Logic and Logical Philosophy
Online First Articles (2024)
DOI: 10.12775/LLP.2024.032

Diego Fernandes

Comparing Expressiveness of Logics Defined within Different Classes of Models

An analysis of some formal conditions

Abstract. It is possible to understand the expressive power of a logic as issuing from its capacity to express properties of its models. There are some ways to formally capture whether a property of models is expressible, among them is one based on the notion of definability, and one based on the notion of discrimination. If the logics to be compared are defined within the same class of models, one can employ the notions of definability and discrimination directly to obtain formal conditions for relative expressiveness. This paper studies generalizations of these formal conditions to cases where the compared logics are defined within different classes of models. There have been proposed in the literature formal conditions of two main kinds: with forward and with backward model-mappings. It is shown that none of them is adequate, despite their initial reasonableness. Moreover, we argue that general and reasonable formal conditions for relative expressiveness involving forward mappings are not likely to be found, given that they turn out to be highly dependent on specific features of the compared logics. On the other hand, it will be argued that there is a reasonable formal condition involving backward model-mappings.

Keywords: model-theoretic logics; model-mappings; relative expressiveness; formal conditions

1. Introduction

In (Fernandes, 2023), comparisons of expressiveness of logics defined within the same classes of models are studied. The purpose of this paper is to expand these investigations for comparisons of expressiveness between logics defined within different classes of models.

Received January 10, 2024. *Accepted* November 28, 2024. *Published online* December 13, 2024

Here, the term 'logic' is to mean 'model-theoretic logic', that is

DEFINITION 1.1 (Model-theoretic Logic). A model-theoretic logic $\mathcal{L}_i$, for $i \in \mathbb{N}$, is a sequence $(\mathcal{M}_i, \mathcal{S}_i, \Vdash_i)$, where $\mathcal{M}_i$ and $\mathcal{S}_i$ are classes and $\Vdash_i \subseteq \mathcal{M}_i \times \mathcal{S}_i$.

Here, $\mathcal{M}_i$ is intended to be the class of models for $\mathcal{L}_i$, $\mathcal{S}_i$ the class of well formed sentences of $\mathcal{L}_i$ in every vocabulary, and $\Vdash_i$ the corresponding satisfaction relation. All logics $\mathcal{L}_i$ considered in this paper are supposed to satisfy the usual basic properties for model-theoretic logics, as listed in (Ebbinghaus, 1985, p. 28), e.g.:

Isomorphism property: *Isomorphic models cannot be distinguished by* $\mathcal{L}_i$

Let us fix the remaining notation that will be used in the paper:

$\mathscr{P}(X)$ – the power-set of X,
$\mathfrak{A}_i, \mathfrak{B}_i$ – arbitrary models,
$\mathbf{Mod}_{\mathcal{L}}(\phi)$ – the class of models satisfying ϕ in $\mathcal{L}$,
$\equiv_i$ – equivalence of models under $\mathcal{L}_i$,
$\preccurlyeq_Y$ – expressiveness relation on logics with respect to condition Y.

DEFINITION 1.2 (Property of models). Let $\mathcal{M}$ be a class of models. A property P of models will be taken to be a subclass of $\mathcal{M}$.

Following Fernandes (2023), the relation '$\mathcal{L}_2$ is at least as expressive as $\mathcal{L}_1$' will be refined as

E^*: Every property of models expressible in $\mathcal{L}_1$ is also expressible in $\mathcal{L}_2$.

As it was argued in (Fernandes, 2023), there are some ways to understand when a property of models is expressible in a logic $\mathcal{L}$, among them are *definability* and *discrimination*. From each one we can obtain precise renderings of E^*:

PROPOSITION 1.1 (Definability – $\preccurlyeq_{\mathrm{DC}}$). *$P$ is expressible in $\mathcal{L}_i$ when it is* definable *in $\mathcal{L}_i$, that is, whenever $P = \mathbf{Mod}_{\mathcal{L}_i}(\phi)$, for $\phi \in \mathcal{S}_i$. Then a formal rendering of E^* is obtained*:

($\mathcal{L}_1 \preccurlyeq_{\mathrm{DC}} \mathcal{L}_2$) Every definable property in $\mathcal{L}_1$ is also definable in $\mathcal{L}_2$.

PROPOSITION 1.2 (Discrimination – $\preccurlyeq_{\mathrm{EQ}}$). *$P$ is expressible in $\mathcal{L}_i$ when it can be discriminated from its complement in $\mathcal{L}_i$, that is, whenever*

$$\text{If } \mathfrak{A} \in P \text{ and } \mathfrak{B} \notin P, \text{ then } \mathfrak{A} \not\equiv_i \mathfrak{B}.$$

Another formal rendering of E^* is thus obtained:

($\mathcal{L}_1 \preccurlyeq_{\mathrm{EQ}} \mathcal{L}_2$) *Every property that can be discriminated from its complement in $\mathcal{L}_1$, can also be discriminated from its complement in $\mathcal{L}_2$.*

Notice that these conditions of expressiveness belong to a framework for comparing expressive power where the logics involved are defined within the same class of models, the so-called "uni-class" framework. Now when one is dealing with logics defined withing different classes of models, it is not possible to directly compare them with respect to their capacity of expressing properties of their models. The establishment of some sort of congruence between properties of models in each logic is needed beforehand. One way to do this is by defining model mappings f or g such that:

DESIDERATA 1.1 (Model mappings and expressibility of properties).

(a) $f\colon \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ *is such that for every* $P \subseteq \mathcal{M}_1$, *it holds that* P *is to* $\mathcal{L}_1$ *as* $f[P]$ *is to* $\mathcal{L}_2$, *or*
(b) $g\colon \mathcal{M}_2 \longrightarrow \mathcal{M}_1$ *is such that for every* $P \subseteq \mathcal{M}_1$ *of the form* $g[Q]$, *it holds that* $g[Q]$ *is to* $\mathcal{L}_1$ *as* Q *is to* $\mathcal{L}_2$.[1]

In the case (b), it is reasonable to require that g be surjective or at least satisfy a weaker variant (see Definition 2.2), in order to make sure that for every property in $\mathcal{M}_1$, there corresponds a property in $\mathcal{M}_2$.

Such a framework of expressiveness comparisons involving model translations will be called here 'multi-class expressiveness'. In the sequence some formal conditions in this framework which generalize $\preccurlyeq_{\mathrm{EQ}}$ and $\preccurlyeq_{\mathrm{DC}}$ will be explored.

2. Multi-class expressiveness

2.1. Generalizations of Discrimination

Let the logics $\mathcal{L}_1 = (\mathcal{M}_1, \mathcal{S}_1, \Vdash_1)$ and $\mathcal{L}_2 = (\mathcal{M}_2, \mathcal{S}_2, \Vdash_2)$ be defined within different classes of models $\mathcal{M}_1$ and $\mathcal{M}_2$. There are at least two ways to generalize $\preccurlyeq_{\mathrm{EQ}}$ to the multi-class framework, based on forward or

[1] One may also consider cases where the class of models of $\mathcal{L}_2$ must be restricted, so that it makes sense to compare it with the other logic. Such a situation is considered in Section 2.10 below.

backward mappings (Desiderata 1.1). In what follows, a generalization based on forward mappings is analysed.

Let $f\colon \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ be a mapping.

DEFINITION 2.1 (Discrimination preservation). The mapping f is a discrimination preservation from $\mathcal{L}_1$ to $\mathcal{L}_2$ whenever

- For each $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{M}_1$, if $\mathfrak{A}_1 \not\equiv_1 \mathfrak{A}_2$, then $f(\mathfrak{A}_1) \not\equiv_2 f(\mathfrak{A}_2)$.

Kocurek (2018, p. 151) gives a generalization for $\preccurlyeq_{\mathrm{EQ}}$, called 'model-coarsening', which is based on backward mappings:

DEFINITION 2.2. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be logics and $g\colon \mathcal{M}_2 \longrightarrow \mathcal{M}_1$. Then g is a model-coarsening from $\mathcal{L}_2$ to $\mathcal{L}_1$ if:

- For each $\mathfrak{A} \in \mathcal{M}_1$, there is a $\mathfrak{B} \in M_2$ such that $\mathfrak{A} \equiv_1 g(\mathfrak{B})$;
- For each $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{M}_2$, if $g(\mathfrak{B}_1) \not\equiv_1 g(\mathfrak{B}_2)$, then $\mathfrak{B}_1 \not\equiv_2 \mathfrak{B}_2$.

FACT 2.1. *If g is a model coarsening of $\mathcal{L}_2$ to $\mathcal{L}_1$, then there is a discrimination preservation f from $\mathcal{L}_1$ to $\mathcal{L}_2$.*

PROOF. Define $f\colon \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ as follows:

- For $\mathfrak{A} \in \mathcal{M}_1$, $f(\mathfrak{A}) = \mathfrak{B}$, for some $\mathfrak{B}$ such that $g(\mathfrak{B}) \equiv_1 \mathfrak{A}$.

Take $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{M}_1$ such that $\mathfrak{A}_1 \not\equiv_1 \mathfrak{A}_2$. Take $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{M}_2$ such that $f(\mathfrak{A}_i) = \mathfrak{B}_i$, $i \in \{1, 2\}$. As $g(\mathfrak{B}_1) \equiv_1 \mathfrak{A}_1$ and $g(\mathfrak{B}_2) \equiv_1 \mathfrak{A}_2$, we have that $g(\mathfrak{B}_1) \not\equiv_1 g(\mathfrak{B}_2)$, which implies by the second item of Definition 2.2 that $\mathfrak{B}_1 \not\equiv_2 \mathfrak{B}_2$, and thus, $f(\mathfrak{A}_1) \not\equiv_2 f(\mathfrak{A}_2)$. ⊣

FACT 2.2. *If f is a discrimination preservation from $\mathcal{L}_1$ to $\mathcal{L}_2$, then there is a model coarsening g from $\mathcal{L}_2$ to $\mathcal{L}_1$.*

PROOF. Take f and select a partial function g^* from f^{-1}. Expand g^* to a total function g obeying the rule that if $\mathfrak{B}_1 \equiv_2 \mathfrak{B}_2$, then $g(\mathfrak{B}_1) \equiv_1 g(\mathfrak{B}_2)$. That g satisfies both items of Definition 2.2 is clear. ⊣

FACT 2.3. *If $f\colon \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ is a surjective discrimination preservation and P is expressible by discrimination in $\mathcal{L}_1$, then $f[P]$ is expressible by discrimination in $\mathcal{L}_2$.*

Let

$$[\mathcal{M}_1]^{\sim}_{\mathcal{L}_1} = \{X \in \mathscr{P}(\mathcal{M}_1) \mid \text{for all } \mathfrak{A}_1, \mathfrak{A}_2 \in X,\ \mathfrak{A}_1 \equiv_1 \mathfrak{A}_2 \text{ holds}\}.$$

FACT 2.4. *If $|[\mathcal{M}_1]^{\sim}_{\mathcal{L}_1}| \leq |[\mathcal{M}_2]^{\sim}_{\mathcal{L}_2}|$, then there is a discrimination preservation $f\colon \mathcal{M}_1 \longrightarrow \mathcal{M}_2$.*

Fact 2.5. *Let $\mathcal{L}$ be any logic. Then there is a surjective discrimination preservation from $\mathcal{L}$ to a propositional logic defined in a language having no operators and only m propositional variables, for some number m.*

The adequacy of discrimination preservation and model coarsening as formal measures of relative expressiveness depends entirely on the prior reasonability of the congruence relation established by the model mappings between the logics. However, to obtain a reasonable notion of congruence of models in different logics only by imposing restrictions on model-mappings — besides very general properties of functions — is a tricky enterprise. As it will be seen below, when expressibility of properties is understood on the basis of definability, the situation is improved, since one can control much more the formula translations and thus regulate better how the satisfability of the target logic must behave.

2.2. Generalizations of Definability

As with the case of $\preccurlyeq_{\mathrm{EQ}}$, there are also at least two ways to generalize $\preccurlyeq_{\mathrm{DC}}$ to the multi-class framework. The first, based on forward mappings, is as follows:

Definition 2.3. ($\mathcal{L}_1 \preccurlyeq_{\mathrm{DC_{G1}}} \mathcal{L}_2$) There is a map $f\colon \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ and a map $\mathcal{T}\colon \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ such that for every $\mathfrak{A} \in \mathcal{M}_1$ and $\phi \in \mathcal{S}_1$:

- $\mathfrak{A} \Vdash_1 \phi$ iff $f(\mathfrak{A}) \Vdash_2 \mathcal{T}(\phi)$.

Definition 2.4. ($\mathcal{L}_1 \preccurlyeq_{\mathrm{DC_{G2}}} \mathcal{L}_2$) There is a map $g\colon \mathcal{M}_2 \longrightarrow \mathcal{M}_1$ and a map $\mathcal{T}\colon \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ such that for every $\mathfrak{B} \in \mathcal{M}_2$ and $\phi \in \mathcal{S}_1$:

- $g(\mathfrak{B}) \Vdash_1 \phi$ iff $\mathfrak{B} \Vdash_2 \mathcal{T}(\phi)$.

A couple of immediate facts on $\preccurlyeq_{\mathrm{DC_{G1}}}$ and $\preccurlyeq_{\mathrm{DC_{G2}}}$:

Fact 2.6. *If $\mathcal{L}_1 \preccurlyeq_{\mathrm{DC_{G1}}} \mathcal{L}_2$, f is surjective and ϕ is $\mathcal{L}_1$-valid, then $\mathcal{T}(\phi)$ is $\mathcal{L}_2$-valid.*

Fact 2.7. *If $\mathcal{L}_1 \preccurlyeq_{\mathrm{DC_{G2}}} \mathcal{L}_2$ and ϕ is $\mathcal{L}_1$-valid, then $\mathcal{T}(\phi)$ is $\mathcal{L}_2$-valid.*

2.2.1. On the conditions of the form $\preccurlyeq_{\mathrm{DC_{G1}}}$

The condition $\preccurlyeq_{\mathrm{DC_{G1}}}$ is not intuitively sufficient to qualify as capturing relative expressiveness, as one can easily come up with a counterexample:

A trivial translation. Let $\mathcal{L}_1 = (\mathcal{M}_1, \mathcal{S}_1, \Vdash_1)$ be any logic with a countable language and let $\mathcal{L}_2 = (\mathcal{M}_2, \mathcal{S}_2, \Vdash_2)$ be a propositional logic where

$\mathcal{S}_2$ is a countable set of propositional variables but with no connectives and where $\mathcal{M}_2$ is set of valuations on S_2. Define $\mathcal{T}$ to map every $\phi \in \mathcal{S}_1$ to a propositional variable $p_\phi \in \mathcal{S}_2$, and define f_t to map a model $\mathfrak{A} \in \mathcal{M}_1$ to a valuation $v \in \mathcal{M}_2$ such that $v \Vdash_2 p_\phi$ iff $\mathfrak{A} \Vdash_1 \phi$. Then clearly it holds that $\mathcal{L}_1 \preccurlyeq_{\mathrm{DC_{G1}}} \mathcal{L}_2$.

Defining $\preccurlyeq_{\mathrm{k}}$. Call 'truth-preserving' all the involved translation pairs in expressive comparisons which are either of the form of Definition 2.3 or 2.4. In his doctorate thesis, Kuijer (2014) studied the expressiveness of various logics of knowledge and action, and selected some results in the literature involving truth-preserving translations as prototypical comparisons in the multi-class framework (cf., e.g., Broersen et al., 2006a; Broersen et al., 2006b; Gasquet and Herzig, 1996; Goranko and Jamroga, 2005; Thomason, 1974). With the aim of investigating further common features, the author went through a number of conditions, among which the following is considered necessary:

DEFINITION 2.5 (Model based). A translation $(\mathcal{T}, f)$ is model based if there are two functions f_1, f_2 such that for all $(\mathfrak{M}, w) \in \mathcal{M}_1$, we have that $f(\mathfrak{M}, w) = (f_1(\mathfrak{M}), f_2(\mathfrak{M}, w))$.

A model based translation would force f to preserve some structure of $\mathfrak{M}$ and prevent that the pointed models $(\mathfrak{M}, w)$ and $(\mathfrak{M}, w')$ be translated to completely unrelated models.

As for the formula-translation, an intuitive condition that came up to the front was that a reasonable translation should be given by a finite number of clauses, and be defined inductively through the formation of formulas. A standard precise way to capture this intuitive desideratum is to require that the translation be *schematic*:

DEFINITION 2.6 (Schematic translation). For each n-ary operator $*$ in $\mathcal{L}_1$ and $\mathcal{L}_1$-formulas $\phi_1, \ldots, \phi_n$, there is a formula $\theta^*(p_1, ..., p_n)$ of $\mathcal{L}_2$ such that $\mathcal{T}(*(\phi_1, \ldots, \phi_n)) = \theta^*(\mathcal{T}(\phi_1)/p_1, \ldots, \mathcal{T}(\phi_n)/p_n)$.

However, in such translations each operator must be translated one at a time, and some of the prototypical translations selected by Kuijer require a contextual translation of operators, for example, in the presented formula translation of the public announcement logic to basic modal logic (Kuijer, 2014, p. 109) there are clauses such as

$$\mathcal{T}([\phi_1]\Box_a\phi_2) = \mathcal{T}(\phi_1) \to \Box_a\mathcal{T}([\phi_1]\phi_2),$$
$$\mathcal{T}([\phi_1][\phi_2]\phi_3) = \mathcal{T}([\phi_1 \wedge [\phi_1]\phi_2]\phi_3).$$

To capture the general form of this sort of mappings, a rather complex notion of a *finitely generated* translation is given. Instead of reproducing it here, a more perspicuous and intuitive reading of it will be presented.

What seems to be the underlying idea of the finitely generated translations — which is not explicit in Kuijer's proposal — is that in order to allow some mapping $\mathcal{T}$ do be defined with respect to a block of operators, one needs auxiliary mappings $\mathcal{T}_1$, $\mathcal{T}_2$, ... of arbitrary arities. They are responsible for parsing the relevant parts of the translated sentence into the appropriate schema with respect to which either another auxiliary or the main translation will be called. For example, the clauses above can be formulated more precisely in these terms as:

$$\begin{aligned}\mathcal{T}([\phi_1]\Box_a\phi_2) &= \mathcal{T}_{[\cdot]}(\phi_1, \Box_a\phi_2)\\ &= \mathcal{T}(\phi_1) \to \Box_a\mathcal{T}([\phi_1]\phi_2),\\ \mathcal{T}([\phi_1][\phi_2]\phi_3) &= \mathcal{T}_{[\cdot]}(\phi_1, [\phi_2]\phi_3)\\ &= \mathcal{T}([\phi_1 \wedge [\phi_1]\phi_2]\phi_3).\end{aligned}$$

For formulas $*(\phi_1, \ldots, \phi_n)$ in which the main operator $*$ do not need special treatment, the translation clause for it will be schematic.

In order to define the general form of finitely generated translations $\mathcal{T}\colon \mathcal{L}_1 \longrightarrow \mathcal{L}_2$, consider the following definition.

Definition 2.7. (Translation metalanguage $\mathcal{L}^{\mathcal{T}}$) For $\mathcal{L}_1$ operators $\#_i$, $1 \leq i \leq n$, of arity $l(i)$ and their respective auxiliary mappings $\mathcal{T}_1, \ldots, \mathcal{T}_n$, define the translation metalanguage $\mathcal{L}^{\mathcal{T}}$ as containing the source and target languages $\mathcal{L}_1, \mathcal{L}_2$, formulas of the sort $\mathcal{T}(\phi)$ and $\mathcal{T}_i(\phi_1, \ldots, \phi_{l(i)})$, where $\phi, \phi_1, \ldots, \phi_{l(i)} \in \mathcal{L}^{\mathcal{T}}$, and is closed under of operators $\#_i$ from $\mathcal{L}_1$ and the operators $*_j$, $1 \leqslant j \leqslant m$, from $\mathcal{L}_2$.

In this way, the general form of the of the compound translation $\mathcal{T}\colon \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ of an n-ary operator $\#$ is:

$$\mathcal{T}(\#(\phi_1, \ldots, \phi_n)) = \mathcal{T}_{\#}(\phi_1, \ldots, \phi_n),$$

where $\mathcal{T}_{\#}\colon \mathcal{L}^{\mathcal{T}} \overbrace{\times \cdots \times}^{n} \mathcal{L}^{\mathcal{T}} \longrightarrow \mathcal{L}^{\mathcal{T}}$.

By being inductively defined and thus respecting the structure of the formulas, the finitely generated translations would be essentially different from the ones appearing in the trivial cases analysed. Thus, Kuijer concludes that the truth-preserving translations that are finitely generated and model-based could capture relative expressiveness, so the final criterion given for multi-class expressiveness is (Kuijer, 2014, p. 111):

DEFINITION 2.8 ($\preccurlyeq_{\mathrm{k}}$). $\mathcal{L}_1 \preccurlyeq_{\mathrm{k}} \mathcal{L}_2$ iff there is a translation $(\mathcal{T}, f)$ from $\mathcal{L}_1$ to $\mathcal{L}_2$ that is model based, finitely generated and truth preserving.

A problem with $\preccurlyeq_{\mathrm{k}}$. Kuijer had no pretensions that his multi-class definition were to be *the* generalization of expressiveness as given by the uni-class framework. The aim was to find only *a* "reasonable generalization" (Kuijer, 2014, p. 83). While keeping this in mind, we would like to argue that his proposal is still not good enough as a formal condition for multi-class expressiveness. This is because it is not hard to find 'natural' logics $\mathcal{L}$ arguably *more* expressive than classical propositional logic ($\mathcal{CPL}$), although $\mathcal{L} \preccurlyeq_{\mathrm{k}} \mathcal{CPL}$.

One of them is modal logic KT. Let L be the classical propositional language over the vocabulary $\tau = \mathcal{P} \cup \{\neg, \wedge\}$, where $\mathcal{P} = \{p_1, p_2, \ldots\}$; and let L^{KT} be the closure of L with respect to the unary operator $\Box$. Let Δ be a vocabulary $\{\delta_{\Box\phi} \mid \Box\phi \in L^{\mathrm{KT}}\}$ of propositional variables disjoint from $\mathcal{P}$, and $L^{\mathcal{CPL}}$ be the closure of L with respect to variables in Δ. Let $\mathcal{M}$ be the class of reflexive Kripke models over the modal signature $\sigma = \{\Box\}$ and propositional signature $\mathcal{P}$, and let $\mathcal{M}^*$ be the class of Kripke models over the empty modal signature and the propositional signature $\mathcal{P} \cup \Delta$.

Define $\mathcal{T} \colon L^{\mathrm{KT}} \longrightarrow L^{\mathcal{CPL}}$:

- $\mathcal{T}(\Box\phi) = \mathcal{T}(\phi) \wedge \mathcal{T}_1(\Box\phi)$, where $\mathcal{T}_1(\Box\phi) = \delta_{\Box\phi}$.
- Literal for $\neg, \wedge$ and atomic formulas.

As the translation is defined inductively through the formation of formulas by a finite number of clauses, it is *finitely generated.*

Define $f \colon \mathcal{M} \longrightarrow \mathcal{M}^*$ as follows.

DEFINITION 2.9. For $(W, R, V, w) \in \mathcal{M}$, define $f(W, R, V, w)$ to be equal to $(f_1(W, R, V), f_2(W, R, V, w)) \in \mathcal{M}^*$, where

- $f_1(W, R, V) = (W, V^*)$, with $V^* = V \cup \{(\delta_{\Box\phi}, \{w \mid (W, R, V, w) \Vdash_{\mathrm{KT}} \Box\phi\})\}$, and
- $f_2(W, R, V, w) = w$.

FACT 2.8. *By the definition above it holds that*:

- $f(W, R, V, w) \Vdash_{\mathcal{CPL}} p_i$ *if and only if* $(W, R, V, w) \Vdash_{\mathrm{KT}} p_i$*;*
- $f(W, R, V, w) \Vdash_{\mathcal{CPL}} \delta_{\Box\phi}$ *if and only if* $(W, R, V, w) \Vdash_{\mathrm{KT}} \Box\phi$*.* ⊣

Clearly f is *model-based.* It holds also that $(\mathcal{T}, f)$ is *truth-preserving*:

FACT 2.9. *For all* $(W, R, V, w) \in \mathcal{M}$ *and* $\phi \in L^{\mathrm{KT}}$, $(W, R, V, w) \Vdash_{\mathrm{KT}} \phi$ *if and only if* $f(W, R, V, w) \Vdash_{\mathcal{CPL}} \mathcal{T}(\phi)$.

PROOF. By induction on the degree of formulas. Only the case where $\phi = \Box\psi$ is shown, as the base and cases for $\neg, \wedge$ are immediate.

Suppose that $(W, R, V, w) \Vdash_{\mathrm{KT}} \Box\psi$, then, by reflexivity of R we have that $(W, R, V, w) \Vdash_{\mathrm{KT}} \psi$. Thus, one can infer from the inductive hypothesis that $f(W, R, V, w) \Vdash_{\mathcal{CPL}} \mathcal{T}(\psi)$. Also by definition of the satisfaction of $\delta_{\Box\psi}$, it follows that $f(W, R, V, w) \Vdash_{\mathcal{CPL}} \delta_{\Box\psi}$. Therefore $f(W, R, V, w) \Vdash_{\mathcal{CPL}} \mathcal{T}(\Box\psi)$

Now suppose that $f(W, R, V, w) \Vdash_{\mathcal{CPL}} \mathcal{T}(\Box\psi)$, then it follows that $f(W, R, V, w) \Vdash_{\mathcal{CPL}} \delta_{\Box\psi}$ and, by definition, $(W, R, V, w) \Vdash_{\mathrm{KT}} \Box\psi$. ⊣

COROLLARY 2.1. KT $\preccurlyeq_{\mathrm{k}} \mathcal{CPL}$.

It is not difficult to construct an analogous translation from Epstein's relatedness logic $\mathcal{R}$ to $\mathcal{CPL}$, so that $\mathcal{R} \preccurlyeq_{\mathrm{k}} \mathcal{CPL}$ (2013, p. 300). The main question now is: does these translations show that $\mathcal{CPL}$ is at least as expressive as KT and $\mathcal{R}$? It is not reasonable to say so since the extra expressiveness brought about the necessity operator in the one case, and by the relevant implication in the other, is only poorly mimicked in $\mathcal{CPL}$ by variables whose interpretation must be sustained by the model-translation. Therefore Kuijer's formal condition does not give an intuitively adequate account of expressiveness.

The use of forward mappings seems to be essential to the counterexamples above. If we were considering a expressiveness comparison of the form $\preccurlyeq_{\mathrm{DC_{G2}}}$, where backward mappings are used, there would be no way to construct the accessibility relation in models of KT out of a valuation, and the same holds for the relatedness predicate $\mathscr{R}$ in the mentioned case of Epstein's relevance logic. Kuijer did not consider backward mappings since they imply that any truth-preserving translation is also validity preserving (Fact 2.7), and some of his paradigmatic examples of multi-class expressiveness are not validity preserving.

On the formula mapping. Notice that in the counterexample presented, in the translation clause for $\Box$, an auxiliary mapping $\mathcal{T}_1$ is called, having the translated formula as an argument. It then outputs a corresponding propositional variable. This introduction of parameters is perfectly legitimate, indeed, the well known standard translation from modal logic to first-order logic is finitely generated and also uses parameters, to represent the accessibility relation (e.g., consider the usual clause for $\Box$:

$ST_x(\Box\phi) = \forall y(Rxy \rightarrow ST_y(\phi)))$. Thus, ruling out parameters in general is not a reasonable option to capture multi-class expressivity.

Therefore, we are here in a quite dire situation: to avoid counterexamples such as the one above, one would have to impose a condition for formula mappings which is stricter than being finitely generated, disallowing the introduction of parameters by auxiliary mappings. However, this restriction is *ad hoc* and would leave out very natural translations across logics defined within different classes of models.

On the model mapping. One perhaps would consider requiring surjectivity, based on the following fact.

FACT 2.10. *Suppose that $\mathcal{L}_1 \preccurlyeq_{\mathrm{DC_{G1}}} \mathcal{L}_2$ and that $f\colon \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ is surjective. If P is expressible in $\mathcal{L}_1$ by definability, then $f[P]$ is expressible in $\mathcal{L}_2$ by definability.*

PROOF. Let $P = \mathbf{Mod}_{\mathcal{L}_1}(\phi)$. Suppose that $\mathfrak{B} \in f[P]$. Then $\mathfrak{B} = f(\mathfrak{A})$, with $\mathfrak{A} \in P$ and thus $\mathfrak{A} \Vdash_1 \phi$. As $\mathcal{L}_1 \preccurlyeq_{\mathrm{DC_{G1}}} \mathcal{L}_2$, we can infer that $\mathfrak{B} \Vdash_2 \mathcal{T}(\phi)$. Now suppose that $\mathfrak{B} \Vdash_2 \mathcal{T}(\phi)$. As f is surjective, there's an $\mathfrak{A} \in \mathcal{M}_1$ with $f(\mathfrak{A}) = \mathfrak{B}$. As $\mathcal{L}_1 \preccurlyeq_{\mathrm{DC_{G1}}} \mathcal{L}_2$, it follows that $\mathfrak{A} \Vdash_1 \phi$. Then $\mathfrak{A} \in P$, which implies that $\mathfrak{B} \in f[P]$. Thus $f[P] = \mathbf{Mod}_{\mathcal{L}_2}(\mathcal{T}(\phi))$. ⊣

However, notice that the function $f\colon \mathcal{M} \longrightarrow \mathcal{M}^*$ in Definition 2.9 is already surjective. Together with the above fact, this implies that surjectiveness is not sufficient to establish an intuitively adequate congruence relation between the classes of models $\mathcal{M}$ and $\mathcal{M}^*$. Moreover, surjectivity is neither necessary, since Thomason's model mapping in the translation from temporal logic to modal logic is not surjective.[2]

There is little hope of finding other reasonable restrictions to impose on model mappings. Naturally, one cannot require for there to be the usual relations between the mapped structures such as a homomorphism, since they may not share the same signature. Moreover, such requirement cannot be made even when the signature is the same. In the mentioned example above of the translation of temporal logic into

[2] In Thomason's tranlation from temporal logic to modal logic, the model mapping $f\colon \mathcal{M}_{TL} \longrightarrow \mathcal{M}_{ML}$ with $(W, R, V) \mapsto (W', R', V')$ is defined as follows:

- $W' = W \uplus W \uplus \{w_0\}$,
- $R' = \{(w^+, w^-), (w^-, w^+), (w^+, w_0) \mid w \in W\} \cup \{(w_1^+, w_2^+), (w_2^-, w_1^-) \mid w_1 < w_2\}$,
- $V'(p) = \{w^+, w^- \mid w \in V(p)\}$.

modal logic, the members of the original and translated model are not related in a functional way.

Given this, it seems that the establishment of a plausible congruence relation using a condition of de form $\preccurlyeq_{\mathrm{DC_{G1}}}$ is tightly linked with the specificities of the logics involved, and it may arguably be inferred that no general and reasonable condition of this form is there to be found. In the next section it will be seen that the situation with respect to conditions of the form $\preccurlyeq_{\mathrm{DC_{G2}}}$ is much better.

2.2.2. On the conditions of the form $\preccurlyeq_{\mathbf{DC_{G2}}}$

García-Matos and Väänänen (2007, p. 21) propose a wider version of $\preccurlyeq_{\mathrm{DC_{G2}}}$ to capture the concept of relative expressiveness. In this version, it is allowed that the class of models $\mathcal{M}_2$ of the target logic $\mathcal{L}_2$ be restricted by an $\mathcal{L}_2$-sentence. The idea behind it is that only models that are significant from the viewpoint of $\mathcal{L}_1$ should be translated. This is a common approach in the area on comparisons of formal systems, indeed, it is often allowed that the $\mathcal{M}_2$ be restricted by a set of $\mathcal{L}_2$-sentences.[3]

Defining $\preccurlyeq_{\mathrm{gv}^+}$. Consider the wider condition for multi-class expressiveness, which is identical with Garcia-Matos and Väänänen's except that it allows $\mathcal{M}_2$ to be restricted by a recursive set of $\mathcal{L}_2$-sentences:

DEFINITION 2.10. Let $\mathcal{L}_1 = (\mathcal{M}_1, \mathcal{S}_1 \Vdash_1)$ and $\mathcal{L}_2 = (\mathcal{M}_2, \mathcal{S}_2, \Vdash_2)$ be logics. Then $\mathcal{L}_1 \preccurlyeq_{\mathrm{gv}^+} \mathcal{L}_2$ if there is a recursive set of sentences $\Theta \subseteq \mathcal{S}_2$ and functions $g\colon \mathbf{Mod}_{\mathcal{L}_2}(\Theta) \longrightarrow \mathcal{M}$ and $\mathcal{T}\colon \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ such that:

(a) For every $\mathfrak{A} \in \mathcal{M}_1$ exists a $\mathfrak{B} \in \mathcal{M}_2$ such that $f(\mathfrak{B}) = \mathfrak{A}$ and $\mathfrak{B} \Vdash_2 \Theta$;
(b) For every $\phi \in \mathcal{S}_1$ and for every $\mathfrak{B} \in \mathcal{M}_2$, if $\mathfrak{B} \Vdash_2 \Theta$, then ($\mathfrak{B} \Vdash_2 \mathcal{T}(\phi)$ if and only if $g(\mathfrak{B}) \Vdash_1 \phi$).

As mentioned before, the above condition is not susceptible to counterexamples as those of Section 2.2.1, due to the direction of the model mapping. However, we argue that $\preccurlyeq_{\mathrm{gv}^+}$ is still materially inadequate.

A problem with $\preccurlyeq_{\mathrm{gv}^+}$. The following fact is a generalization of a result due to Mossakowski et al. (2009, p. 107).

FACT 2.11. *Let $\mathcal{L}$ be any many-valued logic whose semantics can be stated in classical propositional language. Then $\mathcal{L} \preccurlyeq_{\mathrm{gv}^+} \mathcal{CPL}$.*

[3] *E.g.* (Manzano, 1996, p. 270).

SKETCH OF PROOF. Define $\mathcal{T}\colon \mathcal{S}_{\mathcal{L}} \longrightarrow \mathcal{S}_{\mathcal{CPL}}$:

- For every $\phi \in \mathcal{S}_{\mathcal{L}}$, let $\mathcal{T}(\phi) = p_\phi$, where p_ϕ is a new propositional variable.

By Suszko's thesis, $\mathcal{L}$ can be obtained by a collection $\mathcal{C}$ of bivalued semantic clauses. Each such clause will be rendered as a material conditional, where every occurrence of "the valuation attributes 'true' (or 'false') to a formula ϕ", is written as "ϕ" (or "$\neg\phi$"), and the logical expressions are written in the propositional language, as expected.

Let $\Theta \subseteq \mathcal{S}_{\mathcal{CPL}}$ be obtained by the addition, for every n-ary operator $\divideontimes$ of $\mathcal{L}$ and every $\phi_1 \ldots \phi_n \in \mathcal{S}_{\mathcal{L}}$, of at least one of the following items, which express the semantic clauses for $\divideontimes$:

- $\mathcal{T}(\divideontimes(\phi_1 \ldots \phi_n)) \to A_1$,
- $A_2 \to \mathcal{T}(\divideontimes(\phi_1 \ldots \phi_n))$,

where A_1 and A_2, are boolean combinations of $\mathcal{T}(\phi_1) \ldots \mathcal{T}(\phi_n)$.[4]

Now, for $\phi \in \mathcal{L}$ and $v \in \mathbf{Mod}_{\mathcal{CPL}}(\Theta)$, define $g\colon \mathbf{Mod}_{\mathcal{CPL}}(\Theta) \longrightarrow \mathcal{M}_1$ as follows:

- $g(v) \Vdash_{\mathcal{L}} \phi$ iff $v \Vdash_{\mathcal{CPL}} \mathcal{T}(\phi)$

That $(\mathcal{T}, g)$ satisfies items (a) and (b) of Definition 2.10 is clear. Thus, we have that $\mathcal{L} \preccurlyeq_{\mathrm{gv}+} \mathcal{CPL}$. ⊣

The main point of the above result is that, by encoding semantics of $\mathcal{L}$ into Θ and then restricting the $\mathcal{CPL}$-models to those that satisfy Θ, one is able to fetch back $\mathcal{L}$-models via the model-translation. Fact 2.9 appears to show more of a "cheating" allowed by $\preccurlyeq_{\mathrm{gv}+}$, than that $\mathcal{CPL}$ is maximally expressive among such propositional logics. Intuitively, it appears that more things are expressible, e.g., in intuitionistic logic or relevance logics than in classical propositional logic.

An improved version of $\preccurlyeq_{\mathrm{gv}+}$. The modification of García-Mattos and Väänänen's condition, allowing Θ to be a recursive set of sentences instead of a sentence, looks at least as "natural" as the original one. Allowing Θ to be only a single sentence will probably not prevent counterexamples among more expressive logics. However, it is essential for the

[4] For example, if $\mathcal{L}$ contains truth-functional conjunction $\wedge$, Θ will contain all items below:

- $\mathcal{T}(\phi_1 \wedge \phi_2) \to (\mathcal{T}(\phi_1) \wedge \mathcal{T}(\phi_2))$,
- $(\mathcal{T}(\phi_1) \wedge \mathcal{T}(\phi_2)) \to \mathcal{T}(\phi_1 \wedge \phi_2)$.

counterexample to hold that $\mathcal{T}$ does not preserve any eventual structure of $\mathcal{L}$-sentences, and that they be translated into propositional variables.

Now, one can fix the above issue and obtain an improved condition by imposing the requirement that the formula-mapping $\mathcal{T}\colon \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ be finitely generated. This will block the trick of artificially smuggling the semantics of $\mathcal{L}_1$ into Θ.

3. Conclusion

As it was mentioned before, in the uni-class framework it is very simple to define relative expressiveness, since there is a common ground — the class of models — in order to compare the expressibility of properties in them.

In the multi-class framework, where $\mathcal{L}_1 = (\mathcal{M}_1, \mathcal{S}_2, \Vdash_1)$ and $\mathcal{L}_2 = (\mathcal{M}_2, \mathcal{S}_2, \Vdash_2)$ are defined on possibly different classes of structures, the comparison of expressible properties of models requires a certain congruence between properties in $\mathcal{M}_1$ and in $\mathcal{M}_2$, establishing which properties in the one logic correspond to properties in the other.

Such a congruence will be established by either forward model mappings $f\colon \mathcal{M}_1 \longrightarrow \mathcal{M}_2$, or backward model mappings $\mathcal{M}_2 \longrightarrow \mathcal{M}_1$. Fact 2.5 shows that if no restrictions on the model-mappings are made, the generalizations of discrimination to the multi-class framework are trivial. Now, to place restrictions on the model mappings in order to achieve a reasonable congruence seems to be a virtually hopeless enterprise, given the variety of model classes. Even for the restricted case of possible world semantics, we are no better off. Plausible requirements such as being model based are still too general to limit undesirable cases, and thus fails to provide the appropriate congruence.

The counterexample to Kuijer's condition in Section 2.2.1 would not work for backward model mappings. However, requiring only that these mappings be backward, would not provide the proper congruence between models either, as the counterexample in Section 2.2.2 has shown.

For conditions of the form $\preccurlyeq_{\mathrm{DC}_{\mathrm{G1}}}$, placing stricter restrictions on formula translations is efficient to avoid such counterexamples, but it can also block some intuitively reasonable expressiveness comparisons, as it was mentioned on Section 2.2.1. On the other hand, for conditions of the form $\preccurlyeq_{\mathrm{DC}_{\mathrm{G2}}}$, considered in Section 2.2.2, imposing that formula translations be finitely generated is efficient to block counterexamples like the ones presented. Given the general character and naturalness of such translations, the resulting condition arguably will not undergenerate. Nevertheless, it is not completely clear whether the resulting expressiveness condition would still overgenerate, as this might hinge on the difficulty of establishing general conditions for there being a congruence between models in different classes of structures, which can be highly dependent of the characteristics of the compared logics.

References

Broersen, J., A. Herzig and N. Troquard, 2006a, "From coalition logic to stit", *Electronic Notes in Theoretical Computer Science*, 157: 23–35. DOI: 10.1016/j.entcs.2006.02.011

Broersen, J., A. Herzig, and N. Troquard, 2006b, "Embedding alternating-time temporal logic in strategic stit logic of agency", *Journal of Logic and Computation*, 16: 559–578. DOI: 10.1093/logcom/exl025

Ebbinghaus, H., 1985, "Extended logics: The general framework", in J. Barwise and S. Feferman (eds.), *Model-theoretic Logics*, Perspectives in Mathematical Logic, Springer-Verlag.

Epstein, R., 2013, *The Semantic Foundations of Logic, Volume 1: Propositional Logics*, contributors: W. Carnielli, I. M. d'Ottaviano, S. Krajewski and R. D. Maddux, Nijhoff International Philosophy Series. Springer Netherlands. DOI: 10.1007/978-94-009-0525-2

Fernandes, D., 2023, "On comparisons of logics in terms of expressive power", *Manuscrito*, 46: 1–45. DOI: 10.1590/0100-6045.2023.V46N4.DF

García-Matos, M., and J. Väänänen, 2007, "Abstract model theory as a framework for universal logic", pages 19–33 in J.-Y. Beziau (ed.), *Logica Universalis*, Birkhäuser Basel. DOI: 10.1007/978-3-7643-8354-1_2

Gasquet, O., and A. Herzig, 1996, "From classical to normal modal logics", pages 293–311 in H. Wansing (ed.), *Proof Theory of Modal Logic*, Springer Netherlands, Dordrecht. DOI: 10.1007/978-94-017-2798-3_15

Goranko, V., and W. Jamroga, 2005, "Comparing semantics of logics for multi-agent systems", pages 77–116 in *Information, Interaction and Agency*, Springer Netherlands, Dordrecht. DOI: 10.1007/1-4020-4094-6_3

Kocurek, A., 2018, "What can you say? Measuring the expressive power of languages", PhD thesis, University of California, Berkeley, USA. https://philpapers.org/archive/KOCWCY.pdf

Kuijer, L. B., 2014, "Expressivity of logics of knowledge and action", PhD thesis, University of Groningen, Groningen, Netherlands. http://www.rug.nl/research/portal/files/15492763/Complete_dissertation.pdf

Manzano, M., 1996, *Extensions of First-Order Logic*, Cambridge Tracts in Theoretical Computer Science, Cambridge University Press.

Mossakowski, T., R. Diaconescu and A. Tarlecki, 2009, "What is a logic translation?", *Logica Universalis*, 3(1): 95–124. DOI: 10.1007/s11787-009-0005-2

Thomason, S. K., 1974, "Reduction of tense logic to modal logic. I", *Journal of Symbolic Logic*, 39(3): 549–551. DOI: 10.2307/2272895

DIEGO FERNANDES
Universidade Federal da Paraíba, Brasil
dpf@academico.ufpb.br