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A Note on Contradictions in Francez-Weiss Logics

Abstract. It is an unusual property for a logic to prove a formula and its negation without ending up in triviality. Some systems have nonetheless been observed to satisfy this property: one group of such non-trivial negation inconsistent logics has its archetype in H. Wansing's constructive connexive logic, whose negation-implication fragment already proves contradictions. N. Francez and Y. Weiss subsequently investigated relevant subsystems of this fragment, and Weiss in particular showed that they remain negation inconsistent. In this note, we take a closer look at this phenomenon in the systems of Francez and Weiss, and point out two types of necessary conditions, one proof-theoretic and one relevant, which any contradictory formula must satisfy. As a consequence, we propose a ninefold classification of provable contradictions for the logics.

Keywords: connexive logic; contradictory logic; relevant logic; sequent calculus; strong negation

1. Introduction

The system **C** by H. Wansing [23] is arguably one of the most important connexive logics [12, 16, 24]. This logic achieves connexivity in a simple manner, by tweaking the falsity condition of a constructive paraconsistent logic **N4** [1]. Later, N. Francez [7] introduced a subsystem $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ in the implication-negation fragment on the basis of the relevant logic **R** [see, e.g., 6, 11].[1] This was followed by Y. Weiss [26], who formulated A. Urquhart's semilattice semantics [20–22] for $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ and the implication-negation fragment $\mathcal{C}\mathbf{C}^{\neg}_{\rightarrow}$ of **C**, as well as for an intermediate system $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$

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[1] For other approaches to combine connenxive and relevant logic, see [4, 14, 15].

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based on the implicational system $\mathbf{RM0}_{\to}$, a subsystem of the implicational fragment of the logic **R**-Mingle [see, e.g., 5, for the details].

Another important characteristics of **C** (indeed, of $\mathcal{C}\mathbf{C}^{\neg}_{\to}$) is that it is *negation inconsistent*: there is a contradictory pair of a formula A and its negation $\neg A$ which are both provable in the system. Such A is called a *provable contradiction*. In his paper, Weiss established that this result can be extended to the weaker $\mathcal{C}\mathbf{R}^{\neg}_{\to}$ as well.

One way to represent **C**-type systems proof-theoretically is to use a bilateral[2] sequent calculus in the style of calculi **SN4** and **DN4** in [9, 10]. In this framework, a sequent has the form $\Gamma : \Delta \Rightarrow^{*} C$ where $* \in \{+, -\}$. The sign $-$ $(+)$ may be interpreted as representing the notion of *falsification* (*verification*), with Γ (Δ) being a multiset of *falsified* (*verified*) formulas. It can happen then that there is a *verification* (derivation w.r.t. $\Rightarrow^{+}$) and *falsification* (derivation w.r.t. $\Rightarrow^{-}$) for one and the same formula. (As we shall see, this is equivalent to saying that the formula is a provable contradiction). It has been pointed out by Wansing that for some such instances, this perspective reveals a kind of symmetry in their derivations: see, e.g., the next pair of derivations.

$$\begin{array}{c} A : A \Rightarrow^{-} A \\ \hline : A, \neg A \Rightarrow^{-} A \\ \hline : A, \neg A \Rightarrow^{+} \neg A \\ \hline : \neg A \Rightarrow^{+} A \to \neg A \\ \hline : \neg A \Rightarrow^{-} \neg(A \to \neg A) \\ \hline : \Rightarrow^{-} \neg A \to \neg(A \to \neg A) \end{array} \qquad \begin{array}{c} A : A \Rightarrow^{+} A \\ \hline : A, \neg A \Rightarrow^{+} A \\ \hline : A, \neg A \Rightarrow^{-} \neg A \\ \hline : \neg A \Rightarrow^{-} A \to \neg A \\ \hline : \neg A \Rightarrow^{+} \neg(A \to \neg A) \\ \hline : \Rightarrow^{+} \neg A \to \neg(A \to \neg A) \end{array}$$

Such symmetry may suggest that a formula is contradictory only if it treats verification and falsification ‘on a par’. It is then of interest to analyse ways provable contradictions necessitate an interaction between verification and falsification.

In this note, we shall investigate $\mathcal{C}\mathbf{R}^{\neg}_{\to}$ and $\mathcal{C}\mathbf{M}^{\neg}_{\to}$ in two aspects. The first is the extent to which the aforementioned symmetry for provable contradictions holds in these systems. We shall first introduce sequent calculi for these systems, and show cut-elimination results for them. Then we shall transform these systems into a type of hypersequent calculus called *tableaux* [13], by means of which we can construct a corresponding pair of verification/falsification for provable contradictions.

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[2] The term is understand here in the sense of *logical multilateralism* [25], namely as a theory of two derivability relations.

The second aspect is that of relevance. Building on Weiss' results on the variable sharing property, we shall propose a finer property based on whether a variable contributes verificatory and/or falsificatory. This will provide a classification of provable contradictions in the system into nine groups, and also gives a somewhat nuanced view as to the necessity of the interaction between verification and falsification for provable contradictions.

2. Preliminaries

Let $p_1, p_2, \ldots,$ be a countably infinite supply of *propositional variables*. We shall use $p, q, r, \ldots$ as metavariables for propositional variable. An implication-negation language $\mathcal{L}$ is then defined by the following clause for formulas:

$$A ::= p \mid \neg A \mid (A \rightarrow A).$$

$A, B, C, \ldots$ will be used as metavariables for formulas.

2.1. Hilbert-style calculi

The logics $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ and $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$ were formulated in [26] by the next Hilbert-style systems.

Definition 2.1. The system $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ is defined by the following axiom schemata and a rule.

$$\begin{array}{cr}
A \rightarrow A & \text{(I)} \\
(A\rightarrow B)\rightarrow((C\rightarrow A)\rightarrow(C\rightarrow B)) & \text{(B)} \\
(A\rightarrow(B\rightarrow C))\rightarrow(B\rightarrow(A\rightarrow C)) & \text{(C)} \\
(A\rightarrow(A\rightarrow B))\rightarrow(A\rightarrow B) & \text{(W)} \\
(A\rightarrow\neg B)\rightarrow\neg(A\rightarrow B) & \text{(NI1)}
\end{array}$$

$$\begin{array}{cr}
\neg(A\rightarrow B)\rightarrow(A\rightarrow\neg B) & \text{(NI2)} \\
A \rightarrow \neg\neg A & \text{(NN1)} \\
\neg\neg A \rightarrow A & \text{(NN2)} \\
\dfrac{A \qquad A \rightarrow B}{B} & \text{(MP)}
\end{array}$$

The system $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$ is then defined by an additional axiom schema.

$$A \rightarrow (A \rightarrow A) \qquad \text{(M)}$$

A derivation of a formula A from a finite multiset Γ of formulas in these systems is defined as a finite sequence $B_1, \ldots, B_n \equiv A$ such that:

1. Each B_i for $1 \leqslant i \leqslant n$ is either:
    - an instance of an axiom schema,
    - an element of Γ,
    - or obtained from preceding elements by (MP).

In addition, for $i < n$, B_i must be used in an application of (MP).

2. Each element of $\varGamma$ must occur in the sequence.

When there is such a derivation, we write $\varGamma \vdash_{hx} A$, where $x \in \{r, m\}$.

LEMMA 2.1. *$\varGamma, A \vdash_{hx} B$ if and only if $\varGamma \vdash_{hx} A \to B$.*

PROOF. The 'if' direction follows from (MP). For the 'only if' direction, we argue by induction on the depth of derivation.

If the derivation is an instance of an axiom schema, then the multiset $\varGamma \uplus \{A\} = \emptyset$ (where $\uplus$ represents the multiset union), so the statement holds vacuously.

If it is an instance of an assumption, then $\varGamma \uplus \{A\} = \{B\}$; so an instance of (I) gives a desired derivation of $B \to B$.

If B is obtained by an instance of (MP) with C and $C \to B$, then there are multisets $\varGamma_1, \varGamma_2 \subseteq \varGamma \uplus \{A\}$ such that $\varGamma_1 \vdash_{hx} C$ and $\varGamma_2 \vdash_{hx} C \to B$ given by sublists of the derivation for $\varGamma, A \vdash_{hx} B$. Either $\varGamma_1$ or $\varGamma_2$ contains A, and if it does we can apply the inductive hypothesis (I.H.) to conclude $\varGamma_1 \setminus \{A\} \vdash_{hx} A \to C$ or $\varGamma_2 \setminus \{A\} \vdash_{hx} A \to (C \to B)$ (where $\varDelta \setminus \{A\}$ removes one occurrences of A in $\varDelta$.) Consider here the case when the I.H. is applied in both. Then use (C) to get $\varGamma_2 \setminus \{A\} \vdash_{hx} C \to (A \to B)$. Then (B) implies $\varGamma_1 \setminus \{A\}, \varGamma_2 \setminus \{A\} \vdash_{hx} A \to (A \to B)$, so by (W), $\varGamma_1 \setminus \{A\}, \varGamma_2 \setminus \{A\} \vdash_{hx} A \to B$ follows. Now $\varGamma \vdash_{hx} A \to B$ by rewriting the derivation in such a way that we assume a formula only from one of $\varGamma_1 \setminus \{A\}$, $\varGamma_2 \setminus \{A\}$ if is shared between the two. The other cases are analogous. ⊣

2.2. Sequent calculi

Next, we introduce sequent calculi corresponding to $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ and $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$. As already mentioned in the introduction, a *sequent* has the form $\varGamma : \varDelta \Rightarrow^{*} C$, where $\varGamma$ and $\varDelta$ are each finite multisets of formulas, and $* \in \{-, +\}$.

DEFINITION 2.2. The system $\mathbf{G}\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ is defined by the following rules.

$$p : \Rightarrow^{-} p \ (\mathrm{rAx}^{-}) \qquad\qquad : p \Rightarrow^{+} p \ (\mathrm{rAx}^{+})$$

$$\frac{A, A, \varGamma : \varDelta \Rightarrow^{*} C}{A, \varGamma : \varDelta \Rightarrow^{*} C}\ (\mathrm{LC}^{-}) \qquad\qquad \frac{\varGamma : \varDelta, A, A \Rightarrow^{*} C}{\varGamma : \varDelta, A \Rightarrow^{*} C}\ (\mathrm{LC}^{+})$$

$$\frac{\varGamma : \varDelta \Rightarrow^{+} A \qquad B, \varGamma' : \varDelta' \Rightarrow^{*} C}{A \to B, \varGamma, \varGamma' : \varDelta, \varDelta' \Rightarrow^{*} C}\ (\mathrm{L}{\to}^{-}) \qquad \frac{\varGamma : \varDelta, A \Rightarrow^{-} C}{\varGamma : \varDelta \Rightarrow^{-} A \to C}\ (\mathrm{R}{\to}^{-})$$

$$\frac{\varGamma : \varDelta \Rightarrow^{+} A \qquad \varGamma' : \varDelta', B \Rightarrow^{*} C}{\varGamma, \varGamma' : \varDelta, \varDelta', A \to B \Rightarrow^{*} C}\ (\mathrm{L}{\to}^{+}) \qquad \frac{\varGamma : \varDelta, A \Rightarrow^{+} C}{\varGamma : \varDelta \Rightarrow^{+} A \to C}\ (\mathrm{R}{\to}^{+})$$

$$\frac{\Gamma : \Delta, A \Rightarrow^* C}{\neg A, \Gamma : \Delta \Rightarrow^* C}\,(\mathrm{L}\neg^-) \qquad \frac{\Gamma : \Delta \Rightarrow^+ C}{\Gamma : \Delta \Rightarrow^- \neg C}\,(\mathrm{R}\neg^-)$$

$$\frac{A, \Gamma : \Delta \Rightarrow^* C}{\Gamma : \Delta, \neg A \Rightarrow^* C}\,(\mathrm{L}\neg^+) \qquad \frac{\Gamma : \Delta \Rightarrow^- C}{\Gamma : \Delta \Rightarrow^+ \neg C}\,(\mathrm{R}\neg^+)$$

$$\frac{\Gamma : \Delta \Rightarrow^- A \qquad A, \Gamma' : \Delta' \Rightarrow^* C}{\Gamma, \Gamma' : \Delta, \Delta' \Rightarrow^* C}\,(\mathrm{Cut}^-)$$

$$\frac{\Gamma : \Delta \Rightarrow^+ A \qquad \Gamma' : \Delta', A \Rightarrow^* C}{\Gamma, \Gamma' : \Delta, \Delta' \Rightarrow^* C}\,(\mathrm{Cut}^+)$$

The system $\mathbf{G}\mathcal{C}\mathbf{M}^{\neg}_{\to}$ is defined from $\mathbf{G}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ by replacing (rAx$^-$) and (rAx$^+$) with:

$$p, \Gamma^p : \Rightarrow^- p \ (\mathrm{mAx}^-) \qquad : \Delta^p, p \Rightarrow^+ p \ (\mathrm{mAx}^+),$$

where Γ^A and Δ^A are finite multisets consisting only of A. We shall use $\vdash_{gr}$ and $\vdash_{gm}$ for the derivability of the systems.

Remark 2.1. The formulation for $\mathbf{G}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ is based on that of the implication of **R** [see, e.g., 3, 6, for details] with the use of the style of calculi in [9, 10] suitable for obtaining the subformula property. The rules (mAx$^-$) and (mAx$^+$) for $\mathbf{G}\mathcal{C}\mathbf{M}^{\neg}_{\to}$ are due to A. Avron [2]. ⊣

We begin our discussion of these calculi by noting that the *axioms*, i.e. (rAx*) and (mAx*) are generalisable to all formulas.

LEMMA 2.2. *The following statements hold.*

(i) $\vdash_{gr} A : \Rightarrow^- A$ *and* $\vdash_{gr} \ : A \Rightarrow^+ A$.
(ii) $\vdash_{gm} A, \Gamma^A : \Rightarrow^- A$ *and* $\vdash_{gm} \ : \Delta^A, A \Rightarrow^+ A$.

PROOF. By induction on the complexity of A. Here we treat the second statement for (ii) when A has the form $B \to C$. By the I.H., we have $\vdash C, \Gamma^C : \Rightarrow^- C$. Then the following derivation gives the desired sequent. (A double line indicates multiple applications of a rule.)

$$\begin{array}{c}
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\begin{array}{c}
: B \Rightarrow^+ B \qquad \begin{array}{c} : B \Rightarrow^+ B \qquad C, \Gamma^C : \Rightarrow^- C \\ \hline B \to C, \Gamma^C : B \Rightarrow^- C \end{array}\,(\mathrm{L}{\to}^-) \\ \hline\hline
B \to C, \Gamma^{B \to C} : B, \ldots, B \ \Rightarrow^- C
\end{array}\,(\mathrm{L}{\to}^-) \\ \hline\hline
B \to C, \Gamma^{B \to C} : B \Rightarrow^- C
\end{array}\,(\mathrm{LC}^+) \\ \hline
B \to C, \Gamma^{B \to C} : \Rightarrow^- B \to C
\end{array}\,(\mathrm{R}{\to}^-)$$

With this lemma, it is possible to establish the equivalence between the Hilbert-style systems and the sequent calculi. Given a multiset Γ, let us write $\neg\Gamma$ to denote the multiset of negations of elements in Γ. For a formula A, we shall use A^- for $\neg A$, and A^+ for A itself.

PROPOSITION 2.1. *The following statements hold for* $x \in \{r, m\}$.

(i) *If* $\Gamma \vdash_{hx} A$ *then* $\vdash_{gx} : \Gamma \Rightarrow^+ A$.
(ii) *If* $\vdash_{gx} \Gamma : \Delta \Rightarrow^* A$ *then* $\neg\Gamma, \Delta \vdash_{hx} A^*$.

PROOF. (i) By induction on the depth of derivation in $\mathcal{C}\mathbf{R}^{\neg}_{\to}$ ($\mathcal{C}\mathbf{M}^{\neg}_{\to}$). If the derivation is an instance of an axiom schema, e.g., (NI2), we must show $\vdash_{gx} : \Rightarrow^+ \neg(A \to B) \to (A \to \neg B)$. This is established by the next derivation, using Lemma 2.2.

$$\begin{prooftree}
\AxiomC{$: A \Rightarrow^+ A$}
\AxiomC{$B : \Rightarrow^- B$}
\RightLabel{(L$\to^-$)}
\BinaryInfC{$A \to B : A \Rightarrow^- B$}
\doubleLine
\RightLabel{(L$\neg^+$, R$\neg^+$)}
\UnaryInfC{$: A, \neg(A \to B) \Rightarrow^+ \neg B$}
\doubleLine
\RightLabel{(R$\to^+$)}
\UnaryInfC{$: \Rightarrow^+ \neg(A \to B) \to (A \to \neg B)$}
\end{prooftree}$$

If the derivation is an instance of an assumption, then we must show $\vdash_{gx} : A \Rightarrow^+ A$, and this follows from Lemma 2.2.

If the derivation ends with an instance of (MP) obtaining A from B and $B \to A$, then there are multisets $\Gamma_1, \Gamma_2 \subseteq \Gamma$ such that $\Gamma_1 \vdash_{hx} B$ and $\Gamma_2 \vdash_{hx} B \to A$ given by sublists of the derivation for $\Gamma \vdash_{hx} B$. Thus we may use the I.H. that $\vdash_{gx} : \Gamma_1 \Rightarrow^+ B$ and $\vdash_{gx} : \Gamma_2 \Rightarrow^+ B \to A$. Then a derivation of $: \Gamma \Rightarrow^+ A$ is constructed in the next way.

$$\begin{prooftree}
\AxiomC{$: \Gamma_1 \Rightarrow^+ A$}
\AxiomC{$: \Gamma_2 \Rightarrow^+ A \to B$}
\AxiomC{$: A \Rightarrow^+ A$}
\AxiomC{$: B \Rightarrow^+ B$}
\RightLabel{(L$\to^+$)}
\BinaryInfC{$: A, A \to B \Rightarrow^+ B$}
\RightLabel{(Cut$^+$)}
\BinaryInfC{$: \Gamma_2, A \Rightarrow^+ B$}
\RightLabel{(Cut$^+$)}
\BinaryInfC{$: \Gamma_1, \Gamma_2 \Rightarrow^+ B$}
\doubleLine
\RightLabel{(LC$^+$)}
\UnaryInfC{$: \Gamma \Rightarrow^+ B$}
\end{prooftree}$$

(ii) By induction on the depth of derivation in $\mathbf{G}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ ($\mathbf{G}\mathcal{C}\mathbf{M}^{\neg}_{\to}$). If the last rule applied is an instance of (LC$^-$), e.g.:

$$\begin{prooftree}
\AxiomC{$A, A, \Gamma : \Delta \Rightarrow^+ C$}
\RightLabel{(LC$^-$)}
\UnaryInfC{$A, \Gamma : \Delta \Rightarrow^+ C$}
\end{prooftree}$$

then by the I.H., $\neg A, \neg A, \neg\Gamma, \Delta \vdash_{hx} B$. Clearly any such derivation needs only one $\neg A$ as assumption, i.e. $\neg A, \neg\Gamma, \Delta \vdash_{hx} B$. If the last rule applied is an instance of (L$\neg^-$), e.g.:

$$\begin{prooftree}
\AxiomC{$\Gamma : \Delta, A \Rightarrow^+ C$}
\RightLabel{(L$\neg^-$)}
\UnaryInfC{$\neg A, \Gamma : \Delta \Rightarrow^+ C$}
\end{prooftree}$$

then by the I.H. $\neg\Gamma, \Delta, A \vdash_{hx} C$. By Lemma 2.1, $\neg\Gamma, \Delta \vdash_{hx} A \to C$. On the other hand, it follows by (NN2) that $\neg\neg A \vdash_{hx} A$. Thus by (MP), $\neg\neg A, \neg\Gamma, \Delta \vdash_{hx} C$, as required. Other cases are shown in an analogous way. ⊣

Next, we shall show cut-elimination for $\mathcal{C}\mathbf{R}^{\neg}_{\to}$ and $\mathcal{C}\mathbf{M}^{\neg}_{\to}$ via the standard technique of eliminating a more general rule called *fusion* or *extended cut* [6, 17, 19].

DEFINITION 2.3. We shall define systems $e\mathbf{G}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ and $e\mathbf{G}\mathcal{C}\mathbf{M}^{\neg}_{\to}$ by replacing (Cut$^-$) and (Cut$^+$) with the following rules.

$$\frac{\Gamma : \Delta \Rightarrow^- A \qquad A, \Gamma' : \Delta' \Rightarrow^* C}{\Gamma, \Gamma'_A : \Delta, \Delta' \Rightarrow^* C}\ (\mathrm{eCut}^-)$$

$$\frac{\Gamma : \Delta \Rightarrow^+ A \qquad \Gamma' : \Delta', A \Rightarrow^* C}{\Gamma, \Gamma' : \Delta, \Delta'_A \Rightarrow^* C}\ (\mathrm{eCut}^+)$$

where Γ'_A and Δ'_A are each obtained by removing arbitrary instances of A from Γ' and Δ', respectively. The derivability relations of these systems are written by $\vdash^e_{gr}$ and $\vdash^e_{gm}$.

As usual, the general rules do not increase the strength of the systems.

PROPOSITION 2.2. *For $x \in \{r, m\}$, $\vdash^e_{gx} \Gamma : \Delta \Rightarrow^* A$ iff $\vdash_{gx} \Gamma : \Delta \Rightarrow^* A$.*

PROOF. For the 'if' direction, any instance of (Cut$^-$) or (Cut$^+$) is an instance of (eCut$^-$) or (eCut$^+$) as well. For the 'only if' direction, (eCut$^-$) or (eCut$^+$) can be replicated with multiple applications of (Cut$^-$) or (Cut$^+$). ⊣

In order to show that applications of (eCut$^-$) and (eCut$^+$) are dispensable, we need to introduce a couple of notions first. Give an instance of (eCut$^-$) or (eCut$^+$) in a derivation, we call the complexity of the cutformula its *grade*, and the length of the subderivation ending in the instance of (eCut$^-$) or (eCut$^+$) the *height*. Moreover, define $\vdash^f_{gr}$ and $\vdash^f_{gm}$ to be the derivability relations in $\mathbf{G}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ and $\mathbf{G}\mathcal{C}\mathbf{M}^{\neg}_{\to}$ without an application of (Cut$^-$) or (Cut$^+$).

THEOREM 2.1. *For $x \in \{r, m\}$ and $* \in \{+, -\}$, $\vdash^e_{gx} \Gamma : \Delta \Rightarrow^* A$ iff $\vdash^f_{gx} \Gamma : \Delta \Rightarrow^* A$.*

PROOF. The 'if' direction is immediate. The 'only if' direction is shown by establishing that the statement holds in a derivation in which (eCut$^-$)

or (eCut$^+$) occurs only at the last step: then (eCut$^-$)/(eCut$^+$) are eliminable in general by successively removing ones which are topmost.

The claim is proved by induction on the grade of the (eCut$^-$) and (eCut$^+$), with subinduction on the height of the (eCut$^-$) and (eCut$^+$). Here we treat as an example the case of (eCut$^-$) when the cutformula is an implication which is principal on both premises.

$$\dfrac{\dfrac{\Gamma : \Delta, A \Rightarrow^- B}{\Gamma : \Delta \Rightarrow^- A{\to}B}\,(\mathrm{R}{\to}^-) \qquad \dfrac{\Gamma' : \Delta' {\Rightarrow}^+ A \qquad B, \Gamma'' : \Delta'' {\Rightarrow}^* C}{A{\to}B, \Gamma', \Gamma'' : \Delta', \Delta'' \Rightarrow^* C}\,(\mathrm{L}{\to}^-)}{\Gamma, \Gamma'_{A\to B}, \Gamma''_{A\to B} : \Delta, \Delta', \Delta'' \Rightarrow^* C}$$

Then, e.g., we can construct the derivation (I) on Figure 1, when $A \to B$ occurs in Γ'' but not in Γ'.

The subderivation up to the upper right (eCut$^-$) can be replaced with a cut-free one because of the I.H. on height; then we can use the I.H. on the grade to eliminate the other extended cut. $\dashv$

2.3. Tableau calculi

The sequent calculi we introduced in the previous subsection are not ideal for our purpose of constructing a pair of derivations which show a degree of correspondence. This is because some rules of the calculi are not *invertible*, i.e. the derivability of their conclusion does not guarantee the derivability of their premises. In order to assure this property, we shall expand the framework to have a sequence of sequents as the basic unit, i.e. to hypersequent calculi. We shall in particular employ a relatively simple system used by G. Mints [13] for intuitionistic propositional logic. We therefore adopt his terminology to call the systems we introduce *tableau calculi*.

A *tableau* $s_1|\dots|s_n$ ($n \geq 1$) is a finite sequence of sequents. We shall use $\sigma, \tau, \dots$ for (possibly empty) finite sequences of sequents. We define two tableaux calculi $\mathbf{T}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ and $\mathbf{T}\mathcal{C}\mathbf{M}^{\neg}_{\to}$ from $\mathbf{G}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ and $\mathbf{G}\mathcal{C}\mathbf{M}^{\neg}_{\to}$ by modifying the rules to allow sequents on the side: e.g., $\sigma|p : \Rightarrow^- p|\tau$ for (rAx$^-$) and

$$\dfrac{\sigma|\Gamma : \Delta, A \Rightarrow^+ B|\tau}{\sigma|\Gamma : \Delta \Rightarrow^+ A \to B|\tau}$$

for (R$\to^+$). The modified rules will be called (trAx$^-$), (tR$\to^+$) and so on. For the left implication rules and contraction rules, we modify them as in (II) on Figure 1.

Non-displayed sequents in the conclusion of a rules are called *contexts*. Like sequent calculi, a *derivation* of a tableau is a finite tree, whose leaves are instance of a 0-premise rule, each of whose other nodes is a tableau obtained from the tableaux immediately above by one of the rules, in particular the root is the tableau in question. We shall denote the derivabilities in $\mathbf{T}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ and $\mathbf{T}\mathcal{C}\mathbf{M}^{\neg}_{\to}$ by $\vdash_{tr}$ and $\vdash_{tm}$.

Let us first observe some basic properties of $\mathbf{T}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ and $\mathbf{T}\mathcal{C}\mathbf{M}^{\neg}_{\to}$. We shall call a rule *depth-preserving admissible*, if the derivability of its premises with derivations of depth $\leqslant n$ implies the derivability of its conclusion with a derivation of depth $\leqslant n$. A rule will be called *depth-preserving invertible*, if the derivability of the conclusion with a derivation of depth $\leqslant n$ implies that of the premises with derivations of depth $\leqslant n$.

By induction on the depth of derivation we obtain:

PROPOSITION 2.3. *The following rule is depth-preserving admissible in* $\mathbf{T}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ *and* $\mathbf{T}\mathcal{C}\mathbf{M}^{\neg}_{\to}$.

$$\frac{\sigma|\varGamma : \varDelta \Rightarrow^{*} C|\tau}{\sigma|\sigma'|\varGamma : \varDelta \Rightarrow^{*} C|\tau'|\tau}\ (rW)$$

LEMMA 2.3. *All rules of* $\mathbf{T}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ *and* $\mathbf{T}\mathcal{C}\mathbf{M}^{\neg}_{\to}$ *are depth-preserving invertible.*

PROOF. For (tLC^{-}), (tLC^{+}), (tL$\to^{-}$) and (tL$\to^{+}$), the invertibility is assured by Proposition 2.3. For (tR$\to^{-}$), we argue by induction on the depth of derivation. If $\sigma|\varGamma : \varDelta \Rightarrow^{-} A \to B|\tau$ is obtained as an instance of (trAx^{-}), (trAx^{+}), (tmAx^{-}) or (tmAx^{+}), then $\varGamma : \varDelta \Rightarrow^{-} A \to B$ is in the context; so $\sigma|\varGamma : \varDelta, A \Rightarrow^{-} B|\tau$ is another instance of the same rule. Similarly, if it is obtained in a rule in which $\varGamma : \varDelta \Rightarrow^{-} A \to B$ is in the context, then one may apply the I.H. to the premises and then apply the same rule.

If $\varGamma : \varDelta \Rightarrow^{-} A \to B$ is not in the context but the rule is an instance of (tR$\to^{-}$), then the premise is the desired tableau. Otherwise, consider, e.g., the case when the last rule is an instance of (tL$\to^{+}$); see (III) on Figure 1.

Then by the I.H., $\sigma|\varGamma, \varGamma' : \varDelta, \varDelta', C{\to}D, A \Rightarrow^{-} B|\varGamma : \varDelta \Rightarrow^{+} C|\tau$ and $\sigma|\varGamma, \varGamma' : \varDelta, \varDelta', C{\to}D, A \Rightarrow^{-} B|\varGamma' : \varDelta', D, A \Rightarrow^{-} B|\tau$: for the latter, the I.H. must be applied twice, and this is justified by depth-preservation. Now apply (tL$\to^{+}$) to obtain the desired tableau. The cases for the remaining rules are treated analogously. $\dashv$

(I)

$$\dfrac{\dfrac{\Gamma' : \Delta'\Rightarrow^{+} A \qquad \Gamma : \Delta, A\Rightarrow^{-} B}{\Gamma, \Gamma' : \Delta, \Delta'\Rightarrow^{-} B}\,(\mathrm{eCut}^{+}) \qquad \dfrac{\Gamma : \Delta'\Rightarrow^{-} A\to B \qquad B, \Gamma'' : \Delta''\Rightarrow^{*} C}{B, \Gamma, \Gamma''_{A\to B} : \Delta, \Delta''\Rightarrow^{*} C}\,(\mathrm{eCut}^{-})}{\dfrac{\Gamma, \Gamma, \Gamma', \Gamma''_{A\to B} : \Delta, \Delta, \Delta', \Delta''\Rightarrow^{*} C}{\Gamma, \Gamma', \Gamma''_{A\to B} : \Delta, \Delta', \Delta''\Rightarrow^{*} C}\,(\mathrm{LC}^{-}),(\mathrm{LC}^{+})}\,(\mathrm{eCut}^{-})$$

(II)

$$\dfrac{\sigma|A, \Gamma : \Delta \Rightarrow^{*} C|A, A, \Gamma : \Delta \Rightarrow^{*} C|\tau}{\sigma|A, \Gamma : \Delta \Rightarrow^{*} C|\tau}\,(\mathrm{tLC}^{-})$$

$$\dfrac{\sigma|\Gamma : \Delta, A \Rightarrow^{*} C|\Gamma : \Delta, A, A \Rightarrow^{*} C|\tau}{\sigma|\Gamma : \Delta, A \Rightarrow^{*} C|\tau}\,(\mathrm{tLC}^{+})$$

$$\dfrac{\sigma|A\to B, \Gamma, \Gamma' : \Delta, \Delta' \Rightarrow^{*} C|\Gamma : \Delta \Rightarrow^{+} A|\tau \qquad \sigma|A\to B, \Gamma, \Gamma' : \Delta, \Delta' \Rightarrow^{*} C|B, \Gamma' : \Delta' \Rightarrow^{*} C|\tau}{\sigma|A\to B, \Gamma, \Gamma' : \Delta, \Delta' \Rightarrow^{*} C|\tau}\,(\mathrm{tL}\!\to^{-})$$

$$\dfrac{\sigma|\Gamma, \Gamma' : \Delta, \Delta', A\to B \Rightarrow^{*} C|\Gamma : \Delta \Rightarrow^{+} A|\tau \qquad \sigma|\Gamma, \Gamma' : \Delta, \Delta', A\to B \Rightarrow^{*} C|\Gamma' : \Delta', B \Rightarrow^{*} C|\tau}{\sigma|\Gamma, \Gamma' : \Delta, \Delta', A\to B \Rightarrow^{*} C|\tau}\,(\mathrm{tL}\!\to^{+})$$

(III)

$$\dfrac{\sigma|\Gamma, \Gamma' {:} \Delta, \Delta', C\to D\Rightarrow^{-} A\to B|\Gamma {:} \Delta\Rightarrow^{+} C|\tau \qquad \sigma|\Gamma, \Gamma' {:} \Delta, \Delta', C\to D\Rightarrow^{-} A\to B|\Gamma' {:} \Delta', D\Rightarrow^{-} A\to B|\tau}{\sigma|\Gamma, \Gamma' {:} \Delta, \Delta', C\to D\Rightarrow^{-} A\to B|\tau}$$

Figure 1.

PROPOSITION 2.4. *For* $x \in \{r, m\}$, $\vdash_{tx} \mathit{\Gamma} : \mathit{\Delta} \Rightarrow^{*} C$ *iff* $\vdash_{gx} \mathit{\Gamma} : \mathit{\Delta} \Rightarrow^{*} C$.

PROOF. For the 'if' direction, it follows by induction on the depth of derivation in $\mathbf{G}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ (or $\mathbf{G}\mathcal{C}\mathbf{M}^{\neg}_{\to}$). All cases proceed by mimicking sequent rules with corresponding tableau rules. For (L$\to^{-}$), (L$\to^{+}$), (LC^{-}) and (LC^{+}), we must apply (rW) in order to make the corresponding tableau rule applicable.

For the 'only if' direction, we shall show by induction on the depth of derivation in $\mathbf{T}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ (or $\mathbf{T}\mathcal{C}\mathbf{M}^{\neg}_{\to}$). In each step, we must show that there is a sequent in the derived tableau which is derivable in $\mathbf{G}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ (or $\mathbf{G}\mathcal{C}\mathbf{M}^{\neg}_{\to}$). For instance, in the case of (tL$\to^{-}$) we have (I) on Figure 2.

Then by the I.H., both premises have a sequent which is derivable in $\mathbf{G}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ (or $\mathbf{G}\mathcal{C}\mathbf{M}^{\neg}_{\to}$). If they are $\mathit{\Gamma} : \mathit{\Delta} \Rightarrow^{+} A$ and $B, \mathit{\Gamma}' : \mathit{\Delta}' \Rightarrow^{*} C$ respectively, we apply (L$\to^{-}$) to obtain $A{\to}B, \mathit{\Gamma}, \mathit{\Gamma}' : \mathit{\Delta}, \mathit{\Delta}' \Rightarrow^{*} C$. Otherwise, one of the derivable sequents occur in the conclusion tableau as well.

Now, if a derivation ends with a tableau consisting of a single sequent, then it follows from what has been established that the sequent must be derivable in the correlated sequent calculus. $\dashv$

3. Establishing a correspondence

With $\mathbf{T}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ and $\mathbf{T}\mathcal{C}\mathbf{M}^{\neg}_{\to}$ at hand, it is now possible to observe that a formula is a provable contradiction in $\mathcal{C}\mathbf{R}^{\neg}_{\to}$ or $\mathcal{C}\mathbf{M}^{\neg}_{\to}$ only if there are derivations in the corresponding tableau calculus which show the kind of relationship exhibited by the example in the introduction. For this observation, we first introduce a few notions.

We shall say two tableaux $s_1 | \ldots | s_n$ and $s'_1 | \ldots | s'_n$ *correspond* if each s'_i ($1 \leqslant i \leqslant n$) is obtained from s_i by alternating the signs. Then we shall say two finite sequences $\tau_1, \ldots, \tau_n$ and $\tau'_1, \ldots, \tau'_n$ of (non-empty) tableaux *correspond* if for each $1 \leqslant i \leqslant n$, τ_i and τ'_i correspond. Finally, two derivations d and d' in $\mathbf{T}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ ($\mathbf{T}\mathcal{C}\mathbf{M}^{\neg}_{\to}$) are said to *correspond* if there are branches b in d and b' in d' which correspond. We may observe that the derivations in the introduction satisfy this relationship, as intended. We will find another example at the end of this section.

We shall adopt conventions that $\dagger = +\ (-)$ if and only if $* = -\ (+)$. In addition, τ and τ' will denote the same tableau except for the flipped signs. We also introduce a notion called *pseudo-derivation* in $\mathbf{T}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ and $\mathbf{T}\mathcal{C}\mathbf{M}^{\neg}_{\to}$. It is almost a derivation in these systems, except that there is one leaf whose tableau is derivable, but possibly not an instance of (trAx*) or (tmAx*). The notion of correspondence is extended to pseudo-derivations as well.

THEOREM 3.1. *For* $x \in \{r, m\}$, *if* $\vdash_{tx} : \Rightarrow^+ A$ *and* $\vdash_{tx} : \Rightarrow^- A$ *then there are derivations of the tableaux which correspond.*

PROOF. Take a derivation d_1 of $: \Rightarrow^+ A$ (one may equally start with a derivation of $: \Rightarrow^- A$). Let τ_1 and τ_1' be the tableaux $: \Rightarrow^+ A$ and $: \Rightarrow^- A$, respectively.

We shall first construct a pseudo-derivation of $: \Rightarrow^- A$ corresponding to d_1. We start with the single node τ_1', which is a pseudo-derivation of the tableau. Suppose that there are corresponding sequences $\tau_1, \dots, \tau_i$ and $\tau_1', \dots, \tau_i'$ such that:

- τ_{j+1} is a premise of τ_j in d_1 for $j \geq 1$.
- we have a pseudo-derivation in which $\tau_1', \dots, \tau_i'$ is a branch ending (when seen from the root) with a derivable tableau τ_1' that may not be an instance of (trAx*) or (tmAx*).

Then if τ_i is not an initial tableau, we choose a new tableau from the premises of τ_i, which at the same time determines its corresponding tableau τ_i', so that there is a pseudo-derivation of τ_i' which has a branch $\tau_1', \dots, \tau_i', \tau_{i+1}'$ with a derivable tableau τ_1' that may not be an instance of (trAx*) or (tmAx*). We divide into cases depending on the rule applied to obtain τ_i.

If it is an instance of (tL$\to^-$), then the step is of the form (II) on Figure 2. Then we choose τ_{i+1} to be the right premise. Now since τ_i' is

$$\sigma' | A \to B, \varGamma, \varGamma' : \varDelta, \varDelta' \Rightarrow^\dagger C | \tau',$$

we infer from the admissibility of (rW) that both

$$\sigma' | A \to B, \varGamma, \varGamma' : \varDelta, \varDelta' \Rightarrow^\dagger C | \varGamma : \varDelta, \varDelta' \Rightarrow^+ A | \tau'$$

and

$$\sigma' | A \to B, \varGamma, \varGamma' : \varDelta, \varDelta' \Rightarrow^\dagger C | B, \varGamma' : \varDelta' \Rightarrow^\dagger C | \tau'$$

are derivable. The desired pseudo-derivation having $\tau_1', \dots, \tau_i', \tau_{i+1}'$ as branch is obtained by attaching to the pseudo-derivation up to τ_i' a step of (tL$\to^-$), along with a derivation of the former tableau (see (III) on Figure 2).

(I)

$$\frac{\sigma|A{\to}B,\varGamma,\varGamma':\varDelta,\varDelta'\Rightarrow^{*}C|\varGamma:\varDelta\Rightarrow^{+}A|\tau \qquad \sigma|A{\to}B,\varGamma,\varGamma':\varDelta,\varDelta'\Rightarrow^{*}C|B,\varGamma':\varDelta'\Rightarrow^{*}C|\tau}{\sigma|A{\to}B,\varGamma,\varGamma':\varDelta,\varDelta'\Rightarrow^{*}C|\tau}$$

(II)

$$\frac{\sigma|A{\to}B,\varGamma,\varGamma':\varDelta,\varDelta'\Rightarrow^{*}C|\varGamma:\varDelta\Rightarrow^{+}A|\tau \qquad \sigma|A{\to}B,\varGamma,\varGamma':\varDelta,\varDelta'\Rightarrow^{*}C|B,\varGamma':\varDelta'\Rightarrow^{*}C|\tau}{\sigma|A{\to}B,\varGamma,\varGamma':\varDelta,\varDelta'\Rightarrow^{*}C|\tau}$$

(III)

$$\frac{\dfrac{\vdots}{\sigma'|A{\to}B,\varGamma,\varGamma':\varDelta,\varDelta'\Rightarrow^{\dagger}C|\varGamma:\varDelta\Rightarrow^{+}A|\tau'} \qquad \sigma'|A{\to}B,\varGamma,\varGamma':\varDelta,\varDelta'\Rightarrow^{\dagger}C|B,\varGamma':\varDelta'\Rightarrow^{\dagger}C|\tau'}{\sigma'|A{\to}B,\varGamma,\varGamma':\varDelta,\varDelta'\Rightarrow^{\dagger}C|\tau'}$$

Figure 2.

The sequences $\tau_1, \ldots, \tau_i$ and $\tau'_1, \ldots, \tau'_i$ are each extended now by a node, while preserving the two conditions for them. The cases for other rules are similarly treated.

Continuing this process, we end up with a pseudo-derivation d'_1 corresponding to d_1. This pseudo-derivation is in general not a derivation, as there is an initial tableau v' which may not be an instance of (trAx*) or (tmAx*). v' is however assured to be derivable; so take a derivation d'_2 of the tableau. Then we can repeat the above process with respect to d'_2 to obtain a pseudo-derivation d_2 of v (tableau corresponding to v') which corresponds to d'_2. Now, v must an instance of (trAx*) or (tmAx*) because it is an initial tableau in the derivation d_1. In other words, it contains a sequent of the form (rAx*) or (mAx*). This sequent is then preserved when one goes up the pseudo-derivation d_2. This assures that all its initial sequents are instances of (trAx*) or (tmAx*). Hence d_2 is in fact a derivation.

The desired corresponding derivations of $\vdash_{tx}\, : \Rightarrow^+ A$ and $\vdash_{tx}\, : \Rightarrow^- A$ are now obtained by attaching d_2 on top of d_1 (connected by v) on one hand, and d'_2 on top of d_2 (connected by v') on the other hand. $\dashv$

The following example illustrates the process described in the proof.

Example 3.1. We have

$$(\neg p {\to} p) {\to} (\neg(p {\to} p) {\to} (\neg(p {\to} p) {\to} (p {\to} (p {\to} p))))$$

as a provable contradiction in $\mathcal{C}\mathbf{M}^{\neg}_{\to}$. The derivation (I) on Figure 3 verifies this formula in $\mathbf{T}\mathcal{C}\mathbf{M}^{\neg}_{\to}$. Then applying the process in Theorem 3.1, we obtain the pseudo-derivation (II) on Figure 3. The rightmost tableau is not an instance of (tmAx$^-$) or (tmAx$^+$). For a proper derivation, we attach a derivation of $\sigma'_3|\ : p \Rightarrow^- p$, for example:

$$\cfrac{\cfrac{\cfrac{\sigma'_2|\tau|p{:}\ \Rightarrow^- p|\sigma'_4}{\sigma'_2|\tau|p{:}\ \Rightarrow^+ \neg p|\sigma'_4}\,(\mathrm{tR}\neg^+) \qquad \sigma'_2|\tau|\ {:}p, p{\Rightarrow^+} p|\sigma'_4}{\sigma'_2|p{:}\neg p{\to}p, p{\Rightarrow^+} p|\sigma'_4}\,(\mathrm{tL}{\to}^+) \qquad \sigma'_2|p{:}\ \Rightarrow^- p|\sigma'_4}{\sigma'_3|{:}p{\Rightarrow^-} p}\,(\mathrm{tL}{\to}^+)$$

where:

- $\sigma'_4 := p, p : \neg p \to p \Rightarrow^- p|\ \ : p \Rightarrow^- p$,
- $\tau := p : \neg p \to p, p \Rightarrow^+ p$.

We can also extend the rightmost initial sequent of the derivation of the verification with:

(I)

$$\begin{array}{c}
\dfrac{\sigma_1|\ \ :p\Rightarrow^+p \qquad \dfrac{\sigma_2|\ \ :p\Rightarrow^+p \qquad \dfrac{\dfrac{\sigma_3|p,p:\ \Rightarrow^-p}{\sigma_3|p,p:\ \Rightarrow^+\neg p}\,(\mathrm{tR}\neg^+) \qquad \sigma_3|\ \ :p\Rightarrow^+p}{\sigma_2|p,p:\neg p\to p\Rightarrow^+p}\,(\mathrm{tL}\to^+)}{\sigma_1|p,p\to p:\neg p\to p,p\Rightarrow^+p}\,(\mathrm{tL}\to^-)}{p\to p,p\to p:\neg p\to p,p,p\Rightarrow^+p}\,(\mathrm{tL}\to^-) \\ \hline\hline
:\Rightarrow^+(\neg p\to p)\to(\neg(p\to p)\to(\neg(p\to p)\to(p\to(p\to p))))
\end{array}\,(\mathrm{tL}\neg^+,\mathrm{tR}\to^+)$$

where

- $\sigma_1 := p\to p, p\to p : \neg p\to p, p, p\Rightarrow^+ p$,
- $\sigma_2 := \sigma_1|p\to p : \neg p\to p, p\Rightarrow^+ p$,
- $\sigma_3 := \sigma_2|p, p : \neg p\to p\Rightarrow^+ p$.

II

$$\begin{array}{c}
\dfrac{\sigma_1'|\ \ :p\Rightarrow^+p \qquad \dfrac{\sigma_2'|\ \ :p\Rightarrow^+p \qquad \dfrac{\dfrac{\sigma_3'|p,p:\Rightarrow^-p}{\sigma_3'|p,p:\ \Rightarrow^+\neg p}\,(\mathrm{tR}\neg^+) \qquad \sigma_3'|\ \ :p\Rightarrow^-p}{\sigma_2'|p,p:\neg p\to p\Rightarrow^-p}\,(\mathrm{tL}\to^+)}{\sigma_1'|p,p\to p:\neg p\to p,p\Rightarrow^-p}\,(\mathrm{tL}\to^-)}{p\to p,p\to p:\neg p\to p,p,p\Rightarrow^-p}\,(\mathrm{tL}\to^-) \\ \hline\hline
:\Rightarrow^-(\neg p\to p)\to(\neg(p\to p)\to(\neg(p\to p)\to(p\to(p\to p))))
\end{array}\,(\mathrm{tL}\neg^+,\mathrm{tR}\to^-)$$

where $\sigma_1', \ldots, \sigma_3'$ correspond to $\sigma_1, \ldots, \sigma_3$.

Figure 3.

$$\dfrac{\sigma_2|p : \neg p \to p, p \Rightarrow^+ p|\sigma_4 \qquad \sigma_2|p : \Rightarrow^+ p|\sigma_4}{\sigma_3|\ : p \Rightarrow^+ p}\ (\mathrm{tL}{\to}^-)$$

Now the two derivations so constructed[3] can be checked to correspond, by looking at the rightmost branches; the full derivations (Figure 4) can be found in the next page. ⊣

We have thus established that $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ and $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$ prove a contradiction only when the contradictory formula can be verified and falsified in a coordinated manner, as is revealed by $\mathbf{T}\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ and $\mathbf{T}\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$.

4. A relevant classification of contradictions

The contradictory nature of $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ and $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$ may seem so unusual as to give rise to a worry that the relevance of their base systems may be spoiled. In fact, such a worry is not warranted, as Weiss [26] has shown semantically that the *variable sharing property* holds for these systems.

THEOREM 4.1 (26). *For $x \in \{r, m\}$, if $\vdash_{hx} A \to B$ then A and B share a propositional variable.*

A question that may then arise is whether contradictory formulas in these systems can be explicated in terms of relevance: is an implicational formula contradictory *only if* its premise and conclusion are relevant in a specific manner? We shall attempt to give an answer to this type of question in this section, by looking more closely at the relevance of the systems.

We now turn our attention back to $\mathbf{G}\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$ and $\mathbf{G}\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$. A finer view of relevance is achieved through a syntactic method used by Kamide [8].

LEMMA 4.1. *For $x \in \{r, m\}$ and $* \in \{+, -\}$, if $\vdash^f_{gx} : A \Rightarrow^* B$ then there is a branch in its derivation where all sequents have at least one formulas in the antecedent and succedent.*

PROOF. It follows by inspecting the rules that if their conclusion has a non-empty antecedent, then so has at least one of the premises. ⊣

Next, we introduce some classes of propositional variables.

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[3] Also notice that the derivations do not make use of contraction, i.e. (tLC*). Their axiomatic counterpart (W) is shown by Weiss to be essential for the negation inconsistency of $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$. It is thus crucial here that (tmAx*) is used in the derivations, which works as a limited case of weakening [cf. 26, p. 595].

$$\begin{array}{c}
\dfrac{\sigma_1|\ :p\Rightarrow^+ p \qquad \dfrac{\sigma_2|\ :p\Rightarrow^+ p \qquad \dfrac{\dfrac{\sigma_3|p,p:\Rightarrow^- p}{\sigma_3|p,p:\Rightarrow^+ \neg p} \qquad \dfrac{\sigma_2|p:\neg p\to p,p\Rightarrow^+ p|\sigma_4 \qquad \sigma_2|p:\Rightarrow^+ p|\sigma_4}{\sigma_3|\ :p\Rightarrow^+ p}}{\sigma_2|p,p:\neg p\to p\Rightarrow^+ p}}{\sigma_1|p,p\to p:\neg p\to p,p\Rightarrow^+ p}}{p\to p,p\to p:\neg p\to p,p,p\Rightarrow^+ p} \\
\hline\hline
:\Rightarrow^+ (\neg p\to p)\to(\neg(p\to p)\to(\neg(p\to p)\to(p\to(p\to p))))
\end{array}$$

$$\begin{array}{c}
\dfrac{\sigma'_1|\ :p\Rightarrow^+ p \qquad \dfrac{\sigma'_2|\ :p\Rightarrow^+ p \qquad \dfrac{\dfrac{\sigma'_3|p,p:\Rightarrow^- p}{\sigma'_3|p,p:\Rightarrow^+ \neg p} \qquad \dfrac{\dfrac{\dfrac{\sigma'_2|\tau|p:\Rightarrow^- p|\sigma'_4}{\sigma'_2|\tau|p:\Rightarrow^+ \neg p|\sigma'_4} \qquad \sigma'_2|\tau|\ :p,p\Rightarrow^+ p|\sigma'_4}{\sigma'_2|p:\neg p\to p,p\Rightarrow^+ p|\sigma'_4} \qquad \sigma'_2|p:\Rightarrow^- p|\sigma'_4}{\sigma'_3|\ :p\Rightarrow^- p}}{\sigma'_2|p,p:\neg p\to p\Rightarrow^- p}}{\sigma'_1|p,p\to p:\neg p\to p,p\Rightarrow^- p}}{p\to p,p\to p:\neg p\to p,p,p\Rightarrow^- p} \\
\hline\hline
:\Rightarrow^- (\neg p\to p)\to(\neg(p\to p)\to(\neg(p\to p)\to(p\to(p\to p))))
\end{array}$$

Figure 4.

DEFINITION 4.1. Let $V_v^+, V_v^-, V_f^+, V_f^-$ be the following classes of variables.

$$\begin{aligned} V_v^+(p) &= \{p\} & V_v^-(p) &= \emptyset \\ V_v^+(A \to B) &= V_v^+(A) \cup V_v^+(B) & V_v^-(A \to B) &= V_v^+(A) \cup V_v^-(B) \\ V_v^+(\neg A) &= V_v^-(A) & V_v^-(\neg A) &= V_v^+(A) \end{aligned}$$

$$\begin{aligned} V_f^+(p) &= \emptyset & V_f^-(p) &= \{p\}. \\ V_f^+(A \to B) &= V_f^+(A) \cup V_f^+(B) & V_f^-(A \to B) &= V_f^+(A) \cup V_f^-(B) \\ V_f^+(\neg A) &= V_f^-(A) & V_f^-(\neg A) &= V_f^+(A) \end{aligned}$$

The classes are then extended to multisets of formulas: e.g., $V_v^+(\Gamma) := \bigcup_{A \in \Gamma} V_v^+(A)$.

Remark 4.1. Intuitively, $V_v^+(A)$ and $V_f^+(A)$ each collects propositional variables which carries information on verification/falsification with respect to the sequent $: \Rightarrow^+ A$. $V_v^-(A)$ and $V_f^-(A)$ collect propositional variables in a similar manner, but with respect to the sequent $: \Rightarrow^- A$. Thus for instance, a variable p contributes to a sequent $: \Rightarrow^+ \neg(\neg p \to \neg q)$ by carrying falsificatory information, while q carries verificatory information[4]. In contrast, both p and q carry falsificatory information in another sequent $: \Rightarrow^- \neg(\neg p \to \neg q)$. Correspondingly, we have $V_v^+(\neg(\neg p \to \neg q)) = \{q\}$, $V_f^+(\neg(\neg p \to \neg q)) = \{p\}$, $V_v^-(\neg(\neg p \to \neg q)) = \emptyset$ and $V_f^-(\neg(\neg p \to \neg q)) = \{p, q\}$. ⊣

We shall borrow a notation from contsructive mathematics and write $X \between Y$ ("meets" [18]) to compactly indicate that there is an x such that $x \in X \cap Y$. We then introduce a few more preliminary notions.

DEFINITION 4.2. Let $\Gamma : \Delta \Rightarrow^* A$ be a sequent. A *partition* of the sequent is a pair $((\Gamma_1, \Delta_1), (\Gamma_2, \Delta_2, A))$ where $\Gamma_1 \uplus \Gamma_2 = \Gamma$, $\Delta_1 \uplus \Delta_2 = \Delta$ and $\Gamma_1 \uplus \Delta_1 \neq \emptyset$.

DEFINITION 4.3. We say a sequent $\Gamma : \Delta \Rightarrow^* A$ is:

- *verificatory good* if $V_v^-(\Gamma_1) \cup V_v^+(\Delta_1) \between V_v^-(\Gamma_2) \cup V_v^+(\Delta_2) \cup V_v^*(A)$ for any partition $((\Gamma_1, \Delta_1), (\Gamma_2, \Delta_2, A))$ of the sequent.
- *falsificatory good* if $V_f^-(\Gamma_1) \cup V_f^+(\Delta_1) \between V_f^-(\Gamma_2) \cup V_f^+(\Delta_2) \cup V_f^*(A)$ for any partition $((\Gamma_1, \Delta_1), (\Gamma_2, \Delta_2, A))$ of the sequent.

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[4] It helps to recall here that the formula is (strongly) equivalent to $\neg p \to q$.

They represent different ways propositional variables are shared in a sequent. Our first observation connects these with provable sequents of a certain form.

LEMMA 4.2. *If a sequent with a non-empty antecedent is provable in* $\mathbf{G}\mathcal{C}\mathbf{R}^{-}_{\to}$ *and* $\mathbf{G}\mathcal{C}\mathbf{M}^{-}_{\to}$*, then it is either verificatory or falsificatory good.*

PROOF. We show by induction on the depth of derivation. Here we look at the case for $\mathbf{G}\mathcal{C}\mathbf{M}^{-}_{\to}$.

If the derivation ends with an instance of (mAx$^-$):

$$p, \Gamma^p : \Rightarrow^- p$$

then consider a partition $((\Gamma_1, \emptyset), (\Gamma_2, \emptyset, p))$. Since $p \in \Gamma_1$, it follows that $V_f^-(\Gamma_1) \between V_f^-(\Gamma_2) \cup V_f^-(p)$. Thus the sequent is falsificatory good. It is similarly established that an instance of (mAx$^+$) is verificatory good.

Suppose that the derivation ends with an instance of (LC$^-$):

$$\frac{A, A, \Gamma : \Delta \Rightarrow^* C}{A, \Gamma : \Delta \Rightarrow^* C}\ (\mathrm{LC}^-)$$

and a partition $((\Sigma_1 \Cup \Gamma_1, \Delta_1), (\Sigma_2 \Cup \Gamma_2, \Delta_2, C))$ is given, where $\Sigma_1 \Cup \Sigma_2 = \{A\}$, $\Gamma_1 \Cup \Gamma_2 = \Gamma$ and $\Delta_1 \Cup \Delta_2 = \Delta$. Then either $A \in \Sigma_1$ or $A \in \Sigma_2$. In the former case, consider a partition $((\{A\} \Cup \Sigma_1 \Cup \Gamma_1, \Delta_1), (\Sigma_2 \Cup \Gamma_2, \Delta_2, C))$ of the premise. By the I.H., we have one of:

- $V_v^-(\{A\} \Cup \Sigma_1 \Cup \Gamma_1) \cup V_v^+(\Delta_1) \between V_v^-(\Gamma_2) \cup V_v^+(\Sigma_2 \Cup \Delta_2) \cup V_v^*(C)$,
- $V_f^-(\{A\} \Cup \Sigma_1 \Cup \Gamma_1) \cup V_f^+(\Delta_1) \between V_f^-(\Gamma_2) \cup V_f^+(\Sigma_2 \Cup \Delta_2) \cup V_f^*(C)$.

Now since $V_v^-(\{A\} \Cup \Sigma_1 \Cup \Gamma_1) = V_v^-(\Sigma_1 \Cup \Gamma_1)$ and $V_f^-(\{A\} \Cup \Sigma_1 \Cup \Gamma_1) = V_f^-(\Sigma_1 \Cup \Gamma_1)$, we have one of:

- $V_v^-(\Sigma_1 \Cup \Gamma_1) \cup V_v^+(\Delta_1) \between V_v^-(\Gamma_2) \cup V_v^+(\Sigma_2 \Cup \Delta_2) \cup V_v^*(C)$,
- $V_f^-(\Sigma_1 \Cup \Gamma_1) \cup V_f^+(\Delta_1) \between V_f^-(\Gamma_2) \cup V_f^+(\Sigma_2 \Cup \Delta_2) \cup V_f^*(C)$.

On the other hand, if $A \in \Sigma_2$ then we can take a partition $((\Sigma_1 \Cup \Gamma_1, \Delta_1), (\{A\} \Cup \Sigma_2 \Cup \Gamma_2, \Delta_2, C))$ of the premise and argue analogously. The case for (LC$^+$) is similarly shown.

Suppose that the derivation ends with an instance of (L$\to^-$):

$$\frac{\Gamma : \Delta \Rightarrow^+ A \qquad B, \Gamma' : \Delta' \Rightarrow^* C}{A \to B, \Gamma, \Gamma' : \Delta, \Delta' \Rightarrow^* C}\ (\mathrm{L}{\to}^-)$$

and a partition

$$((\Sigma_1 \Cup \varGamma_1 \Cup \varGamma_1', \varDelta_1 \Cup \varDelta_1'), (\Sigma_2 \Cup \varGamma_2 \Cup \varGamma_2', \varDelta_2 \Cup \varDelta_2', C))$$

is given, where $\Sigma_1 \Cup \Sigma_2 = \{A \to B\}$, $\varGamma_1 \Cup \varGamma_2 = \varGamma$, $\varGamma_1' \Cup \varGamma_2' = \varGamma'$, $\varDelta_1 \Cup \varDelta_2 = \varDelta$ and $\varDelta_1' \Cup \varDelta_2' = \varDelta'$. Now if $A \to B \in \Sigma_1$, we partition the right premise to $((\{B\} \Cup \varGamma_1', \varDelta_1'), (\varGamma_2', \varDelta_2', C))$. By the I.H., we have either:

- $V_v^-(\{B\} \Cup \varGamma_1') \cup V_v^+(\varDelta') \between V_v^-(\varGamma_2') \cup V_v^+(\varDelta_2') \cup V_v^+(C)$, or
- $V_f^-(\{B\} \Cup \varGamma_1') \cup V_f^+(\varDelta') \between V_f^-(\varGamma_2') \cup V_f^+(\varDelta_2') \cup V_f^+(C)$,

from which the statement follows, as $V_v^-(A \to B) = V_v^+(A) \cup V_v^-(B)$ and $V_f^-(A \to B) = V_f^+(A) \cup V_f^-(B)$. If on the other hand $A \to B \in \Sigma_2$, then $\varGamma_1 \Cup \varGamma_1' \Cup \varDelta_1 \Cup \varDelta_1'$ must be non-empty. If $\varGamma_1$ or $\varDelta_1$ is non-empty, use $((\varGamma_1, \varDelta_1), (\varGamma_2, \varDelta_2, A))$ as a partition of the left premise, which by the I.H. implies one of:

- $V_v^-(\varGamma_1) \cup V_v^+(\varDelta_1) \between V_v^-(\varGamma_2) \cup V_v^+(\varDelta_2) \cup V_v^+(A)$.
- $V_f^-(\varGamma_1) \cup V_f^+(\varDelta_1) \between V_f^-(\varGamma_2) \cup V_f^+(\varDelta_2) \cup V_f^+(A)$.

Hence the statement follows. In the other case, we can use a partition $((\varGamma_1', \varDelta_1'), (\{B\} \Cup \varGamma_2', \varDelta_2', C))$ of the right premise to establish the statement. The case for (L$\to^+$) is analogous.

Suppose that the derivation ends with an instance of (R$\to^-$):

$$\frac{\varGamma : \varDelta, A \Rightarrow^- B}{\varGamma : \varDelta \Rightarrow^- A \to B}\,(\mathrm{R}{\to}^-)$$

and a partition $((\varGamma_1, \varDelta_1), (\varGamma_2, \varDelta_2, A \to B))$ of the conclusion is given. Then we may partition the premise as $((\varGamma_1, \varDelta_1), (\varGamma_2, \{A\} \cup \varDelta_2, B))$. By the I.H., one of the following holds.

- $V_v^-(\varGamma_1) \cup V_v^+(\varDelta_1) \between V_v^-(\varGamma_2) \cup V_v^+(\{A\} \Cup \varDelta_2) \cup V_v^+(B)$.
- $V_f^-(\varGamma_1) \cup V_f^+(\varDelta_1) \between V_f^-(\varGamma_2) \cup V_f^+(\{A\} \Cup \varDelta_2) \cup V_f^+(B)$.

Now since

- $V_v^+(\{A\} \Cup \varDelta_2) \cup V_v^-(B) = V_v^+(\varDelta_2) \cup V_v^-(A \to B)$ and
- $V_f^+(\{A\} \Cup \varDelta_2) \cup V_f^-(B) = V_f^+(\varDelta_2) \cup V_f^-(A \to B)$,

the statement follows. The case for (R$\to^+$) is similar.

Suppose that the derivation ends with an instance of (L$\to^-$):

$$\frac{\varGamma : \varDelta, A \Rightarrow^* C}{\neg A, \varGamma : \varDelta \Rightarrow^* C}\,(\mathrm{L}{\to}^-)$$

and a partition $((\Sigma_1 \Cup \varGamma_1, \varDelta_1), (\Sigma_2 \Cup \varGamma_2, \varDelta_2, C))$ is given, where $\Sigma_1 \Cup \Sigma_2 =$

$\{\neg A\}$. If $\neg A \in \Sigma_1$, we take a partition $((\Gamma_1, \Delta_1 \Cup \{A\}), (\Gamma_2, \Delta_2, C))$. By the I.H. we have either:

- $V_v^-(\Gamma_1) \cup V_v^+(\Delta_1 \Cup \{A\}) \between V_v^-(\Gamma_2) \cup V_v^+(\Delta_2) \cup V_v^*(C)$, or
- $V_f^-(\Gamma_1) \cup V_f^+(\Delta_1 \Cup \{A\}) \between V_f^-(\Gamma_2) \cup V_f^+(\Delta_2) \cup V_f^*(C)$.

Then as $V_v^-(\neg A) = V_v^+(A)$ and $V_f^-(\neg A) = V_f^+(A)$, the statement follows. Similarly, if $\neg A \in \Sigma_2$, then take a partition $((\Gamma_1, \Delta_1), (\Gamma_2, \Delta_2 \Cup \{A\}, C))$. The case for (L$\neg^+$) is analogous.

Suppose that the derivation ends with an instance of (R$\neg^-$):

$$\frac{\Gamma : \Delta \Rightarrow^+ C}{\Gamma : \Delta \Rightarrow^- \neg C}\,(\mathrm{R}\neg^-)$$

and a partition $((\Gamma_1, \Delta_1), (\Gamma_2, \Delta_2, \neg C))$ be given. Then we take a partition $((\Gamma_1, \Delta_1), (\Gamma_2, \Delta_2, C))$ of the premise. By the I.H. either:

- $V_v^-(\Gamma_1) \cup V_v^+(\Delta_1) \between V_v^-(\Gamma_2) \cup V_v^+(\Delta_2) \cup V_v^+(C)$.
- $V_f^-(\Gamma_1) \cup V_f^+(\Delta_1) \between V_f^-(\Gamma_2) \cup V_f^+(\Delta_2) \cup V_f^+(C)$.

hold. The statement then follows, as $V_v^-(\neg C) = V_v^+(C)$ and $V_f^-(\neg C) = V_f^+(C)$. The case for (R$\neg^+$) is analogous. ⊣

This immediately means the following.

THEOREM 4.2. *For* $x \in \{r, m\}$,

- *If* $\vdash_{gx} : A \Rightarrow^+ B$ *then* $V_v^+(A) \between V_v^+(B)$ *or* $V_f^+(A) \between V_f^+(B)$.
- *If* $\vdash_{gx} : A \Rightarrow^- B$ *then* $V_v^+(A) \between V_v^-(B)$ *or* $V_f^+(A) \between V_f^-(B)$.

PROOF. (i) It follows from Lemma by partitioning $: A \Rightarrow^* B$ into $((\emptyset, \{A\}), (\emptyset, \emptyset, \{B\}))$. (ii) Similar. ⊣

If this theorem is applied to provable contradictions, some different possibilities are discerned.

COROLLARY 4.1. *For* $x \in \{r, m\}$, *if* $\vdash_{hx} A \to B$ *and* $\vdash_{hx} \neg(A \to B)$ *then one of the following cases holds.*

(i) $V_v^+(A) \between V_v^+(B)$ *and* $V_v^+(A) \between V_v^-(B)$,
(ii) $V_v^+(A) \between V_v^+(B)$ *and* $V_f^+(A) \between V_f^-(B)$,
(iii) $V_f^+(A) \between V_f^+(B)$ *and* $V_v^+(A) \between V_v^-(B)$,
(iv) $V_f^+(A) \between V_f^+(B)$ *and* $V_f^+(A) \between V_f^-(B)$.

PROOF. If $\vdash_{hx} A \to B$ and $\vdash_{hx} \neg(A \to B)$, then $\vdash^f_{gx} : A \Rightarrow^+ B$ and $\vdash^f_{gx} : A \Rightarrow^- B$ by Proposition 2.1 and Theorem 2.1. Then Theorem 4.2 provides the four possibilities. ⊣

This gives 4 (or 9 mutually exclusive) classes of provable contradictions for any subsystem of $\mathcal{C}\mathbf{M}^{\neg}_{\rightarrow}$ (in $\mathcal{L}$). Let us apply the classification to some simple cases of provable contradictions.

Example 4.1. Weiss' contradictory formula [26]

$$((\neg p {\rightarrow} p) {\rightarrow} \neg p) \rightarrow \neg((\neg p {\rightarrow} p) {\rightarrow} \neg p)$$

satisfies all of (i)-(iv). Observe that we have for the premise:

- $V^+_v((\neg p {\rightarrow} p) {\rightarrow} \neg p) = V^+_v(\neg p) \cup V^+_v(p) \cup V^-_v(p) = \{p\}$.
- $V^+_f((\neg p {\rightarrow} p) {\rightarrow} \neg p) = V^+_f(\neg p) \cup V^+_f(p) \cup V^-_f(p) = \{p\}$.

while for the conclusion:

- $V^+_v(\neg((\neg p {\rightarrow} p) {\rightarrow} \neg p)) = V^+_v(\neg p {\rightarrow} p) \cup V^-_v(\neg p) = \{p\}$.
- $V^+_f(\neg((\neg p {\rightarrow} p) {\rightarrow} \neg p)) = V^+_f(\neg p) \cup V^+_f(p) \cup V^-_f(\neg p) = \{p\}$.
- $V^-_v(\neg((\neg p {\rightarrow} p) {\rightarrow} \neg p)) = V^+_v((\neg p {\rightarrow} p) {\rightarrow} \neg p) = \{p\}$.
- $V^-_f(\neg((\neg p {\rightarrow} p) {\rightarrow} \neg p)) = V^+_f((\neg p {\rightarrow} p) {\rightarrow} \neg p) = \{p\}$. ⊣

Hence Weiss' contradictory formula can be viewed to satisfy the necessary conditions given by Corollary 4.1 in the *strongest* sense. The opposite of this would be to satisfy the conditions of Corollary 4.1 in the *weakest* sense: i.e. to satisfy only one of the four cases.

Example 4.2. The formula

$$p \rightarrow ((p {\rightarrow} \neg p) {\rightarrow} ((\neg p {\rightarrow} p) {\rightarrow} p))$$

is a provable contradiction in $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$:

$$\begin{prooftree}
\AxiomC{$: p \Rightarrow^+ p$}
\AxiomC{$p : \Rightarrow^- p$}
\doubleLine
\UnaryInfC{$: \neg p \Rightarrow^+ \neg p$}
\BinaryInfC{$: p, p {\rightarrow} \neg p \Rightarrow^+ \neg p$}
\AxiomC{$: p \Rightarrow^+ p$}
\BinaryInfC{$: p, p {\rightarrow} \neg p, \neg p {\rightarrow} p \Rightarrow^+ p$}
\doubleLine
\UnaryInfC{$: \Rightarrow^+ p {\rightarrow} ((p {\rightarrow} \neg p) {\rightarrow} ((\neg p {\rightarrow} p) {\rightarrow} p))$}
\end{prooftree}$$

$$\begin{prooftree}
\AxiomC{$\vdots$}
\doubleLine
\UnaryInfC{$: p, p {\rightarrow} \neg p, \neg p {\rightarrow} p \Rightarrow^+ p$}
\AxiomC{$p : \Rightarrow^- p$}
\UnaryInfC{$: \neg p \Rightarrow^- p$}
\BinaryInfC{$: p, p {\rightarrow} \neg p, p {\rightarrow} \neg p, \neg p {\rightarrow} p \Rightarrow^- p$}
\UnaryInfC{$: p, p {\rightarrow} \neg p, \neg p {\rightarrow} p \Rightarrow^- p$}
\UnaryInfC{$: \Rightarrow^- p {\rightarrow} ((p {\rightarrow} \neg p) {\rightarrow} ((\neg p {\rightarrow} p) {\rightarrow} p))$}
\end{prooftree}$$

This formula satisfies only (i) of Corollary 4.1. Since $V_f^+(p) = \emptyset$, it is not possible that (ii)–(iv) hold. To check that (i) holds, note that $V_v^+(p) = \{p\}$ and:

- $V_v^-((p\to\neg p)\to((\neg p\to p)\to p)) =$ $V_v^+(p) \cup V_v^-(p) \cup V_v^+((\neg p\to p)\to p) = \{p\}$
- $V_f^-((p\to\neg p)\to((\neg p\to p)\to p)) =$ $V_f^+(p) \cup V_f^-(p) \cup V_f^+((\neg p\to p)\to p) = \{p\}$

We can similarly check that the formula

$$\neg p \to ((p\to\neg p)\to((\neg p\to p)\to p))$$

is a provable contradiction in $\mathcal{C}\mathbf{R}_{\to}^{\neg}$ and satisfies only (iv).

Are there provable contradictions which witnesses the other 6 possibilities? In order to answer this question affirmatively, the current language appears insufficient. For this reason, we shall slightly expand it with conjunction in the next section.

5. Expansion with conjunction

There are different ways to formulate conjunction in a relevant setting, and different opinions have been expressed when it comes to a suitable option for $\mathcal{C}\mathbf{R}_{\to}^{\neg}$ and $\mathcal{C}\mathbf{M}_{\to}^{\neg}$. Francez [7] suggests that the *intensional conjunction (fusion)* is more satisfactory for relevance, while Weiss [26] remarks that the addition of the *extensional conjunction* is of interest for a comparison with $\mathcal{C}\mathbf{C}_{\to}^{\neg}$. An accompanying question, independent of which approach is to be adopted, is to decide on the falsification condition.

Our focus here is not so much to contribute to this question, but to use conjunction as an instrument to generate provable contradiction of desired forms. The type of conjunction we adopt here — the intensional one — is chosen solely for a pragmatic reason. We will not even specify when a conjunction is false, because that is not necessary for our objective. For the same reason we shall only consider an expansion for $\mathcal{C}\mathbf{R}_{\to}^{\neg}$.

Let $\mathcal{L}_\circ$ be an expansion of $\mathcal{L}$ with a binary connective $\circ$. We define a system $\mathcal{C}\mathbf{R}_{\to,\circ}^{\neg}$ in $\mathcal{L}_\circ$ by adding the following axiom schemata to $\mathcal{C}\mathbf{R}_{\to}^{\neg}$.

$$A \to (B \to (A \circ B)) \tag{CI}$$

$$(A \to (B \to C)) \to ((A \circ B) \to C) \tag{CE}$$

A sequent calculus $\mathbf{G}\mathcal{C}\mathbf{R}^{\neg}_{\to,\circ}$ is introduced as well, by supplementing $\mathbf{G}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ with the rules below.

$$\dfrac{\mathit{\Gamma} : \mathit{\Delta}, A, B \Rightarrow^{*} C}{\mathit{\Gamma} : \mathit{\Delta}, A \circ B \Rightarrow^{*} C}\,(\mathrm{L}\circ^{+}) \qquad \dfrac{\mathit{\Gamma} : \mathit{\Delta} \Rightarrow^{+} A \qquad \mathit{\Gamma}' : \mathit{\Delta}' \Rightarrow^{+} B}{\mathit{\Gamma}, \mathit{\Gamma}' : \mathit{\Delta}, \mathit{\Delta}' \Rightarrow^{+} A \circ B}\,(\mathrm{R}\circ^{+})$$

$$\dfrac{A : \Rightarrow^{-} A \qquad B : \Rightarrow^{-} B}{A \circ B : \Rightarrow^{-} A \circ B}\,(\circ^{-})$$

We shall use $\vdash_{hr\circ}$ and $\vdash_{gr\circ}$ for their derivabilities. Then as before, $\vdash^{f}_{gr\circ}$ denotes the cut-free derivability.

We first observe that basic properties of $\mathcal{C}\mathbf{R}^{\neg}_{\to}$ and $\mathbf{G}\mathcal{C}\mathbf{R}^{\neg}_{\to}$ hold for $\mathcal{C}\mathbf{R}^{\neg}_{\to,\circ}$ and $\mathbf{G}\mathcal{C}\mathbf{R}^{\neg}_{\to,\circ}$ as well.

PROPOSITION 5.1. *The following statements hold.*

(i) $\mathit{\Gamma}, A \vdash_{hr\circ} B$ *if and only if* $\mathit{\Gamma} \vdash_{hr\circ} A \to B$.

(ii) $\vdash_{gr\circ} A : \Rightarrow^{-} A$ *and* $\vdash_{gr\circ} : A \Rightarrow^{+} A$.

(iii) *If* $\mathit{\Gamma} \vdash_{hr\circ} A$ *then* $\vdash_{gr\circ} : \mathit{\Gamma} \Rightarrow^{+} A$.

(iv) *If* $\vdash_{gr\circ} \mathit{\Gamma} : \mathit{\Delta} \Rightarrow^{*} A$ *then* $\neg\mathit{\Gamma}, \mathit{\Delta} \vdash_{hr\circ} A^{*}$.

PROOF. For each statement, it suffices to check cases involving $\circ$. The argument for (i) is the same as that of Lemma 2.1. (ii) easily follows, but note that the presence of $(\circ^{-})$ is required. (iii) follows standardly from (ii) and the added rules. (iv) can again be shown by using (i). $\dashv$

THEOREM 5.1. *For* $* \in \{+, -\}$, $\vdash_{gr\circ} \mathit{\Gamma} : \mathit{\Delta} \Rightarrow^{*} A$ *iff* $\vdash^{f}_{gr\circ} \mathit{\Gamma} : \mathit{\Delta} \Rightarrow^{*} A$.

PROOF. In view of Proposition 2.2 and Theorem 2.1, it suffices to consider a system with (eCut*), and check cases where the cutformula is conjunction. As an example, if it is an instance of (eCut^{+}) and the cutformula is principal in both of the premises

$$\dfrac{\dfrac{\mathit{\Gamma} : \mathit{\Delta} \Rightarrow^{+} A \qquad \mathit{\Gamma}' : \mathit{\Delta}' \Rightarrow^{+} B}{\mathit{\Gamma}, \mathit{\Gamma}' : \mathit{\Delta}, \mathit{\Delta}' \Rightarrow^{+} A \circ B}\,(\mathrm{R}\circ^{+}) \qquad \dfrac{\mathit{\Gamma}'' : \mathit{\Delta}'', A, B \Rightarrow^{*} C}{\mathit{\Gamma}'' : \mathit{\Delta}'', A \circ B \Rightarrow^{*} C}\,(\mathrm{L}\circ^{+})}{\mathit{\Gamma}, \mathit{\Gamma}', \mathit{\Gamma}''_{A\circ B} : \mathit{\Delta}, \mathit{\Delta}', \mathit{\Delta}''_{A\circ B} \Rightarrow^{*} C}\,(\mathrm{eCut}^{+})$$

then we can use two (eCut^{+}) of lower grade for the case when $\mathit{\Delta}''_{A\circ B} = \mathit{\Delta}''$:

$$\dfrac{\mathit{\Gamma}' : \mathit{\Delta}' \Rightarrow^{+} B \qquad \dfrac{\mathit{\Gamma} : \mathit{\Delta} \Rightarrow^{+} A \qquad \mathit{\Gamma}'' : \mathit{\Delta}'', A, B \Rightarrow^{*} C}{\mathit{\Gamma}, \mathit{\Gamma}'' : \mathit{\Delta}, \mathit{\Delta}'', B \Rightarrow^{*} C}\,(\mathrm{eCut}^{+})}{\mathit{\Gamma}, \mathit{\Gamma}', \mathit{\Gamma}'' : \mathit{\Delta}, \mathit{\Delta}', \mathit{\Delta}'' \Rightarrow^{*} C}\,(\mathrm{eCut}^{+})$$

Otherwise, there is an occurrence of $A \circ B$ in $\mathit{\Delta}''$. Then we have the following derivation whose topmost (eCut^{+}) is of less height, and the others of less grade.

$$\begin{array}{c}\dfrac{\Gamma', \Delta' \Rightarrow^{+} B \qquad \dfrac{\Gamma : \Delta \Rightarrow^{+} A \qquad \dfrac{\Gamma, \Gamma' : \Delta, \Delta' \Rightarrow^{+} A \circ B \qquad \Gamma'' : \Delta'', A, B \Rightarrow^{*} C}{\Gamma, \Gamma', \Gamma'' : \Delta, \Delta', \Delta''_{A \circ B}, A, B \Rightarrow^{*} C}\,(\mathrm{eCut}^{+})}{\Gamma, \Gamma, \Gamma', \Gamma'' : \Delta, \Delta, \Delta', \Delta''_{A \circ B}, B \Rightarrow^{*} C}\,(\mathrm{eCut}^{+})}{\Gamma, \Gamma, \Gamma', \Gamma', \Gamma'' : \Delta, \Delta, \Delta', \Delta', \Delta''_{A \circ B} \Rightarrow^{*} C}\,(\mathrm{eCut}^{+}) \\ \hline\hline \Gamma, \Gamma', \Gamma'' : \Delta, \Delta', \Delta''_{A \circ B} \Rightarrow^{*} C \end{array}\;(\mathrm{LC}^{+},\mathrm{LC}^{-})$$

If it is an instance of (eCut^{-}) then the conclusion and premises are identical, so we may instead take a derivation up to one of the premises. ⊣

It now follows from an inspection of the rules that a sequent in $\mathcal{L}$ can be derived in $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow,\circ}$ only if there is a cut-free derivation that does not use rules related to $\circ$. Therefore $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow,\circ}$ is a conservative expansion of $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$.

As for the classes $V_v^{+}, V_v^{-}, V_f^{+}$ and V_f^{-}, we extend them with the next clauses for conjunction.

$$V^{+}(A \circ B) = V_v^{+}(A) \cup V_v^{+}(B).\ V_v^{-}(A \circ B) = V_v^{-}(A) \cup V_v^{-}(B).$$
$$V_f^{+}(A \circ B) = V_f^{+}(A) \cup V_f^{+}(B).\ V_f^{-}(A \circ B) = V_f^{-}(A) \cup V_f^{-}(B).$$

The same classification of provable contradictions as Corollary 4.1 is then obtained for $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow,\circ}$.

PROPOSITION 5.2. *If* $\vdash_{hr\circ} A \rightarrow B$ *and* $\vdash_{hr\circ} \neg(A \rightarrow B)$ *then one of the following cases holds*:

(i) $V_v^{+}(A) \between V_v^{+}(B)$ *and* $V_v^{+}(A) \between V_v^{-}(B)$,
(ii) $V_v^{+}(A) \between V_v^{+}(B)$ *and* $V_f^{+}(A) \between V_f^{-}(B)$,
(iii) $V_f^{+}(A) \between V_f^{+}(B)$ *and* $V_v^{+}(A) \between V_v^{-}(B)$,
(iv) $V_f^{+}(A) \between V_f^{+}(B)$ *and* $V_f^{+}(A) \between V_f^{-}(B)$.

PROOF. The outline is the same as in the proof of Lemma 4.1–Corollary 4.1. Here we check the statement of Lemma 4.2 for cases when the applied rule is (L$\circ^{+}$), (R$\circ^{+}$) or ($\circ^{-}$).

If the derivation ends with an instance of (L$\circ^{+}$):

$$\frac{\Gamma : \Delta, A, B \Rightarrow^{*} C}{\Gamma : \Delta, A \circ B \Rightarrow^{*} C}\,(\mathrm{L}\circ^{+})$$

and a partition

$$((\Gamma_1, \Delta_1 \uplus \Sigma_1), (\Gamma_2, \Delta_2 \uplus \Sigma_2, C)).$$

where $\Sigma_1 \uplus \Sigma_2 = \{A \circ B\}$, $\Gamma_1 \uplus \Gamma_2 = \Gamma$ and $\Delta_1 \uplus \Delta_2 = \Delta$. Now if $A \circ B \in \Sigma_1$, we partition the premise to $((\Gamma_1, \Delta_1 \uplus \{A, B\}), (\Gamma_2, \Delta_2, C))$.

By the I.H., we have either:

- $V_v^-(\Gamma_1) \cup V_v^+(\Delta_1 \Cup \{A, B\}) \between V_v^-(\Gamma_2) \cup V_v^+(\Delta_2) \cup V_v^+(C)$, or
- $V_f^-(\Gamma_1) \cup V_f^+(\Delta_1 \Cup \{A, B\}) \between V_f^-(\Gamma_2) \cup V_f^+(\Delta_2) \cup V_f^+(C)$.

from which the statement follows, as $V_v^+(A \circ B) = V_v^+(A) \cup V_v^+(B)$ and $V_f^+(A \circ B) = V_f^+(A) \cup V_f^+(B)$. The argument is analogous when $A \to B \in \Sigma_2$.

If the derivation ends with an instance of (R$\circ^+$):

$$\frac{\Gamma : \Delta \Rightarrow^+ A \qquad \Gamma' : \Delta' \Rightarrow^+ B}{\Gamma, \Gamma' : \Delta, \Delta' \Rightarrow^+ A \circ B}\,(\mathrm{R}\circ^+)$$

and a partition

$$((\Gamma_1 \Cup \Gamma_2, \Delta_1 \Cup \Delta_1'), (\Gamma_2 \Cup \Gamma_2', \Delta_2 \Cup \Delta_2', A \circ B)).$$

where $\Gamma_1 \Cup \Gamma_2 = \Gamma$, $\Gamma_1' \Cup \Gamma_2' = \Gamma'$ and $\Delta_1 \Cup \Delta_2 = \Delta$ and $\Delta_1' \Cup \Delta_2' = \Delta'$. Then one of $\Gamma_1 \Cup \Delta_1$ or $\Gamma_1' \Cup \Delta_1'$ must be non-empty. Consider as an example the former case. In this case, take a partition $((\Gamma_1, \Delta_1), (\Gamma_2, \Delta_2, A))$ of the left premise. By the I.H. either:

- $\mathcal{V}_v^-(\Gamma_1) \cup \mathcal{V}_v^+(\Delta_1) \between \mathcal{V}_v^-(\Gamma_2) \cup \mathcal{V}_v^+(\Delta_2) \cup \mathcal{V}_v^*(A)$, or
- $\mathcal{V}_f^-(\Gamma_1) \cup \mathcal{V}_f^+(\Delta_1) \between \mathcal{V}_f^-(\Gamma_2) \cup \mathcal{V}_f^+(\Delta_2) \cup \mathcal{V}_f^*(A)$.

This implies that either:

- $\mathcal{V}_v^-(\Gamma_1 \Cup \Gamma_1') \cup \mathcal{V}_v^+(\Delta_1 \Cup \Delta_1') \between \mathcal{V}_v^-(\Gamma_2 \Cup \Gamma_2') \cup \mathcal{V}_v^+(\Delta_2 \Cup \Delta_2') \cup \mathcal{V}_v^*(A \circ B)$, or
- $\mathcal{V}_f^-(\Gamma_1 \Cup \Gamma_1') \cup \mathcal{V}_f^+(\Delta_1 \Cup \Delta_1') \between \mathcal{V}_f^-(\Gamma_2 \Cup \Gamma_2') \cup \mathcal{V}_f^+(\Delta_2 \Cup \Delta_2') \cup \mathcal{V}_f^*(A \circ B)$

as desired. Other cases are analogously argued. The case for ($\circ^-$) is also similar. $\dashv$

We now provide an example formula for each of the 9 classes given by Proposition 5.2. These are classes of provable contradictions which satisfy:

1. only (i).
2. only (ii).
3. only (iii).
4. only (iv).
5. only (iii) and (iv).
6. only (i) and (ii).
7. only (ii) and (iv).
8. only (i) and (iii).
9. all of (i), (ii), (iii) and (iv).

We have already seen examples for classes 1, 4 and 9, which work for $\mathcal{C}\mathbf{R}^{\neg}_{\to,\circ}$ as well. For the remaining cases, let us first check 2 and 3.

Example 5.1. Contradictions satisfying only (ii)/(iii) are now obtainable, respectively by:

$$(p\circ(p\to\neg p)\circ(\neg p\to p))\to p \text{ and } (p\circ(p\to\neg p)\circ(\neg p\to p))\to\neg p.$$

Here we look at the former case. It easily follows from the penultimate steps in the derivations of Example 4.2 that it is a provable contradiction of $\mathcal{C}\mathbf{R}^{\neg}_{\to,\circ}$. It is also straightforward to check that $V_v^+(p\circ(p\to\neg p)\circ(\neg p\to p)) = V_f^+(p\circ(p\to\neg p)\circ(\neg p\to p)) = \{p\}$. On the other hand $V_v^+(p) = V_f^-(p) = \{p\}$ and $V_f^+(p) = V_v^-(p) = \emptyset$, so only (ii) is satisfied. $\dashv$

For classes 5–8, we have the following examples.

Example 5.2. We claim that

- $(p\circ(p\to\neg p)\circ(\neg p\to p))\to((\neg p\to\neg p)\to\neg p)$
- $(p\circ(p\to\neg p)\circ(\neg p\to p))\to((p\to p)\to p)$
- $(p\circ(p\to\neg p)\circ(\neg p\to p))\to((\neg p\to\neg p)\to p)$
- $(p\circ(p\to\neg p)\circ(\neg p\to p))\to((p\to p)\to\neg p)$

are provable contradictions of $\mathcal{C}\mathbf{R}^{\neg}_{\to,\circ}$ and each satisfies only (iii)&(iv); (i)&(ii); (ii)&(iv); and (i)&(iii) of Proposition 5.2. Let us treat here the first case. It is straightforwardly verified that $(p\circ(p\to\neg p)\circ(\neg p\to p))\to((\neg p\to\neg p)\to\neg p)$ is a provable contradiction of $\mathcal{C}\mathbf{R}^{\neg}_{\to,\circ}$. Also as before, $V_v^+(p\circ(p\to\neg p)\circ(\neg p\to p)) = V_f^+(p\circ(p\to\neg p)\circ(\neg p\to p)) = \{p\}$. On the other hand,

- $V_v^+((\neg p\to\neg p)\to\neg p) = V_v^+(\neg p)\cup V_v^+(\neg p)\cup V_v^+(\neg p) = \emptyset.$
- $V_f^+((\neg p\to\neg p)\to\neg p) = V_f^+(\neg p\to\neg p)\cup V_f^+(\neg p) = \{p\}.$
- $V_v^-((\neg p\to\neg p)\to\neg p) = V_v^+(\neg p\to\neg p)\cup V_v^-(\neg p) = \{p\}.$
- $V_f^-((\neg p\to\neg p)\to\neg p) = V_f^+(\neg p\to\neg p)\cup V_f^-(\neg p) = \{p\}.$

Hence only (iii) and (iv) are satisfied. The other cases can be checked in a similar manner. $\dashv$

6. Concluding remarks

In this note, we have explored the phenomenon of provable contradictions in the systems of Francez and Weiss, using criteria of (i) correspondence in derivation and (ii) relevance. For (i), it has been established that provable contradictions in $\mathcal{C}\mathbf{R}^{\neg}_{\to}$ and $\mathcal{C}\mathbf{M}^{\neg}_{\to}$ accompany tableaux derivations which — generalising Wansing's observation — show a type of correspondence to each other. For (ii), it has been observed that (implicational) provable contradictions in the logics can be classified into 9 types, based on the way propositional variables are shared between their premise and conclusion. In particular, we have found a witness for each of the classes by expanding the language with conjunction $\mathcal{C}\mathbf{R}^{\neg}_{\rightarrow}$, as listed below:

1. $p{\rightarrow}((p{\rightarrow}\neg p){\rightarrow}((\neg p{\rightarrow}p){\rightarrow}p))$
2. $(p \circ (p{\rightarrow}\neg p) \circ (\neg p{\rightarrow}p)){\rightarrow}p$
3. $(p \circ (p{\rightarrow}\neg p) \circ (\neg p{\rightarrow}p)){\rightarrow}\neg p$
4. $\neg p{\rightarrow}((p{\rightarrow}\neg p){\rightarrow}((\neg p{\rightarrow}p){\rightarrow}p))$
5. $(p \circ (p{\rightarrow}\neg p) \circ (\neg p{\rightarrow}p)){\rightarrow}((\neg p{\rightarrow}\neg p){\rightarrow}\neg p)$
6. $(p \circ (p{\rightarrow}\neg p) \circ (\neg p{\rightarrow}p)){\rightarrow}((p{\rightarrow}p){\rightarrow}p)$
7. $(p \circ (p{\rightarrow}\neg p) \circ (\neg p{\rightarrow}p)){\rightarrow}((\neg p{\rightarrow}\neg p){\rightarrow}p)$
8. $(p \circ (p{\rightarrow}\neg p) \circ (\neg p{\rightarrow}p)){\rightarrow}((p{\rightarrow}p){\rightarrow}\neg p)$
9. $((\neg p{\rightarrow}p){\rightarrow}\neg p) \rightarrow \neg((\neg p{\rightarrow}p){\rightarrow}\neg p)$

Can we relate the viewpoints (i) and (ii) to each other? On one hand, the correspondence in (i) (for a contradictory formula) suggests a close connection between verification and falsification. On the other hand, we also observe a close connection in (ii) between the premise and the conclusion of an implicational provable contradiction: the four possibilities in Corollary 4.1 have either $V_v^+(A) \between V_v^-(B)$ or $V_f^+(A) \between V_f^-(B)$ for contradictions of the form $A \rightarrow B$.

At the same time, the four classes can also be divided in terms of v and f, which may be a more ‘internal’ way of looking at verification and falsification. ($+$ and $-$, being signs for sequents, can be understood to represent a more ‘external’ viewpoint.) From this perspective, one finds classes of contradictions in which variable-sharing happens with respect to verification alone/falsification alone (as represented by v and f): the case 1. above concerns only v, and 4. concerns only f.

Therefore, the relevantistic viewpoint appears to offer a more nuanced response to the idea that provable contradictions necessitate a tight relationship between verification and falsification. This depends on whether one takes the internal/external viewpoint. A task for the future then would be to clarify if one of the viewpoints is to be preferred, and if so under which circumstances.

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References

[1] Almukdad, A., and D. Nelson, "Constructible falsity and inexact predicates", *The Journal of Symbolic Logic*, 49(1), 1984: 231–233. DOI: 10.2307/2274105

[2] Avron, A., " Relevant entailment—semantics and formal systems", *The Journal of Symbolic Logic*, 49(2), 1984: 334–342. DOI: 10.2307/2274169

[3] Bimbó, K., *Proof Theory: Sequent Calculi and Related Formalisms*, CRC Press, 2014. DOI: 10.1201/b17294

[4] Brady, R. T, "A Routley-Meyer affixing style semantics for logics containing Aristotle's thesis", *Studia Logica*, 48, 1989: 235–241. DOI: 10.1007/BF02770514

[5] Dunn, J M., "R-Mingle is nice, and so is Arnon Arnon", pages 141–165 in *Arnon Avron on Semantics and Proof Theory of Non-Classical Logics*, Springer, 2021. DOI: 10.1007/978-3-030-71258-7_7

[6] Dunn, J M., and G. Restall, "Relevance logic", pages 1–128 in *Handbook of Philosophical Logic*, vol. 6, Springer, 2002. DOI: 10.1007/978-94-017-0460-1_1

[7] Francez, N., "Relevant connexive logic", *Logic and Logical Philosophy*, 28(3), 2019: 409–425. DOI: 10.12775/LLP.2019.007

[8] Kamide, N., "Substructural logics with mingle", *Journal of Logic, Language and Information*, 11, 2002: 227–249. DOI: 10.1023/A:1017586008091

[9] Kamide, N., and H. Wansing, "Proof theory of Nelson's paraconsistent logic: A uniform perspective", *Theoretical Computer Science*, 415, 2012: 1–38. DOI: 10.1016/j.tcs.2011.11.001

[10] Kamide, N. and H. Wansing, *Proof Theory of N4-related Paraconsistent Logics*, College Publications London, 2015.

[11] Mares, E., "Relevance logic", in E. N. Zalta and U. Nodelman (eds.), *The Stanford Encyclopedia of Philosophy*, Stanford University, Fall 2022 edition. https://plato.stanford.edu/entries/logic-relevance/

[12] McCall, S., "A history of connexivity", pages 415–449 in D. M. Gabbay, F. J. Pelletier and J. Woods (eds.), *Logic: A History of its Central Concepts*, volume 11 of *Handbook of the History of Logic*, North-Holland, 2012.

[13] Mints, G., *A Short Introduction to Intuitionistic Logic*, Kluwer Academic Publishers, 2000.

[14] Mortensen, C., "Aristotle's thesis in consistent and inconsistent logics", *Studia Logica*, 43, 1984: 107–116. DOI: 10.1007/BF00935744

[15] Omori, H., "A simple connexive extension of the basic relevant logic BD", *IFCoLog Journal of Logics and their Applications*, 3(3), 2016: 467–478.

[16] Omori, H., and H. Wansing, "Connexive logic, connexivity, and connexivism: Remarks on terminology", *Studia Logica* 112, 2024: 1–35. DOI: 10.1007/s11225-023-10082-1

[17] Ono, H., *Proof Theory and Algebra in Logic*, Springer, Singapore, 2019. DOI: 10.1007/978-981-13-7997-0

[18] Sambin, G., "Some points in formal topology", *Theoretical Computer Science*, 305(1), 2003: 347–408. DOI: 10.1016/S0304-3975(02)00704-1

[19] Tamura, S., "The implicational fragment of R-mingle", *Proceedings of the Japan Academy*, 47(1), 1971: 71–75. DOI: 10.3792/pja/1195520115

[20] Urquhart, A. I. F., "Completeness of weak implication", *Theoria*, 37(3), 1971: 274–282. DOI: 10.1111/j.1755-2567.1971.tb00072.x

[21] Urquhart, A., "Semantics for relevant logics", *The Journal of Symbolic Logic*, 37(1), 1972: 159–169. DOI: 10.2307/2272559

[22] Urquhart, A., "The semantics of entailment", PhD thesis, University of Pittsburgh, 1973.

[23] Wansing, H., "Connexive modal logic", pages 387–399 in R. Schmidt, I. Pratt-Hartmann, M. Reynolds and H. Wansing (eds.), *Advances in Modal Logic*, vol. 5, College Publications, 2005.

[24] Wansing, H., "Connexive logic", in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*, Stanford University, Summer 2023 edition. https://plato.stanford.edu/entries/logic-connexive/

[25] Wansing, H., and S. Ayhan, "Logical multilateralism", *Journal of Philosophical Logic*, 52, 2023: 1603–1636. DOI: 10.1007/s10992-023-09720-9

[26] Weiss, Y., "Semantics for pure theories of connexive implication", *The Review of Symbolic Logic*, 15(3), 2022: 591–606. DOI: 10.1017/S1755020320000374

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