Logic and Logical Philosophy
Volume 34 (2025), 321–364
DOI: 10.12775/LLP.2025.001

Gilberto Gomes, Claudio Pizzi,
Eric Raidl

Consequential Implication and the Implicative Conditional

Abstract. This paper compares two logical conditionals which are strengthenings of the strict conditional and avoid the paradoxes of strict implication. The logics of both may be viewed as extensions of KT, and the two conditionals are interdefinable in KT. The implicative conditional requires that its antecedent and consequent be both contingent. The consequential conditional may be viewed as a weakening of the implicative conditional, insofar as it also admits the case in which the antecedent and the consequent are strictly equivalent (either both necessary or both impossible). The two conditionals share a number of properties, among them Transitivity, Contraposition, Aristotle's Thesis, Weak Boethius' Thesis and Aristotle's Second Thesis. They also share some restricted principles such as Possibilistic Monotonicity, Possibilistic Simplification and Possibilistic Right Weakening. They differ in relation to Identity, which is validated by consequential implication, while the implicative conditional only validates the restricted principle of Possibilistic Identity. The relations between the two conditionals are represented by two Aristotelian cubes of opposition, one involving the contrariety between *If A, then B* and *If A, then* $\neg B$, according to Weak Boethius' Thesis, and the other the contrariety between *If A, then B* and *If* $\neg A$*, then B*, according to Aristotle's Second Thesis. We also explore the relations between the two logical conditionals and natural language conditionals, emphasizing the dependence of the latter on the context, and the need to distinguish natural language conditionals which may be viewed as consequential or implicative, on one side, and concessive and some other types of conditionals, on the other.

Keywords: consequential implication; implicative conditional; Aristotle's thesis; Boethius' thesis; contraposition; connexivity; definable conditionals; strong super-strict implication; square of oppositions

Received January 21, 2024. *Accepted* November 28, 2024. *Published online* January 10, 2025

1. Introduction

When we say (or think) *If A, then B*, we are talking (or thinking) of a relation between A and B that different theories define in different ways. According to the theory that the material conditional $A \supset B$ provides a correct representation of *If A, then B*, a sentence of this form means that $A \wedge \neg B$ is false. This theory has the merit of simplicity, but there is wide agreement that *If A, then B* usually makes a stronger assertion. According to the theory that *If A, then B* is adequately represented by the strict conditional $\Box(A \supset B)$, the conjunction $A \wedge \neg B$ must be not only false, but also impossible. In C. I. Lewis' original version of this theory, only the *logical* impossibility of $A \wedge \neg B$ would make a strict conditional true. Later versions of the strict conditional account of natural language conditionals, however, also refer to other kinds of impossibility, such as physical or contextual impossibility. Regardless of the kind of impossibility involved, a problem with the strict conditional analysis is that $A \wedge \neg B$ may be impossible just because A is impossible, or just because B is necessary, even if there is no intelligible connection between A and B. Such anomalies are known as the *paradoxes of strict implication*.

Consequential implication and the implicative conditional are two different but related ways of strengthening strict implication which avoid the mentioned paradoxes. Context-independent (also known as *analytic*) consequential implication (symbolised by $\rightarrow$) is defined as:

def CI $\quad A \rightarrow B \;:=\; \Box(A \supset B) \wedge (\Box B \supset \Box A) \wedge (\Diamond B \supset \Diamond A)$

Any conditional whose antecedent and consequent satisfy the relation defined in *def CI* will here be called a *consequential conditional*.[1]

The first conjunct in *def CI* is the strict conditional stating that $A \wedge \neg B$ is impossible. Given that in every normal modal logic $\Box(A \supset B)$ implies $(\Box A \supset \Box B) \wedge (\Diamond A \supset \Diamond B)$, the additional conjuncts in *def CI* ensure that A and B have the same modal status,[2] i.e. $(\Box A \equiv \Box B) \wedge (\Diamond A \equiv \Diamond B)$. This is called the *equimodality condition* and means that when the strict conditional is true, the consequential connection between

[1] The weaker conditional $A \rightarrow^{\circ} B \;:=\; \Box(A \supset B) \wedge (\Diamond B \supset \Diamond A)$ will not be considered here (Pizzi, 2018). Context-dependent (also known as *synthetic*) consequential implication $\twoheadrightarrow$ is discussed in Section 9.

[2] By modal status we mean here the four modalities of the modal square of opposition: necessity, impossibility, non-necessity, possibility.

A and B is lacking only when one of them is necessarily true and the other possibly false, or when one of them is necessarily false and the other possibly true.

The implicative conditional (symbolised as $\Rightarrow$) is defined as:

$$def IC \qquad A \Rightarrow B \ := \ \Box(A \supset B) \wedge \Diamond A \wedge \Diamond \neg B$$

Just as is the case for the consequential conditional, the first conjunct of *defIC* is the strict conditional, which can also be expressed as $\neg \Diamond(A \wedge \neg B)$. The latter expression highlights the relation between the first and the two additional conjuncts. When an implicative conditional $A \Rightarrow B$ holds, although the conjunction $A \wedge \neg B$ is impossible, A and $\neg B$ themselves are possible. It is just the directional connection between A and B that makes the conjunction $A \wedge \neg B$ impossible. In fact, *defIC* entails that an implicative conditional is true only when A and B are both contingent (see Appendix A). When A and B are both necessary, or both impossible, a consequential conditional always holds, but an implicative conditional never does.

The theories of the consequential and the implicative conditional do not imply, however, that a sentence with the linguistic form *If A, B* cannot occur in natural language or thought when, for instance, A is impossible and B is possible. What each theory will say in this case is that such a sentence does not express a consequential or an implicative conditional, respectively.

Two examples may be useful here:

(1) If it rains tomorrow, the picnic will be cancelled.

Assume that in the situation in which this sentence is uttered it is considered impossible to have the picnic in the rain. In this context, (1) comes out true on all four accounts considered above. This is the case for the material conditional, because $A \wedge \neg B$ is false; for the strict conditional, because $A \wedge \neg B$ is impossible; and for the consequential and the implicative conditionals, because in addition to this, both events are contingent.

(2) If London is the capital of France, Martha lives in Buenos Aires.

This sentence seems *prima facie* quite absurd. It might perhaps be viewed as a bizarre way of saying that Martha does not live in Buenos Aires. According to the material conditional account, however, (2) is true, and it is the *negation* of (2) that would imply that Martha does not live in Buenos Aires. According to the strict conditional, if we consider it impossible that London is the capital of France in the context of what we know of the present political and geographical reality, this is sufficient to make (2) true, which seems inadequate. By contrast, in the same context, the sentence is false according to the implicative conditional (because the antecedent is considered impossible) and it is also false according to the consequential conditional (given that the antecedent and the consequent have a different modal status, since the former is considered impossible and the latter possible). The application of the latter two conditionals thus seems to be adequate for this example.

Both conditionals ensure a connection between the antecedent and the consequent and both give rise to a weakly connexive logic[3] within a simple framework of normal modal logic. That is, they are non-symmetrical, and validate *Aristotle's thesis*:

(AT) $\neg(A \rightarrowtail \neg A)$

and *Weak Boethius thesis*:

(wBT) $(A \rightarrowtail B) \supset \neg(A \rightarrowtail \neg B)$

but invalidate *Strong Boethius' Thesis*:

(sBT) $(A \rightarrowtail B) \rightarrowtail \neg(A \rightarrowtail \neg B)$

This stands in contrast to standard approaches to connexive logic, which validate sBT and are mainly axiomatically motivated, usually associated to many-valued and less intuitive semantics (if a semantics is known at all), and cannot in general be modally interpreted. Thus, the two logics presented here may be viewed as attractive alternatives to standard approaches to connexivity. This motivates a detailed comparison between the two conditionals, and a study of their relation to other conditionals.

Throughout the paper, we use $\neg, \wedge, \vee, \supset$ and $\equiv$ for classical negation, conjunction, disjunction, the material conditional, and material equivalence, respectively. We use $:=$ as equality by definition. A, B, C, ... denote well formed formulas, $\top$ is any classical tautology, and $\bot := \neg\top$. $\Box$ is necessity, sometimes defined by the conditionals $\rightarrow$ or $\Rightarrow$, and $\Diamond := \neg\Box\neg$ is possibility. As a generic conditional symbol, we use $\rightarrowtail$. We model both conditionals in reflexive Kripke semantics (with

[3] According to the terminology of (Wansing and Omori, 2024, §4.1).

the modal logic KT[4]). Studying both conditionals in weaker semantics is outside the scope of this paper, and not very appealing from our perspective, as we would loose connexive principles, such as wBT, and the interdefinability of the two conditionals.

We start by introducing the consequential conditional (Section 2) and the implicative conditional (Section 3). We then show that they are interdefinable (Section 4), and that their logics can be axiomatised as two different extensions of a common core (Section 5). We then discuss similarities and differences in their logical principles (Sections 6 and 7). We further investigate how they are related through Aristotelian cubes of opposition (Section 8). Finally, we apply our two conditionals to the analysis of conditional sentences from natural language (Section 9). All proofs are given in Appendices A and B.

2. The Consequential Conditional

The consequential conditional was introduced by C. Pizzi in (1991) and investigated in a series of papers (see, e.g., Pizzi, 1991; Pizzi and Williamson, 1997; Pizzi, 2018). It was originally proposed as a weakened form of a connexive conditional. While the most prominent characteristic theses of a connexive conditional are Aristotle's Thesis (AT) and Strong Boethius' Thesis (sBT), the consequential conditional also validates AT, but validates Boethius Thesis only in its weak form (wBT).

Three alternative definitions of interest are possible, which we prove to be equivalent to *def CI* (Proposition 1, p. 351):

$$def\,IC' \quad A \to B \;:= \;\Box(A \supset B) \wedge ((\nabla A \wedge \nabla B) \vee (\Box A \wedge \Box B) \vee (\neg \Diamond A \wedge \neg \Diamond B))$$

where the nabla symbol ∇ stands for *contingency*, i.e. $\nabla A \;:=\; \Diamond A \wedge \Diamond \neg A$. *def CI′* makes it clear that $A \to B$ expresses that A strictly implies B, and A and B are either the two contingent, or both necessary, or both impossible.

$$def\,IC'' \quad A \to B \;:= \;(\Box(A \supset B) \wedge \Diamond A \wedge \Diamond \neg B) \vee (\Box A \wedge \Box B) \vee (\neg \Diamond A \wedge \neg \Diamond B)$$

[4] KT is the propositional calculus (all propositional tautologies and Modus Ponens for $\supset$), extended by the modal axioms: K $\Box(A \supset B) \supset (\Box A \supset \Box B)$ and T $\Box A \supset A$, and the rule of Necessitation N: If $\vdash A$ then $\vdash \Box A$.

The first disjunct in *def CI″* is the implicative conditional. Thus, the consequential conditional $A \rightarrow B$ is either an implicative conditional, or reduces to A and B being both necessary or both impossible. The implicative conditional is again present in the following alternative definition:

$$def\, IC''' \qquad A \rightarrow B \;:=\; (\Box(A \supset B) \wedge \Diamond A \wedge \Diamond \neg B) \vee \Box(A \equiv B)$$

Thus, the consequential conditional is either an implicative conditional or a strict equivalence. (In the latter case, $A \rightarrow B$ implies its inverse $\neg A \rightarrow \neg B$ and its converse $B \rightarrow A$.) The close relationship between $\rightarrow$ and $\Rightarrow$, expressed in *def CI″* and *def CI‴*, will be further explored in Section 4.

It is proved in (Pizzi and Williamson, 1997) that every system of the consequential conditional may be translated into a system of normal modal logic and vice versa, thanks to the definition of necessity in terms of the consequential conditional provided by:

$$def\Box_{\rightarrow} \qquad \Box A \;:=\; \top \rightarrow A$$

In his chapter *A History of Connexivity*, Storrs McCall (2012, p. 443) writes that the definition of the consequential conditional 'has the immense advantage of suggesting ordinary modal logic — the Lewis systems S1–S5, plus the system [K]T — as a conceptual and formal basis for connexive semantics'. The possibility of a modal translation has a strong impact on the methods used to obtain completeness and decidability results for some special classes of connexive systems, as for example McCall's CFL with a conditional definable as $\Box(A \supset B) \wedge (A \equiv B)$, which also lacks strong Boethius' Thesis.[5] These remarks apply similarly to the systems of the consequential and the implicative conditional studied here.[6]

3. The Implicative Conditional

The implicative conditional as a formal representation of natural language conditionals of a defined type was proposed by Gomes (2005; 2013; 2020) and its logic was axiomatised and investigated by Raidl and Gomes

[5] See Anderson and Belnap (1975, pp. 441–452) and Meyer (1977).

[6] A general theory to derive completeness for definable conditionals and other conditional constructions was developed in (Raidl, 2021a).

(2024).[7] Like the consequential conditional, the implicative conditional validates AT and wBT but does not validate sBT.

Two alternative definitions of interest are possible which we prove to be equivalent to *defIC* (Proposition 2, p. 352). If we want to think in terms of the possibility or impossibility of conjunctions involving the affirmation or negation of A and of B, it can be equivalently defined as:

$$def IC' \qquad A \Rightarrow B \;:=\; \Box(A \supset B) \wedge \Diamond(A \wedge B) \wedge \Diamond(\neg A \wedge \neg B)$$

Since $\Box(A \supset B)$ is equivalent to $\neg\Diamond(A \wedge \neg B)$, only the conjunction $\neg A \wedge B$ is left out of this definition. If the conditional *if A, then B* is asymmetrical ($A \Rightarrow B$ and $\neg(B \Rightarrow A)$), the conjunction $\neg A \wedge B$ is also possible. If, however, the implicative biconditional $A \Leftrightarrow B$ holds, the conjunction $\neg A \wedge B$ is impossible, as required by the strict conditional $\Box(B \supset A)$.

The implicative conditional requires that A and B both be contingent and thus a third equivalent definition is:

$$def IC'' \qquad A \Rightarrow B \;:=\; \Box(A \supset B) \wedge \nabla A \wedge \nabla B$$

Taking $\Rightarrow$ as primitive, one can show (Raidl and Gomes, 2024, Fact 2) that $\Box$ is definable in terms of $\Rightarrow$ (in reflexive Kripke models):

$$def\Box_{\Rightarrow} \qquad \Box A \;:= A \wedge \neg(A \Rightarrow A)$$

The negation of $A \Rightarrow A$ in this definition (which may seem surprising) derives from the fact that the implicative identity conditional $A \Rightarrow A$ is equivalent to A being contingent. Thus, $\neg(A \Rightarrow A)$ expresses that A is not contingent, while A excludes the impossibility of A, defining in this way the necessity of A.

Before Gomes' independent proposal, $\Rightarrow$ had already been studied by two different authors. Burks (1955) called it *non-paradoxical causal implication*,[8] and restricted it to the analysis of sentences of the form *If A, then B* which represent a situation in which A causes B. However, he faced the problem that this conditional can also be used in cases in which A represents an effect of B and B a necessary cause of A, and we obviously cannot say in relation to such cases that A *causes* its cause B. He was also troubled by the apparent failure of contraposition in causal

[7] Gherardi and Orlandelli (2021) studied $\Rightarrow$ under the name of *strong super-strict implication*.

[8] *Non-paradoxical* because it does not involve the paradoxes of strict implication.

conditionals, since the contrapositive often seems to invert the temporal relation between A and B, making $\neg B$ appear to precede $\neg A$. Thus, he considered $\Rightarrow$ only as an approximation to the concept of cause (Burks, 1955, p. 178).

Gomes, by contrast, applies $\Rightarrow$ to a wider range of uses of natural language conditionals, explains cases in which contraposition seems to invert the temporal order (Gomes, 2019), and has no difficulty with cases in which A represents an effect, rather than a cause, of B (Gomes, 2024).

Priest (1999) also investigated a semantics for $\Rightarrow$ as expressing the concept of negation as cancellation, which is compatible with connexivist principles. However, he rejects both the concept of negation as cancellation and connexivist principles, which means that he neither endorses nor explores $\Rightarrow$ as an analysis of natural language conditionals or of the notions of consequence, implication or inference.

4. Interdefinability of the two Conditionals

The consequential conditional and the implicative conditional are interdefinable. We prove this (see Appendix A, Propositions 3 and 4), using the normal modal system KT, as background system, and consider two definitional extensions: $\mathsf{KT}+\mathit{def\,CI}$ and $\mathsf{KT}+\mathit{def\,IC}$.[9]

In $\mathsf{KT}+\mathit{def\,IC}$, the consequential conditional $\rightarrow$ can be defined in terms of the implicative conditional $\Rightarrow$ (compare $\mathit{def\,CI''}$ and $\mathit{def\,CI'''}$):

(def1) $A \rightarrow B \;:=\; (A \Rightarrow B) \vee (\neg\Diamond A \wedge \neg\Diamond B) \vee (\Box A \wedge \Box B)$

(def2) $A \rightarrow B \;:=\; (A \Rightarrow B) \vee \Box(A \equiv B)$

Conversely, in $\mathsf{KT}+\mathit{def\,CI}$, the implicative conditional $\Rightarrow$ can be defined in terms of the consequential conditional $\rightarrow$ in the following equivalent ways:

(def3) $A \Rightarrow B \;:=\; (A \rightarrow B) \wedge \neg(\neg\Diamond A \wedge \neg\Diamond B) \wedge \neg(\Box A \wedge \Box B)$

(def4) $A \Rightarrow B \;:=\; (A \rightarrow B) \wedge ((\Diamond A \wedge \Diamond\neg A) \vee (\Diamond A \wedge \Diamond\neg B) \vee (\Diamond B \wedge \Diamond\neg A) \vee (\Diamond B \wedge \Diamond\neg B))$

Since each of the disjuncts in (def4) in conjunction with $A \rightarrow B$ entails each of the others, the *definiens* may be simplified as using only the first one:

(def5) $A \Rightarrow B \;:=\; (A \rightarrow B) \wedge \nabla A$

[9] Semantically this means that we here consider $\rightarrow$ and $\Rightarrow$ to be defined operators in the modal system KT arising from reflexive Kripke models.

These definitions show that $\Rightarrow$ is stronger than $\rightarrow$. Furthermore, since necessity may be defined in terms of either $\rightarrow$ or $\Rightarrow$, both $\rightarrow$ and $\Rightarrow$ may also be defined just in terms of each other (without $\Box$, or $\Diamond$).

5. Axiomatisations for CI and IC

Axiomatisations for the logics arising from $\rightarrow$ and $\Rightarrow$ interpreted in reflexive Kripke models have been developed by Pizzi (1991) and Raidl and Gomes (2024), respectively: $\rightarrow$ is axiomatised by the system $\mathsf{CI.0}$, and $\Rightarrow$ is axiomatised by the system IC or equivalents $\mathsf{IC.1}$–$\mathsf{IC.11}$.

Here we provide new axiomatisations for both conditionals, built on a common ground system I.

DEFINITION 1. System I has as rules:

Left Logical Equivalence
If $\vdash A \equiv B$, then $\vdash (A \rightarrowtail C) \supset (B \rightarrowtail C)$ LLE

Modus Ponens for the Material Conditional
$A \supset B, A \vdash B$ MP

As axioms (schemata), it has all instances of propositional tautologies (PT) and:

Aristotle's Thesis
$\neg(A \rightarrowtail \neg A)$ AT

Contraposition
$(A \rightarrowtail B) \supset (\neg B \rightarrowtail \neg A)$ C

Transitivity
$((A \rightarrowtail B) \wedge (B \rightarrowtail C)) \supset (A \rightarrowtail C)$ TR

Conditional-to-Material Implication
$(A \rightarrowtail B) \supset (A \supset B)$ MI

Possibilistic Factor Law
$((A \rightarrowtail B) \wedge \Diamond(A \wedge C)) \supset ((A \wedge C) \rightarrowtail (B \wedge C))$ PFL

Implicative Law
$(\Diamond A \wedge \Diamond \neg B \wedge \Box(A \supset B)) \supset (A \rightarrowtail B)$ IL

Consider the following additional axioms:

Identity
$A \rightarrowtail A$ ID

Isolation of Contradictions
$(\bot \rightarrowtail A) \equiv (A \rightarrowtail \bot)$ IOC

Possible Antecedent and Consequent
$(A \rightarrowtail C) \supset \Diamond(A \wedge C)$ PAC
No Explosion
$\neg(\bot \rightarrowtail A)$ NE

We define two different extensions of I:

DEFINITION 2.
$\mathsf{CI.0} := \mathsf{I} + \text{ID} + \text{IOC}$.
In $\mathsf{CI.0}$ we write the conditional as $\rightarrow$, and define $\Box A := \top \rightarrow A$.[10]
$\mathsf{IC.0} := \mathsf{I} + \text{PAC} + \text{NE}$.[11]
In $\mathsf{IC.0}$ we write the conditional as $\Rightarrow$, and define $\Box A := A \wedge \neg(A \Rightarrow A)$.[12]

$\mathsf{CI.0}$ is our proposed axiomatisation for the consequential conditional, and only differs slightly from the initial proposal by Pizzi (1991). $\mathsf{IC.0}$ is our proposed axiomatisation for the implicative conditional, and differs essentially from the equivalent ones offered by Raidl and Gomes (2024). By C, every rule or law has an equivalent *dual* (see Raidl, 2021c, p. 207, and Rott, 2020, p. 6). The dual of LLE is Right Logical Equivalence (RLE): if $\vdash B \equiv C$ then $\vdash (A \rightarrowtail B) \supset (A \rightarrowtail C)$. The dual of IOC is Isolation of Tautologies $(A \rightarrowtail \top) \equiv (\top \rightarrowtail A)$ (IOT), the dual of NE is No Implosion $\neg(A \rightarrowtail \top)$ (NI), and the dual of PAC is Possible Negated Antecedent and Negated Consequent $(A \rightarrowtail B) \supset \Diamond(\neg A \wedge \neg B)$ ($\text{P}\overline{\text{A}}\overline{\text{C}}$).

As a semantics for $\Box$ and hence for $\Rightarrow$ and $\rightarrow$, we use reflexive Kripke models. Thus that A is *valid*, $\vDash A$, will here mean valid in reflexive Kripke models (true in all worlds of all these models). That A follows (locally) from Γ, $\Gamma \vDash A$, means that for all worlds w of all Kripke models, if all B from Γ are true in w then A is true in w. The axiomatic system S is *sound* iff $\Gamma \vdash A$ (in S) implies $\Gamma \vDash A$ (in reflexive Kripke models); and it is *complete* iff the reverse implication holds.

THEOREM 1. $\mathsf{IC.0}$ *is sound and complete for* $\Rightarrow$.

THEOREM 2. $\mathsf{CI.0}$ *is sound and complete for* $\rightarrow$.

By these theorems, a law (wff) is derivable (in KT or our two systems, respectively) if and only if it is valid. And a rule (e.g., MP or LLE)

[10] An alternative is $\Box A := \neg A \rightarrow \bot$ by C, or $\Box A := \bot \rightarrow \neg A$ by IOC.

[11] In $\mathsf{IC.0}$, MI is redundant and IOC derivable.

[12] An alternative is $\Box A := A \wedge \neg(\neg A \Rightarrow \neg A)$ by C. According to the first definition, $\Diamond A = A \vee (\neg A \Rightarrow \neg A)$. According to the second, $\Diamond A = A \vee (A \Rightarrow A)$.

holds for $\vdash$ iff it holds for $\models$ (we then also use 'derivable' and 'valid' in an extended manner). Both strict implication and weak super-strict implication (see Gherardi and Orlandelli, 2021; Gherardi et al., 2024) validate almost all laws from I, the first only lacks AT, the second only lacks C.

6. Valid and Invalid Principles of the two Conditionals

We compare the two conditionals ($\rightarrow$ and $\Rightarrow$) with respect to the principles (i.e. laws and inference rules) that are valid or invalid for them. We show the valid principles to be derivable in $\mathsf{CI.0}$ and $\mathsf{IC.0}$, respectively. For invalid principles, we either prove their invalidity semantically, or show that adding such a principle to the logics would lead to a contradiction. In addition to the principles from I, the two conditionals also share the principles listed in the following theorem.

THEOREM 3. *The following principles are derivable for $\rightarrow$ and $\Rightarrow$ in $\mathsf{CI.0}$ and $\mathsf{IC.0}$:*

Right Logical Equivalence
If $\vdash A \equiv B$, *then* $\vdash (C \rightarrowtail A) \supset (C \rightarrowtail B)$ RLE

Modus Ponens for the Conditional
$A \rightarrowtail B, A \vdash B$ MPC

Weak Boethius' Thesis
$(A \rightarrowtail B) \supset \neg(A \rightarrowtail \neg B)$ wBT

Aristotle's Second Thesis
$(A \rightarrowtail B) \supset \neg(\neg A \rightarrowtail B)$ AT2

Cumulative Transitivity
$((A \rightarrowtail B) \wedge ((A \wedge B) \rightarrowtail C)) \supset (A \rightarrowtail C)$ CUT

Conjunction of Consequents
$((A \rightarrowtail B) \wedge (A \rightarrowtail C)) \supset (A \rightarrowtail (B \wedge C))$ AND

Disjunction of Antecedents
$((A \rightarrowtail C) \wedge (B \rightarrowtail C)) \supset ((A \vee B) \rightarrowtail C)$ OR

Cautious Monotonicity
$((A \rightarrowtail C) \wedge (A \rightarrowtail B)) \supset ((A \wedge B) \rightarrowtail C)$ CM

By completeness it also follows that the above are valid for $\rightarrow$ and $\Rightarrow$.[13]

[13] In both systems wBT is equivalent to Abelard's (First) Principle (Martin, 2004; McCall, 2012): $\neg((A \rightarrowtail B) \wedge (A \rightarrowtail \neg B))$, also known as Principle of Subjunctive Contrariety (Angell, 1962), Strawson's Thesis (Routley, 1978; Mortensen, 1984), and Principle of Conditional Non-Contradiction (Gibbard, 1981). In (Pizzi, 2018), AT2 is

The consequential and the implicative conditionals share with D. Lewis' (1973) variably strict conditional (with weak centring) the laws LLE, RLE, MPC, AND, OR, CUT. These, together with TR and C, are also shared with strict implication. However, the consequential and the implicative conditional disagree with both the variably strict conditional and the strict conditional, since TR, C, wBT, AT, AT2 are invalid for the former, and wBT, AT, AT2 are invalid for the latter. Thus the consequential and the implicative conditional preserve many (although not all) central laws known from variably strict and strict implication. What is added to the common core are the connexive principles AT, wBT and AT2. In particular, by validity of AT2, which has received little attention in the connexivist community (Rott, 2024b, p. 406), both our conditionals are akin to Rott's difference-making conditional,[14] that is, the truth of the antecedent makes a difference to the truth of the consequent.

As for the shared invalid principles:

THEOREM 4. *The following are invalid for both $\rightarrow$ and $\Rightarrow$:*

Material-to-Conditional

$(A \supset B) \supset (A \rightarrowtail B)$ MC

Explosion

$\bot \rightarrowtail A$ E

Implosion

$A \rightarrowtail \top$ I

Necessary Consequent

$\Box C \supset (A \rightarrowtail C)$ NC

Impossible Antecedent

$\neg \Diamond A \supset (A \rightarrowtail C)$ IA

Residuation

If $A, B \vdash C$, *then* $A \vdash B \rightarrowtail C$ R

Simplification

$(A \wedge B) \rightarrowtail A$ SI

Addition

$A \rightarrowtail (A \vee B)$ ADD

Conditional Excluded Middle

$(A \rightarrowtail B) \vee (A \rightarrowtail \neg B)$ CEM

called Secondary Boethius Thesis. And CUT is also known as Limited Transitivity (Bennett, 2003, p. 161).

[14] See also (Raidl, 2021a,c) on this conditional.

Strengthening the Antecedent
$(A \rightarrowtail C) \supset ((A \wedge B) \rightarrowtail C)$ SA
Rational Monotonicity
$((A \rightarrowtail C) \wedge \neg(A \rightarrowtail \neg B)) \supset ((A \wedge B) \rightarrowtail C)$ RM
Right Weakening
If $\vdash B \supset C$, *then* $\vdash (A \rightarrowtail B) \supset (A \rightarrowtail C)$ RW
Factor Law
$(A \rightarrowtail B) \supset ((A \wedge C) \rightarrowtail (B \wedge C))$ FL
Strong Boethius' Thesis
$(A \rightarrowtail B) \rightarrowtail \neg(A \rightarrowtail \neg B)$ sBT
Symmetry
$(A \rightarrowtail B) \supset (B \rightarrowtail A)$ S

By invalidity of MC, the conditionals are non-trivial, since they do not reduce to the material conditional. The failure of Residuation follows. Thereby there is also a clear distinction between the conditional ($\rightarrowtail$) and the consequence or derivability relation ($\models$, $\vdash$). By invalidity of E, a contradiction does not imply (in the sense of $\rightarrow$ or $\Rightarrow$) everything, and by invalidity of I, a tautology is not implied from everything. By invalidity of E, I, NC and IA, both conditionals avoid the crucial paradoxes of strict implication. By invalidity of SI and ADD one can generally neither simplify a conjunction in the antecedent to one of the conjuncts in the consequent, nor can one add an arbitrary disjunct in the consequent of an identity conditional. By invalidity of SA, the conditionals fail to be antecedent monotonic, and by failure of RW, the conditionals fail to be consequent monotonic. In fact, due to C, SA is dual of *Disjunctive Weakening* (DW): $(A \rightarrowtail B) \supset (A \rightarrowtail (B \vee C))$, and the latter is equivalent to RW, by RLE. Hence, a conditional validating C and admitting RLE cannot satisfy RW without validating SA. By invalidity of RM, our conditionals also diverge essentially from mainstream variably strict conditionals.[15] By invalidity of CEM, the converse of wBT: $\neg(A \rightarrowtail \neg B) \supset (A \rightarrowtail B))$ is also invalid. By invalidity of S, a conditional is not equivalential, i.e. the conditional is not in general equivalent to its converse conditional (McCall, 1966, p. 417). Although the logics of our conditionals are classical conditional logics, in the sense of (Chellas, 1975), due to PT, MP, LLE, RLE, and although the normal law AND is derivable, they are not normal conditional logics since RW and $A \rightarrowtail \top$ fail.

[15] For variably strict conditionals invalidating RM, see (Burgess, 1981; Raidl and Rott, 2023; Raidl, 2025a,b).

Invalidity of E and I (as well as NC and IA) distinguishes the two conditionals from many other conditionals, such as variably strict conditionals, and the strict and material conditionals. These invalidities are also shared with relevance conditionals, i.e., conditionals as treated in relevance logic. However, our two conditionals differ from relevance conditionals in that they invalidate the two principles SI and ADD central to the latter. Invalidity of SA — the hallmark of non-monotonic conditionals —, on the other hand, is shared with variably strict conditionals. But contrary to these and due to validity of C, our conditionals invalidate RW as well.[16] By invalidity of CEM, the conditionals do not endorse the reduction of wide scope to narrow scope negation (contrary to Stalnaker's 1968 conditional). By invalidity of sBT, the conditionals differ essentially from connexive conditionals. However, by invalidity of S, they satisfy one of the crucial features of connexivity. Overall, by wBT and AT, and the failure of S, both logics, although only partially connexive, are weakly connexive (in the sense of Wansing and Omori, 2024) or Boethian (in the sense of Pizzi, 2024).[17]

Concerning shared restricted principles, we have:

THEOREM 5. *The following are derivable for* $\rightarrow$ *and* $\Rightarrow$:

Possibilistic Monotonicity

$$((A \rightarrowtail C) \wedge \Diamond(A \wedge B) \wedge \Diamond \neg C) \supset ((A \wedge B) \rightarrowtail C) \qquad \text{PM}'$$

Possibilistic Simplification

$$(\Diamond(A \wedge B) \wedge \Diamond \neg A) \supset ((A \wedge B) \rightarrowtail A) \qquad \text{PSI}$$

Possibilistic Identity

$$(\Diamond A \wedge \Diamond \neg A) \supset (A \rightarrowtail A) \qquad \text{PID}$$

Possibilistic Right Weakening

$$\text{If} \vdash B \supset C \text{ then} \vdash ((A \rightarrowtail B) \wedge \Diamond A \wedge \Diamond \neg C) \supset (A \rightarrowtail C) \qquad \text{PRW}'$$

These restricted laws provide weak replacements for some of the invalid laws (SA, SI, RW, and, for $\mathsf{IC.0}$, ID). The general procedure to obtain them is to add as premise the possibility of the antecedent and

[16] For other conditionals invalidating RW, see (Raidl, 2021c,a; Raidl et al., 2021; Raidl and Rott, 2024).

[17] An LLE-RLE logic is *partially connexive* iff it contains one but not both AT and sBT, and does not contain S (Wansing and Omori, 2024, §3.1). It is *weakly connexive* iff it admits BT in rule form but does not admit S in rule form (Wansing and Omori, 2024, §4.1). It is *Boethian* when it contains wBT, but does not contain S (Pizzi, 2024). In logics with $\supset$ and $\vdash$ extending classical logic, like our logics, 'weakly connexive' is equivalent to 'Boethian'.

of the negated consequent of the conditional figuring in the consequent position of the law in question. Although both conditionals, like variably strict conditionals but unlike strict implication, fail to be monotonic in the antecedent (due to invalidity of SA), they are restrictedly antecedent monotonic (PM′). Although both conditionals, unlike strict or variably strict conditionals, fail to be monotonic in the consequent (due to invalidity of RW), they are restrictedly consequent monotonic (PRW′). And although both conditionals, unlike most conditionals, in particular relevance conditionals, fail to validate SI, they validate a restricted version (PSI). The implicative conditional validates the stronger laws PM and PRW (Raidl and Gomes, 2024)

$$((A \rightarrowtail C) \wedge \Diamond(A \wedge B)) \supset ((A \wedge B) \rightarrowtail C) \qquad \text{PM}$$
$$\text{If } \vdash B \supset C \text{ then } \vdash ((A \rightarrowtail B) \wedge \Diamond\neg C) \supset (A \rightarrowtail C) \qquad \text{PRW}$$

in which the last possibility premise is removed from PM′ (since it is implied by P$\overline{\text{A}}\overline{\text{C}}$), and the second from PRW′ (since it is implied by PAC). For the implicative conditional these premises are redundant, but they are needed for the consequential conditional.

Among the principles not shared by $\rightarrow$ and $\Rightarrow$, the most important one concerns Identity:

Theorem 6. ID *is valid for* $\rightarrow$, *but invalid for* $\Rightarrow$.

This means that $A \rightarrow A$ is a theorem in KT, but $A \Rightarrow A$ is not, only the restricted version PID is one (see Theorem 5).

A second important difference concerns contradictions. Whereas $\Rightarrow$ validates NE $\neg(\bot \rightarrowtail A)$ and its converse $\neg(A \rightarrowtail \bot)$, these are invalid for $\rightarrow$.[18] These principles are crucial for the implicative conditional, as they tell us that contradictions are to be precluded. Thus no contradiction ever appears as a clause in a true implicative conditional. The situation is different for the consequential conditional, as there can be true consequential conditionals in which contradictions occur (for example $\bot \rightarrow \bot$). Thereby, IOC is non-trivial for the consequential conditional, but trivial for the implicative conditional (both sides are always false). Similar remarks hold for tautologies.

[18] These were called *No Antilogical Antecedent* (NAA) and *No Antilogical Consequent* (NAC) by Raidl and Gomes (2024) and Raidl (2023).

7. Logical Similarities and Differences

In this section we discuss logical similarities and differences between our two conditionals. We draw a contrast between both our conditionals and variably strict conditionals and discuss (i) alleged counterexamples to the central valid principles (TR, C), (ii) the most important invalid principle (SI), and (iii) the crucial distinguishing feature (ID). We close by a remark on strong laws and logical strength.

Divergence from variably strict conditionals. Although both our conditionals share many features with variably strict conditionals ($>$), they also diverge essentially from the latter. Variably strict conditionals were mainly motivated by avoiding SA and some paradoxes of material implication (Stalnaker, 1968; Lewis, 1973). Both our conditionals share this aspect but achieve it in a different manner: $>$ weakens the strict conditional whereas our conditionals strengthen it. Our conditionals also preserve other strict-conditional laws (C, TR) than variably strict conditionals do (RW, SI). But TR and SI imply SA, and similarly C and RW imply SA (see Raidl and Gomes, 2024). Hence the two kinds of conditionals bifurcate. Since variably strict conditionals preserve RW and SI, they must drop C and TR. Since our conditionals preserve C and TR, they must drop RW and SI.

Further differences follow: essentially due to RW and ID, variably strict conditionals are still subject to the crucial paradoxes of strict implication (E, I, NC, IA).[19] Both our conditionals avoid the latter, but in a slightly different manner. The implicative conditional avoids them *strongly*, by validating the negation of E and I (NE and NI) and the narrow scope negation of NC and IA (by PAC and its dual), whereas the consequential conditional avoids them *weakly*, by a restriction to 'equimodality', i.e. accepting only those instances where both the antecedent and consequent have the same modal status (due to IOC and its dual). Variably strict conditionals also offer weak replacements for SA, in the form of CM and RM. While CM holds, RM fails for our conditionals. Another kind of restricted antecedent monotonicity is however restored by PM′.

Transitivity. Transitivity seems to be very naturally accepted in the common use of natural language conditionals, and both our conditionals

[19] Where $\Box$ in NC and IA is the outer modality of $>$, $\Box A := \neg A > \bot$. E and I follow from RW. IA follows from adding ID. And NC follows from adding CM and OR.

validate it. Existing counterexamples are controversial, since various authors have argued that they require a different context for evaluating each of the premises. This is the reason why Transitivity is invalid for variably strict conditionals. The following is such a counterexample:[20]

(3) If Brown had been appointed (A), Jones would have resigned (R).
(4) If Jones had died (D), Brown would have been appointed (A).
(5) ♯ If Jones had died (D), he would have resigned (R).

As we can accept (3) and (4) but reject (5), this seems to speak against Transitivity. However, the consequent of (4) is accepted in a context where Jones has died (D), while the antecedent of (3) is accepted in a context where he is alive ($\neg D$). If we make the contextual factors explicit before the logical analysis, we get: (4) $D \rightarrowtail (A \wedge D)$, (3) $(A \wedge \neg D) \rightarrowtail R$; therefore (5) $D \rightarrowtail R$. This argument does not have the logical form of Transitivity. Hence the premises may be true and the conclusion false, without contradicting Transitivity. When the context of both premises is the same (as well as when they are context-independent), transitivity holds for both our conditionals, and it is very difficult to find counterexamples to transitivity for these cases.

Contraposition. Counterexamples to contraposition often involve overt or co-vert concessive conditionals, or can be explained by context shift, or still in other ways (see Gomes, 2019). A recent alleged counterexample is offered by Rott (2022, p. 148) (see also Rott, 2024a, and Raidl and Rott, 2024, §6.1):

> I believe that Pam works on the project (p) and that the project will be successful (q), because Pam is an excellent and dedicated researcher, and if she is missing, the project might fail. So I think 'If Pam works on the project, [it] will be successful'. On the other hand, there is a (slightly remote) possibility that Pam will not perform well, and this is why I reject 'If the project is not successful, Pam has not worked on [it]'.

However, the possibility (however remote) that Pam will not perform well seems to provide as much (or as little) reason to reject the former conditional as it does to reject the latter.

Simplification. Failure of Simplification is a surprising feature shared by our two conditionals. How can such an intuitive and seemingly innocent

[20] See (Edgington, 1995, p. 253). This example is modeled on a well-known counterexample from (Stalnaker, 1968), discussed on p. 344.

principle as $(A \wedge B) \rightarrowtail A$ fail? The requirements of the two conditionals are clearly incompatible with Simplification. Indeed, if $(A \wedge B) \rightarrow A$ were a theorem for the consequential conditional, we would obtain the instance $(A \wedge \neg A) \rightarrow A$ by substitution. But $A \wedge \neg A$ is impossible and, if A is contingent, the two clauses would have a different modal status, violating the equimodality condition. Similarly, for the implicative conditional, since both A and $A \wedge B$ can be either necessary or impossible. Furthermore, on an axiomatic level, if we accept Transitivity and failure of Strengthening the Antecedent, then Simplification must fail as well. As a matter of fact, it is questionable whether Simplification is really universally acceptable in the natural language use of conditionals. From the well-known proverb *Yo no creo en brujas, pero que las hay, las hay,*[21] for example, one can hardly conclude that the speaker does not believe in witches (as stated in the first conjunct of the sentence). Furthermore, B can be something that blocks the bridge from A as premise to A as conclusion (Thompson, 1991, pp. 252–253). An instance of this is $(A \wedge C \wedge (C \rightarrowtail \neg A)) \rightarrowtail A$, as in:

(6) If I want it but she doesn't want it, and I don't want it if she doesn't, then I want it.

If we find that (6) is wrong, we have reason to deny that Simplification is unrestrictedly valid. However, if we require that both $A \wedge B$ and $\neg A$ be possible (according to PSI), then $(A \wedge B) \rightarrowtail A$ unquestionably holds. This explains why (6) can be a false instance of Simplification: its antecedent is impossible. We thus conclude that Possibilistic Simplification is all we need to satisfy the intuition that if A *and* B is true, then A is true.[22]

Identity. Identity is the most salient point of disagreement between our two conditionals. Identity is one of the basic features of connexive logic and as such it has been inherited by the consequential conditional. However, it fails for IC.0 ($\neg(\bot \Rightarrow \bot)$ is a theorem), as well as for some other Boethian logics. An example is Lowe's system D1, which admits Identity only in a weakened form: $\Diamond A \supset (A \rightarrowtail A)$, which is slightly stronger than Possibilistic Identity (PID, Theorem 5). In fact, Identity has been the object of many discussions. It was rejected on philosophical grounds by

[21] Roughly translatable as *I do not believe in witches, but exist they do*, and similar to Moore's paradox *It's raining but I do not believe it is.*

[22] The restricted principles PM, PID, and PRW were discussed by Raidl and Gomes (2024).

Strawson (1948) and Blanshard (1969). Kielkopf (1977, p. 33) finds that Identity is not valid for his *material entailment* (i.e., entailment that is valid thanks to the meaning of non-logical terms). If one maintains that the implication relation is essentially ampliative, in the sense that the consequent must say something more than the antecedent says, as it is typically the case in inductive or abductive inference, Identity does not satisfy this principle. The same holds if one maintains that the consequent must convey information which is a proper part of the information conveyed by the antecedent. If so, since the content of A is not a proper part of A, it could not be true that A implies A.

The basic question for a modally defined implication is whether it is meaningful to hypothesise something which is known to be impossible or contradictory. The idea that it is not is hinted at, for example, in Bolzano's (1978) philosophy of logic. It lies at the basis of the implicative conditional and other logical systems that include $\Diamond A$ as a defining condition for $A \rightarrowtail B$.[23]

Strong laws. As it is the case with logics of variably strict conditionals, the material conditional appears as the main connective in most laws of our conditionals. Scholars of connexive or relevance logics might ask whether we could not replace $\supset$ by $\Rightarrow$ or $\rightarrow$, respectively, giving rise to strong versions (say sBT, sC, sTR, etc.). The reason why this is impossible is as follows. In $\mathsf{IC.0}$, all strong laws are invalid, since no implicative conditional is valid (Raidl and Gomes, 2024, Fact 3). Thus adding sBT or any strong axiom version would lead to inconsistency. For example, sBT would imply that any conditional $A \Rightarrow B$ is possible, contradicting the impossibility of $\bot \Rightarrow B$.[24] This is also the deeper reason why ID, SI and ADD fail for the implicative conditional. For the consequential conditional, the situation is slightly different. In $\mathsf{CI.0}$, we have $\vdash A \rightarrow B$ iff $\vdash A \equiv B$ iff $\vdash B \rightarrow A$ (Pizzi and Williamson, 1997, p. 581). Thus sC and C, as well as sS and S, are equivalent, since both

[23] For example, systems that conjoin the possibility of the antecedent with the *strict conditional* (Gherardi et al., 2024), or with the *variably strict conditional* (Raidl, 2019, 2021b, 2023). Other conditionals invalidating ID include Rott's (2022) difference-making or Spohn's (2013; 2015) reason relations (see Raidl, 2021c, p. 216), Rott's (1986) *because* (see Raidl and Rott, 2024, §7), and Douven's (2015) evidential support.

[24] Compare the proof from (Raidl, 2023) and the remarks in (Gherardi et al., 2024).

C and S are self-symmetrical. Hence sC is valid and sS invalid. This is also the deeper reason why SI and ADD fail and why ID is valid for the consequential conditional. However, whereas wBT is valid, sBT is not, since in CI.0 it would lead to considering $A \rightarrow B$ and $\neg(A \rightarrow \neg B)$ as equivalent, leading to the collapse of $\Diamond A$ into $\Box A$, and of both into A, and would also contradict the invalidity of CEM.

We could, however, satisfy some purist desire, without change in logic. $\vdash$ extends the classical consequence relation, and thus satisfies Modus Ponens and the Deduction Theorem for $\supset$. Hence, any material law $\vdash X \supset Y$ is equivalent to its rule form $X \vdash Y$ — in particular, both our conditionals admit BT and AT2 in rule form. Thus we could state all material axioms in rule form with $\vdash$ replacing $\supset$ in the main position. Thereby a rewrite of our axiomatisations without the material conditional is possible.

Strength. Concerning the implication relations between the two operators, one is stronger than the other: $A \Rightarrow B$ implies $A \rightarrow B$, but not vice versa. In a suitable semantics, we can prove the following relations (where $\prec$ stands for the strict conditional and $>$ stands for the Lewisian variably strict conditional):[25] $A \Rightarrow B \vDash A \rightarrow B \vDash A \prec B \vDash A > B \vDash A \supset B$.

8. Aristotelian Squares and Cubes

The well-known Aristotelian square of opposition involving quantifiers gives rise to the modal square of opposition, and also to the square of opposition involving conditionals shown in Figure 1.

This square of oppositions among conditionals is preserved if the relation *If ... then ...* is replaced by either $\rightarrow$ or $\Rightarrow$. The following two cubes of opposition (Figures 2 and 3) describe the relations between the Aristotelian square for $\rightarrow$ (at the front) and the Aristotelian square for $\Rightarrow$ (at the back). The first cube (Figure 2), just like the square in Figure 1, is based on the opposition between the pair of contraries $A \rightarrowtail B$ and $A \rightarrowtail \neg B$, according to Weak Boethius' Thesis. The second cube (Figure 3) is based on the opposition between the pair of contraries $A \rightarrowtail B$ and $\neg A \rightarrowtail B$, according to Aristotle's Second Thesis.

[25] The semantics needs to be such that the outer necessity of $>$, $\Box A := \neg A > \bot$ is taken as the underlying necessity for $\prec, \rightarrow$ and $\Rightarrow$, and $>$ needs to satisfy weak centring.

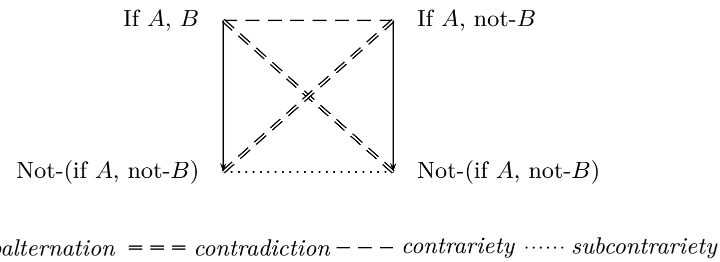

Figure 1. Square of opposition involving conditionals

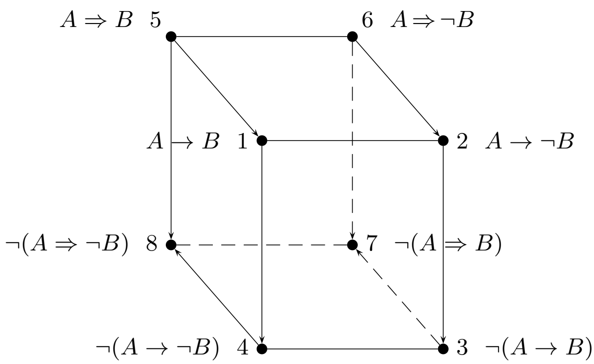

Figure 2. Cube based on the opposition between the pair of contraries $A \rightarrowtail B$ and $A \rightarrowtail \neg B$, according to weak Boethius' Thesis wBT

The relations of entailment are indicated by arrows. The diagonals on the front ($\rightarrow$) and back ($\Rightarrow$) squares (not drawn) represent the opposition of contradictories, as in Figure 1. The top edges of the front and back squares represent the opposition of contraries (which cannot both be true but can both be false) and the bottom edges that of subcontraries (which cannot both be false but can both be true). Note that the section rectangles $\langle 5, 6, 3, 4\rangle$ and $\langle 1, 2, 7, 8\rangle$ also represent Aristotelian squares of opposition.

9. Application to Natural Language Conditionals

The theories of $\Rightarrow$ and $\rightarrow$ also provide accounts of how to analyse natural language conditionals — let us call these the IC- and the CI-accounts. Here we discuss their agreement on excluding certain natural language

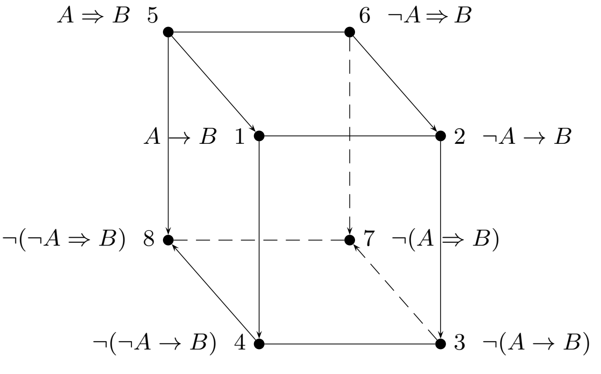

Figure 3. Cube based on the opposition between the pair of contraries $A \rightarrowtail B$ and $\neg A \rightarrowtail B$, according to Aristotle's Second Thesis AT2

conditionals (in particular concessive conditionals) from the analysis, and their divergence on how to disambiguate context-dependent conditionals (in particular counterfactuals) and on handling Identity conditionals or conditionals involving tautologies or contradicitions.

Excluding concessive (and other) conditionals. The application of both our conditionals to the analysis of natural language sentences having the form *If A, B* (or related forms) requires the exclusion of some such sentences, notably concessive conditionals such as (7)–(9) (paraphrasable with *even if*), biscuit conditionals such as (10), and whether-or-not conditionals (also called *unconditionals*) such as (11) (see Raidl and Gomes, 2024; Pizzi, 1980).

(7) I wouldn't marry you if you were the last man on earth.
(8) If you stretch out your hand to kill me, I won't stretch out my hand to kill you.
(9) If they talked, I didn't hear anything.
(10) If you need me, I'll be in my office this afternoon.
(11) If you like it or not, you'll have to do it.

The implicative conditional $\Rightarrow$ is to be applied only to what have been called *standard* conditionals (Ducrot, 1972/1991), *strong* conditionals (Davis, 1983), *robust* conditionals (Lycan, 2001) and *conditional* conditionals (Douven, 2017). The consequential conditional $\rightarrowtail$ may additionally be applied when the antecedent and the consequent are both necessary or both impossible.

Standard accounts of conditionals such as those of Stalnaker, Lewis, Adams and Bennett try to provide a semantics that applies equally well to concessive and non-concessive conditionals. However, several authors have argued that concessive conditionals present a different logical behaviour and should be studied separately (Pollock, 1976; Gomes, 2020; Crupi and Iacona, 2022). We side with the latter. Consider the following two examples:

(12) *I'd like to be with Vanessa, but I know the party will be lousy.* If she invites me to go, I won't go.

(13) *I know it will be a nice party, but I don't want to have Vanessa hanging around me.* If she invites me to go, I won't go.

The conditional is the same in (12) and (13), but the context (in italics) is different. Due to this, it may have two different meanings. In (13), it is naturally interpreted as a regular, non-concessive, conditional. Vanessa's invitation is the reason for the speaker not going to the party. It is natural to infer that if Vanessa does not invite him, he may go. In (12), by contrast, the speaker has a different reason for not going. He will not go to the party in any case, and Vanessa's invitation is insufficient to make him change his mind, although he would like to be with her. In (12), the sentence can be paraphrased with *Even if*, and should be classified as a concessive conditional. According to both the IC- and the CI-account, concessive conditionals require a different analysis and should not be analysed as $A \Rightarrow B$ or $A \rightarrow C$.[26]

Context-dependence. Many natural language conditionals are context-dependent, as highlighted by Quine's famous example (quoted by Lewis, 1973, p. 66) 'If Caesar had been in command [in Korea], he would have used . . . '. A common reply is that conditionals state elliptic consequence relations and should be disambiguated before being subject to a logical analysis. The CI- and the IC-accounts use different strategies for this purpose.

The CI-account makes a sharp distinction between context-independent and context-dependent conditionals, the latter notably including counterfactual conditionals.[27] Whereas the former should be analysed

[26] For similar remarks and examples, see Pizzi (1980). For recent accounts of concessive conditionals, see (Gomes, 2020; Crupi and Iacona, 2022; Raidl et al., 2023).

[27] For a logic of consequential counterfactuals, see (Pizzi, 2022).

by the context-independent (*analytic*) version $\rightarrow$, as studied throughout the article, the latter should be analysed by a context-dependent (*synthetic*) version:[28]

$$def\,CI_{\mathrm{CD}} \qquad A \twoheadrightarrow B \;:=\; \Box(*A \supset B) \wedge (\Box A \equiv \Box B) \wedge (\Diamond A \equiv \Diamond B)$$

Here $*$ is a circumstantial operator, suggested by Åqvist (1973), representing a condition intuitively understood as *ceteris paribus* (other things being equal) or *rebus sic stantibus* (things standing thus) or as *at the selected closest worlds.*

Consider the following well-known example:

(14) If this match had been struck (A), it would have lighted (B).

(14) is to be analysed by $\twoheadrightarrow$, so that here the context constrains only the antecedent of the underlying strict conditional. Since $*A$ is assumed to imply A, $\rightarrow$ is stronger than $\twoheadrightarrow$. It can further be shown that under a standard analysis of $*$ ($w \models^i *A$ iff $w \in f(i, A)$), $\Box(*A \supset B)$ expresses a variably strict conditional $A > B$ (see Lewis, 1973, p. 62, Humberstone, 1978), so that $A \twoheadrightarrow B$ strengthens $A > B$ by an equimodality assumption. Due to this, $\twoheadrightarrow$ also has different logical principles than $\rightarrow$ (Pizzi, 1991, 2022), and is more similar to variably strict conditionals. In particular, $\twoheadrightarrow$ lacks principles such as Contraposition and Transitivity. For example, Contraposition seems to fail for context-dependent conditionals when negligible or improbable facts cause important effects:

(15) If King Alexander of Greece had not been bitten by a little monkey, he would not have died in 1920.

It seems inadequate to contrapositively say that the death of King Alexander of Greece in 1920 entails that he was bitten by a little monkey.

The question of Transitivity is more controversial, but Stalnaker's famous example involving a hypothetical Communist director of the FBI can always be quoted against transitivity. From *If Hoover were a Communist, then he would be a traitor* and *If Hoover had been born a Russian, he would be a Communist*, we cannot infer *If Hoover had been born a Russian, he would be a traitor* (Stalnaker, 1968). This failure can be explained using variably strict conditionals or $\twoheadrightarrow$: if we admit an implicit *ceteris paribus* in the antecedents of the premises, the set of *ceteris* (other things) of one would not be equal to the set of *ceteris* of

[28] For this definition see the conditional $\twoheadrightarrow'''$ from (Pizzi, 2022). For two other definitions, see (Pizzi, 1991, 2018).

the other — and it was for this reason that Stalnaker had to present the premises in inverted order. 'A Communist' refers to an American Communist in the first premise but to a Russian Communist in the second (Pizzi, 1993). And thus, due to a context-shift, Transitivity fails.

The IC-account, by contrast, uses the same logical conditional for the analysis of context-independent and context-dependent conditionals. It therefore has a different approach to explain apparent failures of contraposition and of transitivity in natural language conditionals.

According to the tradition of Mill, Ramsey and Tichý (see Ramsey, 1929, p. 156, and Tichý, 1984, pp. 164–166) concerning accessory conditions, the context in which the sentence is uttered complements the information explicitly present in the antecedent in a way that is necessary to guarantee the conditional connection:

> In general we can say with Mill that 'If p then q' means that q is inferrible from p, that is, of course, from p together with certain facts and laws not stated but in some way indicated by the context.
>
> (Ramsey, 1929, p. 156)

On this view, a conditional 'If A, B' is elliptic for 'we may infer B from A in the context X (or in the presence of the contextual factors X)'.

Consider (14) again. This conditional is of course only true if certain conditions (X) are met, such as the presence of oxygen in the atmosphere and the match being dry. The implicative account recommends that such implicit conditions should be rendered explicit in the antecedent (Raidl and Gomes, 2024), and thus (14) should be analysed as $(X \wedge A) \Rightarrow B$. Hence, if (14) is true, it would have been possible for this match to have been struck (and to light), in the presence of the contextual factors, it would be also have been possible for it (not to have been struck and) not to light, but it would have been impossible for it to have been struck (in the presence of the contextual factors) and not to light.

The antecedent and the consequent themselves may function as contextual factors for the determination of the meaning of one another:

(16) If he doesn't live in Boston (A), he lives somewhere in New England (B).

This has been used as a counterexample to contraposition (Jackson, 1979, p. 587). However, the antecedent here functions as a contextual factor for the consequent, namely excluding Boston from the places in New England where he may live, so that the proposition really expressed by the consequent in the context provided by the antecedent is that he lives somewhere *else* in New England (than in Boston). Conditional (16) may thus be rendered as $A \Rightarrow (B \wedge A)$. Then contraposition does not fail: if he does not live somewhere else in New England (than in Boston), he lives in Boston.[29] Other alleged counterexamples to principles of the implicative conditional can be similarly explained by explicitation (Gomes, 2009, 2019, 2020; Raidl and Gomes, 2024).

When forming the contrapositive of a conditional in which the antecedent denotes a cause and the consequent an effect, it may be necessary to explicitate the direction of the causal relation itself (now presented in converse order), with the help of the words *it is because* before the new consequent (Gomes, 2019). For example, a contrapositive that preserves the meaning of (15) would be: *If King Alexander of Greece died in 1920, it was because he had been bitten by a little monkey.*

While both strategies recognise the role of context, implicit contextual factors are either to be added explicitly to the antecedent or to the consequent, or implicitly used to restrict *solely* the antecedent of the underlying strict conditional. Alleged counterexamples to central principles (such as Contraposition or Transitivity) are thereby treated differently. Explicitation results in explaining away the counterexamples by showing that these have only superficially but not deeply the form of the law in question. Circumstantialisation on the other hand admits the superficial form but recommends applying a restrictor analysis to this form. In any case, both strategies conclude that alleged counterexamples are neither counterexamples to the logic of $\Rightarrow$ (due to a logical form mismatch) nor to the logic of $\rightarrow$ (due to a difference in logical kind).

Equimodality vs. Contingency. A further crucial difference between the consequential and the implicative conditional is already present in their defining conditions. Whereas the implicative conditional (*def IC″*) adds the contingency of the antecedent and consequent to the strict conditional (*contingency condition*), the consequential conditional (*def CI′*) requires only that both the antecedent and the consequent have the same modal status (*equimodality condition*). This is the crucial reason for their divergence with respect to Identity, contradictions and tautologies.

[29] Using the conjunction of the explicit consequent with the antecedent, we would have, for the contrapositive: *If it is not the case that he lives somewhere in New England and does not live in Boston, then he lives in Boston.*

In natural language, we hardly ever utter conditional identity statements. Thus, the disagreement between $\Rightarrow$ and $\rightarrow$ on ID may seem superfluous. However, we sometimes find a conditional with identical antecedent and consequent. Literally interpreted, they provide no information. Natural language, however, is not intended to be uninformative, so a literal interpretation seems inadequate. The intended conditional connection in such cases seems to be between the antecedent and something else that is suggested by its repetition in the consequent.

(17) If he did it, then he did it.

Depending on the context, the sentence may be intended to mean that if *he* did *it*, nothing can be done about that, or he will have to face the consequences of what he did, or we have to accept that he did it, or something similar. All these suggested interpretations have to do with accepting some consequence of a surprising or unwanted fact expressed by the antecedent/consequent. But all such examples agree with Possibilistic Identity, since they involve a contingent antecedent/consequent.[30] Thus Identity may very well occur naturally only in its possibilistic version.

The two accounts differ, however, for examples in which the antecedent/ consequent is either impossible or necessary:

(18) If two plus two is four, then two plus two is four.
(19) If this square is round, then this square is round.

We may find that these are artificial or that it is hard to imagine a situation in which they might be used. And even if considered true (which is more difficult to imagine for 19), we may find that they are both uninformative. This is in line with the IC-account, according to which both conditionals fail to be implicative conditionals. By contrast, according to the CI-account, both conditionals are valid consequential conditionals, even if non-informative — and this is important because the validity of Identity is considered as a precondition of the truth of any conditional stating a consequence relation between the clauses.

The difference between the contingency condition and the equimodality condition extends to the treatment of contradictions and tautologies. By equi-modality, a contradiction such as $2 + 2 = 5$ gives plausibility to

[30] Of course, they also agree with Identity *tout court*.

other akin contradictions such as $3 + 3 = 7$, so that we may accept:[31]

(20) If $2 + 2 = 5$, then $3 + 3 = 7$.

However, it is not the case that a contradiction gives plausibility to any proposition whatsoever, including its own negation, and thus (in line with AT), it seems natural to reject:

(21) If two plus two is five, then two plus two is not five.

This suggests that in natural language a contradiction is not perceived as something that is meaningless, but rather as having a meaning — say, the meaning of being something that is absurd or impossible. From this perspective, it is not the case that a contradiction 'says nothing', as is implicit in the subtraction (or cancellation) concept of negation (Routley and Routley, 1985; Priest, 1999; Wansing and Skurt, 2018). According to the CI-account, a contradiction or impossibility says the same thing as another contradiction or impossibility.

By Contraposition, similar remarks apply when the consequent is a tautology, a truism or a necessity. If someone says, for example, 'Rain is wet', someone else might comment that this is like saying that two plus two is four. Such a comment would seem intended to mean that a truism has the same meaning as a tautology or a mathematical truth, so that it makes sense to maintain that the former implies and is implied by the latter. Thus, the following conditionals would be acceptable:

(22) If rain is wet, then two plus two is four.
(23) If two plus two is four, then rain is wet.

The equimodality condition is thus viewed by the CI-account as describing a well-rooted feature of natural language. The IC-account, by contrast, rejects as false not only identity conditionals with an impossible or necessary antecedent/consequent, such as (18) and (19), but also any conditional with a necessary or impossible, tautological or contradictory antecedent or consequent, such as (20), (22) and (23). This is not to say that the latter can have no meaning in natural language, but rather that they are not implicative conditionals, that is, they do not express the same relation as encoded by $\Rightarrow$.

[31] For a discussion of such sentences, see (Raidl, 2019, pp. 862–864).

10. Conclusion

The present paper investigated the implicative conditional ($\Rightarrow$) and the consequential conditional ($\rightarrow$) from a logical, philosophical and linguistic perspective. We axiomatised the logic of each of them ($\mathsf{IC.0}$ and $\mathsf{CI.0}$), based on a common core, and proved these systems to be sound and complete for $\Rightarrow$ and $\rightarrow$ interpreted in reflexive Kripke models. We also showed how both conditionals are interdefinable, discussed shared valid and invalid principles, as well as crucial differences between them, and contrasted them with other conditionals, such as the material, the strict, the variably strict, relevance and connexive conditionals. Finally, we highlighted their main differences and similarities in the analysis of natural language conditionals.

We showed that $A \Rightarrow B$ is equivalent to $(A \rightarrow B) \wedge \nabla A$. Thus the implicative conditional is the special case of the consequential conditional in which the antecedent is contingent. Conversely, $A \rightarrow B$ is equivalent to $(A \Rightarrow B) \vee \Box(A \equiv B)$. Thus the consequential conditional extends the implicative conditional to the cases in which the antecedent and the consequent are strictly equivalent. The two conditionals differ most strikingly when the antecedent and consequent are identical: Identity $A \rightarrow A$ is a theorem of $\mathsf{CI.0}$, but $A \Rightarrow A$ is not one of $\mathsf{IC.0}$, since the latter requires that A be contingent.[32]

Both conditionals strengthen strict implication, and hence have the impossibility of the conjunction of the antecedent with the negated consequent at their core. Thereby they both mirror the Chrysippean idea that a genuine implication involves a (modal) conflict between antecedent and negated consequent.[33] But while the implicative conditional strengthens strict implication by a contingent antecedent and consequent, the consequential conditional requires only their equimodality. From a doxastic or epistemic perspective, the acceptance of an implicative conditional thus involves judging both the antecedent and the consequent as contingent, while the acceptance of a consequential conditional may also be just due to judging both as strictly equivalent.

Despite these differences, our conditionals share many logical principles. In particular, they preserve the laws of Transitivity (TR) and Con-

[32] The same holds for $A \rightarrowtail B$ when B is provably equivalent to A.

[33] See (Nasti de Vincentis, 2006) for the consequential conditional, (Gomes, 2013) for the implicative conditional, and (Lenzen, 2021) for both.

traposition (C) from material and strict implication. But they avoid the paradoxes of both. Our conditionals also share with the variably strict conditional ($>$) the failure of Strengthening the Antecedent. However, due to C and TR (rejected by $>$), our conditionals reject Simplification and Right Weakening (accepted by $>$). Furthermore, our conditionals are not connexive, as they invalidate Strong Boethius Thesis (sBT). However, they validate crucial connexive principles, such as Weak Boethius' Thesis (wBT) and Aristotle's First and Second Thesis (AT, AT2). By wBT and AT2, $A \rightarrowtail \neg B$ and $\neg A \rightarrowtail B$ are both contraries to $A \rightarrowtail B$ (figures 2 and 3). Due to the additional invalidity of Symmetry (S), both our conditionals are weakly connexive, and in fact Boethian. Overall, both provide a modally interpretable alternative to standard approaches to connexivity.

A crucial difference concerns the analysis of natural language conditionals. While the CI-account distinguishes context-dependent and -independent conditionals (to be analysed by $\rightarrow$ and $\twoheadrightarrow$, respectively), the IC-account treats context-dependency as implicit assumptions, which should be made explicit for the logical analysis of the sentences. This has consequences for the analysis of context-dependent conditionals and the treatment of counterexamples to central principles, such as Contraposition or Transitivity. The CI-account accepts the counterexamples but suggests to analyse the involved conditionals by $\twoheadrightarrow$. The IC-account rejects the counterexamples by showing that once the implicit assumptions are made explicit, the counterexamples fail to have the form of the law in question.

The comparison of the two conditionals highlights that, beyond having a common kernel, they grasp divergent intuitions about implication. One may prefer one or the other, depending on one's theoretical perspective or intended application. The consequential conditional is more in line with traditional logic, since it unrestrictedly validates Identity, and allows for impossible or necessary antecedents or consequents (provided they are either both impossible or both necessary). Whenever the latter condition is satisfied, however, the implication relation collapses into strict equivalence. The implicative conditional departs more sharply from traditional logic, since it only admits contingent antecedents and consequents,[34] even for the identity conditional. Empirical studies might

[34] A similar attitude is taken by Spohn (2013; 2015) for his ranking-based reason relations.

address the question of whether conditionals with impossible or necessary antecedent and consequent are acceptable in natural language, and if they are, whether their acceptance is restricted to cases of equimodality. However, we take this to be independent of the question of the logical application of both conditionals. Overall, the difference between the two conditionals may give rise to further empirical, logical and philosophical questions.

A. Proofs of Equivalences

In all proofs, we assume standard reasoning in KT.

Proposition 1. def CI, def CI′, def CI″ *and* def CI‴ *are equivalent.*

Proof. We prove that the following are equivalent in KT:

(a) $\Box(A \supset B) \wedge (\Box B \supset \Box A) \wedge (\Diamond B \supset \Diamond A)$ CI
(b) $(\Box(A \supset B) \wedge \Diamond A \wedge \Diamond \neg B) \vee (\Box A \wedge \Box B) \vee (\neg \Diamond A \wedge \neg \Diamond B)$ CI″
(c) $\Box(A \supset B) \wedge ((\nabla A \wedge \nabla B) \vee (\Box A \wedge \Box B) \vee (\neg \Diamond A \wedge \neg \Diamond B))$ CI′
(d) $(\Box(A \supset B) \wedge \Diamond A \wedge \Diamond \neg B) \vee \Box(A \equiv B)$ CI‴

(a) implies (b): Suppose (a). Either $\Diamond A \wedge \Diamond \neg B$ or $\neg(\Diamond A \wedge \Diamond \neg B)$. In the first case, we obtain (b) from (a). In the second case, we have $\Box B \vee \neg \Diamond A$. If $\Box B$, we obtain $\Box A$ by $\Box B \supset \Box A$. Thus $\Box A \wedge \Box B$. Hence (b). If $\neg \Diamond A$, then this together with $\Diamond B \supset \Diamond A$ implies $\neg \Diamond B$. And thus $\neg \Diamond A \wedge \neg \Diamond B$. Thus again (b).

(b) implies (c): Suppose (b). Thus we have $\Box(A \supset B)$ and $(\Diamond A \wedge \Diamond \neg B)$, or $(\Box A \wedge \Box B) \vee (\Box \neg A \wedge \Box \neg B)$. If the first disjunct holds, $\Diamond A \wedge \Diamond \neg B$, we also obtain $\Diamond B$ from $\Box(A \supset B)$ and $\Diamond A$. Similarly, $\Diamond \neg A$ follows from $\Box(\neg B \supset \neg A)$ and $\Diamond \neg B$. Thus $\nabla A \wedge \nabla B$, and hence (c). If the second or the third disjunct holds, we also have (c).

(c) implies (d): Suppose (c). That is $\Box(A \supset B) \wedge \nabla A \wedge \nabla B$ or $\Box A \wedge \Box B$ or $\neg \Diamond A \wedge \neg \Diamond B$. The first disjunct of (c) implies the first disjunct of (d), and thus (d). From the second disjunct of (c), $\Box A \wedge \Box B$, we obtain $\Box(A \supset B)$ and $\Box(B \supset A)$, thus $\Box(A \equiv B)$ and hence (d). From the third disjunct of (c), $\Box \neg A \wedge \Box \neg B$, we obtain $\Box(\neg A \supset \neg B)$ and $\Box(\neg B \supset \neg A)$. Thus again $\Box(A \supset B)$ and $\Box(B \supset A)$, that is $\Box(A \equiv B)$ and hence (d).

(d) implies (a): Suppose (d). Thus $\Box(A \supset B) \wedge \Diamond A \wedge \Diamond \neg B$ or $\Box(A \equiv B)$. Assume the first. Since $\Diamond A$, we get $\Diamond B \supset \Diamond A$. Similarly, since

$\Diamond\neg B$, we get $\Diamond\neg A \supset \Diamond\neg B$. Thus $\Box B \supset \Box A$. Thus (a). Assume the second, $\Box(A \equiv B)$. Then $\Box(A \supset B)$ and $\Box(B \supset A)$. Hence $\Box B \supset \Box A$ and $\Diamond B \supset \Diamond A$. Thus again (a). QED.

PROPOSITION 2. *defIC*, *defIC′* and *defIC″* *are equivalent.*

PROOF. We prove the equivalence of the following, in KT:

(a) $\Box(A \supset B) \wedge \Diamond A \wedge \Diamond\neg B$ IC
(b) $\Box(A \supset B) \wedge \Diamond(A \wedge B) \wedge \Diamond(\neg A \wedge \neg B)$ IC′
(c) $\Box(A \supset B) \wedge \Diamond A \wedge \Diamond B \wedge \Diamond\neg A \wedge \Diamond\neg B$ IC″

For this we show that (a) implies (b) which implies (c) which implies (a).

(a) implies (b): Suppose (a). For reductio, assume $\Box(A \supset \neg B)$. This together with $\Box(A \supset B)$ implies $\Box(A \supset \bot)$. That is, $\Box\neg A$, or equivalently $\neg\Diamond A$, contradicting $\Diamond A$. Similarly, $\neg\Diamond(\neg A \wedge \neg B)$ yields $\Box(\neg B \supset A)$. This together with $\Box(A \supset B)$, that is $\Box(\neg B \supset \neg A)$, would imply $\Box\neg\neg B$, i.e., $\neg\Diamond\neg B$. This contradicts $\Diamond\neg B$.

(b) implies (c): $\Diamond(A \wedge B)$ implies $\Diamond A$; $\Diamond(\neg A \wedge \neg B)$ implies $\Diamond\neg B$.

(c) implies (a): trivial. QED.

PROPOSITION 3. $\rightarrow$ *is definable by* $\Rightarrow$ *as follows*:

$$A \rightarrow B \;:=\; (A \Rightarrow B) \vee (\neg\Diamond A \wedge \neg\Diamond B) \vee (\Box A \wedge \Box B) \qquad \text{(def1)}$$
$$A \rightarrow B \;:=\; (A \Rightarrow B) \vee \Box(A \equiv B) \qquad \text{(def2)}$$

PROOF. It suffices to prove the equivalence of:

(a) $\Box(A \supset B) \wedge (\Box B \supset \Box A) \wedge (\Diamond B \supset \Diamond A)$
(b) $(A \Rightarrow B) \vee (\neg\Diamond A \wedge \neg\Diamond B) \vee (\Box A \wedge \Box B)$
(c) $(A \Rightarrow B) \vee \Box(A \equiv B)$

(a) implies (b): Suppose (a). Either $\Diamond A \wedge \Diamond\neg B$, or not. In the first case we have established the disjunct $A \Rightarrow B$ of (b). So let us consider the second case and suppose $\neg(\Diamond A \wedge \Diamond\neg B)$. Thus $\Box\neg A \vee \Box B$. If $\Box B$, then by $\Box B \supset \Box A$, we also have $\Box A$, and hence the third disjunct of (b). If $\Box\neg A$, that is $\neg\Diamond A$, then by contraposing $\Diamond B \supset \Diamond A$, we also have $\neg\Diamond B$, and hence the second disjunct of (b).

(b) implies (c): Suppose (a) and assume $(\neg\Diamond A \wedge \neg\Diamond B) \vee (\Box A \wedge \Box B)$. In both cases we obtain $\Box(A \equiv B)$.

(c) implies (a): Suppose (c). The first disjunct of (c) implies (a), since $\Rightarrow$ implies $\rightarrow$. Assume $\Box(A \equiv B)$. Then we can derive $\Box(A \supset B)$, $\Box(B \supset A)$ and thus $\Box B \supset \Box A$, as well as $\Diamond B \supset \Diamond A$. QED.

PROPOSITION 4. $\Rightarrow$ *is definable by* $\rightarrow$, *as follows*:

$$A \Rightarrow B \ := \ A \rightarrow B \wedge \Diamond A \wedge \Diamond \neg A \qquad \text{(def5)}$$
$$A \Rightarrow B \ := \ A \rightarrow B \wedge \neg(\neg \Diamond A \wedge \neg \Diamond B) \wedge \neg(\Box A \wedge \Box B) \qquad \text{(def3)}$$
$$A \Rightarrow B \ := \ A \rightarrow B \wedge ((\Diamond A \wedge \Diamond \neg A) \vee (\Diamond A \wedge \Diamond \neg B) \vee (\Diamond B \wedge \Diamond \neg A) \vee (\Diamond B \wedge \Diamond \neg B)) \qquad \text{(def4)}$$

PROOF. It suffices to prove the equivalence of:

(a) $\Box(A \supset B) \wedge \Diamond A \wedge \Diamond \neg B$
(b) $(A \rightarrow B) \wedge \Diamond A \wedge \Diamond \neg A$
(c) $(A \rightarrow B) \wedge \neg(\neg \Diamond A \wedge \neg \Diamond B) \wedge \neg(\Box A \wedge \Box B)$
(d) $A \rightarrow B \wedge ((\Diamond A \wedge \Diamond \neg A) \vee (\Diamond A \wedge \Diamond \neg B) \vee (\Diamond B \wedge \Diamond \neg A) \vee (\Diamond B \wedge \Diamond \neg B))$

We prove that (a) implies (b) which implies (c) which implies (d) which implies (a).

(a) implies (b): Suppose (a). From $\Diamond A$ we get $\Diamond B \supset \Diamond A$. From $\Diamond \neg B$, that is $\neg \Box B$, we get $\Box B \supset \Box A$. Hence together with $\Box(A \supset B)$, we have $A \rightarrow B$. Furthermore, $\Diamond A$ holds by assumption, and $\Diamond \neg A$ is obtained as follows: Since $\Box(A \supset B)$, we also have $\Box A \supset \Box B$. Thus $\Diamond \neg B \supset \Diamond \neg A$. Since $\Diamond \neg B$, we get $\Diamond \neg A$. Thus, we have (b).

(b) implies (c): Suppose (b). $\Diamond A$ implies $\Diamond A \vee \Diamond B$ and thus $\neg(\neg \Diamond A \wedge \neg \Diamond B)$. $\Diamond \neg A$ implies $\Diamond \neg A \vee \Diamond \neg B$ and thus $\neg(\Box A \wedge \Box B)$. Hence (c).

(c) implies (d): Suppose (c). $\neg(\neg \Diamond A \wedge \neg \Diamond B)$ implies $\Diamond A \vee \Diamond B$; $\neg(\Box A \wedge \Box B)$ implies $\Diamond \neg A \vee \Diamond \neg B$. Thus, $(\Diamond A \vee \Diamond B) \wedge (\Diamond \neg A \vee \Diamond \neg B)$. This implies $(\Diamond A \wedge \Diamond \neg A) \vee (\Diamond A \wedge \Diamond \neg B) \vee (\Diamond B \wedge \Diamond \neg A) \vee (\Diamond B \wedge \Diamond \neg B)$. But from (d) we also have $\Box(A \supset B)$. Hence we have (d).

(d) implies (a): Assume (d). That is, $A \rightarrow B$ and one of: $\Diamond A \wedge \Diamond \neg A$ or $\Diamond A \wedge \Diamond \neg B$ or $\Diamond B \wedge \Diamond \neg A$ or $\Diamond B \wedge \Diamond \neg B$. From $A \rightarrow B$ we get $\Box(A \supset B)$ which implies $\Box A \supset \Box B$. But any of the disjuncts of (d) with $A \rightarrow B$ implies $\Diamond A \wedge \Diamond \neg B$: Suppose $\Diamond A \wedge \Diamond \neg A$. Then $\Diamond A$, and $\Diamond \neg A$ together with $\Box B \supset \Box A$ also implies $\Diamond \neg B$. When $\Diamond A \wedge \Diamond \neg B$ the result is trivial. Suppose $\Diamond B \wedge \Diamond \neg A$. From $\Diamond B$ we get $\Diamond A$, due to $\Diamond B \supset \Diamond A$ (which is implied by $A \rightarrow B$). From $\Diamond \neg A$, we get $\Diamond \neg B$, due to $\Box B \supset \Box A$ (also implied by $A \rightarrow B$). Suppose $\Diamond B \wedge \Diamond \neg B$. Then $\Diamond \neg B$, and from $\Diamond B$ we get again $\Diamond A$. So in each case, we have (a). QED.

B. Proofs of Theorems

PROOF OF THEOREM 1. KT+*defIC* is sound and complete for $\Rightarrow$ (defined from $\Box$) in reflexive Kripke models (Raidl and Gomes, 2024, Theorem 1). Thus it suffices to prove that the former axiomatisation is equivalent to IC.0.

IC.0 *derives* KT+*defIC*: We prove some intermediate results (in non-bold), the laws and rules to be established are written in bold.

RLE: from LLE by C.

T: $\Box A \supset A$. Suppose $\Box A$. That is $A \wedge \neg(A \Rightarrow A)$. Thus A.

W: $\Diamond(A \wedge B) \supset \Diamond A$ [contrapositively: $\Box A \supset \Box(A \vee B)$]. Suppose $\Diamond(A \wedge B)$. That is $A \wedge B$ or $\neg(A \wedge B) \Rightarrow \neg(A \wedge B)$. Either A or $\neg A$. If A, we have $\Diamond A$ by contraposing T. If $\neg A$, we must have $\neg(A \wedge B) \Rightarrow \neg(A \wedge B)$, that is $(\neg A \vee \neg B) \Rightarrow (\neg A \vee \neg B)$ by LLE and RLE. From $\neg A$ we also obtain $\Diamond \neg A$ by T. Thus $\Diamond((\neg A \vee \neg B) \wedge \neg A)$ by LLE and RLE. Hence, by PFL, $((\neg A \vee \neg B) \wedge \neg A) \Rightarrow ((\neg A \vee \neg B) \wedge \neg A)$. Thus $\neg A \Rightarrow \neg A$ by LLE and RLE. Hence $\Diamond A$.

PA: $(A \Rightarrow C) \supset \Diamond A$. Follows from PAC and W.

P$\overline{\mathbf{C}}$: $(A \Rightarrow C) \supset \Diamond \neg C$. Follows from C and PA.

N: If $\vdash A$, then $\vdash \Box A$. Suppose $\vdash A$. Hence $\vdash A \equiv \top$. But $\vdash \Box \top$ (i.e. $\top \wedge \neg(\top \Rightarrow \top)$ by NE and C). Thus we obtain $\Box A$ by LLE and RLE.

CSI: Assume $A \Rightarrow B$ and for reductio suppose $\neg \Box(A \supset B)$. Then $\Diamond \neg(A \supset B)$. That is $\Diamond(A \wedge \neg B)$ by LLE and RLE. Hence $(A \wedge \neg B) \Rightarrow (B \wedge \neg B)$ by PFL. Therefore $(A \wedge \neg B) \Rightarrow \bot$ (by RLE). Hence $\Diamond \bot$ by PC, that is $\bot \Rightarrow \bot$, contradicting NE.

AND: Suppose $A \Rightarrow B$ and $A \Rightarrow C$. Thus $\Diamond A$ by PA. Therefore $A \Rightarrow (A \wedge C)$ by PFL and LLE. But $\Diamond(A \wedge C)$ by PAC. Hence $(A \wedge C) \Rightarrow (B \wedge C)$ by PFL. Therefore $A \Rightarrow (B \wedge C)$ by TR.

CC: $\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$ [contrapositively: $(\Box A \wedge \Box B) \supset \Box(A \wedge B)$]. Assume $\Diamond(A \vee B)$, and for reductio $\neg \Diamond A$ and $\neg \Diamond B$. Thus $\neg \Diamond(A \wedge B)$ by W. Hence $\neg((A \vee B) \Rightarrow (A \wedge B))$ by LLE, RLE, and PAC. Therefore $\neg((A \vee B) \Rightarrow A)$ or $\neg((A \vee B) \Rightarrow B)$ by AND. Suppose the first. Thus by IL, $\neg \Diamond(A \vee B)$ or $\neg \Diamond \neg A$ or $\neg \Box((A \vee B) \supset A)$. The first contradicts $\Diamond(A \vee B)$. The second would yield $\Box A$, contradicting $\Box \neg A$ by T. Hence we must have the third, i.e. $\Diamond \neg((A \vee B) \supset A)$. Therefore $\Diamond(B \wedge \neg A)$ by LLE and RLE. Thus $\Diamond B$ by W, contradicting $\neg \Diamond B$.

The second disjunctive assumption yields similarly $\Diamond A$, contradicting $\neg\Diamond A$. Therefore $\Diamond A \vee \Diamond B$.

K: Assume $\Box(A \supset B)$ and $\Box A$. Hence $\Box((\neg A \vee B) \wedge A)$ by CC. That is $\Box(A \wedge B)$. Therefore $\Box B$ by W.

KT+*def IC derives* $\mathsf{IC.0}$: MP is a part of KT, IL follows from *def IC*. LLE, C, TR, AT, MI, PAC, NE were proven derivable by Raidl and Gomes (2024, Theorem 2, Fact 4).

PFL: Assume $A \Rightarrow B$ and $\Diamond(A \wedge C)$. But then $\Diamond A$, $\Diamond\neg B$ and $\Box(A \supset B)$ by *def IC* [PA, P$\overline{\text{C}}$, CSI]. Hence $\Diamond\neg(B \wedge C)$, and $\Box((A \wedge C) \supset (B \wedge C))$ by normal reasoning with $\Box$. So, $(A \wedge C) \Rightarrow (B \wedge C)$ by *def IC* [IL]. QED.

PROOF OF THEOREM 2. For labels, see Proof of Theorem 1.

$\mathsf{CI.0}$ *derives* KT*def CI*: We prove some intermediary results (in non-bold), the laws and rules to be established are written in bold.

RLE: from LLE by C.

N: Suppose $\vdash A$. Hence $\vdash A \equiv \top$. But $\vdash \Box\top$ (i.e. $\top \rightarrow \top$ by ID). Thus we obtain $\Box A$ by LLE and RLE.

T: By MI $(\top \rightarrow A) \supset (\top \supset A)$. That is $\Box A \supset A$.

W: Suppose $\Box A$, i.e. $\top \rightarrow A$. For reductio, assume $\neg\Box(A \vee B)$. That is $\Diamond(\neg A \wedge \neg B)$. Hence by PFL and RLE, $(\neg A \wedge \neg B) \rightarrow \bot$. Thus $\top \rightarrow (A \vee B)$ by C and RLE. That is $\Box(A \vee B)$, contradicting the reductio assumption.

CC: Suppose $\Box A$ and $\Box B$. From the first, $\Diamond A$, by T ($\Box A$, hence A thus $\neg\neg A$, hence $\neg\Box\neg A$), that is $\Diamond(\top \wedge A)$ by LLE and RLE. Thus, using $\top \rightarrow B$, we get $A \rightarrow (A \wedge B)$ by PFL and LLE. Therefore, $\top \rightarrow (A \wedge B)$ by TR.

K: by W and CC.

CSI: Suppose $A \rightarrow B$ and for reductio that $\neg\Box(A \supset B)$, i.e. $\Diamond(A \wedge \neg B)$. Hence $(A \wedge \neg B) \rightarrow \bot$ by PFL and RLE. Thus $\top \rightarrow (A \supset B)$ by C and RLE. That is $\Box(A \supset B)$, contradicting the reductio assumption.

NCNA (*Necessary Consequent to Necessary Antecedent*): $(A \rightarrow B) \supset (\Box B \supset \Box A)$. Suppose $A \rightarrow B$ and $\Box B$. That is $\neg B \rightarrow \neg A$ and $\neg B \rightarrow \bot$ by C. Thus $\bot \rightarrow \neg B$ by IOC. Hence $\bot \rightarrow \neg A$ by TR. Therefore $A \rightarrow \top$ by C and LLE.

PCPA (*Possible Consequent to Possible Antecedent*): $(A \rightarrow B) \supset (\Diamond B \supset \Diamond A)$: Suppose $A \rightarrow B$. Thus $\neg B \rightarrow \neg A$ by C. Hence $\Box\neg A \supset \Box\neg B$ by NCNA. Thus $\Diamond B \supset \Diamond A$.

CL (*Consequential Law*): $(\Box(A \supset B) \wedge (\Box B \supset \Box A) \wedge (\Diamond B \supset \Diamond A)) \supset (A \to B)$. If $\Diamond A \wedge \Diamond \neg B$, we obtain $A \to B$ by IL. Thus suppose $\neg(\Diamond A \wedge \Diamond \neg B)$, that is $\Box B$ or $\Box \neg A$. First suppose $\Box B$. Hence $\Box A$ by the second antecedent conjunct. Thus $\neg B \to \bot$ and $\neg A \to \bot$ by C, and therefore also $\bot \to \neg A$ by IOC. TR yields $\neg B \to \neg A$ and thus $A \to B$ by C, LLE and RLE. Second suppose $\Box \neg A$. Hence $\Box \neg B$ by the third antecedent conjunct. Thus by a similar reasoning as above, $A \to B$.

KT+*def CI derives* CI.0: That KT+*def CI* simulates Substitution of provable Equivalents (SE), hence LLE, and derives wBT, TR, C, MI, ID, PFL, IL, IOA was proven by Pizzi (1991). AT follows from ID and wBT. QED.

PROOF OF THEOREM 3. For CUT, AND, and CM, we distinguish the reasoning for $\Rightarrow$ and $\to$. The remaining are proved generically for $\rightarrowtail$, using facts and labels from the proofs of Theorem 1 and 2.

RLE: from LLE by C (see Theorem 1 and 2).

MPC: follows from MI.

wBT: Assume $A \rightarrowtail B$. We have $\neg(A \rightarrowtail \neg A)$ by AT. Contraposing TR yields $\neg((A \to B) \wedge (B \rightarrowtail \neg A))$. Hence $(A \rightarrowtail B) \supset \neg(B \rightarrowtail \neg A)$. By C and LLE, we obtain $(A \to B) \supset \neg(A \to \neg B)$. This is wBT.

AT2: Suppose $A \rightarrowtail B$. Thus $\neg B \rightarrowtail \neg A$ by C. Hence $\neg(\neg B \rightarrowtail A)$ by wBT and RLE. Therefore $\neg(\neg A \rightarrowtail B)$ by C and RLE.

CUT $\Rightarrow$: Suppose $A \Rightarrow B$, $(A \wedge B) \Rightarrow C$. Thus $\Diamond(A \wedge B)$ by PAC, hence $\Diamond A$ by W. Hence $A \Rightarrow (A \wedge B)$ by PFL and LLE. By TR we get $A \Rightarrow C$.

CUT $\to$: Suppose $A \to B$ and $(A \wedge B) \to C$. If $\Diamond(A \wedge B)$, the reasoning is the same as for $\Rightarrow$. Assume $\neg \Diamond(A \wedge B)$, i.e. $\Box(\neg A \vee \neg B)$. Thus $\Box(A \supset \neg B)$, and since $\Box(A \supset B)$ by CSI, we have $\Box \neg A$ by CC and LLE and RLE. Also $\top \to (\neg A \vee \neg B)$. Thus $(A \wedge B) \to \bot$ by C and LLE. Therefore $\bot \to (A \wedge B)$ by IOC. Since $A \to \bot$ we obtain $A \to (A \wedge B)$ by TR. Therefore $A \to C$ by RLE and TR.

AND $\Rightarrow$. See proof of Theorem 1.

AND $\to$. Suppose $A \to B$ and $A \to C$. If $\Diamond(A \wedge B)$, the reasoning is the same as for $\Rightarrow$. If $\neg \Diamond(A \wedge B)$, we obtain, by the same reasoning as for CUT, $\Box \neg A$, thus $\Box \neg B$ by contraposing PCPA, and thus $A \to (B \wedge C)$ (replacing first A by B, and then $A \wedge B$ by $B \wedge C$).

OR: follows from AND, using C, LLE and RLE.

CM $\Rightarrow$. Suppose $A \Rightarrow B$ and $A \Rightarrow C$. [Then $\Diamond(A \wedge B)$ by PAC and $(A \wedge B) \Rightarrow (B \wedge C)$ by PFL.] Similarly $\Diamond(B \wedge C)$, and $\Diamond \neg C$ by C and PA. Hence $(B \wedge C) \Rightarrow C$ by PSI. Therefore $(A \wedge B) \Rightarrow C$ by TR.

CM $\rightarrow$. Suppose $A \rightarrow B$ and $A \rightarrow C$. $A \rightarrow B$ is equivalent to $(A \Rightarrow B) \vee \Box(A \wedge B) \vee \Box(\neg A \wedge \neg B)$. Similarly for $A \rightarrow C$. In the case $(A \Rightarrow B) \wedge (A \Rightarrow C)$, the reasoning is the same as for $\Rightarrow$. If $\Box(A \wedge B)$, we must have $\Box(A \wedge C)$. And thus $\Box C$. Hence $(A \wedge B) \rightarrow C$. If $\Box(\neg A \wedge \neg B)$, we must have $\Box(\neg A \wedge \neg C)$. Thus $\Box \neg(A \wedge B)$ and $\Box \neg C$. Hence again $(A \wedge B) \rightarrow C$. QED.

PROOF OF THEOREM 4. First we prove the *invalidities for CI.*

MC: If MC were valid, we would have $(A \supset B) \supset \Box(A \supset B)$ by CSI, which is invalid in KT.

E: If E were valid, we would have $\bot \rightarrow \top$, contradicting AT.

I: I is invalid, since E is invalid and C valid.

NC: If NC were valid, we would have $A \rightarrow \top$, contradicting I.

IA: If IA were valid, NC would be valid, due to C.

R: Assume for reductio that R holds. Then we would have: $A, B \vdash C$ iff $A \vdash B \rightarrow C$ by MPC. Hence $A \vdash B \rightarrow C$ iff $A \vdash B \supset C$ (since MP and R hold for $\supset$). Hence $\vdash (B \rightarrow C) \equiv (B \supset C)$. This contradicts the invalidity of MC.

SI: If SI were valid, $(A \wedge \neg A) \rightarrow A$, contradicting the invalidity of E.

ADD: If ADD were valid, then $\neg A \rightarrow (\neg A \vee \neg B)$. Thus by C, LLE and RLE, we would get $(A \wedge B) \rightarrow A$, contradicting the invalidity of SI.

CEM: If CEM were valid, we would have $(\top \rightarrow A) \vee \neg(\top \rightarrow \neg A)$. Thus, by (def $\Box_{\rightarrow}$) $\Box A \vee \Box \neg A$, which is equivalent to the invalid $\Diamond A \equiv \Box A$.
SA: If SA were valid, we would have $(A \rightarrow A) \supset ((A \wedge B) \rightarrow A)$. But since ID, we would obtain the invalid SI.

RM (example from Raidl and Gomes (2024)): Consider a world w and $R(w) = \{w, v\}$ such that w is a $p, \neg q, r$ world and v is a $\neg p, \neg q, \neg r$ world. Then we have $\Box(p \supset r)$, $\Diamond p$, $\Diamond \neg r$ in w, and we also have $\Diamond p$ and $\Box \neg q$ in w. Therefore $p \rightarrow r$ and $\neg(p \rightarrow \neg q)$ in w. However, neither $\Box r$, nor $\Box \neg r$ in w, and $\Diamond(p \wedge q)$ is false in w. Therefore $(p \wedge q) \rightarrow r$ is false in w.

RW: If RW were sound, DW (Disjunctive Weakening) would be valid. Indeed, since $\vdash B \supset (B \vee C)$, we would obtain $(A \rightarrow B) \supset (A \rightarrow (B \vee C))$

(DW) by RW. But by C, LLE and RLE, we would then obtain SA, which we proved invalid.

sBT: Pizzi and Williamson (1997) proved that in CI.0: $\vdash X \to Y$ iff $\vdash X \equiv Y$. Hence, if sBT were valid, we would have $(A \to B) \equiv \neg(A \to \neg B)$, contradicting the invalidity of CEM.

S: By CSI, S implies $(A \to B) \supset \Box(B \supset A)$. A countermodel is the reflexive model with worlds w, v, u, such that wRv, wRu, where w is an $A \wedge B$-world, v a $\neg A \wedge B$-world, u a $\neg A \wedge \neg B$-world. Then $A \supset B$ is true in w, v, u. Thus in w: $\Box(A \supset B)$, $\Diamond A, \Diamond \neg B$. Hence $A \Rightarrow B$ and $A \to B$. But $B \to A$ (and hence $B \Rightarrow A$) are false (since $\Box(B \supset A)$ is false in w, due to $B \supset A$ being false in v).

FL: FL would also imply $(A \to B) \supset \Box(B \supset A)$. Indeed, suppose $A \to B$. Hence $\bot \to (B \wedge \neg A)$ by FL and LLE. Thus is $(B \wedge \neg A) \to \bot$ by IOC. Hence $\top \to (B \supset A)$ by C and RLE. This is $\Box(B \supset A)$. To disprove FL, we can thus take the same countermodel as for S.

We now prove the *invalidities for IC* (cf. Raidl and Gomes, 2024, Fact 5).

If one of MC, E, I, R, ADD, or CEM were valid for $\Rightarrow$, they would also be valid for $\to$ (since $\Rightarrow$ implies $\to$), contradicting the invalidity results for $\to$. For NC, IA, RW, S and RM, the reasoning is the same as for $\to$. SA: If SA were valid, we would have $(A \Rightarrow A) \supset ((A \wedge \bot) \Rightarrow A)$. Consider A contingent, then $A \Rightarrow A$ (by PID, see later). Hence $\bot \Rightarrow A$ by LLE. But then $\Diamond \bot$ by PA. That is $\bot \Rightarrow \bot$, contradicting NE.

sBT: If sBT were valid, we would have $\Diamond(A \Rightarrow B)$ for any A, B (by PA). Thus $\Diamond(\bot \Rightarrow \bot)$. But $\vdash \neg(\bot \Rightarrow \bot)$ by NE. Thus $\Box\neg(\bot \Rightarrow \bot)$ by N, i.e. $\neg\Diamond(\bot \Rightarrow \bot)$. This contradicts $\Diamond(\bot \Rightarrow \bot)$.

FL: FL would imply $(A \Rightarrow A) \supset (A \wedge \neg A) \Rightarrow (A \wedge \neg A)$. For A contingent, we have $A \Rightarrow A$ (see PID). Assuming there is a contingent A, we could derive $\bot \Rightarrow \bot$ by LLE and RLE, contradicting NE. QED.

PROOF OF THEOREM 5. PM′: By IL and CSI.

PSI: By IL and $\Box((A \wedge B) \supset A)$ (N).

PRW′: Suppose $\vdash B \supset C$, and assume $A \to B$, $\Diamond A$ and $\Diamond \neg C$. By CSI, $\Box(A \rightarrowtail B)$. Hence $\Box(A \to C)$. By IL, we obtain $A \rightarrowtail C$.

PID: By PSI, LLE and RLE. QED.

PROOF OF THEOREM 6. ID is part of CI.0. But ID is invalid for IC (Raidl and Gomes, 2024, Fact 5), since A might not be contingent. QED.

Acknowledgments. We are grateful to two anonymous reviewers, whose detailed comments led to substantial improvement in the paper.

Authors' Contributions. GG had the idea of the paper, outlined its plan and wrote most of the Introduction. CP suggested the cubes of opposition and the interdefinability of the two conditionals. ER found the common core axiomatisation of the two conditionals. CP and GG wrote most of Sections 2–4 and 8–9. ER wrote Section 5, most of Section 6, and Appendices A–B. All authors have contributed equally to Sections 7 and 10, and to the revision of all sections.

Funding. Eric Raidl's work was funded by the Deutsche Forschungsgemeinschaft (EXC number 2064/1, project no. 390727645).

References

Anderson, A. R., and N. D. Belnap, 1975, *Entailment: The Logic of Relevance and Necessity*, volume 1, Princeton University Press, Princeton.

Angell, R. B., 1962, "A propositional logic with subjunctive conditionals", *The Journal of Symbolic Logic*, 27(3): 327–343. DOI: 10.2307/2964651

Åqvist, L., 1973, "Modal logic with subjunctive conditionals and dispositional predicates", *Journal of Philosophical Logic*, 2: 1–76. DOI: 10.1007/bf02115609

Bennett, J., 2003, *A Philosophical Guide to Conditionals*, Oxford University Press, New York. DOI: 10.1093/0199258872.001.0001

Blanshard, B., 1969, *The Nature of Thought*, G. Allen & Unwin, London.

Bolzano, B., 1978, *Grundlegung der Logik. Wissenschaftslehre I/II*, Felix Meiner Verlag, Hamburg. French translation: *Théorie de la Science* Paris: Gallimard, 2011. DOI: 10.28937/978-3-7873-2626-6

Burgess, J. P., 1981, "Quick completeness proofs for some logics of conditionals", *Notre Dame Journal of Formal Logic*, 22(1): 76–84. DOI: 10.1305/ndjfl/1093883341

Burks, A. W., 1955, "Dispositional statements", *Philosophy of Science*, 22(3): 175–193. DOI: 10.1086/287422

Chellas, B. F., 1975, "Basic conditional logic", *Journal of Philosophical Logic*, 4(2): 133–153. DOI: 10.1007/BF00693270

Crupi, V., and A. Iacona, 2022, "On the logical form of concessive conditionals", *Journal of Philosophical Logic*, 51: 633–651. DOI: 10.1007/s10992-021-09645-1

Davis, W. A., 1983, "Weak and strong conditionals", *Pacific Philosophical Quarterly*, 64(1): 57–71. DOI: 10.1111/j.1468-0114.1983.tb00184.x

Douven., I., 2015, *The Epistemology of Indicative Conditionals: Formal and Empirical Approaches*, Cambridge University Press, Cambridge.

Douven, I., 2017, "How to account for the oddness of missing-link conditionals", *Synthese*, 194: 1541–1554. DOI: 10.1007/s11229-015-0756-7

Ducrot, O., 1972/1991, *Dire et ne pas dire* Hermann, Paris.

Edgington, D., 1995, "On conditionals", *Mind*, 104(414): 235–329. DOI: 10.1093/mind/104.414.235

Gherardi, G., and E. Orlandelli, 2021, "Super-strict implications", *Bulletin of the Section of Logic*, 50(1): 1–34. DOI: 10.18778/0138-0680.2021.02

Gherardi, G., E. Orlandelli, and E. Raidl, 2024, "Proof systems for super-strict implication", *Studia Logica*, 112: 249–294. DOI: 10.1007/s11225-023-10048-3

Gibbard, A., 1981, "Two recent theories of conditionals", pages 211–247 in W. L. Harper, R. Stalnaker and G. Pearce (eds.), *Ifs: Conditionals, belief, decision, chance and time*, The University of Western Ontario Series in Philosophy of Science, vol. 15, Springer, Dordrecht. DOI: 10.1007/978-94-009-9117-0_10

Gomes, G., 2005, "Ordinary language conditionals", Manuscript. https://www.academia.edu/93865496/Ordinary_Language_Conditionals_2005

Gomes, G., 2009, "Are necessary and sufficient conditions converse relations?", *Australasian Journal of Philosophy*, 87(3): 375–387. DOI: 10.1080/00048400802587325

Gomes, G., 2013, "Pensamento e linguagem nas afirmações condicionais", *DELTA: Documentação de Estudos em Linguística Teórica e Aplicada*, 29 (1): 121–134. DOI: 10.1590/s0102-44502013000100006

Gomes, G., 2019, "Meaning-preserving contraposition of natural language conditionals", *Journal of Pragmatics*, 152: 46–60. DOI: 10.1016/j.pragma.2019.08.003

Gomes, G., 2020, “Concessive conditionals without ‘even if’ and nonconcessive conditionals with ‘even if’ ”, *Acta Analytica*, 35(1): 1–21. DOI: 10.1007/s12136-019-00396-y

Gomes, G., 2024, “Necessary and sufficient conditions, counterfactuals and causal explanations”, *Erkenntnis*, 89(8): 3085–3108. DOI: 10.1007/s10670-023-00668-5

Humberstone, I., 1978, “Two merits of the circumstantial operator language for conditional logics”, *Australasian Journal of Philosophy*, 56(1): 21–24. DOI: 10.1080/00048407812341011

Jackson, F., 1979, “On assertion and conditionals”, *Philosophical Review*, 88: 565–589. DOI: 10.2307/2184845

Kielkopf, C. F., 1977, *Formal Sentential Entailment*, University Press of America, Washington.

Lenzen, W., 2021, “The third and fourth stoic accounts of conditionals”, pages 127–146 in M. Blicha and I. Sedlár (eds.), *The Logica Yearbook 2020*, College Publications, Rickmansworth.

Lewis, D., 1973, *Counterfactuals*, Blackwell, Oxford. DOI: 10.1007/978-94-009-9117-0_3

Lycan, W. G., 2001, *Real Conditionals*, Oxford University Press, Oxford. DOI: 10.1093/oso/9780199242078.001.0001

Martin, C., 2004, “Logic”, pages 158–199 in J. Brower and K. Guilfoy (eds.), *The Cambridge Companion to Abelard*, Handbook of the History of Logic, Cambridge University Press, Cambridge.

McCall, S., 1966, “Connexive implication”, *The Journal of Symbolic Logic*, 31 (3): 415–433. DOI: 10.2307/2270458

McCall, S., 2012, “A history of connexivity”, pages 415–449 in D. M. Gabbay, F. J. Pelletier, and J. Woods (eds.), *Logic: A History of its Central Concepts*, volume 11 of *Handbook of the History of Logic*, North-Holland, Amsterdam. DOI: 10.1016/b978-0-444-52937-4.50008-3

Meyer, R. K., 1977, “S5–The poor man’s connexive implication”, *The Relevance Logic Newsletter*, 2: 117–123.

Mortensen, C., 1984, “Aristotle’s thesis in consistent and inconsistent logics”, *Studia Logica*, 43(1/2): 107–116. DOI: 10.1007/BF00935744

Nasti de Vincentis, M., 2006, “Conflict and connectedness: Between modern logic and the history of ancient logic”, pages 229–251 in E. Ballo and M. Franchella (eds.), *Logic and Philosophy in Italy*, Polimetrica International Scientific Publisher, Monza.

Pizzi, C., 1980, "'Since', 'even if', 'as if' ", pages 73–87 in M. L. Dalla Chiara (ed.), *Italian Studies in the Philosophy of Science*, Springer, Dordrecht. DOI: 10.1007/978-94-009-8937-5_6

Pizzi, C., 1991, "Decision procedures for logics of consequential implication", *Notre Dame Journal of Formal Logic*, 32(4): 618–636. DOI: 10.1305/ndjfl/1093635934

Pizzi, C., 1993, "Causality and the transitivity of counterfactuals", *O que nos faz pensar*, 7: 89–103.

Pizzi, C., 2018, "Two kinds of consequential implication", *Studia Logica*, 106: 453–480. DOI: 10.1007/s11225-017-9749-5

Pizzi, C., 2022, "Axioms for a logic of consequential counterfactuals", *Logic Journal of the IGPL*, 31(5): 907–925. DOI: 10.1093/jigpal/jzac052

Pizzi, C., 2024, "An introduction to Boethian logics", pages 79–110 in H. Omori and H. Wansing (eds.), *60 Years of Connexive Logics*, Springer.

Pizzi, C., and T. Williamson, 1997, "Strong Boethius' thesis and consequential implication", *Journal of Philosophical Logic*, 26(5): 569–588. DOI: 10.1023/a:1004230028063

Pollock, J. L., 1976, *Subjunctive Reasoning*, D. Reidel, Dordrecht.

Priest, G., 1999, "Negation as cancellation, and connexive logic", *Topoi*, 18: 14–148. DOI: 10.1023/A:1006294205280

Raidl, E., 2019, "Completeness for counter-doxa conditionals – using ranking semantics", *The Review of Symbolic Logic*, 12(4): 861–891. DOI: 10.1017/S1755020318000199

Raidl, E., 2021a, "Definable conditionals", *Topoi*, 40(1): 87–105. DOI: 10.1007/s11245-020-09704-3

Raidl, E., 2021b, "Strengthened conditionals", pages 139–155 in B. Liao and Y. N. Wáng (eds.), *Context, Conflict and Reasoning. Logic in Asia Series*, Springer, Singapore. DOI: 10.1007/978-981-15-7134-3_11

Raidl, E., 2021c, "Three conditionals: Contraposition, difference-making and dependency", pages 201–217 in M. Blicha and I. Sedlár (eds.), *The Logica Yearbook 2020*, College Publications.

Raidl, E., 2023, "Neutralization, Lewis' doctored conditional, or another note on 'a connexive conditional' ", *Logos & Episteme*, 14(1): 101–118. DOI: 10.5840/logos-episteme20231415

Raidl, E., 2025a (forthcoming), "Between Lewis and Burgess I: Logics without rational monotonicity", *Journal of Philosophical Logic*, pages 1–38.

Raidl, E., 2025b (forthcoming), "Between Lewis and Burgess II : Semantics without rational monotonicity", *Journal of Philosophical Logic*, pages 1–52.

Raidl, E., and G. Gomes, 2024, "The implicative conditional", *Journal of Philosophical Logic*, 53(1): 1–47. DOI: 10.1007/s10992-023-09715-6

Raidl, E., A. Iacona, and V. Crupi, 2021, "The logic of the evidential conditional", *Review of Symbolic Logic*, 15(3): 758–770. DOI: 10.1017/S1755020321000071

Raidl, E., A. Iacona, and V. Crupi, 2023, "An axiomatic system for concessive conditionals," *Studia Logica*, 112: 343–363. DOI: 10.1007/s11225-022-10034-1

Raidl, E., and H. Rott, 2023, "Threshold-based belief change", *Australasian Journal of Logic*, 20(3): 429–477. DOI: 10.26686/ajl.v20i3.7408

Raidl, E., and H. Rott, 2024, "Towards a logic for 'because' ", *Philosophical Studies*, 181: 2247–2277. DOI: 10.1007/s11098-023-01998-4

Ramsey, F. P., 1929, "General propositions and causality", pages 133–151 in H. A. Mellor (ed.), *Philosophical Papers*, Cambridge University Press, Cambridge. DOI: 10.1017/cbo9780511527494.006

Rott, H., 1986, "Ifs, though and because", *Erkenntnis*, 25(3): 345–370. DOI: 10.1007/BF00175348

Rott, H., 2020, "Notes on contraposing conditionals". https://philsci-archive.pitt.edu/17092/

Rott, H., 2022, "Difference-making conditionals and the relevant Ramsey test", *Review of Symbolic Logic*, 15(1): 133–164. DOI: 10.1017/S1755020319000674

Rott, H., 2024a, "Evidential support and contraposition", *Erkenntnis*, 89: 2253–2271. DOI: 10.1007/s10670-022-00628-5

Rott, H., 2024b, "Difference-making conditionals and connexivity", *Studia Logica*, 112: 405–458. DOI: 10.1007/s11225-023-10071-4

Routley, R., 1978, "Semantics for connexive logics. I", *Studia Logica*, 37(4): 393–412. DOI: 10.1007/BF02176171

Routley, R., and V. Routley, 1985, "Negation and contradiction", *Revista Colombiana de Matematicas*, 19(1–2): 201–230. https://repositorio.unal.edu.co/handle/unal/48791

Spohn, W., 2013, "A ranking-theoretic approach to conditionals", *Cognitive Science*, 37(6): 1074–1106. DOI: 10.1111/cogs.12057

Spohn, W., 2015, "Conditionals: A unifying ranking-theoretic perspective", *Philosophers' Imprint*, 15(1): 1–30. http://hdl.handle.net/2027/spo.3521354.0015.001

Stalnaker, R. C., 1968, "A theory of conditionals", pages 98–112 in N. Rescher (ed.), *Studies in Logical Theory*, American Philosophical Quarterly Monographs 2, Blackwell, Oxford. DOI: 10.1007/978-94-009-9117-0_2

Strawson, P. F., 1948, "Necessary propositions and entailment-statements", *Mind*, 57(226): 184–200. DOI: 10.1093/mind/lvii.226.184

Thompson, B. E., 1991, "Why is conjunctive simplification invalid?", *Notre Dame Journal of Formal Logic*, 32(2): 248–254. DOI: 10.1305/ndjfl/1093635749

Tichý, P., 1984, "Subjunctive conditionals: Two parameters vs. three", *Philosophical Studies*, 45(2): 147–179. DOI: 10.1007/bf00372476

Wansing, H., and H. Omori, 2024, "Connexive logic, connexivity, and connexivism: Remarks on terminology", *Studia Logica*, 112: 1–35. DOI: 10.1007/s11225-023-10082-1

Wansing, H., and D. Skurt, 2018, "Negation as cancellation, connexive logic, and qLPm", *The Australasian Journal of Logic*, 15(2): 476–488. DOI: 10.26686/ajl.v15i2.4869

GILBERTO GOMES
PGCL, State University of Norte Fluminense Darcy Ribeiro
Rio de Janeiro, Brazil
ggomes@uenf.br
https://orcid.org/0000-0002-3667-3851

CLAUDIO PIZZI
Prof. Emeritus
Università di Siena, Italy
pizzi4@gmail.com
https://orcid.org/0000-0003-0124-6566

ERIC RAIDL
Cluster of Excellence "Machine Learning for Science"
Eberhard Karls Universität
Tübingen, Germany
eric.raidl@uni-tuebingen.de
https://orcid.org/0000-0001-6153-4979