Logic and **L**ogical **P**hilosophy
Online First Articles (2025)
DOI: 10.12775/LLP.2025.025

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Hyperintensionality, Identity and Excluded Middle

Abstract. We introduce new semantics for the intuitionistic variant of the weakest non-Fregean logic, SCI. This semantics captures the notion of identity within a constructive framework and aligns with our intuitive understanding of it, which is provided by the Brouwer-Heyting–Kolmogorov interpretation. Moreover, we demonstrate how this approach naturally leads to interesting extensions of ISCI.

Keywords: intuitionistic logic; identity; Kripke semantics; hyperintensionality; natural deduction

1. What is propositional identity?

What does it mean for two formulas, ϕ and ψ, to be identical? We understand their equivalence: in classical logic, this simply means they share the same truth value. In a constructive setting, following the Brouwer-Heyting-Kolmogorov (BHK) interpretation of logical connectives, equivalence requires that any proof of ϕ can be *somehow* transformed into a proof of ψ, and *vice versa*. Suszko introduced a class of *non-Fregean logics* based on classical logic, in which a connective stronger than equivalence — *propositional identity* — is analyzed (see, e.g., Bloom and Suszko, 1972; Suszko, 1975). He rejected the Fregean idea that sentences function like names, denoting particular objects such as *Truth* and *Falsity*. Instead, drawing inspiration from Wittgenstein, he proposed that sentences denote *situations*. Equivalence, however, is too weak to represent the identity of situations unless one assumes that only two situations exist.

To address this, Suszko formalized the minimal requirements for propositional identity in the *Sentential Calculus with Identity* (SCI), where identity is a congruence relation between formulas that is stronger

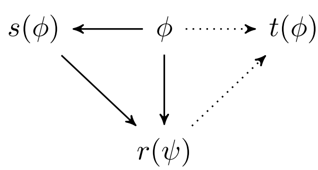
Received July 10, 2024. *Accepted* November 6, 2025. *Published online* December 2, 2025

than equivalence. Further extensions of SCI — the systems WB, WT, and WH — incorporate additional assumptions introduced by Suszko.

How can one interpret the connective of propositional identity in a constructive framework? The notion of a situation — however ambiguous it may seem — does not appear to be the most natural candidate. Instead, reconsidering the BHK-interpretation of intuitionistic connectives, particularly intuitionistic equivalence, proves illuminating.

As noted earlier, a proof of intuitionistic equivalence $\phi \supset\subset \psi$ is a function that *somehow* transforms a proof of ϕ into a proof of ψ, and *vice versa*. If we make the meaning of *somehow* more precise, we can define, in the spirit of BHK, a connective stronger than intuitionistic equivalence. In the weakest intuitionistic variant of SCI, namely ISCI, we assume this transformation is given by the identity function. This interpretation was previously introduced in (Chlebowski and Leszczyńska-Jasion, 2019).

In presenting his approach to non-Fregean logics, Suszko made use of a *Fregean diagram*:



where ϕ is a sentence, $r(\phi)$ is the referent of ϕ, which can be thought of as a situation described by ϕ, $s(\phi)$ is the way in which $r(\phi)$ is given, that is, the sense of ϕ (or the proposition expressed by ϕ). Finally, $t(\phi)$ is the truth value of ϕ (see Suszko, 1975). The Fregean axiom we seek to abolish states that:

$$t(\phi) = t(\psi) \Longrightarrow r(\phi) = r(\psi).$$

In a non-Fregean world, it can happen that two sentences have the same truth value while having different referents.

Let us attempt to reinterpret the diagram in order to approach it from a more constructive perspective. Intuitionistic truth carries an epistemic character — being intuitionistically true requires the existence of a proof. Perhaps it is most clearly visible in the *Curry-Howard Isomorphism*, where sentences/propositions are interpreted as the collections of their proofs (see, e.g., Sørensen and Urzyczyn, 2006; Troelstra and Schwichtenberg, 2000).

The Fregean distinction between sense and reference in the context of intuitionistic logic has previously been addressed in the work of Dummett and Martin-Löf (see, e.g., Dummett, 2021; Martin-Löf, 2021). Our approach aligns with these earlier treatments in that it takes as its point of departure two key observations by Dummett (as cited in Martin-Löf, 2021, p. 502):

> sense will be related to semantic value as a programme to its execution

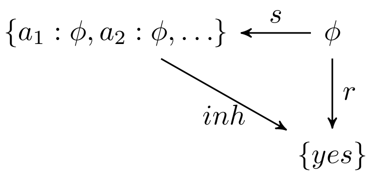
> It is thus correct to regard numerical terms as aiming at natural numbers by varying routes, and hence to apply to each of them a distinction between its reference — the natural number aimed at — and its sense — the particular means for specifying that natural number.

However, it differs from the previous ones by using the notion of truncated type in the explication of the reference of a sentence.

In the light of what has been said, $r(\phi)$ can be thought of as conveying the information whether ϕ is provable or not (without diving into the details of the way ϕ has been proved). $t(\phi)$ is again the truth value of ϕ, determined by $r(\phi)$. The interesting part concerns the function s, which for a given ϕ should represent the way $r(\phi)$ is given, that is, it represents the way in which the provability of ϕ is given — the procedure of establishing the provability of ϕ. It may be thought of as a proof of ϕ in some system. What we are interested in ISCI is precisely this aspect. As the slogan goes: *no entity without identity.* In this light, propositional identity in ISCI can be thought of as representing the identity criteria for $s(\phi)$. What we want to abolish in intuitionistic variant of SCI is this:

$$r(\phi) = r(\psi) \Longrightarrow s(\phi) = s(\psi)$$

We seek to allow for the joint provability of two formulas without requiring that their proofs — as senses — be identical. A more precise way to characterize the distinction under discussion — within a constructive framework — is to interpret $s(\phi)$ as the collection of proofs of ϕ, following the *propositions as types* perspective, and to interpret $r(\phi)$ as the truncated type of ϕ, denoted $inh(\phi)$ (see, e.g., Artemov and Protopopescu, 2016). The truncated type $inh(\phi)$ discards the specific content of the type ϕ — that is $s(\phi)$ — and retains only whether or not the type is inhabited. In other words, it tells us merely whether ϕ has a proof (*yes*) or not (*no*). If $a_1, a_2 \ldots$ is the collection of proofs of ϕ (i.e., the elements of the type), then the fact that this set is nonempty means the truncated type yields a positive answer: *yes*.



Naturally, we can have two formulas ϕ and ψ having identical truncated types, while not having the same set of proofs as their senses (take for example $\phi = p \supset p$, $\psi = p \supset (q \supset p)$).

In this paper, we focus on developing a Kripke-style semantics for ISCI and some of its extensions in a way that reflects these intuitions.

2. Hilbert-style system for ISCI

The language of ISCI, $\mathcal{L}_{\mathsf{ISCI}}$, is defined in a standard manner ($\otimes$ stands for any binary connective of the language):

$$\phi \ ::= \ p_i \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \supset \phi \mid \phi \equiv \phi$$

An axiomatic system for ISCI is defined as an extension of any axiomatic system for Intuitionistic Logic (INT) formulated in a language containing $\bot, \wedge, \vee, \supset$ — for instance, the system presented in (Kleene, 1952).

(I_1) $\phi \supset (\psi \supset \phi)$
(I_2) $(\phi \supset \psi) \supset ((\phi \supset (\psi \supset \chi)) \supset (\phi \supset \chi))$
(I_3) $\phi \supset (\psi \supset (\phi \wedge \psi))$
(I_4) $(\phi \wedge \psi) \supset \phi$
(I_5) $(\phi \wedge \psi) \supset \psi$
(I_6) $\phi \supset (\phi \vee \psi)$
(I_7) $\psi \supset (\phi \vee \psi)$
(I_8) $(\phi \supset \chi) \supset ((\psi \supset \chi) \supset ((\phi \vee \psi) \supset \chi))$
(I_9) $(\phi \supset \psi) \supset ((\phi \supset (\psi \supset \bot)) \supset (\phi \supset \bot))$
(I_{10}) $\bot \supset \psi$
(MP) $\dfrac{\psi \qquad \phi \supset \psi}{\psi}$

by means of the axioms:

($\equiv_1$) $\phi \equiv \phi$ (*reflexivity axiom*)
($\equiv_2$) $(\phi \equiv \psi) \supset (\phi \supset \psi)$ (*special identity axiom*)

$(\equiv_3)$ $(\phi \equiv \psi) \supset ((\chi \equiv \omega) \supset ((\phi \otimes \chi) \equiv (\psi \otimes \omega)))$ (*congruence axiom*)

Note that, as a consequence, $\equiv$ satisfies both symmetry and transitivity.

We denote by $\vdash_{\mathsf{ISCI}}$ the derivability relation in the axiomatic system for ISCI. A formula ϕ is a theorem of ISCI if and only if $\emptyset \vdash_{\mathsf{ISCI}} \phi$.

Note that since ISCI $\subset$ SCI and every provable identity in SCI has the form $\phi \equiv \phi$, it follows that every provable equation in ISCI also has this form.

3. BHK-interpretation

Let us begin with an informal semantics for ISCI. In addition to the standard proof conditions for intuitionistic connectives, we introduce a proof explanation for $\equiv$:

there is no proof of $\bot$	
a is a proof of $\phi \wedge \psi$	$a = (a_1, a_2)$; a_1 is a proof of ϕ and a_2 is a proof of ψ
a is a proof of $\phi \vee \psi$	$a = (a_1, a_2)$; $a_1 =$ left and a_2 is a proof of ϕ or $a_1 =$ right and a_2 is a proof of ψ, where left (right) denotes the left (right) disjunct
a is a proof of $\phi \supset \psi$	a is a construction that converts each proof a_1 of ϕ into a proof $a(a_1)$ of ψ
a is a proof of $\phi \equiv \psi$	a is a construction which shows that the classes of proofs of ϕ and ψ are the same or: a is the identity function id

Let us analyze the following formula:

$$\phi \equiv \phi$$

Our proof-explanation permits these two interpretations of $\phi \equiv \phi$:

(i_1) Two sets of proofs are equal (the set of proofs of ϕ is identical to itself);

(i_2) any proof a of ϕ can be transformed into a proof of ϕ by means of id;

However, this does not entail an interpretation in which

$(*)$ any two proofs of ϕ are equal.

According to (i_1) and (i_2), we adopt an *intensional* (or even *hyperintensional*) view of proofs: knowing that a formula is provable is not sufficient — we are also interested in *how* it is proved. In contrast, according to ($*$), only the fact of provability itself matters.

4. Semantics

We propose an alternative approach to Kripke semantics for ISCI. Previous work in this area relies on the concept of an ISCI-admissible assignment (Chlebowski and Leszczyńska-Jasion, 2019). In this paper, we introduce the notion of a *proof-assignment*, which provides a criterion for determining whether an equation is forced at a given information state.

The present approach to semantics for ISCI may be thought of as a simplification of the previous approach which, at the same time, appears philosophically more plausible.

Definition 1 (Intuitionistic frame). By an *intuitionistic frame* we mean an ordered tuple $\boldsymbol{F} = \langle \mathcal{W}, \leq \rangle$, where $\mathcal{W}$ is a non-empty set, $\leq$ is a reflexive and transitive binary relation on $\mathcal{W}$.

By For_0 we shall mean the set of all propositional variables. If $\boldsymbol{F} = \langle \mathcal{W}, \leq \rangle$ is an intuitionistic frame, then by *truth-assignment in* $\boldsymbol{F}$ we mean a function:

$$v_t \colon For_0 \times \mathcal{W} \longrightarrow \{0, 1\}.$$

By a *proof-assignment* we mean a function:

$$v_p \colon For \times \mathcal{W} \longrightarrow \mathbb{P},$$

where $|\mathbb{P}| \geq 2$ and the following condition is satisfied:

(*con*) for an arbitrary world w, if $v_p(\phi, w) = v_p(\psi, w)$ and $v_p(\chi, w) = v_p(\omega, w)$ then $v_p(\phi \otimes \chi, w) = v_p(\psi \otimes \omega, w)$.

The notion of proof-assignment is introduced to add an additional layer to Kripke semantics — formulas with identity as the main operator cannot be evaluated solely based on co-satisfiability.[1] This aligns with our BHK-interpretation of propositional identity.

These two assignments provide each formula with two dimensions: the dimension of truth (where a formula is either true/proved or false/not proved at a given information state) and the dimension of proof (where each formula is assigned an element from $\mathbb{P}$). Thus, one can think of the set $\mathbb{P}$ as consisting of proofs, though we do not specify its structure, and we impose a minimal condition for its cardinality — there are at least two proofs.[2] A special type of model, at least from the point of view of the basic non-Fregean logic, is the model with a countably infinite set of proofs. The possibility of such a construction supports the claim that each formula has a unique proof, which is indeed needed to prove completeness of ISCI.

Let us now define *forcing relation*:

DEFINITION 2 (Forcing). Let v_t be a truth-assignment in a given frame $\boldsymbol{F}$ and v_p be a proof-assignment. A *forcing relation* $\Vdash$ *determined by* v_t *and* v_p *in* $\boldsymbol{F}$ is a relation between elements of W and elements of $\mathcal{L}_{\mathsf{ISCI}}$ which satisfies, for arbitrary $w \in W$, the following conditions:

(1) $w \Vdash p_i$ iff $v_t(p_i, w) = 1$;
(2) $w \Vdash \phi \equiv \psi$ iff $v_p(\phi, w) = v_p(\psi, w)$;
(3) if $w \Vdash \phi \equiv \psi$ then $w \Vdash \phi \supset \psi$;
(4) $w \nVdash \bot$;
(5) $w \Vdash \phi \wedge \psi$ iff $w \Vdash \phi$ and $w \Vdash \psi$;
(6) $w \Vdash \phi \vee \psi$ iff $w \Vdash \phi$ or $w \Vdash \psi$;
(7) $w \Vdash \phi \supset \psi$ iff for each w' such that $w \leq w'$, if $w' \Vdash \phi$ then $w' \Vdash \psi$;
(*mon*) for any formula ϕ: if $w \Vdash \phi$ and $w \leq w'$, then $w' \Vdash \phi$.

[1] I would like to mention two constructions that are somewhat similar in spirit to the notion of proof-assignment. The first one is used in the field of relating logics (see, e.g., Klonowski, 2021), where one considers an additional function that assigns sets to formulas. The second one is presented by Marek Nowak in his paper on analytic equivalence (Nowak, 2008), where, in addition to truth-valuation, he introduces a certain homomorphism to evaluate formulas in which the identity connective is the main operator.

[2] As it will be visible after the definition of the forcing relation, this cardinality restriction forces propositional identity to be at least as strong as intuitionistic equivalence. In this case, where $|\mathbb{P}| = 2$, we consider only two objects: one of them is assigned to all formulas forced at a given world, the other one to all formulas not forced at a given state.

The forcing condition for identity states that a given equation $\phi \equiv \psi$ is forced at a given information state iff the same proof is assigned to ϕ and ψ.

All of the conditions except (2) express standard requirements for intuitionistic connectives.

Note that not all proof-assignments determine a forcing relation. Assume there is a v_p such that for some w we have $v_p(p, w) = v_p(p \supset \bot, w)$. It follows from the definition of forcing that a given world w is such that:

1. $w \Vdash p \supset (p \supset \bot)$, and
2. $w \Vdash (p \supset \bot) \supset p$ (due to the symmetry of $=$).

Taking these two fact together we have to conclude that there is a world $w* \geq w$ such that $v_t(p, w^*) = 1$ iff $v_t(p, w^*) = 0$, which contradicts the definition of a truth-assignment.

Why, then, do we need to introduce an additional layer to the well-known framework of Kripke semantics? The logic ISCI is *hyperintensional*, meaning that formulas of the form $\phi \equiv \psi$ cannot be interpreted in terms of the joint satisfiability of ϕ and ψ across possible worlds. If this were possible, we would obtain a logic stronger than ISCI, where, for instance, $(\phi \wedge \psi) \equiv (\psi \wedge \phi)$ would be a theorem. Therefore, an additional valuation is needed to assign distinct objects to intuitionistically equivalent formulas. As we mentioned earlier, these objects can be thought of as proofs of a given formula.

Thus, in ISCI (but not in some of its extensions), $\phi \wedge \psi$ does not necessarily share *the same* proof as $\psi \wedge \phi$. The condition (*con*) ensures that proof-assignments respect congruence, a natural assumption about proofs: if ϕ and ψ share the same proof (for example, both are assigned the number 1), and similarly, χ and ω share the same proof (assigned the number 2), then the proof assigned to $\phi \wedge \chi$ will be the same as the proof assigned to $\psi \wedge \omega$.

DEFINITION 3. An ISCI-*model* is a tuple

$$\boldsymbol{M} = \langle \mathcal{W}, \leq, \Vdash \rangle,$$

where $\boldsymbol{F} = \langle W, \leq \rangle$ is an intuitionistic frame, $\Vdash$ is a forcing relation determined some assignments v_t and v_p.

A formula ϕ which is forced by every world of an ISCI-model, that is, such that $w \Vdash \phi$ for each $w \in W$, is called *true in* the model, symbolically $\boldsymbol{M} \Vdash \phi$. A formula true in every ISCI-model is called ISCI-*valid*, $\Vdash \phi$.

By '$\models_{\mathsf{ISCI}}$' we mean the local semantic entailment relation in ISCI. Thus,

$$\Gamma \models_{\mathsf{ISCI}} \phi$$

means that for every model $\boldsymbol{M}$ and every world w from the domain of $\boldsymbol{M}$ we have: if $w \Vdash \psi$ for every $\psi \in \Gamma$ then $w \Vdash \phi$.[3]

Let us state some observations concerning special types of models.

CLAIM 1. *There exists an intensional* ISCI*-model* $\boldsymbol{M}$*, i.e.,* $\boldsymbol{M} \Vdash ((\phi \supset \psi) \wedge (\psi \supset \phi)) \supset (\phi \equiv \psi)$*.*

To illustrate this, let us define v_p as a function specified as follows: for all w, $v_p(\phi, w) = [\phi]_{\approx_w}$, where $\phi \approx_w \psi$ iff $w \Vdash \phi \supset \psi$ and $w \Vdash \psi \supset \phi$. Note that intensional ISCI-model is just an intuitionistic model for a language containing two distinct symbols denoting intuitionistic equivalence.

We call this model intensional since the identity connective is reduced to intuitionistic equivalence, which is itself an intensional connective.

CLAIM 2. *There exists a hyperintensional* ISCI*-model, i.e.,* $\boldsymbol{M} \Vdash \phi \equiv \psi$ *iff* $\phi = \psi$*.*

It is straightforward to construct a model where v_p is an injective function with the set of natural numbers as a codomain, fulfilling (*con*). Thus for an arbitrary w: $v_p(\phi, w) = v_p(\psi, w)$ iff $\phi = \psi$. Notably, in such a model a formula $(\phi \equiv \psi) \supset \bot$ is true whenever ϕ and ψ are syntactically distinct.

One can approach the construction of hyperintensional model in a more *syntactic* manner by putting, for an arbitrary w: $v_p(\phi, w) = \{\phi\}$. This yields an injective function with the set of formulas as its codomain, where each formula in a given information state has a unique proof that is not shared by any other formula.

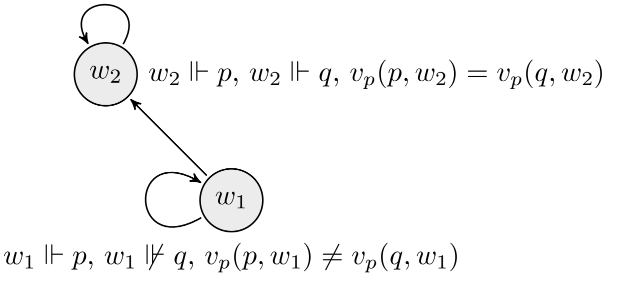
We refer to these models as hyperintensional because propositional identity in them is stronger than intuitionistic equivalence and cannot be defined in terms of co-satisfiability. While there are many different hyperintensional models, the one we have constructed represents an extreme case: in this model, it is impossible to define a connective more

[3] Later on we will use subscripts in order to use the notion of semantic entailment in different extensions of ISCI.

restrictive than propositional identity. As we will demonstrate, the existence of such models makes the logic ISCI somewhat trivial, as the only provable identities are of the form $\phi \equiv \phi$.

CLAIM 3. *A formula* $(p \equiv q) \vee ((p \equiv q) \supset \bot)$ *is not* ISCI-*valid.*

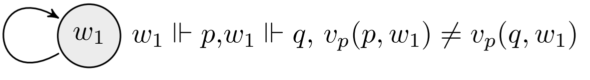
PROOF. Let us consider the following model $\boldsymbol{M} = \langle \{w_1, w_2\}, \leq, \Vdash \rangle$, graphically depicted as follows:



It is easy to check that neither $w_1 \Vdash p \equiv q$ nor $w_1 \Vdash (p \equiv q) \supset \bot$. Thus $w_1 \nVdash (p \equiv q) \vee ((p \equiv q) \supset \bot)$. ⊣

CLAIM 4. *A formula* $((p \supset q) \wedge (q \supset p)) \supset (p \equiv q)$ *is not* ISCI-*valid.*

PROOF. Let us consider the following model $\boldsymbol{M} = \langle \{w_1\}, \leq, \Vdash \rangle$ graphically depicted as follows:



Note that $w_1 \Vdash (p \supset q) \wedge (q \supset p)$ but $w_1 \nVdash p \equiv q$. ⊣

Let us make three important observations concerning validities in the presented semantics.

CLAIM 5. *The only tautological equation in* ISCI *has the following form:* $\phi \equiv \phi$.

It follows from the fact that valid formulas must be true in every ISCI-model, and there exist models based on injective proof-assignments.

CLAIM 6. $(\phi \equiv (\phi \supset \bot)) \supset \bot$ *is* ISCI-*valid.*

Assuming $\phi \equiv (\phi \supset \bot)$ leads to absurdity by an application of the condition (3) of the forcing relation and the fact that $\equiv$ is symmetric.

Claim 7. *$\equiv$ is an equivalence relation, that is, the formulas:*
(ref) $\phi \equiv \phi$
(sym) $(\phi \equiv \psi) \supset (\psi \equiv \phi)$
(trans) $(\phi \equiv \psi) \wedge (\psi \equiv \chi) \supset (\phi \equiv \chi)$
are valid in the presented semantics.

The last claim follows from the definition of proof-assignement and the fact that it obeys the congruence condition.

4.1. Soundness

Let us prove:

Theorem 4.1 (Soundness). If $\vdash_{\mathsf{ISCI}} \phi$, then $\Vdash \phi$.

Proof. Let us consider identity specific axioms:

For $\phi \equiv \phi$. Assume there is a a model and a world w such that $w \nVdash \phi \equiv \phi$. It implies that $v_p(\phi, w) \neq v_p(\phi, w)$, which is impossible.

For $(\phi \equiv \psi) \supset (\phi \supset \psi)$. Assume that $w \nVdash (\phi \equiv \psi) \supset (\phi \supset \psi)$. Thus there is a $w' \geq w$ such that $w' \Vdash \phi \equiv \psi$ and $w' \nVdash \phi \supset \psi$. But from the condition (3) of the definition of forcing we get at the same time $w' \Vdash \phi \supset \psi$.

For $(\phi \equiv \psi) \supset ((\chi \equiv \omega) \supset ((\phi \otimes \chi) \equiv (\psi \otimes \omega)))$. Assume $w \nVdash (\phi \equiv \psi) \supset ((\chi \equiv \omega) \supset ((\phi \otimes \chi) \equiv (\psi \otimes \omega))$. Thus there is a $w' \geq w$ such that $w' \Vdash \phi \equiv \psi$ and $w' \nVdash (\chi \equiv \omega) \supset ((\phi \otimes \chi) \equiv (\psi \otimes \omega))$. Consequently there is a world $w^* \geq w'$ such that $w^* \Vdash \phi \equiv \psi$, $w^* \Vdash \chi \equiv \omega$ and $w^* \nVdash (\phi \otimes \chi) \equiv (\psi \otimes \omega)$. Thus $v_p(\phi, w^*) = v_p(\psi, w^*)$ and $v_p(\chi, w^*) = v_p(\omega, w^*)$. From the condition (*con*) imposed on proof-assignments we get $v_p(\phi \otimes \chi, w^*) = v_p(\psi \otimes \omega, w^*)$. Thus $w^* \Vdash (\phi \otimes \chi) \equiv (\psi \otimes \omega)$.

It is easy to show that all the other axioms, specific for intuitionistic logic, are valid and that *modus ponens* preserve validity. ⊣

4.2. Completeness

In the completeness proof we will use the notion of *canonical* ISCI-*model* build up from maximally consistent sets of formulas. This work is based on our previous work on Kripke semantics for ISCI (Chlebowski and Leszczyńska-Jasion, 2019). Let Γ be a set of formulas and ϕ a single formula of $\mathcal{L}_{\mathsf{ISCI}}$. We stipulate that Γ is *ϕ-consistent* iff $\Gamma \nvdash_{\mathsf{ISCI}} \phi$. If that is not the case, we call Γ *ϕ-inconsistent.* ϕ-consistent set Γ is called *maximally ϕ-consistent* iff there is not proper ϕ-consistent supserset of Γ. If a formula ϕ is irrelevant in a given context we will omit the prefix and simply say about a given set that it is (maximally) consistent.

Let us recall:

Lemma 4.1 (Lindenbaum's lemma). For every ϕ-consistent set Γ there is a maximally ϕ-consistent set $\mathbf{\Gamma} \supseteq \Gamma$.

Proof. We start by enumerating all formulas of $\mathcal{L}_{\mathsf{ISCI}}$: $\psi_1, \psi_2, \ldots$. Assume Γ is ϕ-consistent. We can recursively construct an infinite sequence of sets in the following manner:

$$\Gamma_0 = \Gamma$$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n & \text{if } \Gamma_n \cup \{\psi_{n+1}\} \vdash_{\mathsf{ISCI}} \phi \\ \Gamma_n \cup \{\psi_{n+1}\} & \text{otherwise.} \end{cases}$$

Naturally, each Γ_i is ϕ-consistent and $\mathbf{\Gamma} = \bigcup_{n=0}^{\infty} \Gamma_n$ is maximally ϕ-consistent. ⊣

Note that if a given set Γ is ϕ-consistent then certainly $\phi \notin \Gamma$.

Lemma 4.2. $\emptyset \vdash_{\mathsf{ISCI}} \phi$ if and only if ϕ is an element of each maximally consistent set.

Proof. "⇒" Assume $\emptyset \vdash_{\mathsf{ISCI}} \phi$ but there is a maximally consistent set $\mathbf{\Gamma}$ such that $\phi \notin \mathbf{\Gamma}$. Then there exists a formula ψ such that $\mathbf{\Gamma}$ is ψ-consistent. By the construction of $\mathbf{\Gamma}$, there is a set Γ_n such that $\Gamma_n \cup \{\phi\} \vdash_{\mathsf{ISCI}} \psi$. Naturally, $\Gamma_n \vdash_{\mathsf{ISCI}} \phi \supset \psi$. But since $\emptyset \vdash_{\mathsf{ISCI}} \phi$ then also $\Gamma_n \vdash_{\mathsf{ISCI}} \phi$. By *modus ponens* we get that $\Gamma_n \vdash_{\mathsf{ISCI}} \psi$. Thus Γ_n is ψ-inconsistent and so is its superset $\mathbf{\Gamma}$.

"⇐" Assume $\emptyset \nvdash_{\mathsf{ISCI}} \phi$. Thus the empty set is ϕ-consistent and by Lindenbaum lemma, there is a maximally consistent extension of $\emptyset$ such that ϕ is not its member. ⊣

The central construction needed in the completeness proof is that of a canonical model.

Definition 4 (Canonical ISCI-model). Canonical ISCI-model is a tuple

$$\boldsymbol{M} = \langle \mathcal{W}, \subseteq_{\mathcal{W}}, \in_{\mathcal{W}} \rangle,$$

where $\mathcal{W}$ is the set of all maximally consistent sets of formulas of $\mathcal{L}_{\mathsf{ISCI}}$, $\subseteq_{\mathcal{W}}$ is the set inclusion in $\mathcal{W}$, and $\in_{\mathcal{W}}$ is the membership relation between formulas of $\mathcal{L}_{\mathsf{ISCI}}$ and elements of $\mathcal{W}$. We take truth assignment in $\langle \mathcal{W}, \subseteq_{\mathcal{W}} \rangle$ to be a function $v_t \colon For_0 \times \mathcal{W} \longrightarrow \{0,1\}$ defined: $v(\phi, w) = 1$ iff $\phi \in w$, for all $w \in \mathcal{W}$. Proof-assignment is defined as follows: $v_p(\phi, w) = [\phi]_{\approxeq_w}$, where $\phi \approxeq_w \psi$ iff $\phi \equiv \psi \in w$.

Note that proof assignment defined this way satisfies (con) due to the fact that $(\phi \equiv \psi) \supset ((\chi \equiv \omega) \supset ((\phi \otimes \chi) \equiv (\psi \otimes \omega))) \in w$, for an arbitrary maximally consistent set w.

LEMMA 4.3. The canonical model is an ISCI-model.

PROOF. Let $\boldsymbol{M} = \langle \mathcal{W}, \subseteq_{\mathcal{W}}, \in_{\mathcal{W}} \rangle$ be the canonical ISCI-model. Naturally frame $\langle \mathcal{W}, \subseteq_{\mathcal{W}} \rangle$ is an intuitionistic frame since $\mathcal{W}$ is non-empty (the empty set is a consistent and thus it has a maximally consistent superset (by Lemma 4.1)). Moreover, the relation $\subseteq_{\mathcal{W}}$ is reflexive and transitive.

We now show that the membership relation $\in_{\mathcal{W}}$ satisfies the conditions imposed on the forcing relation. For the standard intuitionistic connectives, this follows from the following observation: if w is a maximally consistent set, then the following conditions hold:

1. $\bot \notin w$;
2. $A \in w$ iff $w \vdash_{\mathsf{ISCI}} A$;
3. $A \wedge B \in w$ iff $A \in w$ and $B \in w$;
4. $A \vee B \in w$ iff $A \in w$ or $B \in w$;
5. if $A \supset B \in w$ and $A \in w$, then $B \in w$;
6. if $A \supset B \notin w$, then $w \cup \{A\}$ is B-consistent (see, e.g., Chlebowski and Leszczyńska-Jasion, 2019 for a detailed proof.)

Let us just show that $\phi \equiv \psi \in w$ iff $v_p(\phi, w) = v_p(\psi, w)$: assume that $\phi \equiv \psi \in w$ for a fixed but arbitrary w. From the definition of the canonical model we know that it happens if and only if $\phi \approxeq_w \psi$. Thus $\psi \in [\phi]_{\approxeq_w}$ and $\phi \in [\psi]_{\approxeq_w}$ (since $\equiv$ is an equivalence relation). Therefore we get that $[\phi]_{\approxeq_w} = [\psi]_{\approxeq_w}$ and in consequence $v_p(\phi, w) = v_p(\psi, w)$. ⊣

Let us recall that by Lemma 4.2 each thesis of ISCI is true in the canonical ISCI-model.

THEOREM 4.2 (completeness). If a formula is ISCI-valid, then it is a theorem of ISCI.

PROOF. We will prove the converse: if a formula is not a theorem of $\mathsf{H}_{\mathsf{ISCI}}$ then it is not ISCI-valid. Assume ϕ is not a thesis of ISCI. Then, by Lemma 4.2, there exists maximally consistent set w such that $\phi \notin w$. Thus there is a world w in the canonical ISCI-model which does not contain ϕ. Therefore ϕ is not ISCI-valid. ⊣

5. ISCI$^-$

To ensure the soundness of ISCI with respect to the proposed semantics, we had to add condition (*con*), which each proof-assignment must satisfy. If we remove this requirement from the definition, we obtain a subsystem of ISCI,[4] namely ISCI$^-$, which can be axiomatized using the following specific axioms for identity:

($\equiv_1$) $\phi \equiv \phi$ (*reflexivity axiom*)
($\equiv_2$) $(\phi \equiv \psi) \supset (\phi \supset \psi)$ (*special identity axiom*)

In order to obtain semantics for this system we erase the condition (con) imposed in proof assignments in ISCI. It is now possible that the following equalities hold:

$$v_p(p, w) = v_p(q, w) \qquad v_p(r, w) = v_p(s, w)$$

while we have

$$v_p(p \wedge r, w) \neq v_p(q \wedge s, w)$$

since p and q (r and s respectively) can be mapped into the same object via proof-assignement, being at the same time syntactically distinct.

Completeness of the system can be proved by a modification of the completeness proof for ISCI. In what follows we will use constructions defined earlier (like that of *maximally consistent set* or Lindenbaum extension) in a modified manner: the derivability relation is in each case defined by an axiomatic system under discussion. In the definition of canonical model for ISCI$^-$ we determine the proof-assignment exactly as in the case of ISCI: $v_p(\phi, w) = [\phi]_{\approxeq_w}$ ($\phi \approxeq_w \psi$ iff $\phi \equiv \psi \in w$). Naturally there are ISCI$^-$ maximally consistent sets which do not contain the congruence axiom. Thus we can have (for a suitable w): $\phi \equiv \psi \in w$, $\chi \equiv \omega \in w$ but $(\phi \otimes \chi) \equiv (\psi \otimes \omega) \notin w$.

[4] Some systems related to classical SCI have been analyzed before (see, e.g., Ishii, 1998).

6. ISCI$^+$ and some other extensions

ISCI with decidable identity (ISCI$^+$). In Section 3 we have introduced an interpretation of the propositional identity in terms of the identity function. In the light of this interpretation, should we also assume that identity is decidable? In his *Remarks on the foundations of mathematics* (part IV-10) Wittgenstein observes:

> When someone hammers away at as with the law of excluded middle as something which cannot be gainsaid, it is clear that there is something wrong with his question.
>
> When someone sets up the law of excluded middle he is as it were putting the two pictures before us to choose from, and saying that one must correspond to a fact. But what if it is questionable whether the pictures can be applied here.
>
> And if you say that the infinite expansion must contain the pattern ϕ or not contain it, you are so to speak shewing us the picture of an unsurveayble series reaching into the distance.
>
> But what if the picture began to flicker in the far distance?
>
> (Wittgenstein, 1956)

The rejection of the law of excluded middle, at least within the constructivist tradition, is deeply tied to skepticism about the existence of actual infinity. This type of skepticism is clearly visible in the quotation above. But it is questionable whether Wittgenstein's picture can be applied here. Is it the case that for every pair of formulas ϕ and ψ we can decide whether $\phi \equiv \psi$ holds or $(\phi \equiv \psi) \supset \bot$ holds? We have shown that it does not hold in ISCI, but there are some reasons to consider it intuitively valid (given our informal interpretation). Assume that the intended reading of the notion of *proof* in BHK-interpretation is *natural deduction proof*. Then certainly equality of two recursively constructed trees is a decidable property.[5] Of course, whether we accept the decidability of identity or not depends on the specification of the vague notion of proof, but it seems to me that proofs, unlike infinite series, cannot flicker in the far distance.[6]

[5] An interesting take on the problem of decidability of identity of proofs can be found in (Kuznets, 2007).

[6] Another source of inspiration for considering this extension is Heyting arithmetic. Naturally $\phi \vee \neg\phi \notin \mathsf{HA}$, for some formula ϕ, but $(a = b) \vee \neg(a = b) \in \mathsf{HA}$, that is equality of natural numbers is decidable.

The logic ISCI^+ results from ISCI by the addition of the following axiom:

$(\equiv^*)$ $(\phi \equiv \psi) \vee ((\phi \equiv \psi) \supset \bot)$

An ISCI^+-model can be obtained from the ISCI-model by the modification of the notion of proof assignment. In ISCI^+ we take it to be a function $v_p \colon For \longrightarrow \mathbb{P}$ (where $|\mathbb{P}| \geq 2$, as in the case of ISCI) satisfying the familiar requirement:
(con) if $v_p(\phi) = v_p(\psi)$ and $v_p(\chi) = v_p(\omega)$ then $v_p(\phi \otimes \chi) = v_p(\psi \otimes \omega)$.
Thus the forcing condition for identity has the following form:

(2^*) $w \Vdash \phi \equiv \psi$ iff $v_p(\phi) = v_p(\psi)$.

Due to this modification, the truth value of identity in ISCI^+ no longer depends on a given information state, introducing a more platonistic view of proofs. As a result, it is easy to see that the new forcing condition for identity (stemming from the decidability of equality, $=$, in the metatheory) implies the validity of $(\phi \equiv \psi) \vee ((\phi \equiv \psi) \supset \bot)$.

An alternative approach to defining the semantics of ISCI^+ is to introduce the proof assignment as a function $v_p \colon For \times \mathcal{W} \longrightarrow \mathbb{P}$ (with $|\mathbb{P}| \geq 2$), which satisfies the (con) requirement and, moreover, the following condition:

(em) If $v_p(\phi, w) \neq v_p(\psi, w)$ and $w \leq v$ then $v_p(\phi, v) \neq v_p(\psi . v)$.

The (em) condition (standing for excluded middle) blocks the counterexample to excluded middle given in Claim 3. For each information state w, either $v_p(\phi, w) = v_p(\psi, w)$, in which case $w \Vdash \phi \equiv \psi$, or $v_p(\phi, w) \neq v_p(\psi, w)$, in which case, by (em), $w \Vdash (\phi \equiv \psi) \supset \bot$.

CLAIM 8. ISCI^+ *is complete with respect to the class of all* ISCI^+*-models.*

PROOF. *Canonical* ISCI^+*-model* is a tuple

$$\boldsymbol{M} = \langle \mathcal{W}, \subseteq_{\mathcal{W}}, \in_{\mathcal{W}} \rangle,$$

where $\mathcal{W}$ is the set of all maximally consistent sets of formulas of $\mathcal{L}^+_{\mathsf{ISCI}}$, $\subseteq_{\mathcal{W}}$ is the set inclusion in $\mathcal{W}$, and $\in_{\mathcal{W}}$ is the membership relation between formulas of $\mathcal{L}_{\mathsf{ISCI}}$ and elements of $\mathcal{W}$. As before, we take truth assignment in $\langle \mathcal{W}, \subseteq_{\mathcal{W}} \rangle$ to be a function $v_t \colon For_0 \times \mathcal{W} \longrightarrow \{0, 1\}$ defined: $v(\phi, w) = 1$ iff $\phi \in w$, for all $w \in \mathcal{W}$.

The proof-assignment is defined as before $v_p(\phi, w) = [\phi]_{\approxeq_w}$, where $\phi \approxeq_w \psi$ iff $\phi \equiv \psi \in w$. It is straightforward to verify that $\phi \equiv \psi \in w$ iff

$v_p(\phi, w) = v_p(\psi, w)$. What remains is to show that the proof assignment defined in this way satisfies condition (em). Assume $v_p(\phi, w) \neq v_p(\psi, w)$, $w \subseteq_{\mathcal{W}} v$, and $v_p(\phi, v) = v_p(\psi, v)$. Then $\phi \equiv \psi \notin w$, $w \subseteq_{\mathcal{W}} v$, and $\phi \equiv \psi \in v$. Since

$$(\phi \equiv \psi) \vee ((\phi \equiv \psi) \supset \bot)$$

is ISCI^+-valid, it belongs to every maximally consistent set, and in particular to w. By the property that $\gamma \vee \delta \in w$ iff $\gamma \in w$ or $\delta \in w$, and given that $\phi \equiv \psi \notin w$, we conclude that $(\phi \equiv \psi) \supset \bot \in w$, and hence $(\phi \equiv \psi) \supset \bot \in v$. But since we also have $\phi \equiv \psi \in v$, it follows that $\bot \in v$, a contradiction.[7]

Returning to completeness, assume that ϕ is not a theorem of ISCI^+. Then there exists a maximally consistent set that does not contain ϕ. Hence, there is a world in the canonical ISCI^+ model in which ϕ is not present, and therefore ϕ is not valid. ⊣

CLAIM 9. ISCI^+ *does not enjoy disjunction property.*

PROOF. Axiom $(\equiv^*)$ is a counterexample, that is

$$\vdash_{\mathsf{ISCI}^+} (\phi \equiv \psi) \vee ((\phi \equiv \psi) \supset \bot)$$

but neither $\vdash_{\mathsf{ISCI}^+} \phi \equiv \psi$ nor $\vdash_{\mathsf{ISCI}^+} (\phi \equiv \psi) \supset \bot$, when ϕ and ψ are syntactically distinct. ⊣

Strong ISCI (ISCI^s). Note that ISCI is sound (but not complete) with respect to the class of all hyperintensional models. In these models, we assume the proof-assignment to be an injective function. As a result, formulas of the following form are true in such models:

(ref) $\phi \equiv \phi$
(nref) $(\phi \equiv \psi) \supset \bot$, for syntactically distinct ϕ and ψ.

While (ref) is provable in ISCI, (nref) is not. However, the axiomatic system for ISCI can be strengthened by adding the following axiom

$(\equiv_4)$ $(\phi \equiv \psi) \supset \bot$, for syntactically distinct ϕ and ψ.

CLAIM 10. ISCI^s *is complete with respect to the class of all hyperintensional* ISCI-*models.*

[7] I would like to thank the anonymous referee for pointing out some nuances concerning the construction of canonical model for ISCI^+.

PROOF. In hyperintensional models proof assignment is assumed to be injective function. Thus in canonical ISCI^s-model $v_p(\phi, w)$ is defined to be a singleton consisting only of ϕ. It follows that in each world of an arbitrary model $\phi \equiv \psi$ is not forced, for syntactically distinct ϕ and ψ. Therefore $(\phi \equiv \psi) \supset \bot$ is forced at every world. ⊣

CLAIM 11. ISCI^s *enjoys disjunction property.*

PROOF. Disjunction property follows from axioms $(\equiv_1)$ and $(\equiv_4)$ — if $(\phi \equiv \psi) \vee ((\phi \equiv \psi) \supset \bot)$ is provable then exactly one of the components is provable as well. ⊣

Logic of propositional isomorphism. In type theory, the notion of type isomorphism is a relation that is weaker than type identity but stronger than equivalence (see, e.g., DiCosmo, 1995). Thus, it is natural to consider it as an extension of ISCI. Logic ISCI^{iso} is obtained by adding the following axioms:

(iso_1) $\phi \wedge \psi \equiv \psi \wedge \phi$
(iso_2) $(\phi \wedge \psi) \wedge \chi \equiv \phi \wedge (\psi \wedge \chi)$
(iso_3) $\phi \vee \psi \equiv \psi \vee \phi$
(iso_4) $(\phi \vee \psi) \vee \chi \equiv \phi \vee (\psi \vee \chi)$
(iso_5) $(\phi \supset (\psi \supset \chi)) \equiv (\psi \supset (\phi \supset \chi))$
(iso_6) $((\phi \wedge \psi) \supset \chi) \equiv (\phi \supset (\psi \supset \chi))$
(iso_7) $\phi \supset (\psi \wedge \chi) \equiv (\phi \supset \psi) \wedge (\phi \supset \chi)$
(iso_8) $(\phi \vee \psi) \supset \chi \equiv (\phi \supset \chi) \vee (\psi \supset \chi)$
(iso_9) $\phi \vee \bot \equiv \phi$
(iso_{10}) $\phi \wedge \bot \equiv \bot$

Building a Kripke model for ISCI^{iso} requires additional conditions on proof assignment v_p, which follows the following strategy: we take all additional axioms and add corresponding conditions on equality in proof assignment. For instance, from axiom (iso_1) we obtain the condition: $v_p(\phi \wedge \psi) = v_p(\psi \wedge \phi)$ and so on.[8]

The completeness of ISCI^{iso} is proved analogously to the completeness proof of ISCI. Thus, proof-assignment is defined as follows: $v_p(\phi, w) = [\phi]_{\approxeq_w}$, where $\phi \approxeq_w \psi$ iff $\phi \equiv \psi \in w$ (naturally, the notion of maximally consistent set is relativized to the derivability relation in ISCI^{iso}). This

[8] It is interesting to note that one can consider subsystems of ISCI^{iso} in which connectives behave in an asymmetrical manner: disjunction may be considered commutative while conjunction is not.

gives us each axiom characterizing propositional isomorphism forced at every world of a canonical model.

Claim 12. ISCI^{iso} *is complete with respect to the class of* ISCI^{iso}*-models.*

Since we added only identity axioms and the background logic is intuitionistic, we have:

Claim 13. ISCI^{iso} *enjoys disjunction property.*

7. Natural deduction systems

7.1. ISCI

To formalize ISCI in a natural deduction system, we need to add the following identity-specific rules to the set of rules defining standard intuitionistic connectives (where $\otimes$ is a fixed binary connective):

$$\dfrac{\begin{matrix}[\phi \equiv \phi] \\ \vdots \\ \psi\end{matrix}}{\psi}\ \equiv_1 \qquad \dfrac{\phi \equiv \psi \qquad \begin{matrix}[\phi \supset \psi] \\ \vdots \\ \chi\end{matrix}}{\chi}\ \equiv_2$$

$$\dfrac{\phi \equiv \chi \quad \psi \equiv \omega \qquad \begin{matrix}[(\phi \otimes \psi) \equiv (\chi \otimes \omega)] \\ \vdots \\ \lambda\end{matrix}}{\lambda}\ \equiv_3$$

The resulting system is denoted N_{ISCI}. This system is naturally sound with respect to the ISCI semantics, and it is also complete. It can simulate Hilbert-style systems (since it proves all axioms and can simulate *modus ponens*), which are known to be complete. A thorough study of this system can be found in (Chlebowski et al., 2022).

7.2. ISCI$^+$

In order to obtain a natural deduction system N_{ISCI^+} for ISCI^+ we need to add the following rule to N_{ISCI}:

$$\dfrac{\begin{matrix}[\phi \equiv \psi] & [(\phi \equiv \psi) \supset \bot] \\ \vdots & \vdots \\ \chi & \chi\end{matrix}}{\chi}\ +$$

The rule resembles the rule of excluded middle, which is commonly used to formalize classical logic within a natural deduction framework. However, in this case, the assumptions have a specified logical form — they must be identities. It is easy to see that the rule is sound with respect to the ISCI^+ semantics: assume we have

$$\phi \equiv \psi \models_{\mathsf{ISCI}^+} \chi \qquad \text{and} \qquad (\phi \equiv \psi) \supset \bot \models_{\mathsf{ISCI}^+} \chi$$

From the fact that $\models_{\mathsf{ISCI}^+} (\phi \equiv \psi) \vee ((\phi \equiv \psi) \supset \bot)$ it follows that $\models_{\mathsf{ISCI}^+} \chi$.

Since the system is based on N_{ISCI}, it can simulate *modus ponens*. Thus, it provides a complete proof system for the logic under consideration.

7.3. ISCI^s

Natural deduction system for strong ISCI can be obtained from the system N_{ISCI} by the addition of the following rule:

$$\frac{\phi \equiv \psi}{\bot}\ s$$

which ca be applied provided ϕ and ψ are syntactically distinct.

Note that the law of excluded middle can be proven in this system. Assume $\phi \neq \psi$.

$$\dfrac{\dfrac{\dfrac{[\phi \equiv \psi]}{\bot}\ s}{(\phi \equiv \psi) \supset \bot}\ I\supset}{(\phi \equiv \psi) \vee ((\phi \equiv \psi) \supset \bot)}\ I\vee$$

If $\phi = \psi$ we have:

$$\dfrac{\dfrac{[\phi \equiv \psi]}{(\phi \equiv \psi) \vee ((\phi \equiv \psi) \supset \bot)}\ I\vee}{(\phi \equiv \psi) \vee ((\phi \equiv \psi) \supset \bot)}\ \equiv_1$$

The rule i is sound with respect to the ISCI^i semantics, and similarly to the previously described systems, the system is complete, which can be shown via simulation of a Hilbert-style system.

7.4. ISCI^{iso}

A natural deduction system for intuitionistic logic with propositional isomorphism can be obtained from ISCI by replacing $\equiv_1$ rule by its more general version:

$$\dfrac{\begin{matrix}[\phi \equiv \psi] \\ \vdots \\ \psi\end{matrix}}{\psi}\ \equiv_1$$

where an equation $\phi \equiv \psi$ is identical to one of the schemes (iso_1)-(iso_{10}), or $\phi = \psi$. The rule is sound with respect to ISCI^{iso} semantics. The system is also complete.

8. Discussion

We presented, on the one hand, an informal semantics for ISCI based on BHK-interpretation and, on the other hand, a formal Kripke semantics for the considered logic, in which the notion of proof assignment plays a central role. These two pictures of propositional identity correspond to each other very well and since BHK interpretation is considered standard informal interpretation of intuitionistic logic, our formal modeling of it seems close to the intuition of founding fathers of intuitionism.

In the proposed framework, it is easy to see that there are uncountably many extensions of ISCI, since there are uncountably many different proof-assignments. We discussed some of them, and Fig. 1 depicts the relations that hold among the introduced systems.

Note that ISCI^i contradicts ISCI^{iso}, since the former rejects formulas that are accepted as theses in the latter. Note also that among these extensions only ISCI^+ does not enjoy disjunction property.[9]

These formal investigations raise an interesting philosophical question: what is the logic of constructive propositional identity? It seems to

[9] Some other variants of intuitionistic logic with propositional identity can be found in (Golińska-Pilarek, 2025).

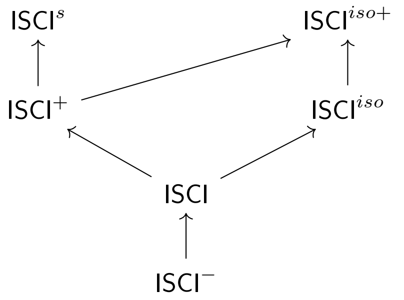


Figure 1. Relations between the systems introduced

me that the logic should be at least as strong as ISCI$^+$ for the following reason: being a proof or justification appears to require something finite and decidable. If we cannot determine whether something qualifies as a proof, it suggests that it was never a proof in the first place. Therefore, given two proofs or justifications, it should be constructively possible to determine whether they are the same proof. This seems to be a minimal requirement.

The interpretation of propositional identity in constructive context is related to the notion of *synonymity* in proof theoretic semantics, which is more fine-grained notion than that of intuitionistic equivalence (see, e.g., Wansing, 2024). Naturally ISCI defines an extreme case, but some extensions of it can grasp this notion more adequately, not on the level of metalanguage, as in proof-theoretic semantics, but on the level of propositional language. In the logic of propositional isomorphism for example, one accepts axiom $(\phi \wedge \psi) \equiv (\psi \wedge \phi)$, which states that syntactically distinct formulas share the same proof.

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