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What is the Principle of (Non-)Contradiction, Precisely? The Struggle at the Dawn of Formal Logic

Abstract. The principle of contradiction, or non-contradiction, is traditionally included as one of the three fundamental principles of logic, together with the principle of identity and the principle of excluded middle. There is a consensus now regarding the shape of the principle of contradiction in modern formal logic. However, a deeper look at the history of its formulation reveals a much more complicated picture. We trace some of such developments from the beginning of the twentieth century when all sorts of formalisms were proposed, and even the name itself was up for debate. Our focal point is the proposals made by Christine Ladd-Franklin, which we describe against the background of other attempts at the time.

Keywords: the principle of non-contradiction; history of logic; Christine Ladd-Franklin

1. Introduction

Aristotle famously stated that *It is impossible for the same thing to belong and not to belong at the same time to the same thing and in the same respect.*[1] This statement is standardly formalised in propositional logic as $\neg(p \wedge \neg p)$ but before this became the norm, the exact formulation varied from author to author. However, once a standard form arose, it became *the sole* formulation. This is the side effect of solidification of logic as a discipline: codification of systems into propositional and

[1] *Metaphysics IV 3 1005b19–*, *The Stanford Encyclopedia of Philosophy* entry on the topic, (Gottlieb , 2023), also mentions the following two formulations: *It is impossible to hold the same thing to be and not to be* and *opposite assertions cannot be true at the same time.*

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first-order with universally accepted syntax and semantics. The working logician of today has a well-established, standard ways of thinking about his or her field of study. This article contributes mostly to the history of the formal treatment of the principle of (non-)contradiction. The aim here is two-fold. The first, broader goal, is to provide a historical analysis the status of the principle of (non-)contradiction in early modern period. The second, narrower, is related to bringing the work of Christine Ladd-Franklin back to the attention of logicians. Ladd-Franklin lived through the formative period of formal logic (she was born in 1847 and died in 1930) and made important contributions to the field. She also had to fight her way through the predominantly male-oriented academia of that era to have her voice heard. The results presented in her thesis (and she could not be awarded a degree because of her gender) were hailed by some as settling the problem posed by Aristotle that baffled logicians for two thousand years (see see Xu, forthcoming, for recent developments, and Uckelman, 2021, for a critical analysis of this claim). In her PhD she was interested in finding a single syllogistic form to which all the valid syllogisms can be reduced. One of the best students of C. S. Peirce, Ladd-Franklin corresponded with many great minds of her time, including Bertrand Russell (for details see Trybus, 2019). Sadly, her voice was later distorted in the writing of Russell himself and Smullyan (see, e.g., Smullyan, 1983). Ladd-Franklin's work also exerted somehow more lasting influence on some centres of logic study: for example, C. I. Lewis and his students at Harvard carefully analysed her antilogism in during the 1930s.[2] Moreover, she contributed to many fields outside formal logic.[3]

We want to put forward the following claim:

Ladd-Franklin is an important figure in the critique of the principle of (non-)contradiction and should be taken into account when describing the pre-history of paraconsistent logic.

We note that although there is no evidence that she ever considered an alternative logic, her approach to the principle shows her to be an early critic of simply accepting it at face value. Contrary to her main

[2] I owe this observation to Francesco Paoli. For more details, see (Mares & Paoli, 2019, p. 409). Her work was also noticed by Theodore de Laguna (see, e.g., 1912, p. 398).

[3] See (Rossi, 2019) for her work on colour theory and (Ladd, 1879) for her more mathematical work.

results, however, the evidence for the above claim is not to be found in any of her academic publications. We focus on *The Dictionary of Philosophy and Psychology*, which contains numerous entries on logic by Peirce and his collaborators, including Ladd-Franklin. We will present the contents of some of the entries she co-wrote, explaining her approach to the history of the principle of (non-)contradiction, the various forms of this principle and its significance. We want to place what she said in a broader context of contemporary treatment (and critique) of the principle. Since a proof from other principles of logic is a sure sign that (at least in some sense) one views the principle of (non-)contradiction as secondary, we emphasise this aspect in the writings of other logically-minded philosophers, mathematicians and amateurs.

We assume familiarity with the standard syntax of propositional logic and rudiments of predicate calculus. Where it does not take away from the results described, we opted — in hope of increasing accessibility — to present formalism anachronistically using modern notation. Moreover, since the principle in question is variously known as either the principle of contradiction (in older texts) and the principle of non-contradiction (in more contemporary treatments), we use both names and from now on refrain from the cumbersome use of parentheses when quoting the full name. We sometimes call it a law instead of a principle as well.

The text is structured as follows. We start with the first mention of the proof of the principle by George Boole, then move on to what Ladd-Franklin and her circle thought of the principle. We also describe Russell's treatment of the topic, starting from 1903 up until *Principia Mathematica*. Finally, we talk about the contributions of Łukasiewicz, who is widely recognised as one of the first to be brave enough to critique the principle.

2. Boole's Forgotten Proof

George Boole's most mature publication on logic is his *An Investigation of the Laws of Thought*, first published in 1854.[4] This is also the first — and only — place where Boole attempts to show that the law of contradiction can be deduced from what he considers the laws of thought. Boole's laws are introduced as governing reasoning about objects and their attributes by analogy with the common laws governing algebraic

[4] We have used the reprint from 1958: (Boole, 1854).

manipulations in mathematics.[5] Astonishingly, this theme does not seem to be mentioned in any work that builds on his approach[6] Moreover, apart from somehow idiosyncratic treatment in (Béziau, 2018), Boole's proof of the principle is not paid much attention to by the historians of logic.[7] As the topic is important to the goal we pursue in this article, we shall make an attempt at reconstructing Boole's proof. Let us start with the formula

$$xx = x,$$

expressing the fact that the intersection of a class of objects with itself is the same as the original class (which is one of the laws in Boole's approach). Boole chooses to abbreviate it as

$$x^2 = x.$$

Then, by the law of transposition, stating that if the two classes are equal, then deducing some other class from both of them also produces equal classes, he obtains

$$x - x^2 = 0.$$

Now, by the fact that

$$x = 1x = x1$$

[5] Like all analogies, this can only get us so far. Boole's efforts — where he justifies his steps not by exhaustive mathematical reasoning but by providing specific examples that — seem flawed with *ad hoc* restrictions and arbitrary decisions designed to make the system work.

[6] One observes a more general theme of ignoring the specifics of Boole's contribution. This includes Ladd-Franklin's entries in the *Dictionary*, where Boole is only mentioned in passing. This might be, as noted in (Corcoran, 2003, p. 279), because "Boole's work is marred by what appear to be confusions, incoherencies, fallacies and glaring omissions".

[7] Corcoran (2003, p. 278) glances over the topic, Burris (2022) does not even mention it, whereas Kneale & Kneale (1962, p. 406) states that on the whole the book is not very original does not add much to what Boole had published before. Later on — Kneale & Kneale (1962, p. 411) — one reads, however, that

> [Boole] writes $(1-x)$ for the complement of the class denoted by x and introduces the sign $\bar{x}$ merely as an abbreviation. This usage enables him to express the special principle of his system in the formula $x(x-1) = 0$. Sometimes he derives this from the equation $x^2 = x$ [...]; but sometimes he prefers to regard it as basic, remarking that it is a formulation of the principle of non-contradiction [...].

and by unraveling the abbreviation x^2, we get

$$x1 - xx = 0.$$

Finally, by the law introduced by analogy to algebra, we obtain

$$x(1 - x) = 0.$$

Treating $(1 - x)$ as the complement of x, we can read this statement as "the common part of the class of objects having the property x and the class of objects not having the property x equal Nothingness," considered by Boole to be an expression of the principle of contradiction stated as "it is impossible for any being to posses a quality and at the same time not to posses it" (see Boole, 1854, p. 49). Passing over the inaccuracies and flaws of Boole's treatment, this seems to be indeed the first *bona fide* attempt at proving the principle, one that, however, did not win the hearts of the logicians.[8] Yet, the form that Boole chose for his representation of the principle seems to be echoed in the formulations presented by others.

3. Ladd-Franklin's Critique

The Dictionary of Philosophy and Psychology is a three-volume set published between 1901 and 1905 and edited by James Mark Baldwin, a well-known early psychologist.[9] It involved a host of collaborating editors, mostly recognised figures from their respective fields. Ladd-Franklin is listed as Associate Editor for Logic and Psychology. One notes the absence of Ladd-Franklin's mentor, Charles Sanders Peirce, among the listed scholars, although a closer inspection reveals that he was involved in writing many entries related to logic. The reason might be that at that time Peirce was a disgraced ex-academic, nearly destitute and without a chance for an academic post. Baldwin, who was friends with Peirce, tried to help him make ends meet by giving him odd writing jobs like composing entries in the dictionary but perhaps adding him to the title page,

[8] I wish to thank Jean-Yves Béziau for pointing out to me the existence of Boole's proof and Tim Madigan for his help in locating sources on Boole's approach.

[9] Green (2004) describes the details of Baldwin's fight for a position at the University of Toronto in Canada.

which he in many ways deserved, was not an option.[10] It what follows, we focus on two entries where Ladd-Franklin contributed her thoughts on the principle of non-contradiction and note in passing a third entry. We note that the main trouble with analysing these was lack of consistency in approach, both in terms of subject-matter and notation. Many entries — which in addition are quite terse as one would expect from dictionary descriptions — have shared authorship and no care is given to unify the accounts. Moreover, the reader is thrown *in medias res* without any preliminaries regarding the notation or other conventions which are often introduced *en passant*.

We start with what was written under *Laws of Thought* (Baldwin, 1901, vol. 1), which was an entry shared by Peirce and Ladd-Franklin. They consider three laws: identity, contradiction and excluded middle showing mostly disdain for the label under which the entry is written. First comes Peirce who in his part mostly considers what the proper way of formulating the principles could be. Initially, the law of identity is given as *A is A*, the law of non-contradiction is *A is not not-A* and the law of excluded middle is stated as *everything is either A or not-A*. Peirce's concern lies in precisely defining terms, especially regarding negation and he agrees that the principle of contradiction and the principle of excluded middle jointly define 'not'. When considering their status, he dismisses the thought that these might be practical maxims as "nobody needs a maxim to remind him that a contradiction, for example, is an absurdity" (p. 641). In the course of his analysis, Peirce considers various other phrasings of the principle of non-contradiction such as: *What is at once A and not-A is nothing* and *Whatever there may be which is both A and not-A is X, no matter what term X may be*, which to a modern logician sounds more like the so-called *ex falso quodlibet* law. Peirce concludes that the formulation of the principle of contradiction as *A is not not-A* and of the principle of excluded middle as *Not not-A is A* is inadequate for defining negation (we note that these are essentially the laws of double negation in today's parlance). Interestingly, he seems to finally settle on the following versions of both principles: *Whatever is both A and not-A is nothing* and *Everything is either A or not-A* (p. 642). The use of 'nothing' and 'everything' places this approach closer to the algebraic tradition of Boole.

[10] Christopher Green in personal communication relayed to me the story behind the collaboration between Baldwin and Peirce on the dictionary.

When it comes to Ladd-Franklin's ideas, it should be noted that she essentially repeats her sentiments from an earlier paper *Some Proposed Reforms in Common Logic* (Ladd-Franklin, 1890). Interestingly in the context of our remark above regarding Peirce's affinity with Boole's approach, what comes forcefully across is her scepticism towards the phrase 'laws of thought', which is also related to Boole. She writes:

> though the doctrine that they are three [...] laws of all thought or of all reasoning has been held by a comparatively small party which hardly survives; and it is not too much to say that the doctrine is untenable. (Ladd-Franklin, 1890, p. 641)

and refers to the three principles as 'the so-called three laws of thought'. Although it should be said that in using the notions of everything and nothing Ladd-Franklin is following in Peirce's footsteps. When it comes to a detailed critique of the subject matter, she first complains about the name itself: the principle of contradiction should not be called that as a contradiction is a *relation* between two statements, such that *one is the negation of the other* and this means that (1) one or the other must be true and (2) both cannot be true (p. 643). Thus we see that two conditions are involved: (1) is usually expressed by the principle of excluded middle and (2) by the principle of contradiction. When it comes to symbolic description of the latter, Ladd-Franklin introduces notation that is a variation on the theme developed by Peirce above and proposes the following (p. 644):

$$xx' < 0.$$

Note that her phrasing echoes that of Boole with the exception that she is using implication (here denoted $<$) instead of equality. How should one understand such a statement? In her words: 'what is at once x and x' does not exist, or in the language of propositions the conjoint occurrence of x and x' does not take place'). Let us pause here for a moment, as this point is crucial yet easy to overlook. Ladd-Franklin provides *two* interpretations of all her formulas *at once* (the same was actually true of her definition of contradiction, which we chose to simplify): when x is considered a class (a set) and when it is considered a proposition. One can argue that the latter connects her to Boole's tradition, while the former to Frege's modernised logic. We will see this thread found and lost in other thinkers we analyse. It also seems like the authors we discuss here, struggle with the understanding of the special symbols, in this case

∞ and 0. The next formula, expressing the principle of excluded middle involves the second constant:

$$\infty < x + x'.$$

Ladd-Franklin reads this as 'everything is either x or x', or in the language of propositions, what can occur is either x or x', or reality entails x or x' — there is no *tertium quid*'. She states that it would be better to call (1) *exclusion* and (2) *exhaustion* adding that 'the mere fact that (1) has been called the principle of contradiction has given it a pretended superiority over the other which it by no means deserves'. Ladd-Franklin circles back to Peirce's criticism and also complains about the name *the laws of thought*, proposing to call them the *laws* (if laws at all) *of negation*. She makes the now-obvious point that all rules of logic are laws of thought in some sense. Interestingly, she says that if any of these laws do deserve distinction it is mostly the *law of transitivity* (or the *law of syllogism*). In her words: 'This is the great law of thought, and everything else is of minor importance in comparison with it'. In her discussion she also mentions the laws of double negation but *says that they follow from exclusion and exhaustion*.

By far the most important, and novel, is the approach presented in the entry on *Symbolic logic* (Baldwin, 1902, vol. 2). There is lot of common ground between what is said here and in Couturat's *L'Algèbre de la logique* (Couturat, 1905b): as we shall see soon, Couturat actually credits Ladd-Franklin with a great discussion on the two principles, which he treats the same way. In this entry, the notation changes twice (we simplify the exposition as it is not essential to our purposes) and in both cases Ladd-Franklin proposes formal treatments being variations on Peirce's work. She also — again — mentions the two possible interpretations of such formal constructions, this time they are called *conceptual* and *propositional*. Let us briefly introduce the first formalism. Here, the expression $a \to b$ can be interpreted as either that a is a subset of b or that a implies b.[11] (To simplify the exposition, from now on, we shall take the propositional interpretation.) Negation is expressed as a bar above the negated formula, so $\overline{(a \to b)}$ simply means $\neg(a \to b)$, whereas conjunction is encoded as concatenation (but sometimes also as $\times$), disjunction is represented by the $+$ sign, and equivalence $a = b$

[11] We note that the entry uses yet another symbol for implication but we chose to override it with $\to$ for reasons of simplicity.

is defined as $(a \to b)(b \to a)$. In addition, Ladd-Franklin introduces constants 1 and 0. The former is called 'the the logical everything' and is understood as '*everything* which *exists* or the universe of discourse'. The latter is called 'the logical zero' and is understood as '*nothing* or *the non-existent*' (p. 641). The use of such constants in the object-level language connects her approach to that of Boole. Ladd-Franklin is interested in the problem of constructing formulas that capture the behaviour of various connectives. In the case of negation, she states that the task is accomplished by the following two formulas:

$$a\bar{a} \to 0$$

and

$$1 \to a + \bar{a}.$$

The reader will, obviously, recognize the former as the principle of contradiction and the latter as the principle of excluded middle. Interestingly, Ladd-Franklin — consistently with her previously quoted opinions — calls these *mutual exclusion* and *conjoint exhaustion*, respectively, and uses their traditional names in scare quotes. After describing the propositional logic in such a manner, she then seamlessly moves to a description of a proof system using yet another notational convention. This time it is something akin to Peirce's existential graphs (see, e.g., Ma & Pietarinen, 2020). This is how in this notation the principle of non-contradiction is formalised (Baldwin, 1902, pp. 647–648):

$$[A(A)].$$

How to untangle this? It all becomes clear when we observe that it is the brackets that serve as negation here and conjunction is still expressed as concatenation (Baldwin, 1902, pp. 646–647). Now comes a very interesting twist. At this point, Ladd-Franklin matter-of-factly states that the principle can be formally *deduced* (Baldwin, 1902, p. 648). This is puzzling as what she is about to describe might be one of the first, if not the *first* formal proofs. No statement is wasted to describe the situation and the reader is simply presented with the derivation. The deduction takes the following shape: we start with X representing any premise[12],

[12] The article (Ma & Pietarinen, 2020) describes a modern development of this proof system. Interestingly, it also contains the same proof but there it is labeled as the proof of the principle of identity.

then by the law of double negation obtaining

$$[(X)],$$

which can be concatenated with the original X, obtaining

$$X[(X)].$$

(The modern way of expressing it is, say, $q \wedge \neg\neg q$.) A rule specific to the system allows then to introduce any formula whatsoever like so:

$$X[A(X)].$$

In modern notation this is

$$q \wedge \neg(p \wedge \neg q).$$

We can do that since in semantic terms, X (or q) 'controls' the value of the formula. However, the rules of the system allow us to add A again, obtaining:

$$X[A(AX)]$$

In modern notation this can be represented as

$$q \wedge \neg(p \wedge \neg(p \wedge q)).$$

We see now that semantically speaking this is true, as q is assumed true and whatever value p is assigned, the second conjunct is false. However, in contrast to the previous transformation, it is now the introduced term (A, or p) that controls that behaviour and q becomes inessential. In fact, our original assumption can now be left out entirely as it is allowed by the rules of the system. In the final step we therefore obtain (Baldwin, 1902, p. 648):

$$[A(A)],$$

which is the formalisation of the principle of non-contradiction we gave above (modern notation is standard: $\neg(p \wedge \neg p)$). Ladd-Franklin does not stop here but rather develops the proof system further, introducing additional rules, but this goes beyond the scope of our article (Baldwin, 1902, pp. 645–650).

Finally, let us say that Christine Ladd-Franklin touches on the notion of laws of thought in her entry on *Syllogism* (Baldwin, 1902, vol. 2) but although she also speaks critically of their role as a category, it is in a different vein that interests us here.

4. Russell's Formal Treatment

Russell introduces a sort of formal approach to propositional calculus in his *The Principles of Mathematics* from 1903 (Russell, 1903), and after listing 9 axioms states that from these

> we can prove the law of contradiction; we can prove [. . .] that p implies not-not-p. (Russell, 1903, p. 17)

Although he neither provides formal proofs of these results nor gives a strict formulation of the principle of contradiction. However, this fragment shows at once two things: first that around 1903 the principle of contradiction was already considered not one of *the first* principles but a statement that follows from others, and second that Russell does distinguish between the principles of non-contradiction and double negation (which is not that clear in Peirce).[13] Confusingly, Russell also states the following after introducing some more machinery: "From this point we can prove the laws of contradiction and excluded middle and double negation" (Russell, 1903, p. 18). It is anybody's guess what 'the laws of contradiction' are here and whether the previously stated law of contradiction is somehow distinct from these. If these are somehow related, it is puzzling why the proof is brought up again at this point, however it confirms that both 'fundamental' laws of thought are considered secondary and different to the laws of double negation. It seems that, contrary to Ladd-Franklin, Russell does not view these as following from the principle of contradiction and the principle of excluded middle but due to the imprecise nature of such deliberations, it is hard to say what it all means formally. Russell's 1906 "The Theory of Implication" (Russell, 1906) contains perhaps the first formal proof of the principle of contradiction outside of what can be found in the *Dictionary.* This is how it is done there, with the principle of contradiction understood as "nothing is both true and not true", and with the notation updated to modern standards. Assume one of the laws of double negation

$$p \to \neg\neg p$$

and substitute in it for p the entire formula itself, obtaining

$$(p \to \neg\neg p) \to \neg\neg(p \to \neg\neg p),$$

[13] I owe this discovery to the discussion among the members of The Bertrand Russell Society, especially Moises Macias Bustos and Gregory Landini.

then by the definition of $p \wedge q$ as $\neg(p \to \neg q)$ we can replace $\neg\neg(p \to \neg\neg p)$ by $\neg(p \wedge \neg p)$ obtaining finally

$$(p \to \neg\neg p) \to \neg(p \wedge \neg p),$$

modus ponens now gives

$$\neg(p \wedge \neg p),$$

the principle of contradiction (Russell, 1906, p. 177). We note that Russell's approach is markedly influenced here by Frege's analyses (see our remarks at the end of the article).

Interestingly, the proof of the principle of contradiction contained in *Principia Mathematica* is essentially different to that presented by Russell in "The Theory of Implication". Partially it has to do with a different choice of primitives. This time assume $p \vee \neg p$, the principle of excluded middle, and substitute $\neg p$ for p in that formula obtaining

$$\neg p \vee \neg\neg p$$

by one of the de Morgan laws we get

$$\neg(p \wedge \neg p).$$

This, in turn, connects Russell's proof to one of the solutions proposed around the same time by Jan Łukasiewicz. We turn to Łukasiewicz after briefly discussing Couturat, who infuenced him greatly.

5. Couturat's Influential Recap

Louis Couturat is perhaps the best known French logician of the early period of the development of modern logic. In his *Les Principes des Mathématiques* from 1905 he mentions (?) 13[]cout1 the two principles — contradiction and excluded middle — as defining negation and we saw that already with Peirce and Ladd-Franklin. However, Couturat also claims that "They are [...] independent of each other, and each independent of the principle of identity." This is of course in stark contrast with Russell's proof of the principle of contradiction that assumes the principle of excluded middle but his proof was published later.[14] In

[14] It is also quite easy to show that the principle of identity can be deduced from the principle of contradiction.

this book, Couturat is not interested in these principles as such so he moves on to other topics, however, the same year saw a publication of yet another book that he authored: *L'Algebre de la Logique*.[15] It reached a much broader audience and was arguably more influential than *Les Principes des Mathématiques*. It is also more important in the story of the principle of contradiction. This is for two reasons. The first is that despite it repeating essentially the same sentiment regarding the role of the principle of contradiction and the principle of excluded middle as defining negation (Couturat, 1905b, pp. 22–24), this time Couturat however, explicitly credits Ladd-Franklin's work in that regard. Let us quote the English translation of the footnote where this happens:

> As Mrs. Ladd-Franklin has truly remarked [in Baldwin's *Dictionary*], the principle of contradiction is not sufficient to define contradictories; the principle of excluded middle must be added which equally deserves the name of principle of contradiction. This is why Mrs. Ladd-Franklin proposes to call them respectively the principle of exclusion and the principle of exhaustion, inasmuch as, according to the first, two contradictory terms are exclusive (the one of the other); and, according to the second, they are exhaustive (of the universe of discourse).[16]
>
> (Couturat, 1914, p. 23)

We mentioned that Couturat's ideas were influential in shaping the views of Jan Łukasiewicz, a philosopher/logician who is widely credited for his critique of the principle of contradiction. We see now that Couturat can be thought of as a bridge between the European and the American developments in logic. In our final section, we turn to Łukasiewicz to see what he made of the principle.

6. Łukasiewicz's Philosophical Analysis

Łukasiewicz's main contribution to the critique of the principle of contradiction is contained in his 1910 book translated into English as *On the Principle of Contradiction in Aristotle*.[17] The publisher of the book was

[15] See (Couturat, 1905b) translated into English as (Couturat, 1914).

[16] Incidentally, Ladd-Franklin is also mentioned in another place of that book but in a different context, that related to her antilogism.

[17] For a number of years, there was no official English translation of this source and only one article summarizing its contents was in circulation. However, we have uncovered a manuscript of the translation of this book into English by Łukasiewicz

The Academy of Learning in Kraków, (then, Austro-Hungarian Empire). Recent archival work at the The Polish Academy of Learning (as it became to be known in the wake of The Second Polish Republic) in Kraków revealed a broader background of Łukasiewicz's efforts to obtain funds for his research on the book. On 28 July 1908 Łukasiewicz submitted an application form for funds from The Osławski Foundation (a benefactor of The Academy of Learning). In the attached *curriculum vitae* dating from 1907 and describing the previous year, he writes:

> And while algebraic logic[18] does contain a large proportion of formalist elements, often of little scientific value, it has, nevertheless, touched on a number of issues that are worthy of attention. This includes, most prominently, the questions of the *foundations of logic*, that is the principles of identity, contradiction and excluded middle. It is known that the logic of Aristotle is based on these; but it is also known that throughout the entire history of philosophy, from its Greek origin to Hegel and to more contemporary developments, there are traces of some other "non-Aristotelian" logic that negates some of these principles. (Łukasiewicz, 1908)

We see that Łukasiewicz already had the idea of non-Aristotelian logics in mind, even though at the time he was only "hoping to publish a book" (the stipend was mostly for his travels) and, curiously, it seems that the theme of the principle of contradiction is not yet at the fore of his efforts. However, the application also contains an addendum dating 24 June 1908, he had already a manuscript of his book — now with the principle of contradiction as a centrepiece — ready for publication.[19]

Le us now briefly state the main results from *On the Principle of Contradiction in Aristotle*, focusing on formalisation and proof. Łukasiewicz argues that there are three distinct versions of the principle to be found in Aristotle: (1) Ontological — No object may at the same time possess

himself from the 1950s. The manuscript (with emendations) is now being prepared for publication. As it turned out, Owen Le Blanc, a researcher from Manchester, knew about Łukasiewicz's translation but in efforts to improve on it, produced his own partial translation somewhere around the late twentieth century (unpublished). Recently, a full translation by Holger Heine appeared (Łukasiewicz, 2021).

[18] This term then simply meant logic in the tradition of Boole.

[19] A research on the background of the publication of this book taking into account a number of other recently uncovered archival sources is under way. It remains to be ascertained to what extent Łukasiewicz's research travels contributed to the final shape of his 1910 opus on the principle of contradiction.

and not possess the same property, (2) Logical — Contradictory statements are not simultaneously true, and (3) Psychological — No one can simultaneously believe contradictory things. He claims that the principle cannot be deduced either from the principle of double negation or from the principle of identity. However, in an interesting twist, the book's additional chapter, titled *The Appendix: The Principle of Contradiction and Symbolic Logic* provides claims to the contrary, but with an interesting caveat. *The Appendix* is devoted to an exposition of logic following Couturat 1905,[20] with the goal of showing that the logical version of the principle of contradiction is not a fundamental principle, but instead is deduced from a number of other logical principles.[21] We read there:

> [...] against the arguments of Aristotle and the commonly accepted opinion — the principle of contradiction is not an ultimate and non-provable law. Symbolic logic proves, however, something more: a large number of logical laws have no relation with the principle of contradiction, hence not only is the principle not an ultimate, it is also not a necessary basis for logical thinking. (Łukasiewicz, 2020)

As we saw, Couturat is not shy about mentioning the influence of Christine Ladd-Franklin in relation to the very topic of Łukasiewicz's study, that is the principle of contradiction. However, Łukasiewicz himself never mentions this name. It is a curious omission and, at the very least, a serious academic oversight on his part. We know from elsewhere that Baldwin's Dictionary was a source accessible to those working on logic in Kraków.[22] Whatever the reasons, it is Christine Ladd-Franklin who should be considered first to critique the principle of contradiction and to show that it is provable. Łukasiewicz's importance is mostly about pushing the non-Aristotelian agenda.

[20] Łukasiewicz was an avid reader of Couturat (and even helped translate his *L'algebra de la Logique* into Polish (see Couturat, 1918).

[21] We note that this claim is not as revolutionary as it might seem. For example, Edwin Bildwel Wilson in his 1908 review of Couturat's book already states that authors can choose their undefined symbols and postulates with considerable arbitrariness and thus "what is a theorem or definition for one author may be an an undefined symbol or postulate for another". See (Wilson, 1908, p. 179). For a discussion of *The Appendix* see (Trybus & Linsky, 2020) and for an English text of this source, see (Łukasiewicz, 2020).

[22] The Jagiellonian Unviersity Archives in Kraków contain a doctoral dissertation of another well-known Polish logician, Leon Chwistek (1906). In this source, dating from 1906, Chwistek mentions the Baldwin's dictionary. The English translation of this dissertation is being prepared for publication.

Łukasiewicz seemingly has the same starting point, namely the formulation of the principle inherited from Couturat (via Ladd-Franklin?):

$$aa' < 0.$$ [23]

We should keep in mind that Łukasiewicz is not talking about propositional logic here (contrary to Couturat and Ladd-Franklin who — as we saw — allow for two interpretations of the formulas). He stays squarely within traditional term logic and on top of that interprets 1 and 0 in a very peculiar manner. The lower-case Latin letters refer to properties but these seem to be relativised to some given object, e.g. a would mean 'P has the property a'. That the nature of P is not specified seems intentional, as P might or might not be an object (and part of what Łukasiewicz says depends heavily on the definition of 'object'.) Now, 1 is defined as 'P is an object, or P is something' and 0 as 'P is not an object, or P is nothing'. The reader also soon discovers, however, that Łukasiewicz does not prove this version of the principle of contradiction. Instead, he first proposes his own interpretation of what Aristotle meant:

> As I mentioned above, none of the principles of symbolic logic presents the principle of contradiction as formulated by Aristotle. This formulation is as follows: "The same cannot both be assigned and not be assigned to the same and in the same way", or "no object can both have and not have the same feature". I assume that these formulations are equivalent with the following conditional: "If P is an object, it cannot both have a and not have a. (Łukasiewicz, 2020)

As expected, this leads him to the following formulation:

$$1 < (aa')',$$

which he takes to truly represent the principle. Now, the endeavour of proving it can begin in earnest. Łukasiewicz proposes two ways of doing so:

> As $[1 < (aa')']$ contains a negation of a product when trying to reduce it to already-known principles or proved laws, one has to make use of either the law of contraposition or a De Morgan formula.
> (Łukasiewicz, 2020)

[23] In modern terms, this is simply rendered as negating whatever implies falsehood, giving us the modern propositional $\neg(p \wedge \neg p)$.

Interestingly, Łukasiewicz also makes use of some heavy-weight principles of logic that are not mentioned in this short introduction to what he is about to prove. Thus, in the first proof he infers the new formulation of the principle by also using the law of double negation and the old (critiqued and rejected as such) principle of contradiction: he reformulates his new formulation as

$$[1 < (aa')'] = (aa' < 0)$$

and then applies *modus ponens* to

$$(aa' < 0) < [1 < (aa')'].$$ [24]

The second derivation Łukasiewicz makes use the principle of excluded middle (and it seems that he has no issue with *that* formulation). He first, again, transforms his new formulation using de Morgan's law:

$$[1 < (aa')'] = (1 < a + a')$$

and then, by *modus ponens* obtains the result from

$$(1 < a + a') < [1 < (aa')'].$$ [25]

Łukasiewicz then goes to great lengths to dismiss the claim that both formulations of the principle of contradiction actually amount to the same thing. And this requires some creativity: in order to achieve this Łukasiewicz rejects the synonymity of "What is a and a' at the same time, is not 1 [hence is 0]" and "What is 1, is not at the same time a and a' [hence is $(aa')'$]". Interestingly, in order to show this, Łukasiewicz seems to lead toward what is now called *paraconsistent* logic as he considers the case where *ex contradictio quodlibet* is not an accepted principle. Here, he ventures into the realm of contradictory objects (no doubt influenced by Meinong). This is very important to him: Łukasiewicz does not reject the principle of contradiction, he limits its scope to non-contradictory objects (Trybus & Linsky, 2020). The question remains open whether what he said should be considered as endorsing a logic where this principle does not hold but he was no doubt one of the first (if not the first!) to start the conversation.

[24] Note that this would not do as a proof in the modern sense.

[25] In both 'proofs' Łukasiewicz also uses the fact that equivalence is implication in both ways, using today's terminology.

7. Conclusions

The following table summarises efforts presented in this article. It is ordered by the date of publication, author's initials and contains information about the formulation of the principle using primitive terms of the system, whether a proof was attempted and on which other formulas it depended. We note that in Frege, the formula is not explicitly present and that in Russell the final forms made use of defined connectives.

Year	**Initials**	**PonC**	**Proof**	**Dependencies**
1854	GB	$(x)(x-1)=0$	+	-
1879	GF	$\neg\neg(p \to \neg\neg p)$	-	-
1901	CLF	a. $xx' < 0$	-	-
1901	CLF	b. $a\bar{a} \to 0$	-	-
1903	CLF	c. $[A(A)]$	+	DN
1905	LC	$aa' = 0$	-	-
1906	BR	a. $\neg\neg(p \to \neg\neg p)$	+	DN,
1910	BR	b. $\neg p \vee \neg\neg p$	+	EM, DM
1910	JŁ	a. $aa' < 0$	-	-
1910	JŁ	b. $1 < (aa')'$	+	(JŁa, CP); (DM, EM)

Only the most notable dependencies were chosen. The dependencies should be understood as follows: DN: double negation, EM: excluded middle, DM: de Morgan, CP: contraposition, JŁa: the first formulation of the principle of contradiction in Łukasiewicz. Also, in case of Russell, we opted for providing the formulations of the principle of contradiction using primitive terms.

Today, of course, all the various formulations of the principle of non-contradiction are standardised into a single statement. In propositional logic we have $\neg(p \wedge \neg p)$, which represents the original sentiment that p and p' imply falsity. Initially we seemed to have two notions of negation (no doubt inherited from the original treatment, where p could also be considered a class). In later works, Russell, realising perhaps that in propositional calculus this distinction does not make sense, in every case of negation talks about implying falsity.[26] Similar remarks can be applied to Łukasiewicz's contorted attempt at rephrasing the principle or his analyses of equivalence and similarity. All this simply does not seem appealing today in the context of the formal semantics for propositional (or first-order) logic. A question arises whether there are some other

[26] Thanks to Gregory Landini for pointing that out.

attempts at a critique or proof of the principle of contradiction that are overlooked in this survey. The impact of Frege's 1879 *Begriffsschrift* (English translation (Frege, 1879)) on the development of logic cannot be overstated. It predates any of the sources analysed here (with the exception of Boole) and is the first formal treatment of propositional logic, making it a natural candidate. However, Frege's system is based on implication and negation as primitives, and the conjunction $p \wedge q$ is defined (Frege, 1879, p. 19) as $\neg(p \to \neg q)$, and while Russell had essentially the same definition, Frege's system lacked 1 and 0 as constants. There seems to be enough of the interference in what Russell wrote between Frege's line of thought and that represented by the algebraic approach so that Russell is able to talk about the principle of contradiction, whereas in Frege's system, it becomes $\neg\neg(p \to \neg\neg p)$, which — again — strikes one as conceptually closer to the law of identity (it is enough to apply the laws of double negation, which are axioms in Frege's system). So although Frege does talk about the proof of the principle of identity, and we saw that there is a thin line in some authors between the two, a *bona fide* proof of the principle of contradiction cannot be found in *Begriffsschrift*.[27] We hope to have convinced the reader that what Christine Ladd-Franklin presented in her entries in the *Dictionary* should be added to the growing history of critique of the principle of contradiction.

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[27] Bernie Linsky pointed out that $\neg\neg(p \to \neg\neg p)$ is provable in Frege's system from the law of identity and the laws of double negation, so technically this version of the principle of contradiction can be proved although it was not done so by Frege himself. We also note that Russell's approach used only one of the laws of double negation. Christian Thiel suggested that perhaps the discussion of the formalisation of the principle of contradiction in Frege's system should be moved to the meta-level.

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