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The Principle of Explosion in the Stoic Logic

Abstract. I argue that the Stoic logic is explosive. The claim applies to the Stoics' syllogistic in the strictest sense, because there is a provable syllogism which qualifies as a principle of explosion. It applies also to the general consequence operation, in the sense that every sentence is derivable from any pair containing both a sentence and the negation of the sentence. Finally, it applies to the connective of implication (conditional), in the sense that any conditional is derivable, providing its antecedent is a conjunction of a sentence and the negation of the sentence. All three claims allow weakening, i.e., additions of extra premises to an inference or extra conjuncts to the antecedent of an implication, respectively. Consequently, no concept of relevance, let alone paraconsistency or connexivity is applicable to the Stoic logic; in particular, the Stoics' connective of implication is either material (Boolean) or formal (strict).

Keywords: ancient; Stoic logic; syllogistic; explosive

Introduction

Starting with Jan Łukasiewicz's seminal lecture *On the Stoics' Logic* (delivered in 1923, exactly a century ago, and published later as (Łukasiewicz, 1927)), throughout the 20th century, conviction prevailed that the propositional logic of the Stoa was simply an early or premature version of classical propositional calculus.

Unfortunately, in Łukasiewicz's argument there clearly are flaws. Particularly, for Łukasiewicz it is an indispensable premise that in the Stoic logic the connective of implication is truth-functional. That, however, cannot be easily determined based on the extant source material. Marek Nasieniewski (1998) demonstrated that a purely implicational

system, based on the Stoics' well known assumptions, is deviant and even relevantistic.

Furthermore, in the endmost quarter of the 20th century interest in deviant logics increased by leaps and bounds, including the search for origins of such logics in the distant past. At the turn of the 20th and the 21st centuries a number of works were released, arguing that the Stoic logic was relevantistic or paraconsistent (Bobzien, 1996; Priest, 1998; Sylvan, 2000). Such a view became widely accepted in the 21st century. In some cases the view is merely historical, in other ones a part of a larger enterprise aimed at revising logic. The possible existence of knowingly deviant logics in antiquity could be taken as evidence that classical propositional calculus does not correspond to the vernacular or to the common sense. It would rather indicate that logic was originally not classical and that deviant logics represent a default position rather than one of extravagance.

The objective of this study is to justify a weakened version of Łukasiewicz's thesis. Without deciding whether or not the Stoic logic is equal to classical propositional calculus, it will be demonstrated that it is at least explosive. A logical system is called explosive if it legitimises deriving every sentence from an inconsistent set of sentences. Theorems which ascertain explosiveness are called principles of explosion. Logics which are not explosive, are called non-explosive. Depending on philosophical motivations or other properties they may be further characterised as relevantistic, paraconsistent, adaptive, sociative, connexive or similarly. Clearly, relevantism, paraconsistency and similar characteristics are irreconcilable with explosiveness.

1. Syntax

The Stoics developed a formal propositional language, containing propositional letters or variables and propositional connectives. The role of the former is played by ordinal numerals: the first, the second, etc., which seem to abbreviate expressions: the first sentence, the second sentence, etc. Compound sentences are constructed from simple ones and connectives, in accordance with the following patterns:

– negation: not the first,
– conjunction: both the first and the second,
– disjunction: either the first or the second,
– implication (conditional): if the first then the second,

with the first and the second being any sentences, either simple or compound. Admittedly, the Stoics do not qualify a sentence as compound unless it contains a binary connective, but this grammatical subtlety makes here no difference. The connective of negation is unary. It has an entire sentence as the scope and takes the form of the prefix "not" ($\dot{\text{o}}\upsilon$, $\dot{\text{o}}\upsilon\varkappa$), quite analogically to the contemporarily employed modifier "it is not the case that". The connectives of conjunction, disjunction and implication are binary. Admittedly, in the Stoics' syntax the arity of the connectives of conjunction and disjunction is flexible, e.g., one can say: both the first and the second and the third, with a single occurrence of the connective. It is, however, of no relevance in this paper, for the connective of conjunction to be employed meets the conditions of commutativity and associativity, and the only occurrences of the connective of disjunction to be in focus are binary.

Nota bene, every connective either simply is a prefix or includes a part which is a prefix. Thanks to it, similarly to the Polish notation, despite the lack of parentheses, the syntax of the Stoic logic is usually rather clear. For example, the sentence "not both the first and not the second" is a negation of a conjunction, the first conjunct of the conjunction is a simple sentence, and the other conjunct is the negation of another, not necessarily different, simple sentence. The sentence "both not the first and not the second" is itself a conjunction and its first conjunct is a negation, Thanks to the presence of the prefix "both", establishing the scope of the connective of negation presents no difficulty.

Hence, the Stoics' propositional syntax may be reliably directly transposed to the contemporary logic. Let then φ, χ, ψ, either with or without numerical subscripts, be any sentences, whether simple or compound. In addition to simple sentences, sentences are also:

– the negation $\neg\varphi$ of φ,
– the conjunction $(\varphi \wedge \psi)$ of φ and ψ,
– the disjunction $(\varphi \mid \psi)$ of φ and ψ, and
– the implication $(\varphi \rightarrow \psi)$ from the antecedent φ to the consequent ψ,

to be read in accordance with the above quoted patterns. Parentheses may be omitted for brevity. In absence of parentheses, homogenous connectives have longer and longer scopes from left to right whereas those heterogenous have longer and longer scopes in the very order they have been just introduced: negation, conjunction, disjunction, implication. Let finally Φ and Ψ be any finite sets of sentences, not excluding the empty set.

2. Semantics

The Stoics unanimously upheld the principle of bivalence (Diogenes, 1925, 7:73; Sextus, 1935, 8:85; cf. Mates, 1953). It is a well established fact because in Antiquity the question of bivalence was an issue within a great philosophical debate on the future contingents. The Stoics, like their predecessors Dialecticians, were leading defendants of bivalence, whereas the Epicureans gave up bivalence for future contingents and the Peripatetics took a somehow uncertain stance. Ancient thinkers generally spoke of truth-values as relative to time. Hence, the Stoics would say that at a given time every sentence has one and only one of the two truth-values, either truth or falsehood. However, in syllogistic the expression "at a given time" means in practice something similar to "in a given describable state of affairs" and constitutes an early version of the idea of truth-value assignment.

The Stoics' account of the connectives of disjunction and implication seems to have been a matter of an unending debate since the very beginning. It remains far from certain whether those connectives are to be understood as truth-functional or as modal.

On the contrary, it is quite clear that connectives of negation (ἀπο-φατικόν) and of conjunction (συμπεπλεγμένον) were thought of as truth-functional (Boolean) and that the way to understand those connectives was no subject of controversy but universally accepted throughout the Stoa (Sextus, 1935; cf. Mates, 1953). The Stoics agree that, at a given time, every negation of a true sentence is false, every negation of a false sentence is true, every conjunction of two true sentences is true, but all remaining conjunctions are false. The truth-functional account of the connectives of negation and of the conjunction is universally accepted in the scholarship, including proponents of the alleged relevantist or connexive character of logical theories of the Stoa (cf. Bobzien, 1996). It seems therefore reasonable to base the case for explosiveness of the Stoic logic on the two connectives with such clear and unchallenged understanding.

In light of the principle of bivalence and truth-functional account of the connectives of negation and of conjunction, it is beyond doubt that, with respect to the sentences containing no connectives except those of negation and of conjunction, all assignments of truth-values within the confines of the Stoic logic and of classical propositional calculus coincide. Consequently, with respect to the sentences in question, the sets of the Stoics' and classical tautologies are identical.

As multiple truth-functional conjunctions of the same conjuncts are equivalent to one another regardless the parenthesis arrangement, it is legitimate in sentences of that type to omit the inner parentheses and write, e.g. $(\varphi \wedge \chi \wedge \psi)$ instead of $(\varphi \wedge (\chi \wedge \psi))$ or $((\varphi \wedge \chi) \wedge \psi)$. It is to be understood that either of latter sentences may stand in the place of the former. As it has been mentioned in the previous section, the Stoics' practice of flexible arities also bears for such a convention.

Among tautologies containing no connectives except those of negation and of conjunction some are of essence here. Particularly, the principle of contradiction:

$$\neg(\varphi \wedge \neg\varphi)$$

qualifies as a tautology on the ground of the Stoic logic, in the sense that it is true for every sentence φ at any time. The same applies to related sentences supplemented with arbitrary extra conjuncts, including

$$\neg(\varphi \wedge \neg\varphi \wedge \neg\psi), \tag{1}$$
$$\neg(\neg\varphi \wedge \varphi \wedge \neg\psi), \tag{2}$$
$$\neg(\psi \wedge \varphi \wedge \neg\psi), \tag{3}$$

and, for every positive integer n,

$$\neg(\varphi_1 \wedge \varphi_2 \wedge \ldots \varphi_n \wedge \varphi \wedge \neg\varphi \wedge \neg\psi), \tag{4}$$

Every Stoic would agree that the above sentences are tautologies, in the sense that they are true at any time or on all valuations.

Observe that the above claim, concerning the identity of the sets of negation-conjunction tautologies in the Stoic logic and classical propositional calculus, does not by itself prejudice the role tautologies play in deduction processes. That is still to be discovered. Hence, it has not been prejudiced here that the negation and conjunction fragments of the Stoic logic and of classical propositional calculus coincide. That would be begging the question because the Stoics' concept of consequence is here under consideration. But it is justified to claim that regarding the truth-values, and consequently the tautologicity, of single formulas built by means of at most the connectives of negation and conjunction the Stoic logic matches perfectly classical propositional calculus.

3. Contradictories

When studying ancient logical texts, one continuously needs to employ the concept of pairs of contradictory sentences (contradictories). Hence, it is convenient for brevity to introduce a purely auxiliary symbol in the metalanguage, let it be the superscript of star, to the effect that

$$\varphi \text{ and } \varphi^\star$$

form a pair of contradictory sentences (contradictories) for every sentence φ, one being reciprocally called contradictory to the other or a contradictory of the other.

The Stoics carefully distinguished the object language concept of negation (ἀποφατικόν) and the metalinguistic concept of a contradictory (ἀντικείμενον). Still, as they endorsed both the principle of bivalence and the truth-functional connective of negation, it comes at no surprise that those concepts are closely related. Two key statements may be found in the extant sources (Alexander, 1883; Diogenes, 1925):

– every sentence is a contradictory of its negation (and conversely),
– the double negation means exactly the same as the initial sentence.

The latter statement is the strongest possible version of the principle of double negation (ὑπεραποφατικόν), one the Stoics endorsed. As it will shortly get evident, the full freedom of mutual replacement of sentences with their double negations matches perfectly the way sentences are being described in the Stoic logic.

According to the Stoics, on the one hand, syntactically, every sentence has a wide variety of contradictories, but on the other, semantically and pragmatically, all contradictories of a given sentence are unrestrictedly interchangeable. As it will turn out, that includes deductive interchangeability. Although in the sources it is unequivocally stated regarding double negations, and consequently any even number of negations, it seems to be in line with Stoicism to generalise the statement over all would be contradictories once they have been established. Still, in this study it is sufficient to remember that any sentence φ, whether simple or compound, on the one hand, prefixed with an odd number of occurrences of the sign of negation is a contradictory of the initial sentence φ and conversely, but on the other, prefixed with an even number of occurrences of the sign of negation is freely interchangeable with the initial sentence φ.

It is therefore not assumed here that in a pair of contradictories the starred sentence is unique or more complex than the other one. It is particularly important to understand that $(\neg\varphi)^\star$ may stand for $\neg\neg\varphi$ as well as for φ.

4. Indemonstrables

A hallmark of the Stoic's syllogistic is the collection of indemonstrable syllogisms — indemonstrables (ἀναποδείκτοι) for short — serving in syllogistic by the way of axioms and very well established (Diogenes 1925, 7:80-81; Sextus 1935, 8:224-226; Mates, 1953, gives a detailed list of independent sources confirming the indemonstrables):

$$(\varphi \to \psi), \varphi \vdash \psi, \tag{5}$$
$$(\varphi \to \psi), \psi^\star \vdash \varphi^\star, \tag{6}$$
$$\neg(\varphi \wedge \psi), \varphi \vdash \psi^\star, \tag{7}$$
$$\neg(\varphi \wedge \psi), \psi \vdash \varphi^\star, \tag{8}$$
$$(\varphi \mid \psi), \varphi^\star \vdash \psi, \tag{9}$$
$$(\varphi \mid \psi), \psi^\star \vdash \varphi, \tag{10}$$
$$(\varphi \mid \psi), \varphi \vdash \psi^\star. \tag{11}$$
$$(\varphi \mid \psi), \psi \vdash \varphi^\star. \tag{12}$$

Theorem (5) represent the Stoics' first indemonstrable, known as modus ponendo ponens: from an implication and its antecedent infer the consequent of the implication. Theorem (6) represent the second indemonstrable, known as modus tollendo tollens: from an implication and a contradictory of its consequent infer any contradictory of the antecedent of the implication. Theorems (7) and (8) combined represent the third indemonstrable, known as the conjunctive syllogism: from the negation of a conjunction and either of its conjuncts infer any contradictory of the other conjunct. Theorems (9) and (10) combined represent the fourth indemonstrable, known as modus tollendo ponens: from a disjunction and a contradictory of either of its disjuncts infer the other disjunct. And theorems (11) and (12) combined represent the fifth indemonstrable, known as modus ponendo tollens: from a disjunction and either of its disjuncts infer any contradictory of the other disjunct.

5. Themata

Alongside with the indemonstrable syllogisms the Stoics endorsed trans-
formation rules. Some of the rules, called themata (ϑέματα), take a
number of syllogisms and return a syllogism. The Stoics endorsed at
least four themata and other primitive rules of inference, however, of the
themata only two have survived: a rule of transposition and a rule of
cut (consult Mates, 1953, and Bobzien, 1996, for details).

The rule of transposition, known as the first thema (τὸ πρῶτον ϑέμα),
is the following one:

$$\frac{\varphi, \psi^\star \vdash \chi,}{\varphi, \chi^\star \vdash \psi.} \tag{13}$$

It is reported in Apuleius:

> if two sentences entail a third sentence, then either of the two sentences
> together with a contradictory of the third sentence entails any contra-
> dictory of the other of the two sentences".   (Apuleius, 1991, p. 209)

The rule of cut is known as the third thema (τὸ τρίτον ϑέμα). In the
system of syllogistic it takes the following form, for any positive integer n:

$$\frac{\begin{array}{c}\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \varphi \\ \varphi, \psi \vdash \chi\end{array}}{\varphi_1, \varphi_2, \ldots, \varphi_n, \psi \vdash \chi} \tag{14}$$

It is reported equivalently by Alexander of Aphrodisias and by Simpli-
cius. Alexander says:

> if from some two $[\varphi, \psi]$ some third $[\chi]$ follows, and one $[\varphi]$ of those two
> $[\varphi, \psi]$ follows from some additional assumptions $[\varphi_1, \varphi_2, \ldots, \varphi_n]$, then
> the very same $[\chi]$ follows from the other one $[\psi]$ and those additional
> ones $[\varphi_1, \varphi_2, \ldots, \varphi_n]$ which entail the first one $[\varphi]$.
>
> (Alexander, 1883, p. 278)

Simplicius changes the order of the premises and seems to assume that
$n = 2$:

> if from some two $[\varphi_1, \varphi_2]$ some third $[\varphi]$ follows, and from that following
> one $[\varphi]$ together with some additional assumption $[\psi]$ it follows yet
> another one $[\chi]$, then that very other one $[\chi]$ follows from the first two
> $[\varphi_1, \varphi_2]$ together with the additional assumption $[\psi]$.
>
> (Simplicius, 1894, 237)

The Stoics' deductive practice shows that Alexander is more accurate and Simplisius should have read "when from a number another follows" instead from "when from two a third follows". Still, this minor difference is completely irrelevant due to the described truth-functional account of the connective of conjunction. Whether a number of premises is to be considered separately or as a conjunction makes practically no difference. Such differences suggest independence of the sources rather than difference of theories. It is therefore safe to say that rule (14) is multiply attested to in the extant sources. Nevertheless, it is worth mentioning that for the key result of this work both accounts of the rule of cut would do.

The outlined axiomatics is far from optimal. For example, theorems (5) and (6) are immediately derivable from each other by means of rule (13). In the extant sources there are some traces of attempts to improve it. Still, the key point here is that theorems (5)–(12), rule (13) and rule (14) were universally endorsed by the Stoics (actually, also by the Peripatetics) and throughout the ancient logic remained beyond any doubt or dispute.

6. Apparent Proofs of Explosiveness

For a highly debatable theorem, the principle of explosion has a rather high number of surprisingly simple, difficult to challenge proofs. Explosion turns out to be simply a borderline case of consequence, like zero in the collection of numbers and a point in the realm of topology. It is a key cause of the fact that attempts to construct a serious relevantist, paraconsistent, adaptive or connexive alternative to classical propositional calculus fail one after another.

However, typical proofs of the principle of explosion usually involve borderline cases of basic principles. As the extant sources material for the ancient history of logic is fragmentary, far from systematic, mostly secondary, of rather low quality, often consists of marginal remarks and comes from non-logicians, it abounds in careless common sense statements apparently contradicting borderline case laws of logic. Therefore, prima facie proofs of principles of explosion happen to make easy targets for challenges from logicians who read ancient fragments as if they were excerpts of contemporary academic journals.

For example, a strikingly obvious proof of explosiveness comes from rule (13). It is a thema of the Stoics as well as a constitutive part of

Aristotle's logical writings. The following transformation rule is a special case of rule (13) and as such belongs to the Stoic logic, and actually to that of Aristotle as well:

$$\frac{\varphi, \neg\psi \vdash \varphi,}{\varphi, \neg\varphi \vdash \psi.} \tag{15}$$

Rule (15) itself gives the impression of a ready-to-use proof of a principle of explosion within the confines of both the Stoic and the Peripatetic logics. However, such a proof rests upon a borderline case of consequence, namely on its reflexivity:

$$\varphi, \Phi \vdash \varphi, \tag{16}$$

which allows to get the premise of rule (15). Supporters of the view that ancient logics are non-explosive claim that the Stoics and the Peripatetics rejected theorem (16). With respect to the Peripatetic logic, the claim rests on Aristotle's manner of speech that in a syllogism "one thing being assumed something else follows", interpreted as if it excluded following a sentence from itself. With respect to the Stoic logic, the claim rests on Sextus' brief comments that the Stoics were not ready to accept syllogisms whose conclusion repeats a premise (ἀδιαφόρως περαίνοντες, cf. Sextus 1933, 2.147; 1935, 8.431). If the operation of consequence were not reflexive, rule (15) would not allow to prove the principle of explosion. There are good reasons to believe that the operation of consequence both in Aristotle and in the Stoicism is certainly reflexive (cf. Tkaczyk, 2024). Still, the above mentioned passages do exist and make a room for a fair controversy.

Hence, it seems to be an interesting question whether on the ground of the Stoic logic there is a less expected, but irrefutable, proof of a principle of explosion. Such a proof should be based on bedrock principles exclusively, namely on those firmly attested to and never denied in the extant source material. It turns out that such a proof does exist.

7. A Syllogism of Explosion

To prove a principle of explosion it is sufficient to accept either of indemonstrable syllogisms (7), (8), in other words, the conjunctive syllogism, and rule (14). Within the confines of the Stoic logic, theorems (7) and (8) are both undoubtedly syllogisms, and transformation rule (14) takes and returns syllogisms. Therefore, the following theorem is

derivable on the ground of the Stoics' pure syllogistic and qualifies itself as a syllogism.

Theorem.

$$\neg(\varphi \wedge \neg\varphi \wedge \neg\psi), \varphi, \neg\varphi \vdash \psi. \tag{17}$$

Proof.

1. $\neg(\varphi \wedge \neg\varphi \wedge \neg\psi), \varphi \vdash \neg(\neg\varphi \wedge \neg\psi)$ indemonstrable (7)
2. $\neg(\neg\varphi \wedge \neg\psi), \neg\varphi \vdash \psi$ indemonstrable (7)
3. $\neg(\varphi \wedge \neg\varphi \wedge \neg\psi), \varphi, \neg\varphi \vdash \psi$ from lines 1 and 2 by rule (14)

An analogical theorem and a proof of its, citing indemonstrable (8) instead of (7) in lines 1 and 2, are too similar to be displayed. Notice that syllogism (17) together with the above proof holds good even under the extremely unlikely assumption that in the case of the rule of cut Simplicus' account is the sole correct one, in other words, that according to the Stoics rule (14) is valid for $n = 2$ only.

    Theorem (17) together with the above derivation unshakeably belong to the Stoics' syllogistic. No Stoic whatsoever could ever think of undermining either theorem (17) or its proof. If there is anything like the Stoic logic at all, theorem (17) does belong to it. It is a syllogism in the strictest possible sense of syllogism and it is provable in the strictest possible sense of provability within the Stoics' syllogistic.

    Theorem (17) is a principle of explosion, one characteristic of the Stoics' pure syllogistic. To understand it, observe that the first premise of syllogism (17) is tautology (1), one recognised by the Stoics. Take then any arbitrary sentences φ, ψ and assume that both sentences $\varphi, \neg\varphi$ are true. Sentence (1) is true for any sentences φ, ψ, including those selected. Hence, every premise of syllogism (17) is true, and so is the conclusion ψ. Syllogism (17) literally allows to infer every sentence once any sentence has been assumed together with its negation. Syllogism (17) qualifies therefore as a principle of explosion, particularly, it introduces explosiveness to the Stoics' syllogistic as a borderline case or perhaps a side effect of the principle of contradiction.

8. Syllogisms and Derivability

The presence of syllogism (17) in the Stoics' system of logic does not at all exhaust the topic of explosiveness, because the scope of valid inferences

is wider than the scope of syllogisms. Actually, mistaking continuously the concept of being a syllogism for the concept of being valid is the foundational error of authors who suspect ancient logics of relevantism or paraconsistency or similar properties. Two properties of the operation of consequence need to be pointed at here:

– derivability of the unrestricted rule of cut,
– derivability of tautologous sentences.

Even though they may exceed the concept of being syllogism, they both undoubtedly belong to the logic of the Stoics, and actually, to that of Aristotle as well.

By the unrestricted rule of cut it is to be understood such a generalisation of rule (14) that is not sensitive to the number of sentences on the left-hand side of the turnstile. It is the rule:

$$\frac{\Phi \vdash \varphi \quad \varphi, \Psi \vdash \chi}{\Phi, \Psi \vdash \chi} \tag{18}$$

Rule (14) is equal to rule (18) with the caveat that

$$\Phi = \{\varphi_i\}_{i=1}^n, \text{ for a positive integer } n,$$
$$\Psi = \{\psi\}.$$

Rule (18) allows any finite number of sentences in the sets Φ, Ψ. But it is claimed here that in the Stoic logic rule (18) undoubtedly remains valid if either of the sets Φ, Ψ or both are empty, in other words, the following cases of rule (18) are perfectly valid:

$$\frac{\vdash \varphi \quad \varphi, \Psi \vdash \chi}{\Psi \vdash \chi} \qquad\qquad \frac{\vdash \varphi \quad \varphi \vdash \chi}{\vdash \chi} \tag{19}$$

Consequently, it is legitimate to treat valid sentences as if they were valid inferences with zero premises. The other claim is that in the Stoic logic tautologies, i.e., sentences true regardless permutations of simple sentences, qualify as valid in the above sense, i.e., may be treated in deduction processes as if they were valid inferences with zero premises. As it has been mentioned, both claims hold good for Aristotle's logic as well.

To substantiate the above two claims it is sufficient to carefully follow the deductive practice of the Stoics or, actually, any ancient thinkers. Observe Sextus' account of the Stoics' way of proving the theorem that signs must exist (1935, 8:281-285,292). The Stoics apply the following syllogism, a case of the constructive dilemma:

$$(\varphi \to \varphi), (\neg\varphi \to \varphi), (\varphi \mid \neg\varphi) \vdash \varphi, \tag{20}$$

specifying that φ is the sentence "a sign exists". The Stoics do not limit themselves to ascertaining that the conclusion follows logically from the premises, but go on to prove the premises. They declare the three premises valid for three different reasons. The first premise "if a sign exists then the sign exists" is declared valid simply because the consequent of the implication follows logically from the antecedent: "for it [the implication] is duplicated, and «a sign exists» follows from «a sign exists»" (Sextus, 1935, 8:281). The proof of the second premise "if no sign exists then a sign exists" goes as follows: suppose no sign exists, by supposing it you have just produced a sentence "no sign exists", but the sentence is a sign, therefore a sign exists. Finally, the third premise "either a sign exists or no sign exists" is declared valid on semantical grounds: it is valid simply because it is true for every φ. It is claimed that in every pair of contradictories one sentence is true and the other false, which is a sufficient condition for a true disjunction.

The Stoics specifically insist that the quoted argument confirms the status of the sentence "a sign exists" as a proven theorem, one necessarily true (Sextus, 1935, 8:284). Syllogism (20) does not suffice to achieve such a conclusion. Rather, the argument is an instance of the following rule:

$$
\begin{array}{c}
\vdash (\varphi \to \varphi) \\
\vdash (\neg\varphi \to \varphi) \\
\vdash (\varphi \mid \neg\varphi) \\
\underline{(\varphi \to \varphi), (\neg\varphi \to \varphi), (\varphi \mid \neg\varphi) \vdash \varphi} \\
\vdash \varphi
\end{array}
\tag{21}
$$

It is crystal clear that rule (21) is underlain by triple application of rule (18) with the set Φ being empty, and that in the final step the set Ψ is empty too.

The extant sources assign many demonstrations following exactly pattern (21) to the Stoics. A similarity to Socrates' refutation of Prothagoras' thesis that all sentences are equally true and to Aristotle's

famous proof that one ought to philosophise is also striking. And Sextus complains that the described reasoning represents a typical mindset of dogmatists, not only of the Stoics. It is therefore likely that in antiquity applying rule (18) with no restriction was a matter of routine.

The Stoics' proof of the theorem "a sign exists" is even more instructive for the proof of the third premise "either a sign exists or no sign exists" boils down to establishing that the premise is a tautology, or more specifically, that the sentence $(\varphi \mid \neg\varphi)$ is true for every φ. But it has been established that on the ground of the Stoic logic sentences: (1), (2), (3) and (4) undoubtedly qualify as tautologies. Therefore, the use of those sentences should be allowed in proofs in the same way as the third premise in rule (21). Consequently, the following lemmas should be considered established:

$$\vdash \neg(\varphi \wedge \neg\varphi \wedge \neg\psi), \tag{22}$$
$$\vdash \neg(\neg\varphi \wedge \varphi \wedge \neg\psi), \tag{23}$$
$$\vdash \neg(\psi \wedge \varphi \wedge \neg\psi), \tag{24}$$

as well as

$$\vdash \neg(\varphi_1 \wedge \varphi_2 \wedge \ldots \varphi_n \wedge \varphi \wedge \neg\varphi \wedge \neg\psi), \tag{25}$$

for every positive integer n. They are perfectly legitimate to be cited in proofs throughout the Stoic logic.

Based on the descriptions of the rule of cut, quoted from Alexander and Simplicius, scholars have claimed that in the Stoic logic rule (18) obtains if and only if the cardinality of Φ is greater than 1 and the cardinality of Ψ is greater than zero. It is, however, a misunderstanding and it should be quite clear that the unrestricted version of rule (18) obtains in the Stoics' logic as well as in Aristotle's. Rule (14) has been quoted in the context of syllogisms as it suffices to deal with complex theorems like syllogisms (5)-(12). It is, however, no evidence whatsoever for the alleged rejection of the stronger rule (18).

9. The Explosiveness of the Operation of Consequence

Quite a general version of the principle of explosion is derivable based exclusively on indemonstrable (7), i.e., the conjunctive syllogism, lemma

(22), i.e., an undoubtedly valid formula, and transformation rule (18), i.e., the unrestricted rule of cut.

Theorem.
$$\varphi, \neg\varphi \vdash \psi. \tag{26}$$

Proof.

1. $\neg(\varphi \wedge \neg\varphi \wedge \neg\psi), \varphi \vdash \neg(\neg\varphi \wedge \neg\psi)$ indemonstrable (7)
2. $\neg(\neg\varphi \wedge \neg\psi), \neg\varphi \vdash \psi$ indemonstrable (7)
3. $\neg(\varphi \wedge \neg\varphi \wedge \neg\psi), \varphi, \neg\varphi \vdash \psi$ from lines 1 and 2 by rule (18)
4. $\vdash \neg(\varphi \wedge \neg\varphi \wedge \neg\psi)$ emma (22)
5. $\varphi, \neg\varphi \vdash \psi$ from lines 4 and 3 by rule (18)

According to theorem (26) every sentence is a consequence of a sentence and its negation combined. It is certainly a basic version of the principle of explosion. The proposed proof is arguably hardly possible to be challenged by any Stoic.

Rule (18), i.e., the unrestricted rule of cut, is cited in the above proof. The rule is well attested to and never denied in the extant sources. As argued, the view to the contrary, that appears occasionally in the literature, is neither substantiated nor even likely. If the above proof of theorem (26) were to be undermined, then so would have to be the demonstration of the sentence "a sign exists", a favourite of the Stoics, and many other demonstrations, following pattern (21) and assigned in the extant sources to thinkers of the Stoa.

10. Explosiveness and the Connective of Implication

To bridge the gap between explosiveness of the consequence operation and of the connective of implication, it is enough to invoke the Stoics' criterion of validity, known as the principle of conditionalisation and universally endorsed in the Stoa: an inference with premises $\varphi_1, \varphi_2, \ldots, \varphi_n$ and conclusion φ is valid if and only if so is the implication with antecedent $(\varphi_1 \wedge \varphi_2 \wedge \ldots \varphi_n)$ and consequent φ (Sextus, 1933, 2:135; 1935, 8:415; cf. Mates, 1953). Therefore, the following rule of inference belongs to the Stoic logic:

$$\frac{\varphi_1, \varphi_2, \ldots, \varphi_n, \vdash \varphi}{\vdash (\varphi_1 \wedge \varphi_2 \wedge \ldots \varphi_n) \to \varphi} \tag{27}$$

for any positive integer n. Anticipating criticism, it is worth emphasising that rule (27) also applies to the case of $n = 1$, taking the following form then:

$$\frac{\varphi \vdash \psi}{\vdash \varphi \to \psi}$$

To decide this, let it suffice to invoke again Sextus' testimony that the Stoics qualified the conditional $(\varphi \to \varphi)$ as valid on the very basis that its consequent follows logically from the antecedent (Sextus, 1935, 8:281, consult section 8 above for details). By applying rule (27) to theorem (26) one can immediately prove the following theorem:

$$\vdash \varphi \wedge \neg\varphi \to \psi, \tag{28}$$

securing that the connective of implication in the Stoic logic is explosive. An even stronger result is available, namely the following transformation rule is derivable on the basis of the Stoic syllogistic.

Theorem.
$$\frac{\vdash \neg(\varphi \wedge \neg\psi),}{\vdash \varphi \to \psi.} \tag{29}$$

Proof.

1.	$\vdash \neg(\varphi \wedge \neg\psi)$	assumption
2.	$\neg(\varphi \wedge \neg\psi), \varphi \vdash \psi$	indemonstrable (7)
3.	$\varphi \vdash \psi$	from lines 1 and 2 by rule (18)
4.	$\vdash \varphi \to \psi$	from line 3 by rule (27)

It facilitates proofs of the two following transformation rules.

Theorem.
$$\frac{\vdash \neg\varphi,}{\vdash \varphi \to \psi.} \tag{30}$$

Proof.

1.	$\vdash \neg\varphi$	assumption
2.	$\neg(\neg\varphi \wedge \varphi \wedge \neg\psi), \neg\varphi \vdash \neg(\varphi \wedge \neg\psi)$	indemonstrable (7)
3.	$\vdash \neg(\neg\varphi \wedge \varphi \wedge \neg\psi)$	lemma (23)
4.	$\neg\varphi \vdash \neg(\varphi \wedge \neg\psi)$	from lines 3 and 2 by rule (18)
5.	$\vdash \neg(\varphi \wedge \neg\psi)$	from lines 1 and 4 by rule (18)
6.	$\vdash \varphi \to \psi$	from line 5 by rule (29)

Theorem.

$$\frac{\vdash \psi,}{\vdash \varphi \to \psi.} \tag{31}$$

Proof.

1. $\vdash \psi$ — assumption
2. $\neg(\psi \wedge \varphi \wedge \neg\psi), \psi \vdash \neg(\varphi \wedge \neg\psi)$ — indemonstrable (7)
3. $\vdash \neg(\psi \wedge \varphi \wedge \neg\psi)$ — lemma (24)
4. $\psi \vdash \neg(\varphi \wedge \neg\psi)$ — from lines 3 and 2 by rule (18)
5. $\vdash \neg(\varphi \wedge \neg\psi)$ — from lines 1 and 4 by rule (18)
6. $\vdash \varphi \to \psi$ — from line 5 by rule (29)

Validity of rules (30) and (31) secures that no relevance, connexivity, containment or similar condition is included in the concept of implication within Stoicism. It is therefore certain that the Stoics' connective of implication is either material or formal (strict). One cannot, however, decide easily between the two.

On the one hand, in the extant sources there is clear and multiple evidence that the Stoics endorsed the concept of material implication (Diogenes, 1925; Sextus, 1933; 1935). On the other, theorems and rules formally derivable based on the extant sources are shared by the material and strict implications (the quoted theorems and rules of inference hold good for both concepts of implication, and hence, that property seems to be invariant with respect to inferences).

Notice in particular that rules (29), (30) and (31) hold good both for material and strict implications. Those rules themselves exclude only relevantistic treatment of the connective of implication. It is the provability of the corresponding theorems which distinguish the concepts of material and strict implication.

An easy way to picture a concept of implication is to relate it to the truth-functional concepts of negation and conjunction. The following two theorems put together are characteristic of the material implication:

$$(\varphi \to \psi) \vdash \neg(\varphi \wedge \neg\psi), \tag{32}$$

$$\neg(\varphi \wedge \neg\psi) \vdash (\varphi \to \psi). \tag{33}$$

Admittedly, theorem (32) is demonstrable. For it is well attested in the extant sources the following theorem, known as the syllogism of two principles (τὸ διὰ δύο τροπικῶν θεώρημα), was endorsed by the Stoics (cf. Sextus, 1933, 2:3):

$$(\varphi \to \psi), (\varphi \to \neg\psi) \vdash \neg\varphi. \tag{34}$$

Simultaneously, even though in antiquity the concept of implication was subject to a fierce controversy, literally each party of the dispute accepted the validity of the following implications:

$$\vdash \varphi \wedge \psi \to \varphi, \tag{35}$$

$$\vdash \varphi \wedge \psi \to \psi. \tag{36}$$

Therefore, the following proof of theorem (32) might be offered.

1. $(\varphi \to \psi), \varphi \vdash \psi$ indemonstrable (5)
2. $\vdash (\varphi \to \psi) \wedge \varphi \to \psi$ from line 1 by (27)
3. $((\varphi \to \psi) \wedge \varphi \to \psi), ((\varphi \to \psi) \wedge \varphi) \vdash \psi$ indemonstrable (5)
4. $((\varphi \to \psi) \wedge \varphi) \vdash \psi$ from lines 2, 3 by (18)
5. $\vdash ((\varphi \to \psi) \wedge \varphi \wedge \neg\psi) \to ((\varphi \to \psi) \wedge \varphi)$ lemma (35)
6. $(((\varphi \to \psi) \wedge \varphi \wedge \neg\psi) \to ((\varphi \to \psi) \wedge \varphi)), ((\varphi \to \psi) \wedge \varphi \wedge \neg\psi)$
    $\vdash ((\varphi \to \psi) \wedge \varphi)$ indemonstrable (5)
7. $((\varphi \to \psi) \wedge \varphi \wedge \neg\psi) \vdash ((\varphi \to \psi) \wedge \varphi)$ from 5, 6 by (18)
8. $((\varphi \to \psi) \wedge \varphi \wedge \neg\psi) \vdash \psi$ from 7, 4 by (18)
9. $\vdash ((\varphi \to \psi) \wedge \varphi \wedge \neg\psi) \to \psi$ from 8 by (27)
10. $\vdash ((\varphi \to \psi) \wedge \varphi \wedge \neg\psi) \to \neg\psi$ lemma (36)
11. $(((\varphi \to \psi) \wedge \varphi \wedge \neg\psi) \to \psi), (((\varphi \to \psi) \wedge \varphi \wedge \neg\psi) \to \neg\psi)$
    $\vdash \neg((\varphi \to \psi) \wedge \varphi \wedge \neg\psi)$ syllogism (34)
12. $(((\varphi \to \psi) \wedge \varphi \wedge \neg\psi) \to \neg\psi) \vdash \neg((\varphi \to \psi) \wedge \varphi \wedge \neg\psi)$
          from lines 9, 11 by (18)
13. $\vdash \neg((\varphi \to \psi) \wedge \varphi \wedge \neg\psi)$ from lines 10, 12 by (18)
14. $\neg((\varphi \to \psi) \wedge \varphi \wedge \neg\psi), (\varphi \to \psi) \vdash \neg(\varphi \wedge \neg\psi)$
          indemonstrable (7)
15. $(\varphi \to \psi) \vdash \neg(\varphi \wedge \neg\psi)$ from lines 13, 14 by (18)

Unfortunately, theorem (32) holds good for any concept of implication. And no unquestionable derivation of theorem (33) seems to be available. The key problem is that the extant sources remain silent on the issue of sufficient conditions to derive an implication. The only exception is the above mentioned principle of conditionalisation, entailing rule (27), unfortunately, shared by the concepts of material and strict implication.

One cannot even claim that of the semantic and axiomatic descriptions of the Stoics' concept of implication the former refers to the material and the latter to the formal implication. For the semantic descriptions, however clear, are not unchallenged. And the axiomatic description is uncertain due to the hopeless paucity of the extant source

material. Particularly, it cannot be ruled out that the full-blooded deduction theorem is endorsed in the lost original works by Chrisippus of Soli (which would allow to easily decide in favour of the material implication) instead of the simplified principle of conditionalisation with rule (27), reported by the historian Diogenes Laertios. Therefore, it seems reasonable to leave it open whether the Stoics' implication is strict or material, i.e. whether or not all theorems the Stoics would be ready to consider valid are formally demonstrable on the ground of their system of logic.

11. The Explosiveness and Structural Rules

Strictly speaking, theorem (26) secures that every sentence is a consequence of any pair of contradictories. It is a principle of explosion in itself because it thwarts any attempt to impose, within the confines of the Stoic logic, any relevance requirements either on the operation of consequence or on the connective of implication. However, the full-blooded description of explosiveness states that every theorem is a consequence of any inconsistent set of sentences. As scholars debate whether or not the presence of redundant premises invalidates otherwise valid inference, it is worth emphasising that within the confines of the Stoic logic inference (26) with extra premises remains valid. To prove it, consider first the following lemma. For any positive integer n the following theorem is provable on the ground of the Stoics' syllogistic:

$$\neg(\varphi_1 \wedge \varphi_2 \wedge \ldots \varphi_n \wedge \varphi), \{\varphi_i\}_{i=1}^{n} \vdash \varphi^{\star}. \tag{37}$$

Of course, the proof goes by induction over the integer n. If $n = 1$ theorem (37) is simply equal with indemonstrable (7). Assume that there is a positive integer k such that theorem (37) obtains for $n = k$. It follows that theorem (37) obtains for $n = k+1$ as well. By assumption, the following syllogism is provable:

$$\neg(\varphi_1 \wedge \varphi_2 \wedge \ldots \varphi_k \wedge \varphi_{k+1} \wedge \varphi), \{\varphi_i\}_{i=1}^{k} \vdash \neg(\varphi_{k+1} \wedge \varphi). \tag{38}$$

The following theorem is simply another wording of indemonstrable (7):

$$\neg(\varphi_{k+1} \wedge \varphi), \varphi_{k+1} \vdash \varphi^{\star}. \tag{39}$$

From theorems (38) and (39) by rule (14) the following syllogism is derivable:

$$\neg(\varphi_1 \wedge \varphi_2 \wedge \ldots \varphi_k \wedge \varphi_{k+1} \wedge \varphi), \{\varphi_i\}_{i=1}^{k+1} \vdash \varphi^\star, \qquad (40)$$

which completes the proof of lemma (37). Notice that what syllogism (37) states is simply a generalised version of indemonstrable (7). It leaves no doubt that it would be willingly accepted by the Stoics.

It follows immediately that, for any positive integer n, the following theorem is provable on the ground of the Stoic syllogistic:

$$\neg(\varphi_1 \wedge \varphi_2 \wedge \ldots \varphi_n \wedge \varphi \wedge \neg\varphi \wedge \neg\psi), \{\varphi_i\}_{i=1}^{n}, \varphi, \neg\varphi \vdash \psi. \qquad (41)$$

It is a special case of lemma (37), and hence, a valid syllogism. The first premise of syllogism (41) is equal to tautology (4) and may serve as lemma (25). By application of rule (18) to lemma (25) and syllogism (41) one proves the following theorem:

$$\{\varphi_i\}_{i=1}^{n}, \varphi, \neg\varphi \vdash \psi. \qquad (42)$$

Theorems (26) and (42) combined make the following one:

$$\Phi, \varphi, \neg\varphi \vdash \psi, \qquad (43)$$

stating directly that every sentence is derivable from any inconsistent set of sentences. It is an unequivocal formulation of explosiveness. Together with rule (27) theorem (43) secures validity of every implication, namely

$$\vdash (\varphi_1 \wedge \varphi_2 \wedge \ldots \varphi_n) \to \psi,$$

for every integer n, providing there are such integers k, l that $1 \leq k \leq n$ and $1 \leq l \leq n$, and $\varphi_k = \neg\varphi_l$. It is a generalisation of theorem (28).

It follows that no relevance requirements are applicable whether on the concept of syllogism or the general concept of consequence or on the connective of implication.

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