Logic and **L**ogical **P**hilosophy
Online First Articles (2024)
DOI: 10.12775/LLP.2024.028

Guanglong Luo

A Paradox for ZF-Class Nominalism

Abstract. In a recent article in this journal, Calemi challenges the Küng-Armstrong trilemma, a well-known objection to traditional class nominalism, by proposing a fusion of class nominalism with Zermelo-Fraenkel set theory (ZF). In this note, we argue that ZF-class nominalism faces significant challenges in the form of incompleteness and potential paradoxes stemming from Gödel's incompleteness theorem. We will explore these issues in detail, highlighting the key implications for the viability of ZF-class nominalism as a philosophical position.

Keywords: class nominalism; Zermelo-Fraenkel set theory; Gödel's incompleteness theorem

1. Introduction

Class nominalism is a nominalism about universals. It claims that properties are just sets or classes of particulars, so property-talk (and relation talk) has to be analyzed in terms of set-membership. For instance, a's having a property F should be analyzed as a's being a member of the class of Fs. However, this analysis suffers from the Küng-Armstrong trilemma (cf. Küng, 1967; Armstrong, 1978) in the sense that either the predicate "being member of" is left unanalyzed (therefore, incomplete) or if it is analyzable, it has to be analyzed in terms of set-membership again. And if it is analyzed in terms of set-membership, then as Armstrong claims, this analyzed type either equals the former type or not. If it does equal former type, then it is circular in that the analysans contains the analysandum, if it does not, then it will become a vicious regress.

Received July 25, 2024. *Accepted* September 26, 2024. *Published online* November 2, 2024

Calemi (2024) has challenged the Küng-Armstrong trilemma by incorporating the principles of Zermelo-Fraenkel (ZF) set theory. Calemi's key insight is that the universal set defined as: $\{\langle x, y\rangle \mid x \in y\}$ which is often used in nominalist analyses to represent the relation "being a member of", does not exist within the framework of ZF set theory. Building on this observation, Calemi argues that "being a member of" is not a property-denoting predicate in the ZF context. Consequently, class nominalists are not compelled to analyze this predicate within their framework, as it does not inherently require such an analysis.

I take this approach to be one of best approaches to resist Küng-Armstrong's trilemma against class nominalism we have so far:[1] instead of refraining from directly denying the status of "being a member of" or "instantiating" as non-property-denoting predicates, this approach seeks to explain their non-property-denoting nature by grounding their consistency within a formal framework, that of ZF set theory. While this approach provides a rigorous foundation for discussing "being a member of" and other non-property-denoting predicates, I will argue below that its reliance on ZF set theory and the associated complexities come at a high cost. In particular, I contend that the price of adopting this approach is too high to sustain it as a viable new form of class nominalism.

2. ZF-Class Nominalism

Assuming, as Calemi (2024) does, that ZF is the appropriate arena for analyzing the predicate "being a member of", Calemi's conclusion that this predicate is not fully analyzable rests on two key principles:

1. Abstraction Principle: For any x and any set y, if x is a member of y, then this relationship can be uniquely captured or abstracted within ZF set theory. Formally, it might be expressed as something like $\forall x \exists y (x \in y \to \varphi(x))$.

2. Completeness Principle: This principle asserts that ZF set theory is complete in the sense that any true statement about sets that can be expressed in ZF's language is provable within ZF. Specifically, if a sentence φ about sets is true, then ZF contains a proof of φ, *vice versa*.

Consider the sentence "5 is prime". In the context of class analysis, this statement can be understood as asserting that 5 belongs to the

[1] A similar problem for resemblance nominalism raised by Russell (1912) and responded (not completely satisfactorily) by Rodriguez-Pereyra (2001, 2004).

class of all prime numbers, denoted as $5 \in \{x \mid x \text{ is prime}\}$. However, Calemi emphasizes that a class analysis, as understood here, is distinct from a ZF-class analysis. A ZF-class analysis determines whether a class denoted by a predicate exists based on whether it can satisfy both the Abstraction Principle and the Completeness Principle above. For instance, the class $\{x \mid x \text{ is prime}\}$ exists in ZF because it can be easily defined by a ZF formula (let's call it φ) and this formula is satisfied within ZF. Thus, both the Abstraction Principle and the Completeness Principle are trivially satisfied for this class.

However, when it comes to the class $\{\langle x, y\rangle \mid x \in y\}$, the situation is different. According to the Abstraction Principle, if this set were to exist based on a ZF-formula (let us call it ψ), then the Completeness Principle would require that be provable in ZF. But here's the catch: the Foundation of ZF dictates that certain sentences involving unrestricted quantification over sets (like "$\forall x \exists y(x \in y \to \varphi(x)$") attempting to capture the universal membership relation) are not provable within ZF. Therefore, if we try to apply the Abstraction Principle to create the class $\{\langle x, y\rangle \mid x \in y\}$, we end up with a formula that contradicts the Foundation of ZF, violating the Completeness Principle.

3. A Paradox

Now, suppose we want to analyze the sentence "ZF is consistent" using ZF-class nominalism. According to this approach, we can represent this sentence as a class assertion, specifically as $\mathrm{ZF} \in \{x \mid x \text{ is consistent}\}$. Since ZF includes a theory of syntax, we can formulate a formula — let's call it $\forall x \mathrm{Con}(x)$ — that captures the notion of consistency. When we apply this formula to ZF itself, we get Con(ZF), which is essentially asserting the consistency of ZF. So the Abstraction Principle is satisfied.

However, Gödel's Incompleteness Theorem poses a problem. It states that it is not the case that $\mathrm{ZF} \vdash \forall x \mathrm{Con}(\mathrm{x})$ as $\mathrm{ZF} \nvdash \mathrm{Con}(\mathrm{ZF})$, if ZF is consistent. Therefore, if we assume that "ZF is consistent" is true, this leads to a contradiction within ZF-class analysis by Completeness Principle, indicating that the assumption must be false.

On the other hand, if we assume that "ZF is consistent" is false, then "ZF is inconsistent" would be true. But, according to ZF-class analysis, even this assertion cannot be proven within ZF due to Gödel's theorem, as it would require proving the negation of Con(ZF). This creates a paradoxical situation where neither "ZF is consistent" nor "ZF is inconsistent" can be definitively asserted within ZF if ZF is consistent.

4. Further disccusions

Three points need to be emphasized here, First and trivially, there is an essential difference between the predicates "being a member of" and "being consistent" within ZF-class analysis. The Completeness Principle excludes the sentence of the form "$\forall x \exists y(x \in y \to \varphi(x))$" as false, which also indicates that the class $\{\langle x, y\rangle \mid x \in y\}$ does not exist within ZF. This might be controversial, but not paradoxical because denying "being member of" as a class denoting does not have to presuppose it as class denoting. However, this is not the case in "being consistent": the Completeness Principle plus Gödel's Theorem excludes the sentence of the form "Con(ZF)" by first supposing "ZF is consistent" is true, and it excludes the sentence of form "$\neg$Con(ZF)" by first supposing "ZF is inconsistent" is true.

Secondly, Gödel's Theorem tells us if ZF is consistent, then both Con(ZF) and $\neg$Con(ZF) cannot be proved in ZF. Now, suppose with ZF-Class analysis is right that this shows that neither "being consistent" nor "being inconsistent" is a class-denoting predicate. Does that mean that Gödel's Theorem does not get off the ground as it already assumes that "ZF is consistent"? No, on the contrary, this is exactly where the paradox of ZF-Class analysis takes place: to draw the conclusion that "being consistent" or "being inconsistent" is not a class-denoting predicate, the ZF-class nominalist has to use Gödel's Theorem, and to use Gödel's Theorem, "ZF is consistent" has to be assumed at first; otherwise, the (in)completeness Principle would have no application. That means that a ZF-class nominalist needs to treat "being consistent" as a class denoting predicate at first to make sense of "ZF is consistent," then conclude that "being consistent" is not a class-denoting predicate.

Thirdly, "being consistent" and "being a member of" are not the only predicates excluded as class-denoting by ZF-Class analysis. Along with these predicates, the predicates "being true," "being satisfiable," etc., are also excluded. Tarski's Indefinability Theorem tells us that ZF cannot define its own truth. So, according to ZF-Class analysis, truth cannot denote a class. Consequently, sentences of the form "this sentence is true" cannot be evaluated as true or false. This is hard to accept on the one hand. On the other hand, it is also paradoxical in the sense that if truth is not a class-denoting predicate, then truth has already been employed as a class-denoting predicate; otherwise, the Completeness Principle cannot get off the ground.

A ZF-class nominalist might propose that the consistency of ZF can be analyzed within a stronger set theory, such as Morse-Kelly (MK).[2] While MK offers a more robust framework that might enable discussions about ZF's consistency, it still encounters a similar obstacle. According to Gödel's Theorem, MK is unable to prove its own consistency, which leads to similar paradoxical considerations. An alternative stance would be to consider the consistency of ZF as a unique and inherently unanalyzable property. However, adopting this position comes with a significant cost. It necessitates offering a separate explanation for why the consistency of other systems, like Peano Arithmetic (PA), which can be derived within ZF, does not share the same status of being unanalyzable.

5. Conclusion

In conclusion, ZF-class nominalism's analysis of "being consistent" either leads to paradoxes or incompleteness, highlighting the complexities and limitations inherent in any formal system's ability to fully analyze and describe properties within its own framework. The need for a more nuanced approach that acknowledges these limitations and seeks alternative explanations for the unanalyzability of certain properties becomes evident.

Acknowledgments. I would like to thank DAAD-CSC for supproting my stay at Konstanz when conducting this research and the editor for the improvement of this paper.

References

Armstrong, D. M., 1978, *Universals and Scientific Realism. Volume 1: Nominalism and Realism*, Cambridge: Cambridge University Press.

[2] Calemi (2024) claims in Section 7 that MK is not optimal in comparison with ZF in other perspectives, such as it also suffers from Küng-Armstrong's trilemma and it is not ontologically clean.

Calemi, F. F., 2024, "ZF-class nominalism and the Küng-Armstrong trilemma: A plea for moderate ineffabilism", *Logic and Logical Philosophy* 33 (2): 205–223. DOI: 10.12775/LLP.2024.005

Küng, G., 1967, *Ontology and the Logical Analysis of Language: An Enquiry into the Contemporary Views on Universals*, Dordrecht: Reidel.

Rodriguez-Pereyra, G., 2001, "Resemblance nominalism and Russell's regress", *Australasian Journal of Philosophy* 79 (3): 395–408. DOI: 10.1080/713659267

Rodriguez-Pereyra, G., 2004, "Paradigms and Russell's resemblance regress", *Australasian Journal of Philosophy* 82 (4): 644–651. DOI: 10.1080/713659904

Russell, B., 1912, "The world of universals", chapter IX in *The Problems of Philosophy*, London: Williams and Norgate. Also, pages 45–50 in D. H. Mellor and A. Oliver (eds.), *Properties*, Oxford: Oxford University Press, 1997. See https://www.gutenberg.org/files/5827/5827-h/5827-h.htm#link2HCH0009 or http://www.sophia-project.org/uploads/1/3/9/5/13955288/russell_universals.pdf

GUANGLONG LUO
School of Philosophy
Nanain University
Tianjin, China
glluo@nankai.edu.cn