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Egocentric Doxastic Logic

Abstract. Originally proposed by Prior, egocentric logics is a class of logical systems that capture properties of agents rather than of possible worlds. The article proposes a doxastic egocentric system with rigid names for reasoning about beliefs that an agent might have about herself.

Keywords: beliefs; egocentric; axiomatisation; non-rigid names

1. Introduction

Prior proposed to consider *egocentric* logical systems that express properties of agents [26]. We can formalise his idea by considering a ternary satisfaction relation $w, a \Vdash \varphi$ between a world w, an agent a, and a formula φ. For example, we can write

$$w, a \Vdash \text{“is sick”}$$

to express the fact that agent a is sick in world w. In this article, we extend Prior's egocentric approach to *egocentric beliefs* or an agent's belief about herself. For example, we write

$$w, a \Vdash \mathsf{B}\text{“is sick”}$$

to express the fact that, in world w, agent a believes that she (agent a) is sick. Egocentric statements can be combined in the usual way using Boolean connectives. For example, one can write

$$w, a \Vdash \mathsf{B}\text{“is sick”} \wedge \mathsf{B}\text{“will recover”}$$

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to state that, in world w, agent a has two beliefs: that she is sick and that she will recover. Modality B can be nested to say that, in world w, agent a believes that she believes that she is sick:

$$w, a \Vdash \mathsf{B}\mathsf{B}\text{“is sick”}$$

or that she believes that she does not believe that she is sick:

$$w, a \Vdash \mathsf{B}\neg\mathsf{B}\text{“is sick”}.$$

Of course, the above statement could be also written in the traditional (non-egocentric) doxastic logic as $w \Vdash \mathsf{B}_a\neg\mathsf{B}_a$“agent a is sick”. Thus, it does not make much sense to introduce a doxastic egocentric logic to do the same. The advantage of the egocentric approach becomes clear if the modality B is combined with agent-dependent names. By such a name, we mean any name whose meaning depends on the agent that uses this name. An example of an agent-dependent name is *ma* (short for “mother”). To use agent-specific names, we consider modality @. For example, the statement

$$w, a \Vdash @_{ma}\text{“is sick”}$$

means that the mother of agent a is sick. Note that nesting modality $@_{ma}$ gives a way to refer to the mother of the mother. For example, the statement

$$w, a \Vdash @_{ma}@_{ma}\text{“is sick”}$$

means that the *grandma* of agent a on the mother's side is sick. When the belief modality B is combined with modality @, the language of the logical system becomes much richer. For example, the statement

$$w, a \Vdash \mathsf{B}@_{ma}\text{“is sick”}$$

means that agent a believes that her mother is sick. At the same time, the statement

$$w, a \Vdash @_{ma}\mathsf{B}\text{“is sick”}$$

means that the mother of agent a believes that she (the mother) is sick. The statement

$$w, a \Vdash \mathsf{B}(@_{ma}\text{“is sick”} \vee @_{pa}\text{“is sick”})$$

means that agent a believes that at least one of her parents is sick. The statement

$$w, a \Vdash @_{ma}\mathsf{B}(@_{ma}\text{“is sick”} \vee @_{pa}\text{“is sick”})$$

means that agent a's mother believes that at least one of her (the mother's) parents is sick. Finally, note that the statement

$$w, a \Vdash @_{ma}\mathsf{B}@_{pa}\mathsf{B}\text{“is sick”}$$

means that agent a's mother believes that her father believes that he is sick.

In this article, we introduce a logical system that describes the interplay between modalities B and @. We give the formal semantics of this modality by combining the standard KD45-style semantics of beliefs [9, 23] with the egocentric setting. Our main technical result is a sound and complete axiomatisation of this logical system.

The rest of the paper is structured as follows. In the next section, we review the existing literature on egocentric logical systems. In Section 3, we describe the class of models that we use. In the section that follows, we give the syntax and the semantics of our logical system. Section 5 lists the axioms and inference rules of the system. The soundness of the system is established in Section 6. To improve readability, the proof of completeness is split into two sections. Section 7 defines the canonical model and Section 8 uses this model to prove completeness. In Section 9, we use a non-standard semantics with non-rigid names to show that one of our inference rules is not derivable. Section 10 describes partial and common names as possible directions for future work. Section 11 concludes the article.

2. Literature Review

Two types of egocentric systems have been considered in the literature: “pure” egocentric systems with a binary satisfaction relation $a \Vdash \varphi$ that captures properties of an agent a and “hybrid” systems (like ours) that use a ternary relation $w, a \Vdash \varphi$.

Seligman, Liu, and Girard [28, 29] proposed a pure egocentric logic of friendship that contains “all friends” modality F. For example, in their language the statement $a \Vdash \mathsf{F}$“is sick” means that all friends of agent a are sick. Modality F is also used in [5, 6]. Jiang and Naumov suggested egocentric modality "likes those who" L [17, 19]. In their language, the statement $a \Vdash \mathsf{L}\varphi$ means that agent a likes those who have property φ.

Grove and Halpern [14–16] proposed a hybrid egocentric logical system for knowledge modality K. The statement $w, a \Vdash \mathsf{K}$"is sick" means in their language that, in world w, agent a knows that she (agent a) is sick. Epstein, Naumov, and Tao considered the modality "know who" [8]. Naumov and Tao also proposed the modality "know how to tell apart" [25].

3. Doxastic Models

In this section, we define the class of models that we use to give the semantics of our logical system. Throughout the rest of the article, we fix the set of propositional variables and a set of names N.

Definition 3.1. A tuple $(W, \mathcal{A}, \{R_a\}_{a\in\mathcal{A}}, \{e_a\}_{a\in\mathcal{A}}, \pi)$ is called a doxastic model when

1. W is a (possibly empty) set of "worlds",
2. $\mathcal{A}$ is a (possibly empty) set of "agents",
3. R_a is a binary "plausibility" relation on W for each $a \in \mathcal{A}$ which is
   (a) *transitive*: for all $w, u, v \in W$, if wR_au and uR_av, then wR_av,
   (b) *Euclidean*: for all $w, u, v \in W$, if wR_au and wR_av, then uR_av,
   (c) *serial*: for any $w \in W$ there is an $u \in W$ such that wR_au,
4. "extension" function e_a is such that $e_a(n) \in \mathcal{A}$ for each name $n \in N$,
5. $\pi(p) \subseteq W \times \mathcal{A}$ for each propositional variable p.

The above definition adds extension functions e_a to the standard KD45 doxastic models [1–4, 10, 11]. In the literature, sphere [20], neighbourhood [12, 13, 30], and trust-based [18] semantics of beliefs have also been considered.

Intuitively, in the above definition, $e_a(n)$ is the agent to which agent a refers by name n. The names that we consider in this article are *total* because function e_a is defined for all names, the names are *proper* because the function has a unique value, and they are *rigid* because the value $e_a(n)$ does not depend on the world. In Section 9 and Section 10, we discuss possible generalisations of our work for *partial*, *common*, and *non-rigid* names.

Note that, unlike the standard semantics of the doxastic logic, in our case, the valuation function π returns a set of pairs. Intuitively, $(w, a) \in \pi(p)$ if propositional variable p is true in world w about agent a.

4. Syntax and Semantics

The language Φ of our logical system is defined by the grammar:

$$\varphi := p \mid \neg\varphi \mid \varphi \to \varphi \mid @_n\varphi \mid \mathsf{B}\varphi,$$

where p is a propositional variable and $n \in N$ is a name. We read $@_n\varphi$ as "statement φ is true about the agent named n". We read $\mathsf{B}\varphi$ as "believes in φ". We assume that biconditional $\leftrightarrow$, disjunction $\vee$ and Boolean constants true $\top$ and false $\bot$ are defined in the usual way. For any integer $k \geq 0$ and any sequence of names $\alpha = n_1, \dots, n_k$ by $\overline{@}_\alpha\varphi$ we denote the formula $@_{n_1} \dots @_{n_k}\varphi$. Note that if α is the empty sequence, then $\overline{@}_\alpha\varphi$ is the formula φ.

Definition 4.1. For any world $w \in W$ and any agent $a \in \mathcal{A}$ of a doxastic model $(W, \mathcal{A}, \{R_a\}_{a\in\mathcal{A}}, \{e_a\}_{a\in\mathcal{A}}, \pi)$ and any formula $\varphi \in \Phi$, the satisfaction relation $w, a \Vdash \varphi$ is defined as follows:

1. $w, a \Vdash p$ if $(w, a) \in \pi(p)$,
2. $w, a \Vdash \neg\varphi$ if $w, a \nVdash \varphi$,
3. $w, a \Vdash \varphi \to \psi$ if either $w, a \nVdash \varphi$ or $w, a \Vdash \psi$,
4. $w, a \Vdash @_n\varphi$, if $w, e_a(n) \Vdash \varphi$,
5. $w, a \Vdash \mathsf{B}\varphi$ if $u, a \Vdash \varphi$ for all worlds $u \in W$ such that wR_au.

5. Axioms

In addition to propositional tautologies in language Φ, our logical system contains the following axioms:

- Distributivity:
  $\mathsf{B}(\varphi \to \psi) \to (\mathsf{B}\varphi \to \mathsf{B}\psi)$,
  $@_n(\varphi \to \psi) \to (@_n\varphi \to @_n\psi)$,
- Positive Introspection: $\mathsf{B}\varphi \to \mathsf{B}\mathsf{B}\varphi$,
- Negative Introspection: $\neg\mathsf{B}\varphi \to \mathsf{B}\neg\mathsf{B}\varphi$,
- Belief Consistency: $\neg\mathsf{B}\bot$,
- Commutativity: $\neg@_n\varphi \leftrightarrow @_n\neg\varphi$.

We write $\vdash \varphi$ and say that formula $\varphi \in \Phi$ is a *theorem* of our logical system if this formula is provable from these axioms using the Modus Ponens, the Necessitation, and the Insertion inference rules:

$$\frac{\varphi, \varphi \to \psi}{\psi} \qquad \frac{\varphi}{@_n\varphi} \qquad \frac{\overline{@}_\alpha\varphi}{\overline{@}_\alpha\mathsf{B}\varphi}$$

We write $X \vdash \varphi$ if formula $\varphi \in \Phi$ is derivable from the *theorems* of our logical system and the additional set of assumptions $X \subseteq \Phi$ using *only* the Modus Ponens inference rule. Note that statements $\vdash \varphi$ and $\varnothing \vdash \varphi$ are equivalent. We say that a set $X \subseteq \Phi$ is consistent if $X \nvdash \bot$.

The Insertion inference rule is the most non-trivial rule in our logical system. Note that a special case of this rule, if the sequence α is empty, is the Necessitation rule for modality B:

$$\frac{\varphi}{\mathsf{B}\varphi}$$

One might naturally wonder if the Insertion rule in its full form is really necessary in our logical system or whether it can be replaced by the Necessitation rule for modality B. We know only a partial answer to this question. Namely, let L^- denote a version of our logical system that contains the Necessitation rule for modality B instead of the full form of the Insertion rule. In Section 9, we use a semantics with non-rigid names to show that the full form of the Insertion rule is not *derivable* in system L^-. The question of whether this rule is *admissible* in L^- remains open.

Next, we show several lemmas that are used later in the proof of completeness. The first of them generalises the Commutativity axiom.

Lemma 5.1. $\vdash \overline{@}_\alpha \neg\varphi \leftrightarrow \neg\overline{@}_\alpha \varphi$.

Proof. We prove the lemma by the induction on the length of sequence α. If the length is zero, then the formula $@_\alpha \neg\varphi \leftrightarrow \neg\overline{@}_\alpha \varphi$ is a propositional tautology. Let α be $n_1, \dots, n_k$, where $k \geq 1$. Then, by the induction hypothesis,

$$\vdash \overline{@}_{n_2,\dots,n_k} \neg\varphi \leftrightarrow \neg\overline{@}_{n_2,\dots,n_k} \varphi.$$

Thus, by the laws of propositional reasoning,

$$\vdash \overline{@}_{n_2,\dots,n_k} \neg\varphi \to \neg\overline{@}_{n_2,\dots,n_k} \varphi,$$
$$\vdash \neg\overline{@}_{n_2,\dots,n_k} \varphi \to \overline{@}_{n_2,\dots,n_k} \neg\varphi.$$

Hence, by the Necessitation inference rule,

$$\vdash @_{n_1}(\overline{@}_{n_2,\dots,n_k} \neg\varphi \to \neg\overline{@}_{n_2,\dots,n_k} \varphi),$$
$$\vdash @_{n_1}(\neg\overline{@}_{n_2,\dots,n_k} \varphi \to \overline{@}_{n_2,\dots,n_k} \neg\varphi).$$

Then, by the Distributivity axiom and the Modus Ponens inference rule,

$$\vdash @_{n_1}\overline{@}_{n_2,\dots,n_k} \neg\varphi \to @_{n_1}\neg\overline{@}_{n_2,\dots,n_k} \varphi,$$
$$\vdash @_{n_1}\neg\overline{@}_{n_2,\dots,n_k} \varphi \to @_{n_1}\overline{@}_{n_2,\dots,n_k} \neg\varphi.$$

Thus, by the Commutativity axiom and propositional reasoning,

$$\vdash @_{n_1}\overline{@}_{n_2,\dots,n_k}\neg\varphi \to \neg @_{n_1}\overline{@}_{n_2,\dots,n_k}\varphi,$$
$$\vdash \neg @_{n_1}\overline{@}_{n_2,\dots,n_k}\varphi \to @_{n_1}\overline{@}_{n_2,\dots,n_k}\neg\varphi.$$

Therefore, by propositional reasoning,

$$\vdash @_{n_1}\overline{@}_{n_2,\dots,n_k}\neg\varphi \leftrightarrow \neg @_{n_1}\overline{@}_{n_2,\dots,n_k}\varphi.$$

This concludes the proof of the lemma. ⊣

LEMMA 5.2. $\vdash \neg\overline{@}_\alpha \mathsf{B}\bot$.

PROOF. By the Belief Consistency axiom, $\vdash \neg\mathsf{B}\bot$. Thus, by the Necessitation inference rule applied multiple times, $\vdash \overline{@}_\alpha\neg\mathsf{B}\bot$. Therefore, $\vdash \neg\overline{@}_\alpha\mathsf{B}\bot$ by Lemma 5.1 and propositional reasoning. ⊣

The next lemma proves the Distributivity axiom for modality @ in a more general form.

LEMMA 5.3. $\vdash \overline{@}_\alpha(\varphi \to \psi) \to (\overline{@}_\alpha\varphi \to \overline{@}_\alpha\psi)$.

PROOF. Let α be the sequence $n_1, \dots, n_k$, where $k \geq 0$. We prove the statement by induction on k. If $k = 0$, then the formula

$$\vdash \overline{@}_\alpha(\varphi \to \psi) \to (\overline{@}_\alpha\varphi \to \overline{@}_\alpha\psi)$$

is a propositional tautology.

Suppose that $k > 0$. By the induction hypothesis,

$$\vdash \overline{@}_{n_2,\dots,n_k}(\varphi \to \psi) \to (\overline{@}_{n_2,\dots,n_k}\varphi \to \overline{@}_{n_2,\dots,n_k}\psi).$$

Then, by the Necessitation inference rule,

$$\vdash @_{n_1}(\overline{@}_{n_2,\dots,n_k}(\varphi \to \psi) \to (\overline{@}_{n_2,\dots,n_k}\varphi \to \overline{@}_{n_2,\dots,n_k}\psi)).$$

Thus, by the Distributivity axiom and the Modus Ponens inference rule,

$$\vdash @_{n_1}\overline{@}_{n_2,\dots,n_k}(\varphi \to \psi) \to @_{n_1}(\overline{@}_{n_2,\dots,n_k}\varphi \to \overline{@}_{n_2,\dots,n_k}\psi)).$$

Therefore,

$$\vdash @_{n_1}\overline{@}_{n_2,\dots,n_k}(\varphi \to \psi) \to (@_{n_1}\overline{@}_{n_2,\dots,n_k}\varphi \to @_{n_1}\overline{@}_{n_2,\dots,n_k}\psi))$$

by the Distributivity axiom and propositional reasoning. ⊣

The next lemma shows that the converse of Lemma 5.3 is also true.

LEMMA 5.4. $\vdash (\overline{@}_\alpha\varphi \to \overline{@}_\alpha\psi) \to \overline{@}_\alpha(\varphi \to \psi)$.

PROOF. Observe that the following two formulae are propositional tautologies:

$$\neg\varphi \to (\varphi \to \psi) \qquad \psi \to (\varphi \to \psi).$$

Thus, by the Necessitation inference rule applied multiple times,

$$\vdash \overline{@}_\alpha(\neg\varphi \to (\varphi \to \psi)) \qquad \vdash \overline{@}_\alpha(\psi \to (\varphi \to \psi)).$$

Hence, by Lemma 5.3 and the Modus Ponens inference rule,

$$\vdash \overline{@}_\alpha\neg\varphi \to \overline{@}_\alpha(\varphi \to \psi) \qquad \vdash \overline{@}_\alpha\psi \to \overline{@}_\alpha(\varphi \to \psi).$$

Then, by Lemma 5.1 and propositional reasoning,

$$\vdash \neg\overline{@}_\alpha\varphi \to \overline{@}_\alpha(\varphi \to \psi) \qquad \vdash \overline{@}_\alpha\psi \to \overline{@}_\alpha(\varphi \to \psi).$$

Hence, $\vdash \neg\overline{@}_\alpha\varphi \vee \overline{@}_\alpha\psi \to \overline{@}_\alpha(\varphi \to \psi)$ by propositional reasoning. Thus, $\vdash (\overline{@}_\alpha\varphi \to \overline{@}_\alpha\psi) \to \overline{@}_\alpha(\varphi \to \psi)$ again by propositional reasoning. ⊣

LEMMA 5.5. $\vdash \overline{@}_\alpha\mathsf{B}(\varphi \to \psi) \to (\overline{@}_\alpha\mathsf{B}\varphi \to \overline{@}_\alpha\mathsf{B}\psi)$.

PROOF. An instance $\mathsf{B}(\varphi \to \psi) \to (\mathsf{B}\varphi \to \mathsf{B}\psi)$ of the Distributivity axiom implies, by multiple applications of the Necessitation inference rule, that

$$\vdash \overline{@}_\alpha(\mathsf{B}(\varphi \to \psi) \to (\mathsf{B}\varphi \to \mathsf{B}\psi)).$$

Therefore,

$$\vdash \overline{@}_\alpha\mathsf{B}(\varphi \to \psi) \to (\overline{@}_\alpha\mathsf{B}\varphi \to \overline{@}_\alpha\mathsf{B}\psi)$$

by Lemma 5.3 and propositional reasoning. ⊣

We give proof of the following standard lemma in the appendix.

LEMMA 5.6 (deduction). *For any* $n > 0$ *and any formulae* $\varphi_1, \dots, \varphi_n, \psi$, *if* $\varphi_1, \dots, \varphi_n \vdash \psi$, *then* $\varphi_1, \dots, \varphi_{n-1} \vdash \varphi_n \to \psi$.

The next lemma rephrases the Insertion inference rule in a more general form which is more convenient to use in the proof of completeness.

LEMMA 5.7. *If* $\overline{@}_\alpha\varphi_1, .., \overline{@}_\alpha\varphi_n \vdash \overline{@}_\alpha\psi$, *then* $\overline{@}_\alpha\mathsf{B}\varphi_1, .., \overline{@}_\alpha\mathsf{B}\varphi_n \vdash \overline{@}_\alpha\mathsf{B}\psi$.

PROOF. Suppose $\overline{@}_\alpha\varphi_1, \dots, \overline{@}_\alpha\varphi_n \vdash \overline{@}_\alpha\psi$. Hence, by Lemma 5.6,

$$\overline{@}_\alpha\varphi_1, \dots, \overline{@}_\alpha\varphi_{n-1} \vdash \overline{@}_\alpha\varphi_n \to \overline{@}_\alpha\psi.$$

Then, by Lemma 5.4 and the Modus Ponens inference rule,

$$\overline{@}_\alpha\varphi_1, \dots, \overline{@}_\alpha\varphi_{n-1} \vdash \overline{@}_\alpha(\varphi_n \to \psi)$$

Thus, by repeating the last two steps $n-1$ more times,

$$\vdash \overline{@}_\alpha(\varphi_1 \to \dots (\varphi_n \to \psi)\dots).$$

Hence, by the Insertion inference rule,

$$\vdash \overline{@}_\alpha \mathsf{B}(\varphi_1 \to \dots (\varphi_n \to \psi)\dots).$$

Then, by Lemma 5.5,

$$\vdash \overline{@}_\alpha \mathsf{B}\varphi_1 \to \overline{@}_\alpha \mathsf{B}(\varphi_2 \to \dots (\varphi_n \to \psi)\dots).$$

Thus, by the Modus Ponens inference rule,

$$\overline{@}_\alpha \mathsf{B}\varphi_1 \vdash \overline{@}_\alpha \mathsf{B}(\varphi_2 \to \dots (\varphi_n \to \psi)\dots).$$

Therefore,

$$\overline{@}_\alpha \mathsf{B}\varphi_1, \dots, \overline{@}_\alpha \mathsf{B}\varphi_n \vdash \overline{@}_\alpha \mathsf{B}\psi$$

by repeating the last two steps $n-1$ more times. ⊣

Finally, by the standard proof of Lindenbaum's lemma [see, e.g., 22, Proposition 2.14], we have:

Lemma 5.8 (Lindenbaum). *Any consistent set of formulae can be extended to a maximal consistent set of formulae.*

6. Soundness

Theorem 6.1. *If $\vdash \varphi$, then $w, a \Vdash \varphi$ for any world w and any agent a of any doxastic model.*

The proofs of soundness of the Distributivity, the Positive Introspection, the Negative Introspection, the Belief Consistency, and the Commutativity axiom as well as of the Modus Ponens and the Necessitation inference rules are straightforward. Below we prove the soundness of the Insertion inference rule as a separate lemma.

Lemma 6.1. *If $w, a \Vdash \overline{@}_\alpha \varphi$ for each agent $a \in \mathcal{A}$ and each world w of each doxastic model, then $w, a \Vdash \overline{@}_\alpha \mathsf{B}\varphi$ for each agent $a \in \mathcal{A}$ and each world w of each doxastic model.*

PROOF. Consider any doxastic model $(W, \mathcal{A}, \{R_a\}_{a\in\mathcal{A}}, \{e_a\}_{a\in\mathcal{A}}, \pi)$, any $w \in W$, and any agent $a \in \mathcal{A}$. It suffices to show that $w, a \Vdash \overline{@}_\alpha \mathsf{B}\varphi$.

Let α be the sequence $n_1, \dots, n_k$. By item 4 of Definition 4.1, to show $w, a \Vdash \overline{@}_{n_1,\dots,n_k}\mathsf{B}\varphi$, it suffices to prove that $w, e_a(n_1) \Vdash \overline{@}_{n_2\dots,n_k}\mathsf{B}\varphi$. To prove the last statement, by the same item 4 of Definition 4.1, it suffices to show that $w, e_{e_a(n_1)}(n_2) \Vdash \overline{@}_{n_3\dots,n_k}\mathsf{B}\varphi$. By repeating the same argument $k-2$ more times, we can observe that it suffices to show that

$$w, e_{\cdot^{\cdot^{\cdot}}{}_{e_{e_a(n_1)}(n_2)}\cdot^{\cdot^{\cdot}}}(n_k) \Vdash \mathsf{B}\varphi.$$

Next, consider any world $u \in W$ such that $wR_a u$. By item 5 of Definition 4.1, it suffices to show that

$$u, e_{\cdot^{\cdot^{\cdot}}{}_{e_{e_a(n_1)}(n_2)}\cdot^{\cdot^{\cdot}}}(n_k) \Vdash \varphi.$$

Then, by the same item 5 of Definition 4.1 applied k times, it suffices to show that $u, a \Vdash \overline{@}_\alpha \varphi$. The last statement is true by the assumption of the lemma. $\dashv$

7. Canonical Model

Towards the proof of the completeness theorem, in this section, we define the canonical model $(W, \mathcal{A}, \{R_a\}_{a\in\mathcal{A}}, \{e_a\}_{a\in\mathcal{A}}, \pi)$ for our logical system. As usual in modal logic, possible worlds are maximal consistent sets of formulae.

DEFINITION 7.1. W is the set of all maximal consistent sets of formulae.

Next, we define the set of agents in the canonical model. In the proofs of completeness, elements of the model are usually specified using *syntactical constructions*. In our case, we define the agents in the canonical model as sequences of names in the language.

DEFINITION 7.2. $\mathcal{A}$ is the set of all finite sequences of names.

Informally, the empty sequence represents an "initial" agent in our model. The sequence "*ma*" represents the mother of the initial agent. The sequence "*ma*,*pa*" represents the father of the mother of the initial agent, and so on. Throughout the paper, by $s :: x$ we mean the concatenation of an element x to the end of a sequence s.

DEFINITION 7.3. $e_a(n) = a :: n$ for any agent $a \in \mathcal{A}$ and any name $n \in N$.

For example, intuitively, $e_{ma,ma}(pa) = (ma, ma) :: pa = ma, ma, pa$ because ma, ma is a grandmother of the initial agent and ma, ma, pa is a great-grandfather of the initial agent.

At the core of most proofs of completeness is a "truth" lemma. In traditional (non-egocentric) modal logics, it usually states that $w \Vdash \varphi$ iff $\varphi \in w$ for each world w and each formula φ. Note that, in our case, satisfaction $w, a \Vdash \varphi$ is a ternary relation that includes agent a. Thus, we need to modify the right-hand side of the truth lemma to somehow include agent a. The solution we found is stated as Lemma 8.2 in the next section: $w, a \Vdash \varphi$ iff $\overline{@}_a\varphi \in w$.

Informally, the statement $\overline{@}_a\mathsf{B}\varphi \in w$ means that, in the world w, agent a believes that agent a has property φ. To model this, we must guarantee that property $\overline{@}_a\varphi$ is satisfied in all worlds that agent a finds plausible. This explains our intuition behind the following definition:

DEFINITION 7.4. wR_au if $\{\overline{@}_a\varphi \mid \overline{@}_a\mathsf{B}\varphi \in w\} \subseteq u$.

The next definition is chosen to satisfy the base case of the truth lemma for our logical system.

DEFINITION 7.5. $\pi(p) = \{(w, a) \mid \overline{@}_a p \in w\}$.

This concludes the definition of the canonical model. In the next three lemmas, we show that relation R_a satisfies conditions 3(a) through 3(c) of Definition 3.1.

LEMMA 7.1. *Relation R_a is transitive.*

PROOF. Consider any worlds $w, u, v \in W$ and a formula $\varphi \in \Phi$ such that wR_au, uR_av, and $\overline{@}_a\mathsf{B}\varphi \in w$. By Definition 3.1(3(a)), it suffices to prove that $\overline{@}_a\varphi \in v$.

Indeed, the formula $\mathsf{B}\varphi \to \mathsf{BB}\varphi$ is an instance of the Positive Introspection axiom. Then, $\vdash \overline{@}_a(\mathsf{B}\varphi \to \mathsf{BB}\varphi)$ by the Necessitation inference rule. Hence, $\vdash \overline{@}_a\mathsf{B}\varphi \to \overline{@}_a\mathsf{BB}\varphi$ by the Distributivity axiom and the Modus Ponens inference rule. Thus, $w \vdash \overline{@}_a\mathsf{BB}\varphi$ by the assumption $\overline{@}_a\mathsf{B}\varphi \in w$. Then, $\overline{@}_a\mathsf{BB}\varphi \in w$ because w is a maximal consistent set. Hence, $\overline{@}_a\mathsf{B}\varphi \in u$ by the assumption wR_au and Definition 7.4. Therefore, $\overline{@}_a\varphi \in v$ by the assumption uR_av and Definition 7.4. ⊣

LEMMA 7.2. *Relation R_a is Euclidean.*

PROOF. Consider any worlds $w, u, v \in W$ and a formula $\varphi \in \Phi$ such that wR_au, wR_av, and $(\star)$: $\overline{@}_a\mathsf{B}\varphi \in u$.

By item 3(a) of Definition 3.1, it suffices to prove that $\overline{@}_a\varphi \in v$.

Suppose that $\overline{@}_a\varphi \notin v$. Thus, $\overline{@}_a\mathsf{B}\varphi \notin w$ by the assumption wR_av and Definition 7.4. Hence, $\neg\overline{@}_a\mathsf{B}\varphi \in w$ because w is a maximal consistent set. Then, by Lemma 5.1 and propositional reasoning, we have ($\star\star$): $w \vdash \overline{@}_a\neg\mathsf{B}\varphi$.

At the same time, note that the formula $\neg\mathsf{B}\varphi \to \mathsf{B}\neg\mathsf{B}\varphi$ is an instance of the Negative Introspection axiom. Thus, $\vdash \overline{@}_a(\neg\mathsf{B}\varphi \to \mathsf{B}\neg\mathsf{B}\varphi)$ by the Necessitation rule. Hence, $\vdash \overline{@}_a\neg\mathsf{B}\varphi \to \overline{@}_a\mathsf{B}\neg\mathsf{B}\varphi$ by Lemma 5.3. Then, by ($\star\star$) and propositional reasoning, $w \vdash \overline{@}_a\mathsf{B}\neg\mathsf{B}\varphi$.Thus, $\overline{@}_a\mathsf{B}\neg\mathsf{B}\varphi \in w$ because w is a maximal consistent set. Hence, $\overline{@}_a\neg\mathsf{B}\varphi \in u$ by the assumption wR_au and Definition 7.4. Then, $u \vdash \neg\overline{@}_a\mathsf{B}\varphi$ by Lemma 5.1 and propositional reasoning. Therefore, $\overline{@}_a\mathsf{B}\varphi \notin u$ because u is consistent, which contradicts assumption ($\star$). ⊣

Lemma 7.3. *Relation R_a is serial.*

Proof. Consider any $w \in W$. By item 3(c) of Definition 3.1, it suffices to show that there is a world $u \in W$ such that wR_au. Indeed, consider the set of formulae ($*$) $X = \{\overline{@}_a\varphi \mid \overline{@}_a\mathsf{B}\varphi \in w\}$. First, we show that it is consistent. Assume the opposite. Then there are formulae ($**$) $\overline{@}_a\mathsf{B}\psi_1, \dots, \overline{@}_a\mathsf{B}\psi_n \in w$, where $\overline{@}_a\psi_1, \dots, \overline{@}_a\psi_n \vdash \bot$. Hence, $\overline{@}_a\psi_1, \dots, \overline{@}_a\psi_n \vdash \overline{@}_a\bot$ by the Modus Ponens rule and the tautology $\bot \to \overline{@}_a\bot$. Thus, $\overline{@}_a\mathsf{B}\psi_1, \dots, \overline{@}_a\mathsf{B}\psi_n \vdash \overline{@}_a\mathsf{B}\bot$, by Lemma 5.7. Hence, by ($**$), we have $w \vdash \overline{@}_a\mathsf{B}\bot$, which contradicts Lemma 5.2 and the consistency of w.

By Lemma 5.8, the set X can be extended to a maximal consistent set u. Note that wR_au by Definition 7.4, equation ($*$), and the choice of u as an extension of X. ⊣

8. Completeness

In this section, we prove the completeness theorem for our logical system. Logical systems in which satisfaction relation $\Vdash$ has more than just a possible world on the left-hand side are sometimes called "multi-dimensional". Multiple completeness results for such systems have been obtained in the past [21, 24]; the closest to us are works [7, 8, 16, 25, 27].

The semantics of logical systems in [7, 8, 16, 25], just like the one in the current article, is defined in terms of the relation $w, a \Vdash \varphi$ between a world w, an agent a, and a formula φ. In such a setting, constructing canonical models in which all agents are present in all worlds has been particularly challenging. In fact, in the semantics used in [16], "each agent exists in just one world" [16, Appendix D]. In [7, 8], the authors prove completeness with respect to the class of models in which not all agents are present in each world. This assumption is essential for the tree-based canonical model construction in these works. In [25], a newly proposed "matrix" technique is used to construct a canonical model that allows all agents to be present in all possible worlds.

Work [27] considers a logical system without agents whose semantics is defined using a ternary relation $w, u \Vdash \varphi$, where w and u are possible worlds of two distinctive types. One can treat the worlds of the second type as agents, which makes [27] similar to [7, 8, 16, 25] as well as the current paper. The distinctive feature of [27] is the inclusion of @ modality. In the presence of such modality, a complicated "matrix" technique of [25] is not needed and the canonical model can be constructed using a single maximal consistent set. This is the approach that we adopt in the current article as well. In fact, the truth lemma in [27] has the form "$i, a \Vdash \varphi$ iff $@_i @_a \in w$", which is very similar to our Lemma 8.2. The most significant difference between our work and [27] is that our language contains names, like ma, whose meanings change depending on the agent that uses this name. Of course, our logical system also captures beliefs. As a result, we assume that relation R_a is transitive, Euclidean, and serial. The modality in [27] captures the property of a general reachability relation.

As usual, we prove the truth lemma by induction. To improve the readability of its proof, we separated the most non-trivial part of the induction step into a separate lemma below.

LEMMA 8.1. *If $\overline{@}_a \mathsf{B}\varphi \notin w$, then $\overline{@}_a \varphi \notin u$ for some world $u \in W$ such that $wR_a u$.*

PROOF. First, consider the set of formulae

$$X = \{\neg \overline{@}_a \varphi\} \cup \{\overline{@}_a \psi \mid \overline{@}_a \mathsf{B}\psi \in w\}. \qquad (*)$$

We show that this set is consistent. Suppose the opposite. Then there are formulae $(**)$ $\overline{@}_a \mathsf{B}\psi_1, \dots, \overline{@}_a \mathsf{B}\psi_n \in w$ such that $\overline{@}_a \psi_1, \dots, \overline{@}_a \psi_n \vdash \overline{@}_a \varphi$. Thus, $\overline{@}_a \mathsf{B}\psi_1, \dots, \overline{@}_a \mathsf{B}\psi_n \vdash \overline{@}_a \mathsf{B}\varphi$, by Lemma 5.7. Hence, $w \vdash \overline{@}_a \mathsf{B}\varphi$, by assumption $(**)$. Then, $\overline{@}_a \mathsf{B}\varphi \in w$ because w is maximal, which contradicts the assumption of the lemma. Therefore, X is consistent.

Second, by Lemma 5.8, the set X can be extended to a maximal consistent set u. Note that $wR_a u$, by Definition 7.4, equation $(*)$, and the choice of u as an extension of X. Finally, $\neg\overline{@}_a\varphi \in X \subseteq u$, by equation $(*)$, and the choice of u as an extension of X. Therefore, $\overline{@}_a\varphi \notin u$ because u is consistent. ⊣

LEMMA 8.2. $w, a \Vdash \varphi$ *iff* $\overline{@}_a\varphi \in w$.

PROOF. We prove the lemma by induction on the structural complexity of formula φ. If φ is a propositional variable, then the statement of the lemma follows from item 1 of Definition 4.1 and Definition 7.5.

Suppose that φ has the form $\neg\psi$. $(\Rightarrow)$: Assume that $w, a \Vdash \neg\psi$. Thus, $w, a \nVdash \psi$ by item 2 of Definition 4.1. Hence, $\overline{@}_a\psi \notin w$ by the induction hypothesis. Then, $\neg\overline{@}_a\psi \in w$ because w is a maximal consistent set of formulae. Hence, $w \vdash \overline{@}_a\neg\psi$ by the Commutativity axiom and propositional reasoning. Therefore, $\overline{@}_a\neg\psi \in w$ because w is a maximal consistent set. $(\Leftarrow)$: Assume that $\overline{@}_a\neg\psi \in w$. Then $w \vdash \neg\overline{@}_a\psi$, by Lemma 5.1. Hence, $\overline{@}_a\psi \notin w$ because w is consistent. Then, $w, a \nVdash \psi$ by the induction hypothesis. Thus, $w, a \Vdash \neg\psi$, by Definition 4.1(2).

Suppose that φ has the form $\psi_1 \to \psi_2$. By Definition 4.1(3), the statement $w, a \Vdash \psi_1 \to \psi_2$ is equivalent to the *disjunction* of the statements $w, a \nVdash \psi_1$ and $w, a \Vdash \psi_2$. By the induction hypothesis, the disjunction of these statements is equivalent to the disjunction of the statements $\overline{@}_a\psi_1 \notin w$ and $\overline{@}_a\psi_2 \in w$. Because w is a maximal consistent set, the latter disjunction is equivalent to the disjunction of the statements $\neg\overline{@}_a\psi_1 \in w$ and $\overline{@}_a\psi_2 \in w$. Finally, the last disjunction is equivalent to $\overline{@}_a(\psi_1 \to \psi_2) \in w$, by Lemma 5.3, Lemma 5.4, and the maximality of w.

Suppose that φ has the form $@_n\psi$. By Definition 4.1(4), the statement $w, a \Vdash @_n\psi$ is equivalent to $w, e_a(n) \Vdash \psi$. By Definition 7.3, the last statement is equivalent to $w, a :: n \Vdash \psi$. By the induction hypothesis, the statement $w, a :: n \Vdash \psi$ is equivalent to $\overline{@}_{a::n}\psi \in w$. The last statement is equivalent to $\overline{@}_a @_n\psi \in w$ by the definition of $\overline{@}$.

Suppose that φ has the form $\mathsf{B}\psi$. $(\Rightarrow)$: Assume that $\overline{@}_a\mathsf{B}\psi \notin w$. Then, by Lemma 8.1, there exists a world $u \in W$ such that $wR_a u$ and $\overline{@}_a\psi \notin u$. Hence, $u, a \nVdash \psi$ by the induction hypothesis. Therefore, $w, a \nVdash \mathsf{B}\psi$ by item 5 of Definition 4.1. $(\Leftarrow)$: Consider any world $u \in W$ such that $wR_a u$. By item 5 of Definition 4.1, it suffices to show that $u, a \Vdash \psi$. Indeed, the assumptions $\overline{@}_a\mathsf{B}\psi \in w$ and $wR_a u$ imply that $\overline{@}_a\psi \in u$ by Defintion 7.4. Therefore, $u, a \Vdash \psi$ by the induction hypothesis. ⊣

Theorem 8.1 (strong completeness). *For any set of formulae $X \subseteq \Phi$ and any formula $\varphi \in \Phi$, if $X \nvdash \varphi$, then there is a world w and an agent a of a doxastic model such that $w, a \Vdash \chi$ for each formula $\chi \in X$ and $w, a \nVdash \varphi$.*

Proof. The assumption $X \nvdash \varphi$ implies that the set $X \cup \{\neg\varphi\}$ is consistent. By Lemma 5.8, this set can be extended to a maximal consistent w. Let ε be the empty sequence of names. Lemma 8.2 implies that $w, \varepsilon \Vdash \chi$ for each formula $\chi \in X$ and $w, \varepsilon \Vdash \neg\varphi$. So, $w, \varepsilon \nVdash \varphi$, by Definition 4.1(2). $\dashv$

9. Non-Rigid Names

In Definition 3.1, we assumed that the value of the extension function $e_a(n)$ depends on agent a, but does not depend on the world. Thus, for example, we assumed that agent a has the same mother $e_a(ma)$ in all possible worlds. The names whose meanings do not depend on the current world are usually called *rigid*. One can also consider a more general setting in which the meaning of the name might be changing from one world to another. Such names are called *non-rigid*. Non-rigid names could be used, for example, to model a situation when agent a believes that the agent's mother is one person, but in actuality, she is a different person. To capture non-rigid names, it suffices to assume in Definition 3.1 that extension function $e^w_a(n)$ has an additional parameter $w \in W$. In Definition 4.1, we only need to modify item 4 as follows:

4′. $w, a \Vdash @_n\varphi$, if $w, e^w_a(n) \Vdash \varphi$.

Recall from Section 5, that by L^- we denote a version of our logical system that contains the Necessitation inference rule for modality @ instead of the Insertion inference rule. Informally, we say that an inference rule is *derivable* in system L^- if it can be represented by a fixed finite combination of the axioms and inference rules from system L^-. An example of a derivable rule in L^- is the rule

$$\frac{@_n\mathsf{B}\varphi}{@_n\mathsf{B}\mathsf{B}\varphi}$$

Indeed, this single rule is equivalent to the following combination of the Positive Introspection and the Distributivity axioms as well as the Modus Ponens and the Necessitation (for modality @) inference rules:

$$\dfrac{@_n\mathsf{B}\varphi \qquad \dfrac{\dfrac{\mathsf{B}\varphi \to \mathsf{BB}\varphi}{@_n(\mathsf{B}\varphi \to \mathsf{BB}\varphi)} \qquad @_n(\mathsf{B}\varphi \to \mathsf{BB}\varphi) \to (@_n\mathsf{B}\varphi \to @_n\mathsf{BB}\varphi)}{@_n\mathsf{B}\varphi \to @_n\mathsf{BB}\varphi}}{@_n\mathsf{BB}\varphi}$$

However, it is not so easy to capture the above intuitive notion of "derivability" in a formal definition. In this article, we formally define derivability as stated in Definition 9.2 below.

DEFINITION 9.1. A set of formulae S is a *theory* of a logical system $\mathcal{L}$ if S contains all axioms of $\mathcal{L}$ and is closed with respect to the inference rules of $\mathcal{L}$.

DEFINITION 9.2. An inference rule is *derivable* in a logical system $\mathcal{L}$ if any *theory* of $\mathcal{L}$ is closed with respect to this inference rule.

The notion of derivability is different from the notion of admissibility.

DEFINITION 9.3. An inference rule is *admissible* in a logical system $\mathcal{L}$ if the set of *theorems* of $\mathcal{L}$ is closed with respect to this inference rule.

Each derivable inference rule is admissible, but an admissible rule is not necessarily derivable. In this section, we use a doxastic model with non-rigid names depicted in Figure 1 to prove that the Insertion inference rule is not derivable in system L^-. We do not know if it is admissible.

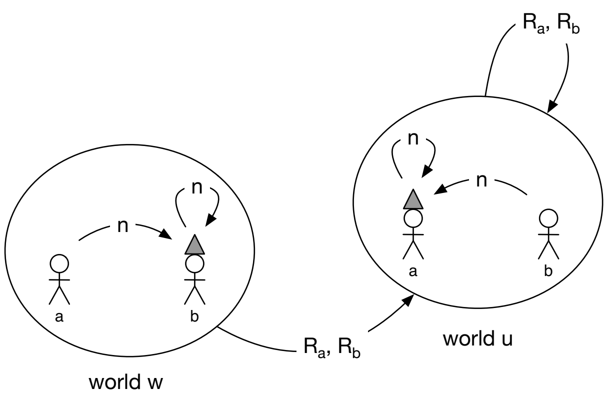

Figure 1. A doxastic model with non-rigid names.

The doxastic model with non-rigid names depicted in Figure 1 contains two possible worlds, w and u, and two agents, a and b. Both agents find world u to be more plausible than world w. In other worlds, $R_a = R_b = \{(w,u),(u,u)\}$. We show this in Figure 1 by directed arrows labelled with R_a and R_b from world w to world u and from world u back to world u. Without loss of generality, we assume that the set of names N contains a single name n. The values of the extension function are specified in Figure 1 by the directed edges *inside* each world:

$$e^w_a(n) = e^w_b(n) = b \quad \text{and} \quad e^u_a(n) = e^u_b(n) = a. \tag{1}$$

In other words, the name n always refers to agent b in world w and it always refers to agent a in world u. Finally, again without loss of generality, we assume that the language Φ has a single propositional variable. To improve the readability of the proof, we assume that the name of the variable is "has a hat". This variable is true in world w for agent b and in world u for agent a:

$$\pi(\text{"has a hat"}) = \{(w,b),(u,a)\}. \tag{2}$$

Figure 1, we visualise this by placing a hat on the agent in the worlds where the propositional variable "has a hat" is true for this agent.

Towards the proof of nonderivability of the Insertion inference rule, let us make two observations about the model depicted in Figure 1.

LEMMA 9.1. *$x, y \Vdash @_n$"has a hat" for each world $x \in \{w,u\}$ and each agent $y \in \{a,b\}$ of the doxastic model depicted in Figure 1.*

PROOF. First, note that $w, b \Vdash$ "has a hat" and $u, a \Vdash$ "has a hat". One can observe the above statements using Figure 1 or, more formally, using Definition 4.1(1) and equation (2). Thus, by Definition 4.1(4), we have $w, y \Vdash @_n$"has a hat" and $u, y \Vdash @_n$"has a hat" for each agent $y \in \{a,b\}$. ⊣

LEMMA 9.2. *$w, a \nVdash @_n\mathsf{B}$"has a hat".*

PROOF. Note that $u, b \nVdash$ "has a hat". Thus, $w, b \nVdash \mathsf{B}$"has a hat", by Definition 4.1(5) and because wR_bu. Hence, $w, e^w_a(n) \nVdash \mathsf{B}$"has a hat" by equation (1). So, $w, a \nVdash @_n\mathsf{B}$"has a hat", by Definition 4.1(4). ⊣

By Definition 9.2, to prove that the Insertion inference rule is not derivable in system L^-, it suffices to construct a theory S of L^- which is not closed with respect to the Insertion rule.

DEFINITION 9.4. $S = \{\varphi \in \Phi \mid \forall x \in \{w,u\}\, \forall y \in \{a,b\} (x, y \Vdash \varphi)\}$.

The next lemma essentially states that the axioms of L^- are sound with respect to the model with non-rigid names depicted in Figure 1. Its proof consists of straightforward verification of the validity of the axioms in the model.

Lemma 9.3. *The set S contains each instance of each axiom of L^-.*

Lemma 9.4. *The set S is closed with respect to the Modus Ponens inference rule and the Necessitation inference rules for modalities B and @.*

Proof. The proof of this lemma is also straightforward. For example, in order to show that S is closed with respect to the Necessitation inference rule for B, it suffices to observe that if a formula φ is satisfied for each agent in each world of the model depicted in Figure 1, then $\mathsf{B}\varphi$ is also satisfied for each agent in each world of the model depicted in Figure 1. ⊣

Lemma 9.5. *The set S is not closed with respect to the Insertion inference rule.*

Proof. Note that $@_n$"has a hat" $\in S$, by Lemma 9.1 and Definition 9.4. At the same time $@_n\mathsf{B}$"has a hat" $\notin S$, by Lemma 9.2 and Definition 9.4. ⊣

Theorem 9.1. *The Insertion rule is not derivable in logical system L^-.*

Proof. By Definition 9.1, Lemma 9.3 and Lemma 9.4 imply that S is a theory of system L^-. Then, the Insertion inference rule is not derivable in L^- by Definition 9.2 and Lemma 9.5. ⊣

10. Other Possible Extensions

10.1. Partial Names

In the main part of this article, we assumed that function $e_a(n)$ is *total*. In other words, for each name n and each agent a there is somebody to whom a refers by name n. One can generalise our work to the setting where function $e_a(n)$ is partial. In such a setting, there are two ways to define modality @:

- $w, a \Vdash @_n\varphi$, if either the value $e_a(n)$ is not defined or $w, e_a(n) \Vdash \varphi$.
- $w, a \Vdash @^\star_n\varphi$, if $e_a(n)$ is defined and $w, e_a(n) \Vdash \varphi$.

Note that, as the lemma below shows, these two modalities are expressible through each other.

Lemma 10.1. 1. *$w, a \Vdash @^{\star}_n\varphi$ iff $w, a \Vdash @_n\varphi \wedge \neg @_n\bot$.*
2. *$w, a \Vdash @_n\varphi$ iff $w, a \Vdash @^{\star}_n\varphi \vee \neg @^{\star}_n\top$.*

Our attempts to generalise our results to partial names led us to a belief that although it is probably possible, the resulting axiomatisation and completeness proof will not be as elegant as those that are presented in the current article. Here, for example, is an additional inference rule that we needed to carry out the completeness argument:

$$\frac{\bigwedge_i \overline{@}_{\alpha_i}\bot \wedge \bigwedge_j \neg\overline{@}_{\beta_j}\bot \to \overline{@}_{\gamma}\varphi}{\bigwedge_i \overline{@}_{\alpha_i}\bot \wedge \bigwedge_j \neg\overline{@}_{\beta_j}\bot \to \overline{@}_{\gamma}\mathsf{B}\varphi}$$

Informally, the assumption of this rule states that in each model and each world-agent combination, if all names α_i are not defined and all names β_i are defined, then an agent γ must have a property φ. The conclusion states that in the same situation, γ must believe that she has φ.

10.2. Common Names

So far, we assumed that each name uniquely identifies a single agent (or, in the case of partial names at most one agent). Such names are known as *proper*. On the other hand, *common* names might be used by a given agent to refer to several agents. An example of a common name is "parent". Common names can be modelled in our setting by assuming that function $e_a(n)$ returns a set of agents. In this setting, modality @ has two distinct versions:

- $w, a \Vdash @^{\forall}_n\varphi$, if $w, b \Vdash \varphi$ for each agent $b \in e_a(n)$,
- $w, a \Vdash @^{\exists}_n\varphi$, if $w, b \Vdash \varphi$ for some agent $b \in e_a(n)$.

These two versions of @ have been studied in [14–16]. A possible direction of future research is to combine these modalities with the belief modality B.

11. Conclusion

In this article, we proposed a logical system that combines the egocentric setting with the standard plausibility-based semantics of beliefs. Our main technical result is a sound and complete logical system that describes the interplay between @ and B. The most interesting inference rule of this system is the Insertion rule. Using a non-standard semantics with non-rigid names, we have shown that this rule is not derivable in the logical system obtained from ours by replacing this rule with the Necessitation rule for @.

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A. The Proof of Deduction Lemma

LEMMA 5.6. *If* $X, \varphi \vdash \psi$, *then* $X \vdash \varphi \to \psi$.

PROOF. Suppose that sequence $\psi_1, \dots, \psi_n$ is a proof from a set $X \cup \{\varphi\}$ and the theorems of our logical system that uses the Modus Ponens inference rule only. In other words, for each $k \leqslant n$,

1. either $\vdash \psi_k$, or
2. $\psi_k \in X$, or
3. ψ_k is equal to φ, or
4. there are $i, j < k$ such that formula ψ_j is equal to $\psi_i \to \psi_k$.

It suffices to show that $X \vdash \varphi \to \psi_k$ for each $k \leqslant n$. We prove this by induction on k by considering the four cases above separately.

Case 1: $\vdash \psi_k$. Note that $\psi_k \to (\varphi \to \psi_k)$ is a propositional tautology, and thus, is an axiom of our logical system. Hence, $\vdash \varphi \to \psi_k$ by the Modus Ponens inference rule. Therefore, $X \vdash \varphi \to \psi_k$.

Case 2: $\psi_k \in X$. Note again that $\psi_k \to (\varphi \to \psi_k)$ is a propositional tautology, and thus, is an axiom of our logical system. Therefore, by the Modus Ponens inference rule, $X \vdash \varphi \to \psi_k$.

Case 3: formula ψ_k is equal to φ. Thus, $\varphi \to \psi_k$ is a propositional tautology. Therefore, $X \vdash \varphi \to \psi_k$.

Case 4: formula ψ_j is equal to $\psi_i \to \psi_k$ for some $i, j < k$. Thus, by the induction hypothesis, $X \vdash \varphi \to \psi_i$ and $X \vdash \varphi \to (\psi_i \to \psi_k)$. Note that the formula $(\varphi \to \psi_i) \to ((\varphi \to (\psi_i \to \psi_k)) \to (\varphi \to \psi_k))$ is a propositional tautology. Therefore, $X \vdash \varphi \to \psi_k$ by applying the Modus Ponens inference rule twice. $\dashv$

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