary$\}$
- $H^T_{\mathcal{S}}(\omega : \varphi^{\checkmark}) = \{\omega : \varphi^{\checkmark}\} \cup \{\omega' : \varphi \mid \bar{S}\omega\omega' \in T$ and $\mathcal{S}$ is hereditary$\}$.

The first clause states that inheritances does not affect negative labelled formulas independently of being marked or not. The second and third clauses establish that the set of inherited labelled formulas of a formula (marked or not) labelled by ω is always composed of that labelled formula plus all formulas labelled with ω' (not marked) for every ω' such that $S\omega\omega'$.

We can extend $H^T_{\mathcal{S}}$ to $\wp(F^{\Omega} \cup F^{\Omega^{\checkmark}})$ by union. We are ready to introduce the *one-step alternative map* $\mathrm{Alt}^T_{\mathcal{S}}$. Each one-step alternative guarantees the satisfaction of the labelled formula in terms of its dependants.

DEFINITION 4.1. Let $T \subseteq S^{\Omega}$ be a finite set and $\mathcal{S}$ a constraint specification over a signature C and a set of propositional symbols Q of a structure $(\{\omega : \omega \text{ occurs in } T\}, \{(\omega, \omega') : S\omega\omega' \in T\}, V)$. The map

$$\mathrm{Alt}^T_{\mathcal{S}} \colon F^{\Omega} \cup F^{\Omega^{\checkmark}} \to \wp\wp\,(F^{\Omega} \cup F^{\Omega^{\checkmark}}),$$

called *one-step alternative map*, is defined as follows: $\mathrm{Alt}^T_{\mathcal{S}}(\omega \bowtie_b q)$ is

$$\{\{\omega \bowtie_b q^{\checkmark}\} \cup \{\omega' \bowtie_b q^{\checkmark} \mid \bar{S}\omega\omega' \in T, b = 1 \text{ and } \mathcal{S} \text{ is hereditary}\}\}$$

and

$$\mathrm{Alt}^T_{\mathcal{S}}(\omega \bowtie_b q^{\checkmark}) = \{\{\omega \bowtie_b q^{\checkmark}\}\}$$

for $q \in Q$, and for $\gamma = \omega \bowtie_b c(\varphi_1, \ldots, \varphi_n)$,

$$\mathrm{Alt}^T_{\mathcal{S}}(\gamma^{\checkmark}) = \{\{\gamma^{\checkmark}\}\}$$

and when $c \in C_n$ with

- local constraints in $\mathcal{S}$ depending on $c_1, \ldots, c_m \in C_1 \cup \{\mathsf{id}\}$ and on $i_1, \ldots, i_m$, then $\mathrm{Alt}^T_{\mathcal{S}}(\gamma)$ is

$$\{H^T_{\mathcal{S}}(\{\omega \bowtie_{b_1} c_1(\varphi_{i_1}), \ldots, \omega \bowtie_{b_m} c_m(\varphi_{i_m}), \gamma^{\checkmark}\}) : (b_1, \ldots, b_m) \in \overline{B}^c_b\}$$

- a universal non-local constraint in $\mathcal{S}$ on $b_1, \ldots, b_n \in \{0, 1\}$, then $\mathrm{Alt}^T_{\mathcal{S}}(\gamma)$ is

$$\{H^T_{\mathcal{S}}(\bigcup_{k=1}^{\ell} A^k \cup \{\gamma^{\checkmark}\}) : A^i \in \{A^{\omega_i, b'_1, \ldots, b'_n} \mid \exists_{j=1}^n b'_j = b_j\}, i = 1, \ldots, \ell\}$$

letting $\{\omega' : S\omega\omega' \in T\} = \{\omega_1, \ldots, \omega_\ell\}$ and $A^{\omega', b''_1, \ldots, b''_n}$ is $\{\omega' \bowtie_{b''_1} \varphi_1, \ldots, \omega' \bowtie_{b''_n} \varphi_n\}$
- an existential non-local constraint in $\mathcal{S}$ on $b_1, \ldots, b_n \in \{0, 1\}$

$$\mathrm{Alt}^T_{\mathcal{S}}(\gamma) = \{H^T_{\mathcal{S}}(\{\omega' \bowtie_{b_1} \varphi_1, \ldots, \omega' \bowtie_{b_n} \varphi_n, \gamma^{\checkmark}\}) : S\omega\omega' \in T\}$$

We can extend $\mathrm{Alt}^T_{\mathcal{S}}$ to a finite set as follows: $\mathrm{Alt}^T_{\mathcal{S}}(\{\gamma\}) = \mathrm{Alt}^T_{\mathcal{S}}(\gamma)$ and

$$\mathrm{Alt}^T_{\mathcal{S}}(\Delta_1 \cup \cdots \cup \Delta_n) = \mathrm{Alt}^T_{\mathcal{S}}(\Delta_1) \star \cdots \star \mathrm{Alt}^T_{\mathcal{S}}(\Delta_n).$$

where

$$\star : \wp\wp\,(F^{\Omega} \cup F^{\Omega^{\checkmark}}) \times \wp\wp\,(F^{\Omega} \cup F^{\Omega^{\checkmark}}) \to \wp\wp\,(F^{\Omega} \cup F^{\Omega^{\checkmark}})$$

is such that

$$\{\Delta_1, \ldots, \Delta_k\} \star \{\Delta'_1, \ldots, \Delta'_m\} = \{\Delta_i \cup \Delta'_j : i = 1, \ldots, k, j = 1, \ldots, m\}.$$

The map $\mathrm{Alt}^T_{\mathcal{S}}$ also extends to a finite set of finite sets of formulas as follows:

$$\mathrm{Alt}^T_{\mathcal{S}}(\{\Delta_1, \ldots, \Delta_m\}) = \mathrm{Alt}^T_{\mathcal{S}}(\Delta_1) \cup \cdots \cup \mathrm{Alt}^T_{\mathcal{S}}(\Delta_m).$$

We can iterate $\mathrm{Alt}^T_{\mathcal{S}}$ as follows. For a finite set Ψ of finite sets of labelled formulas,

$$(\mathrm{Alt}^T_{\mathcal{S}})^n(\Psi)$$

is inductively defined as follows: $(\mathrm{Alt}^T_{\mathcal{S}})^0(\Psi) = \Psi$ and $(\mathrm{Alt}^T_{\mathcal{S}})^{n+1}(\Psi) = (\mathrm{Alt}^T_{\mathcal{S}})^n(\mathrm{Alt}^T_{\mathcal{S}}(\Psi))$. When $(\mathrm{Alt}^T_{\mathcal{S}})^{n+1}(\Psi) = (\mathrm{Alt}^T_{\mathcal{S}})^n(\Psi)$, we denote by

$$(\mathrm{Alt}^T_{\mathcal{S}})^*(\Psi)$$

the set $(\mathrm{Alt}^T_{\mathcal{S}})^n(\Psi)$ which is a fixed point of $\mathrm{Alt}^T_{\mathcal{S}}$. The elements of $(\mathrm{Alt}^T_{\mathcal{S}})^*(\Psi)$ are the *alternatives* of Ψ.

In the sequel we use $|\delta|$ to denote the assertion δ if $\delta \in F^\Omega$ and $\omega \bowtie_b \varphi$ if δ is $\omega \bowtie_b \varphi^\checkmark$. The following auxiliary result states a sufficient condition for a set to be consistent with an alternative of each labelled formula or its dual.

PROPOSITION 4.2 (24). *Let $\Delta \subseteq F^\Omega \cup F^{\Omega^\checkmark}$ be a finite consistent set such that for each $\delta \in \Delta$, there is $\Lambda \in Alt^T_{\mathcal{S}}(|\delta|)$ with $\Lambda \subseteq \Delta$. Then, for each $\omega \bowtie_b \varphi$, there is*

$$\Gamma \in Alt^T_{\mathcal{S}}(\omega \bowtie_b \varphi) \cup Alt^T_{\mathcal{S}}(\omega \bowtie_{1-b} \varphi)$$

such that $\Delta \cup \Gamma$ is consistent.

We are ready to prove the main result of soundness. The main idea is to use a derivation d for inducing a prestructure where the set of worlds is the set of all labels that occur in d and the binary relation is composed by each pair (ω, ω') such that $S\omega\omega'$ is in d. Finally, the valuation function starts by including only the information for satisfying the assertions in d. This prestructure is then extended to the other labelled formulas with labels in d.

PROPOSITION 4.3. *The particular calculus $\mathcal{D}_{\mathcal{S}}$ is sound with respect to $\mathcal{M}_{\mathcal{S}}$.*

PROOF. Assume that $\Theta \vdash^{\exists}_{\mathcal{D}_{\mathcal{S}}} \omega_1 \bowtie_{b_1} \psi_1, \dots, \omega_s \bowtie_{b_s} \psi_s$ with a derivation $d = \eta_1 \dots \eta_t$. Let Ω^d and S^d be the set of labels and the set of relational assertions occurring in d, respectively. Let $V^d : \Omega^d \times F \to \{0, 1\}$ be the partial function

$$V^d(\omega, \varphi) = \begin{cases} 1 & \text{if } \omega : \varphi \text{ occurs in } d \\ 0 & \text{if } \omega \not: \varphi \text{ occurs in } d \\ \text{undefined} & \text{otherwise} \end{cases}$$

Observe that V^d is well defined since, by definition of derivation, a labelled formula and its conjugate cannot appear both in d.

Let $\{\omega_0 \bowtie_{b_0} \varphi_0, \omega_1 \bowtie_{b_1} \varphi_1, \dots\}$ be an enumeration of all labelled formulas that have labels in Ω^d but do not occur in d. Consider the family $\{V_j\}_{j\in\mathbb{N}}$ where each $V_j : \Omega^d \times F \to \{0, 1\}$ is a partial function inductively defined as follows:

- $V_0 = V^d$

- V_{j+1} is an extension of V_j such that if $V_j(\omega_j, \varphi_j)$ is defined then $V_{j+1} = V_j$. Otherwise pick

$$\Lambda_{j+1} \in (\mathrm{Alt}_{\mathcal{S}}^{S^d})^*(\{\{\omega_j \bowtie_{b_j} \varphi_j\}\}) \cup (\mathrm{Alt}_{\mathcal{S}}^{S^d})^*(\{\{\omega_j \bowtie_{1-b_j} \varphi_j\}\})$$

such that $\Lambda_{j+1} \cup V_j$ is consistent and let

$$V_{j+1}(\omega, \varphi) = \begin{cases} 1 & \text{if } \omega : \varphi^{\checkmark} \in \Lambda_{j+1} \\ 0 & \text{if } \omega \not: \varphi^{\checkmark} \in \Lambda_{j+1} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Note that the existence of Λ_{j+1} is a consequence of Proposition 4.2. Let $\overline{V}^d : \Omega^d \times F \to \{0, 1\}$ be the map such that

$$\overline{V}^d = \bigcup_{j \in \mathbb{N}} V_j.$$

Thus $M^d = (\Omega^d, S^d, \overline{V}^d)$ is in $\mathcal{M}_{\mathcal{S}}$ by a proof similar to the one in Proposition 4.15 of [24]. So $M^d \Vdash \theta$ for $\theta \in \Theta$ and $M^d\,\mathrm{id} \Vdash \omega_j \bowtie_{b_j} \varphi_j$ for each $j = 1, \ldots, s$ by definition of V^d. Therefore, $\Theta \vDash^{\exists}_{\mathcal{D}_{\mathcal{S}}} \omega_1 \bowtie_{b_1} \psi_1, \ldots, \omega_s \bowtie_{b_s} \psi_s$. ⊣

5. Completeness of particular reasoning over a theory

The objective of this section is to investigate when an induced particular calculus captures all the semantic particular consequences of a theory.

A particular calculus $\mathcal{D}_{\mathcal{S}}$ is *complete with respect to* $\mathcal{M}_{\mathcal{S}}$ whenever

$$\Theta \vDash^{\exists}_{\mathcal{M}_{\mathcal{S}}} \omega_1 \bowtie_{b_1} \psi_1, \ldots, \omega_s \bowtie_{b_s} \psi_s \Longrightarrow \Theta \vdash^{\exists}_{\mathcal{D}_{\mathcal{S}}} \omega_1 \bowtie_{b_1} \psi_1, \ldots, \omega_s \bowtie_{b_s} \psi_s$$

We begin by proving a finite version of completeness, that is, assuming that there is a finite model of Θ that satisfies $\omega_1 \bowtie_{b_1} \psi_1, \ldots, \omega_s \bowtie_{b_s} \psi_s$ then there is a derivation of $\omega_1 \bowtie_{b_1} \psi_1, \ldots, \omega_s \bowtie_{b_s} \psi_s$ from Θ.

We need the notion of complexity of a labelled formula in the context of a set T of relational assertions. The *complexity of a labelled formula* is 0 when either the formula is a propositional symbol or has a check. When the primary constructor is local and has at least one dependent that is not id, the complexity is calculated as the sum of the complexities of its dependents, plus 1, and the complexity of the labelled formulas generated whenever $\mathcal{S}$ is hereditary. In the other cases it is the usual definition via subformulas taking into account propagation whenever $\mathcal{S}$ is hereditary. Finally, the *complexity of a sequence of labelled formulas* is the sum of the complexities of each element.

We say that (W, R, V) is *finite* whenever W is a finite set. To simplify the presentation, we assume that given (W, R, V), $W \subseteq \Omega$. Furthermore, if $(W, R, V)\rho \Vdash \omega \bowtie_b \varphi$ then $\omega \in W$ and $\rho|_W$ is the identity over W and so we omit the reference to ρ.

PROPOSITION 5.1. *Let $M \in \mathcal{M}_\mathcal{S}$ be a finite model such that $M \Vdash \omega \bowtie_b \psi$. Then $\vdash^\exists_{\mathcal{D}_\mathcal{S}} \omega \bowtie_b \psi$ with a derivation where all the elements are satisfied by M.*

PROOF. Let T be the set $\{S\omega\omega' : (\omega, \omega') \in R\}$ and τ a sequence with all the elements in T. Observe that $M \Vdash S\omega\omega'$ for every $S\omega\omega' \in T$. We show that if $M \Vdash \omega \bowtie_b \psi$ then there is a sequence π such that $\tau H^T_\mathcal{S}(\pi)$ is a derivation of $\omega \bowtie_b \psi$ where all the elements are satisfied by M by induction on the complexity of ψ. Suppose that $M \Vdash \omega \bowtie_b \psi$.

(Basis) Assume that ψ is $q \in Q$. Considering the sequence π

$$1 \quad \omega \bowtie_b q \quad \text{Ax}$$

then $\tau H^T_\mathcal{S}(\pi)$ is a particular derivation for $\vdash^\exists_{\mathcal{D}_\mathcal{S}} \omega \bowtie_b \psi$. Note that $M \Vdash \omega \bowtie_b q$ by hypothesis. Moreover, if $\mathcal{S}$ is hereditary and $b = 1$ then $M \Vdash \omega' \bowtie_b q$ for every ω' such that $(\omega, \omega') \in R$ because $M \in \mathcal{M}_\mathcal{S}$.

(Step) There are several cases:

(a) ψ is $c(\varphi_1, \dots, \varphi_n)$ and c has local constraints in $\mathcal{S}$ depending on $c_1, \dots, c_m \in C_1 \cup \{\mathsf{id}\}$ and on $i_1, \dots, i_m$. Let $(b'_1, \dots, b'_m) \in \overline{B}^c_b$ be such that $M \Vdash \omega \bowtie_{b'_j} c_j(\varphi_{i_j})$ for $j = 1, \dots, m$, which exists since $M \Vdash \omega \bowtie_b c(\varphi_1, \dots, \varphi_n)$. Then, by the induction hypothesis, there is a sequence $\pi_{\omega \bowtie_{b'_j} c_j(\varphi_{i_j})}$ such that $\tau H^T_\mathcal{S}(\pi_{\omega \bowtie_{b'_j} c_j(\varphi_{i_j})})$ is a derivation for $\omega \bowtie_{b'_j} c_j(\varphi_{i_j})$ for every $j = 1, \dots, m$ and M satisfies all the elements in these derivations. Let π be the sequence

$$\begin{array}{ll} \pi_{\omega \bowtie_{b'_1} c_1(\varphi_{i_1})} & \text{IH} \\ \vdots & \\ \pi_{\omega \bowtie_{b'_m} c_m(\varphi_{i_m})} & \text{IH} \\ \omega \bowtie_b c(\varphi_1, \dots, \varphi_n) & c^{b'_1 \dots b'_m, b} \end{array}$$

Then $\tau H^T_{\mathcal{S}}(\pi)$ is a derivation for $\omega \bowtie_b c(\varphi_1, \ldots, \varphi_n)$ and M satisfies all the elements in $H^T_{\mathcal{S}}(\pi)$ since it satisfies the hereditary constraint (if present).

(b) ψ is $c(\varphi_1, \ldots, \varphi_n)$ where c has a universal non-local constraint on $b_1, \ldots, b_n$ and b. Assume that $\{\omega' : S\omega\omega' \in T\} = \{\omega_1, \ldots, \omega_\ell\}$. Then for each $j = 1, \ldots, \ell$ there is $\varphi'_j \in \{\varphi_1, \ldots, \varphi_n\}$ such that $M \Vdash \omega_j \bowtie_{b_j} \varphi'_j$ since $M \Vdash \omega \bowtie_b c(\varphi_1, \ldots, \varphi_n)$. Thus, by the induction hypothesis, for each $j = 1, \ldots, \ell$, there is a sequence $\pi_{\omega_j \bowtie_{b_j} \varphi'_j}$ such that $\tau H^T_{\mathcal{S}}(\pi_{\omega_j \bowtie_{b_j} \varphi'_j})$ is a derivation for $\omega_j \bowtie_{b_j} \varphi'_j$ and M satisfies all the elements in these derivations. Consider the sequence π

$$\begin{array}{ll} \pi_{\omega_1 \bowtie_{b_1} \varphi'_1} & \text{IH} \\ \vdots & \\ \pi_{\omega_\ell \bowtie_{b_\ell} \varphi'_\ell} & \text{IH} \\ \omega \bowtie_b c(\varphi_1, \ldots, \varphi_n) & c_1 \end{array}$$

Then $\tau H^T_{\mathcal{S}}(\pi)$ is a derivation for $\vdash^{\exists}_{\mathcal{D}_{\mathcal{S}}} \omega \bowtie_b c(\varphi_1, \ldots, \varphi_2)$ and M satisfies all the elements in $\tau H^T_{\mathcal{S}}(\pi)$ since it satisfies the hereditary constraint (if present).

(c) ψ is $c(\varphi_1, \ldots, \varphi_n)$ and c has a particular non-local constraint on $b_1, \ldots, b_n$ and b. Since $M \Vdash \omega \bowtie_b c(\varphi_1, \ldots, \varphi_n)$, let $\omega' \in \{\omega' : S\omega\omega' \in T\}$ be such that $M \Vdash \omega' \bowtie_{b_j} \varphi_j$ for $j = 1, \ldots, n$. Hence, by the induction hypothesis, there is a sequence $\pi_{\omega' \bowtie_{b_j} \varphi_j}$ such that $\tau H^T_{\mathcal{S}}(\pi_{\omega' \bowtie_{b_j} \varphi_j})$ is a derivation for $\omega' \bowtie_{b_j} \varphi_j$ for $j = 1, \ldots, n$. Consider the sequence π

$$\begin{array}{ll} \pi_{\omega' \bowtie_{b_1} \varphi_1} & \text{IH} \\ \vdots & \\ \pi_{\omega' \bowtie_{b_n} \varphi_n} & \text{IH} \\ \omega \bowtie_b c(\varphi_1, \ldots, \varphi_n) & c_b \end{array}$$

Then $\tau H^T_{\mathcal{S}}(\pi)$ is a derivation for $\vdash^{\exists}_{\mathcal{D}_{\mathcal{S}}} \omega \bowtie_b c(\varphi_1, \ldots, \varphi_n)$ and M satisfies all the elements in $\tau H^T_{\mathcal{S}}(\pi)$ since it satisfies the hereditary constraint (if present). $\dashv$

The following results are crucial for proving completeness in Proposition 5.3.

Lemma 5.1. *Let $M = (W, R, V)$ be a witness for*

$$\vDash^{\exists}_{\mathcal{S}} \omega_1 \bowtie_{b_1} \delta_1, \ldots, \omega_s \bowtie_{b_s} \delta_s$$

with $W = \{\omega_1, \ldots, \omega_s\}$*. Let* $T = \{S\omega\omega' \mid (\omega, \omega') \in R\}$*. Then there is a finite consistent set*

$$\Lambda \in (Alt_{\mathcal{S}}^T)^*(\{\{\omega_1 \bowtie_{b_1} \delta_1, \ldots, \omega_s \bowtie_{b_s} \delta_s\}\})$$

such that $\{\omega \in \Omega \mid \omega \text{ occurs in } \Lambda\} = W$ *and* $M \Vdash \Lambda$*.*

Proof. We prove the result by induction on the complexity ℓ of the sequence $\omega_1 \bowtie_{b_1} \delta_1, \ldots, \omega_s \bowtie_{b_s} \delta_s$ of labelled formulas.

(Base) $\ell = 0$. Then $\delta_1, \ldots, \delta_s$ are propositional symbols. Then Λ is $\{\omega_1 \bowtie_{b_1} \delta_1, \ldots, \omega_s \bowtie_{b_s} \delta_s\}$.

(Step) $\ell > 0$. Assume without loss of generality that δ_1 is of the form $c(\varphi_1, \ldots, \varphi_n)$. There are several cases to consider.

(1) c has local constraints in $\mathcal{S}$ depending on $c_1, \ldots, c_m \in C_1 \cup \{\mathsf{id}\}$ and on $i_1, \ldots, i_m$. Observe that $\mathrm{Alt}_{\mathcal{S}}^T(\omega_1 \bowtie_{b_1} \delta_1)$ is

$$\bigcup_{(b'_1,\ldots,b'_m)\in\overline{B}^c_{b_1}} \{H_{\mathcal{S}}^T(\{\omega_1 \bowtie_{b'_1} c_1(\varphi_{i_1}), \ldots, \omega_1 \bowtie_{b'_m} c_m(\varphi_{i_m}), \omega_1 \bowtie_{b_1} \delta_1^{\checkmark}\})\}$$

Note also that

$$\begin{aligned}
&(\mathrm{Alt}_{\mathcal{S}}^T)^*(\{\{\omega_1 \bowtie_{b_1} \delta_1, \ldots, \omega_s \bowtie_{b_s} \delta_s\}\}) \\
&\quad = (\mathrm{Alt}_{\mathcal{S}}^T)^*(\{H_{\mathcal{S}}^T(\{\omega_1 \bowtie_{b'_1} c_1(\varphi_{i_1}), \ldots, \omega_1 \bowtie_{b'_m} c_m(\varphi_{i_m}), \omega_1 \bowtie_{b_1} \delta_1^{\checkmark}\}) \\
&\qquad\qquad \cup \{\omega_2 \bowtie_{b_2} \delta_2, \ldots, \omega_s \bowtie_{b_s} \delta_s\} \mid (b'_1, \ldots, b'_m) \in \overline{B}^c_{b_1}\}) \\
&\quad = \bigcup_{(b'_1,\ldots,b'_m)\in\overline{B}^c_{b_1}} (\mathrm{Alt}_{\mathcal{S}}^T)^*(\{H_{\mathcal{S}}^T(\{\omega_1 \bowtie_{b'_1} c_1(\varphi_{i_1}), \ldots, \\
&\qquad\qquad\qquad \omega_1 \bowtie_{b'_m} c_m(\varphi_{i_m}), \omega_1 \bowtie_{b_1} \delta_1^{\checkmark}\}) \\
&\qquad\qquad \cup \{\omega_2 \bowtie_{b_2} \delta_2, \ldots, \omega_s \bowtie_{b_s} \delta_s\}\}).
\end{aligned}$$

Observe that there is $(b'_1, \ldots, b'_m) \in \overline{B}^c_{b_1}$ such that

$$M \Vdash H_{\mathcal{S}}^T(\{\omega_1 \bowtie_{b'_1} c_1(\varphi_{i_1}), \ldots, \omega_1 \bowtie_{b'_m} c_m(\varphi_{i_m}), \omega_1 \bowtie_{b_1} \delta_1^{\checkmark}\})$$

since $M \Vdash \omega_1 \bowtie_{b_1} \delta_1$. Thus, by the induction hypothesis, there is a finite consistent set Λ_1 in

$$(\mathrm{Alt}_{\mathcal{S}}^T)^*(\{H_{\mathcal{S}}^T(\{\omega_1 \bowtie_{b'_1} c_1(\varphi_{i_1}), \ldots, \omega_1 \bowtie_{b'_m} c_m(\varphi_{i_m}), \omega_1 \bowtie_{b_1} \delta_1^{\checkmark}\}) \cup \{\omega_j \bowtie_{b_j} \delta_j \mid j = 2, \ldots, s\}\})$$

such that $\{\omega : \omega \text{ occurs in } \Lambda_1\} = W$ and $M \Vdash \Lambda_1$. It is enough to take Λ to be Λ_1 since Λ_1 is in $(\mathrm{Alt}_{\mathcal{S}}^T)^*(\{\{\omega_1 \bowtie_{b_1} \delta_1, \ldots, \omega_s \bowtie_{b_s} \delta_s\}\})$.

The other cases follow in a similar way. $\dashv$

PROPOSITION 5.2. *Assume that $T \subseteq S^\Omega$ is a finite set closed for the relational constraints in $\mathcal{S}$ and $\Lambda \in (Alt^T_{\mathcal{S}})^*(\{\{\omega_1 \bowtie_{b_1} \psi_1, \ldots, \omega_s \bowtie_{b_s} \psi_s\}\})$ is a finite consistent set. Let τ be a sequence with all the elements of T. Then there is a sequence λ with all the elements of Λ such that $\tau\lambda$ is a derivation for $\omega_1 \bowtie_{b_1} \psi_1, \ldots, \omega_s \bowtie_{b_s} \psi_s$.*

PROOF. Assume that $\Lambda \in (\mathrm{Alt}^T_{\mathcal{S}})^k(\{\{\omega_1 \bowtie_{b_1} \psi_1, \ldots, \omega_s \bowtie_{b_s} \psi_s\}\})$ is a finite consistent set. The proof is by induction on k.

(Basis) $k = 1$. So Λ is $\bigcup_{i=1}^s \{\omega_i \bowtie_{b_i} \psi_i^\checkmark\} \cup \{\omega' \bowtie_{b_i} \psi_i^\checkmark \mid \bar{S}\omega_i\omega' \in T, b_i = 1$ and $\mathcal{S}$ is hereditary$\}$ where $\psi_i \in Q$ for $i = 1, \ldots, s$. Then $\tau\lambda$ is a derivation for $\omega_1 \bowtie_{b_1} \psi_1, \ldots, \omega_s \bowtie_{b_s} \psi_s$ where λ is any sequence with the elements of Λ.

(Step) Observe that

$$\Lambda \in (\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(\mathrm{Alt}^T_{\mathcal{S}}(\{\{\omega_1 \bowtie_{b_1} \psi_1, \ldots, \omega_s \bowtie_{b_s} \psi_s\}\})),$$

that is,

$$\Lambda \in (\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(\mathrm{Alt}^T_{\mathcal{S}}(\{\omega_1 \bowtie_{b_1} \psi_1\}) \star \cdots \star \mathrm{Alt}^T_{\mathcal{S}}(\{\omega_s \bowtie_{b_s} \psi_s\}))$$

and so

$$\Lambda \in (\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(\mathrm{Alt}^T_{\mathcal{S}}(\{\omega_1 \bowtie_{b_1} \psi_1\})) \star \cdots \star (\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(\mathrm{Alt}^T_{\mathcal{S}}(\{\omega_s \bowtie_{b_s} \psi_s\})).$$

Then there is

$$\Lambda_i \in (\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(\mathrm{Alt}^T_{\mathcal{S}}(\{\omega_i \bowtie_{b_i} \psi_i\}))$$

such that $\Lambda = \bigcup_{i=1}^s \Lambda_i$. Note that each Λ_i is a finite consistent set because Λ is a finite consistent set.

Suppose that ψ_i is $c(\varphi_1, \ldots, \varphi_n)$. There are several cases to consider.

(1) c has local constraints in $\mathcal{S}$ depending on $c_1, \ldots, c_m \in C_1 \cup \{\mathsf{id}\}$ and on $i_1, \ldots, i_m$. Then, $\mathrm{Alt}^T_{\mathcal{S}}(\omega_i \bowtie_{b_i} \psi_i)$ is

$$\bigcup_{(b_{i_1}, \ldots, b_{i_m}) \in \overline{B}^c_{b_i}} \{H^T_{\mathcal{S}}(\{\omega_i \bowtie_{b_{i_1}} c_1(\varphi_{i_1}), \ldots, \omega_i \bowtie_{b_{i_m}} c_m(\varphi_{i_m}), \omega_i \bowtie_{b_i} \psi_i^\checkmark\})\}.$$

Hence

$$(\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(\mathrm{Alt}^T_{\mathcal{S}}(\{\omega_i \bowtie_{b_i} \psi_i\}))$$

is the union of the sets

$$(\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(\{H^T_{\mathcal{S}}(\{\omega_i \bowtie_{b_{i_1}} c_1(\varphi_{i_1}), \ldots, \omega_i \bowtie_{b_{i_m}} c_m(\varphi_{i_m}), \omega_i \bowtie_{b_i} \psi_i^\checkmark\})\})$$

for each $(b_{i_1}, \dots, b_{i_m}) \in \overline{B}^c_{b_i}$. Thus, there is $(b_{i_1}, \dots, b_{i_m}) \in \overline{B}^c_{b_i}$ such that Λ_i is in

$$(\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(\{H^T_{\mathcal{S}}(\{\omega_i \bowtie_{b_{i_1}} c_1(\varphi_{i_1}), \dots, \omega_i \bowtie_{b_{i_m}} c_m(\varphi_{i_m}), \omega_i \bowtie_{b_i} \psi_i^{\checkmark}\})\}).$$

On the other hand, $H^T_{\mathcal{S}}(\{\omega_i \bowtie_{b_{i_1}} c_1(\varphi_{i_1}), \dots, \omega_i \bowtie_{b_{i_m}} c_m(\varphi_{i_m}), \omega_i \bowtie_{b_i} \psi_i^{\checkmark}\})$ is

$$\bigcup_{\ell=1}^{m} H^T_{\mathcal{S}}(\omega_i \bowtie_{b_{i_\ell}} c_\ell(\varphi_{i_\ell})) \cup H^T_{\mathcal{S}}(\omega_i \bowtie_{b_i} \psi_i^{\checkmark}).$$

So Λ_i is in

$$\bigstar_{\ell=1}^{m} (\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(\{H^T_{\mathcal{S}}(\omega_i \bowtie_{b_{i_\ell}} c_\ell(\varphi_{i_\ell}))\}) \star (\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(\{H^T_{\mathcal{S}}(\omega_i \bowtie_{b_i} \psi_i^{\checkmark})\}).$$

Then there is

$$\Lambda_{i,\ell} \in (\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(\{H^T_{\mathcal{S}}(\omega_i \bowtie_{b_{i_\ell}} c_\ell(\varphi_{i_\ell}))\})$$

for $\ell = 1, \dots, m$ and

$$\Lambda'_i \in (\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(\{H^T_{\mathcal{S}}(\omega_i \bowtie_{b_i} \psi_i^{\checkmark}) \setminus \{\omega_i \bowtie_{b_i} \psi_i^{\checkmark}\}\})$$

such that

$$\Lambda_i = \bigcup_{\ell=1}^{m} \Lambda_{i,\ell} \cup \Lambda'_i \cup \{\omega_i \bowtie_{b_i} \psi_i^{\checkmark}\}.$$

Observe that each $\Lambda_{i,\ell}$ is consistent as well as Λ'_i since Λ_i is consistent. Thus, by the induction hypothesis, there is a sequence $\lambda_{i,\ell}$ with all the elements of $\Lambda_{i,\ell}$ such that $\tau\lambda_{i,\ell}$ is a derivation for $H^T_{\mathcal{S}}(\omega_i \bowtie_{b_{i_\ell}} c_\ell(\varphi_{i_\ell}))$ for each $\ell = 1, \dots, m$ and there is a sequence λ'_i with all the elements of Λ'_i such that $\tau\lambda'_i$ is a derivation for $H^T_{\mathcal{S}}(\omega_i \bowtie_{b_i} \psi_i^{\checkmark}) \setminus \{\omega_i \bowtie_{b_i} \psi_i^{\checkmark}\}$. The sequence

$$\begin{array}{l} \lambda_{i,1} \\ \vdots \\ \lambda_{i,m} \\ \lambda'_i \\ \omega_i \bowtie_{b_i} \psi_i \end{array}$$

written λ_i, is such that $\tau\lambda_i$ is a derivation for $\omega_i \bowtie_{b_i} \psi_i$.

(2) $c \in C_n$ with an particular non-local constraint in $\mathcal{S}$ and $b_i = 1$. Then $\mathrm{Alt}^T_{\mathcal{S}}(\omega_i : c(\varphi_1, \dots, \varphi_n))$ is

$$\{H^T_{\mathcal{S}}(\{\omega' : \varphi_1, \dots, \omega' : \varphi_n, \omega_i : c(\varphi_1, \dots, \varphi_n)^{\checkmark}\}) \mid S\omega_i\omega' \in T\}.$$

Hence

$$(\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(\mathrm{Alt}^T_{\mathcal{S}}(\{\omega_i : c(\varphi_1,\ldots,\varphi_n)\}))$$

is

$$\bigcup_{\{\omega' \mid S\omega_i\omega' \in T\}} (\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(\{H^T_{\mathcal{S}}(\{\omega'{:}\varphi_1,\ldots,\omega'{:}\varphi_n,\omega_i{:}c(\varphi_1,\ldots,\varphi_n)^{\checkmark}\})\}).$$

Thus, there is ω' such that $S\omega_i\omega' \in T$ and

$$\Lambda_i \in (\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(\{H^T_{\mathcal{S}}(\{\omega' : \varphi_1,\ldots,\omega' : \varphi_n,\omega_i : c(\varphi_1,\ldots,\varphi_n)^{\checkmark}\})\}).$$

On the other hand, $H^T_{\mathcal{S}}(\{\omega' : \varphi_1,\ldots,\omega' : \varphi_n,\omega_i : c(\varphi_1,\ldots,\varphi_n)^{\checkmark}\})$ is

$$\bigcup_{\ell=1}^{n} H^T_{\mathcal{S}}(\omega' : \varphi_\ell) \cup H^T_{\mathcal{S}}(\omega_i : c(\varphi_1,\ldots,\varphi_n)^{\checkmark}).$$

So

$$\Lambda_i \in \bigstar_{\ell=1}^{n}(\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(H^T_{\mathcal{S}}(\omega' : \varphi_\ell)) \star (\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(H^T_{\mathcal{S}}(\omega_i : c(\varphi_1,\ldots,\varphi_n)^{\checkmark}).$$

Then there is

$$\Lambda_{i,\ell} \in (\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(\{H^T_{\mathcal{S}}(\omega' : \varphi_\ell)\})$$

for $\ell = 1,\ldots,n$ and

$$\Lambda'_i \in (\mathrm{Alt}^T_{\mathcal{S}})^{k-1}(\{H^T_{\mathcal{S}}(\omega_i : c(\varphi_1,\ldots,\varphi_n)^{\checkmark}) \setminus \{\omega_i : c(\varphi_1,\ldots,\varphi_n)^{\checkmark}\}\})$$

such that

$$\Lambda_i = \bigcup_{\ell=1}^{n} \Lambda_{i,\ell} \cup \Lambda'_i \cup \{\omega_i : c(\varphi_1,\ldots,\varphi_n)^{\checkmark}\}.$$

Observe that each $\Lambda_{i,\ell}$ is consistent as well as Λ'_i since Λ_i is consistent. Thus, by the induction hypothesis, there is a sequence $\lambda_{i,\ell}$ with all the elements of $\Lambda_{i,\ell}$ such that $\tau\lambda_{i,\ell}$ is a derivation for $H^T_{\mathcal{S}}(\omega' : \varphi_\ell)$ for $\ell = 1,\ldots,n$ and there is a sequence λ'_i with all the elements of Λ'_i such that $\tau\lambda'_i$ is a derivation for $H^T_{\mathcal{S}}(\omega_i : c(\varphi_1,\ldots,\varphi_n)^{\checkmark}) \setminus \{\omega_i : c(\varphi_1,\ldots,\varphi_n)^{\checkmark}\}$. The following sequence

$$\begin{array}{l} \lambda_{i,1} \\ \vdots \\ \lambda_{i,n} \\ \lambda'_i \\ \omega_i : c(\varphi_1,\ldots,\varphi_n) \end{array}$$

written λ_i, is such that $\tau\lambda_i$ is a derivation for $\omega_i : c(\varphi_1,\ldots,\varphi_n)$.

The proof of the other cases is similar. Finally, take λ to be $\lambda_1 \dots \lambda_s$. Therefore, $\tau\lambda$ is a derivation for $\omega_1 \bowtie_{b_1} \psi_1, \dots, \omega_s \bowtie_{b_s} \psi_s$. ⊣

We now establish a completeness result where we assume the existence of a finite witness.

Proposition 5.3. *Let $\Theta \subseteq F$ be a finite set and M a finite witness for $\Theta \vDash^{\exists}_{\mathcal{M}_{\mathcal{S}}} \omega_1 \bowtie_{b_1} \psi_1, \dots, \omega_s \bowtie_{b_s} \psi_s$. Then $\Theta \vdash^{\exists}_{\mathcal{D}_{\mathcal{S}}} \omega_1 \bowtie_{b_1} \psi_1, \dots, \omega_s \bowtie_{b_s} \psi_s$ with a derivation using as labels all the elements of W and no more.*

Proof. Let T be the set $\{S\omega\omega' : (\omega, \omega') \in R\}$, τ a sequence with all the elements of T, $W = \{\omega'_1, \dots, \omega'_\ell\}$ and $\Theta = \{\theta_1, \dots, \theta_k\}$. Observe that M is a witness for

$$\vDash^{\exists}_{\mathcal{M}_{\mathcal{S}}} \omega_1 \bowtie_{b_1} \psi_1, \dots, \omega_s \bowtie_{b_s} \psi_s, \omega'_1 : \theta_1, \dots, \omega'_\ell : \theta_k.$$

Then, by Proposition 5.1, there is a finite consistent set

$$\Lambda \in (\mathrm{Alt}^T_{\mathcal{S}})^*(\{\{\omega_1 \bowtie_{b_1} \psi_1, \dots, \omega_s \bowtie_{b_s} \psi_s, \omega'_1 : \theta_1, \dots, \omega'_\ell : \theta_k\}\})$$

such that $M \Vdash \Lambda$. Hence, by Proposition 5.2, there is a sequence λ with all the elements of Λ such that $\tau\lambda$ is a derivation for

$$\omega_1 \bowtie_{b_1} \psi_1, \dots, \omega_s \bowtie_{b_s} \psi_s, \omega'_1 : \theta_1, \dots, \omega'_\ell : \theta_k.$$

Thus $\Theta \vdash^{\exists}_{\mathcal{D}_{\mathcal{S}}} \omega_1 \bowtie_{b_1} \psi_1, \dots, \omega_s \bowtie_{b_s} \psi_s$. ⊣

Strong finite model property. We now establish a sufficient condition for a logic to be complete by ensuring the existence of a finite witness. We say that $\mathcal{S}$ has the *strong finite model property* whenever for every set $\Gamma \subseteq F$ closed for subformulas and $M \in \mathcal{M}_{\mathcal{S}}$ there are a finite model $M' \in \mathcal{M}_{\mathcal{S}}$ and an onto map $f : W \to W'$ such that

- if $(\omega_1, \omega_2) \in R$ then $(f(\omega_1), f(\omega_2)) \in R'$
- if $(\omega'_1, \omega'_2) \in R'$ then there are $\omega_1, \omega_2 \in W$ such that $(\omega_1, \omega_2) \in R$ and $f(\omega_i) = \omega'_i$ for $i = 1, 2$
- $M \Vdash \omega \bowtie_b \psi$ if and only if $M' \Vdash f(\omega) \bowtie_b \psi$ for every $\omega \bowtie_b \psi \in \Gamma$.

Proposition 5.4. *Assume that $\mathcal{S}$ has the strong finite model property. Then $\mathcal{D}_{\mathcal{S}}$ is complete with respect to $\mathcal{M}_{\mathcal{S}}$.*

Proof. Suppose that $\Theta \vDash^{\exists}_{\mathcal{M}_{\mathcal{S}}} \omega_1 \bowtie_{b_1} \psi_1, \ldots, \omega_s \bowtie_{b_s} \psi_s$ with a witness $M \in \mathcal{M}_{\mathcal{S}}$. Let Γ be the set of all subformulas of $\Theta \cup \{\psi_1, \ldots, \psi_s\}$. Thus, there are a finite model $M' \in \mathcal{M}_{\mathcal{S}}$ and $f : W \to W'$ such that

$$M \Vdash \omega_\ell \bowtie_{b_\ell} \psi_\ell \text{ if and only if } M' \Vdash f(\omega_\ell) \bowtie_{b_\ell} \psi_\ell$$

for each $\ell = 1, \ldots, s$ and

$$M \Vdash \theta \text{ if and only if } M' \Vdash \theta$$

for every $\theta \in \Theta$. Hence M' is a witness for

$$\Theta \vDash^{\exists}_{\mathcal{M}_{\mathcal{S}}} f(\omega_1) \bowtie_{b_1} \psi_1, \ldots, f(\omega_s) \bowtie_{b_s} \psi_s$$

Thus, by Proposition 5.3, there is a derivation $\tau\pi$ for $\Theta \vdash^{\exists}_{\mathcal{D}_{\mathcal{S}}} f(\omega_1) \bowtie_{b_1} \psi_1, \ldots, f(\omega_s) \bowtie_{b_s} \psi_s$ using as labels the elements of M'. Let $\{\omega'_1, \ldots, \omega'_\ell\}$ be the set of all labels that occur in the derivation and $\Theta = \{\theta_1, \ldots, \theta_k\}$. Then $\tau\pi$ is a derivation for

$$\vdash^{\exists}_{\mathcal{D}_{\mathcal{S}}} f(\omega_1) \bowtie_{b_1} \psi_1, \ldots, f(\omega_s) \bowtie_{b_s} \psi_s, \omega'_1 : \theta_1, \ldots, \omega'_\ell : \theta_k.$$

So, by several applications of Proposition 3.1 for replacing $f(\omega_1)$ by $f^{-1}(f(\omega_1)) \cap \{\omega_1, \ldots, \omega_s\}$, $\ldots$, $f(\omega_s)$ by $f^{-1}(f(\omega_s)) \cap \{\omega_1, \ldots, \omega_s\}$, we can conclude that there is a derivation $\tau'\pi'$ for

$$\vdash^{\exists}_{\mathcal{D}_{\mathcal{S}}} f^{-1}(f(\omega_1)) \cap \{\omega_1, \ldots, \omega_s\} \bowtie_{b_1} \psi_1, \ldots, f^{-1}(f(\omega_s)) \cap \{\omega_1, \ldots, \omega_s\} \bowtie_{b_s} \psi_s,$$
$$(\omega'_1)^{-1}:\theta_1, \ldots, (\omega'_\ell)^{-1}:\theta_k$$

where $(\omega'_j)^{-1} = \omega'_j$ if $\omega'_j \neq f(\omega_1), \ldots, f(\omega_s)$ and otherwise $(\omega'_j)^{-1} = f^{-1}(\omega'_j) \cap \{\omega_1, \ldots, \omega_s\}$. Hence $\tau'\pi'$ is a derivation for

$$\vdash^{\exists}_{\mathcal{D}_{\mathcal{S}}} \omega_1 \bowtie_{b_1} \psi_1, \ldots, \omega_s \bowtie_{b_s} \psi_s, (\omega'_1)^{-1} : \theta_1, \ldots, (\omega'_\ell)^{-1} : \theta_k$$

Therefore,

$$\Theta \vdash^{\exists}_{\mathcal{D}_{\mathcal{S}}} \omega_1 \bowtie_{b_1} \psi_1, \ldots, \omega_s \bowtie_{b_s} \psi_s$$

with the derivation $\tau'\pi'$ since each $\theta \in \Theta$ is proved in $\tau'\pi'$ for all labels that occur in that derivation. ⊣

Observe that in the application of Proposition 3.1 in the proof of the result above we should see in each step a partition of the assertions of the theory in two classes: one for the labelled formulas to be replaced and the other with the remaining labelled formulas.

Example 5.1. Recall Examples 2.3 and 3.2. Observe that $\mathcal{S}_{\mathsf{J}}$ has the strong finite model property by filtration [see 3, 25]. Therefore, by Proposition 5.4, $\mathcal{D}_{\mathcal{S}_{\mathsf{J}}}$ is complete with respect to $\mathcal{M}_{\mathcal{S}_{\mathsf{J}}}$. ⊣

Example 5.2. Recall Examples 2.2 and 3.1. Observe that $\mathcal{S}_{\mathsf{K}}$ has the strong finite model property by filtration [see 25]. Therefore, by Proposition 5.4, $\mathcal{D}_{\mathcal{S}_{\mathsf{K}}}$ is complete with respect to $\mathcal{M}_{\mathcal{S}_{\mathsf{K}}}$. ⊣

Example 5.3. Recall Example 2.4. We now establish that $\mathcal{D}_{\mathcal{S}_{\mathsf{N4}}}$ is complete with respect to $\mathcal{M}_{\mathcal{S}_{\mathsf{N4}}}$ using Proposition 5.4 by proving that $\mathcal{S}_{\mathsf{N4}}$ has the strong finite model property. We use the reduction of $\mathsf{N4}$ to J^+ [see 16] and capitalize on the fact that $\mathcal{S}_{\mathsf{J}^+}$ has the strong finite model property (this can be straightforwardly shown taking into account that $\mathcal{S}_{\mathsf{J}}$ has this property). Denote by L_{N4} and L_{J^+} the sets of formulas for $\mathsf{N4}$ and J^+ over sets of propositional symbols $P \cup \{{\sim}p : p \in P\}$ and $P \cup \{p' : p \in P\}$, respectively. Let $\tau_{\mathsf{N4}\to\mathsf{J}^+} : L_{\mathsf{N4}} \to L_{\mathsf{J}^+}$ be a map such that

- $\tau_{\mathsf{N4}\to\mathsf{J}^+}(p) = p$ and $\tau_{\mathsf{N4}\to\mathsf{J}^+}({\sim}p) = p'$
- $\tau_{\mathsf{N4}\to\mathsf{J}^+}(\varphi_1 * \varphi_2) = \tau_{\mathsf{N4}\to\mathsf{J}^+}(\varphi_1) * \tau_{\mathsf{N4}\to\mathsf{J}^+}(\varphi_2)$, for $* \in \{\supset, \wedge, \vee\}$
- $\tau_{\mathsf{N4}\to\mathsf{J}^+}({\sim}{\sim}\varphi) = \tau_{\mathsf{N4}\to\mathsf{J}^+}(\varphi)$
- $\tau_{\mathsf{N4}\to\mathsf{J}^+}({\sim}(\varphi_1 \supset \varphi_2)) = \tau_{\mathsf{N4}\to\mathsf{J}^+}(\varphi_1) \wedge \tau_{\mathsf{N4}\to\mathsf{J}^+}({\sim}\varphi_2)$
- $\tau_{\mathsf{N4}\to\mathsf{J}^+}({\sim}(\varphi_1 \wedge \varphi_2)) = \tau_{\mathsf{N4}\to\mathsf{J}^+}({\sim}\varphi_1) \vee \tau_{\mathsf{N4}\to\mathsf{J}^+}({\sim}\varphi_2)$
- $\tau_{\mathsf{N4}\to\mathsf{J}^+}({\sim}(\varphi_1 \vee \varphi_2)) = \tau_{\mathsf{N4}\to\mathsf{J}^+}({\sim}\varphi_1) \wedge \tau_{\mathsf{N4}\to\mathsf{J}^+}({\sim}\varphi_2)$.

Let $g \colon \mathcal{M}_{\mathcal{S}_{\mathsf{N4}}} \to \mathcal{M}_{\mathcal{S}_{\mathsf{J}^+}}$ be a map defined as follows

$$g(W, R, V) = (W, R, \overline{V})$$

such that $\overline{V}(w, p) = V(w, p)$ and $\overline{V}(w, p') = V(w, {\sim}p)$. We prove that

$$M \Vdash_{\mathsf{N4}} \omega : \varphi \text{ if and only if } g(M) \Vdash_{\mathsf{J}^+} \omega : \tau_{\mathsf{N4}\to\mathsf{J}^+}(\varphi)$$

by induction on the structure of φ.

(Base) φ is p or φ is ${\sim}p$. Then the thesis follows by construction.

(Step) We just consider the case where φ is ${\sim}{\sim}\psi$. Then $M \Vdash_{\mathsf{N4}} \omega : {\sim}{\sim}\psi$ iff $M \Vdash_{\mathsf{N4}} \omega : \psi$ iff $g(M) \Vdash_{\mathsf{J}^+} \omega : \tau_{\mathsf{N4}\to\mathsf{J}^+}(\psi)$ iff $g(M) \Vdash_{\mathsf{J}^+} \omega : \tau_{\mathsf{N4}\to\mathsf{J}^+}({\sim}{\sim}\psi)$.

Let $h \colon \mathcal{M}_{\mathcal{S}_{\mathsf{J}^+}} \to \mathcal{M}_{\mathcal{S}_{\mathsf{N4}}}$ be a map defined as follows

$$h(W, R, V) = (W, R, \underline{V})$$

where $\underline{V}(w, p) = V(w, p)$ and $\underline{V}(w, {\sim}p) = V(w, p')$. Then

$$M' \Vdash_{\mathsf{J}^+} \omega : \tau_{\mathsf{N4}\to\mathsf{J}^+}(\varphi) \text{ if and only if } h(M') \Vdash_{\mathsf{N4}} \omega : \varphi.$$

Capitalizing on these maps we now show that $\mathcal{S}_{\mathsf{N4}}$ has the strong finite model property. Let $\Gamma \subseteq L_{\mathsf{N4}}$ be a set closed for subformulas and $M \in \mathcal{M}_{\mathcal{S}_{\mathsf{N4}}}$. Then $\tau_{\mathsf{N4}\to\mathsf{J}^+}(\Gamma)$ is also closed for subformulas and $g(M) \in \mathcal{M}_{\mathcal{S}_{\mathsf{J}^+}}$. Then, because $\mathcal{S}_{\mathsf{J}^+}$ has the strong finite model property there is a finite model $M' \in \mathcal{M}_{\mathcal{S}_{\mathsf{J}^+}}$ and an onto map $f\colon W \to W'$ with the properties in the definition of strong finite model property. Hence, $g(M) \Vdash \omega : \tau_{\mathsf{N4}\to\mathsf{J}^+}(\gamma)$ if and only if $M' \Vdash f(\omega) : \tau_{\mathsf{N4}\to\mathsf{J}^+}(\gamma)$ for every $\gamma \in \Gamma$. Then $\mathcal{S}_{\mathsf{N4}}$ has the strong finite model property with f and $h(M') \in \mathcal{M}_{\mathcal{S}_{\mathsf{N4}}}$ as we now show. Indeed: $M \Vdash_{\mathsf{N4}} \omega : \gamma$ iff $g(M) \Vdash_{\mathsf{J}^+} \omega : \tau_{\mathsf{N4}\to\mathsf{J}^+}(\gamma)$ iff $M' \Vdash f(\omega) : \tau_{\mathsf{N4}\to\mathsf{J}^+}(\gamma)$ iff $h(M') \Vdash_{\mathsf{N4}} f(\omega) : \gamma$. The remaining two conditions follow straightforwardly. ⊣

6. Outlook

Particular reasoning is extended to accommodate theories. With this objective in mind a new notion of particular derivation within the context of a theory is proposed. An illustration of particular reasoning over a theory is provided by an example involving a robot navigating a square area with obstacles. After that the soundness of the induced particular calculus is analyzed. Completeness of the particular calculus is shown to hold for logics fulfilling a strong finite model property. The results are shown to be applicable to logics with a strong finite model property, including classical, intuitionistic, certain modal logics, and Nelson's N4 logic, among others.

It would be interesting to analyze further properties of particular reasoning in the context of a theory, specifically its decidability and computational complexity and relate them with similar properties of the theory. Furthermore, extending particular reasoning to first-order theories could be worthwhile. It would also be interesting to assess the impact of particular reasoning on abduction and machine learning.

Finally, we intend to investigate particular reasoning over a theory in the context of other deductive calculi, such as Gentzen/Jaśkowski calculi. We think our purpose is closer to Jaśkowski's concept [see 13, 14, 17] since he was concerned with the imitation of practical mathematical proof procedures whereas Gentzen [see 11] concentrated on proof-theoretical considerations in the foundations of mathematics. Additionally, building on this experience, we aim to abstract the notion of particular reasoning over theories to a general particular consequence relation.

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