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Literal and Controllable Paraconsistency

Abstract. The principle of explosion asserts that any formula can be derived from any pair of other contradictory formulas. Paraconsistent logic is typically regarded as a logic in which the universal validity of this principle is questioned. Therefore, a key point is determining when the validity can be considered universal to classify a logic as paraconsistent. A pertinent example to illustrate this point is the calculus $\mathbf{CB}_1$ that admits the principle but only for negated formulas, i.e., from any set $\{\alpha, {\sim}\alpha\}$ any other formula follows if and only if α is of the form ${\sim}\gamma$. Another example is Sette's calculus **P1**, which is paraconsistent at the level of variables but not complex formulas. Both serve as compelling examples of the so-called borderline cases.

In this paper, we examine several calculi expected to be paraconsistent at the level of literals. It means that a pair of formulas, α and ${\sim}\alpha$, can yield any β if, and only if α is neither a propositional variable nor is its iterated negation. Furthermore, it is assumed that in some calculi presented here, β must adhere to specific restrictions. Once these conditions are satisfied, we refer to calculus as paraconsistent in a "controllable manner".

Keywords: paraconsistent logic; Sette's calculus; paraconsistency; paranormal logics

1. Basic Terminology and Definitions

Let Var denote a denumerable set of propositional variables: p, q, r, ..., $p_1, p_2, \ldots$ The set For of formulas is conventionally defined with variables from Var and the symbols $\sim$, $\to$ for negation and implication, respectively. In the following discussion, we will examine several axiomatic propositional calculi in a Hilbert-style formalization that employs *detachment*, (MP) $\alpha \to \beta,\ \alpha\ /\ \beta$, as the sole rule of inference. For all

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$\alpha \in$ For and $\Gamma \subseteq$ For, we say that α is *derivable* from Γ using (MP) iff there is a finite sequence of formulas $\beta_1, \beta_2, \ldots, \beta_n$ such that $\beta_n = \alpha$ and for each $i \leqslant n$, either $\beta_i \in \Gamma$ or $\beta_k = \beta_j \to \beta_i$ for some $j, k \leqslant i$.

A *calculus* $\mathcal{C}$, identified with the triple $\langle \text{For}, \text{Ax}_{\mathcal{C}}, \vdash_{\mathcal{C}} \rangle$, is determined by its set of axioms $\text{Ax}_{\mathcal{C}}$, where $\text{Ax}_{\mathcal{C}} \subseteq$ For. For all $\alpha \in$ For and $\Gamma \subseteq$ For, we say that α is *provable* Γ within $\mathcal{C}$ (in symbols: $\Gamma \vdash_{\mathcal{C}} \alpha$) iff α is derivable from $\Gamma \cup \text{Ax}_{\mathcal{C}}$ using (MP). A formula α is a *thesis* of $\mathcal{C}$ (in symbols: $\alpha \in \text{Th}(\mathcal{C})$) iff $\emptyset \vdash_{\mathcal{C}} \alpha$, i.e., α is derivable from $\text{Ax}_{\mathcal{C}}$ using (MP).

Since (MP) is the only rule of inference of $\mathcal{C} = \langle \text{For}, \text{Ax}_{\mathcal{C}}, \vdash_{\mathcal{C}} \rangle$, if for all $\alpha, \beta, \gamma \in$ For, the following formulas are axioms of $\mathcal{C}$

$$\alpha \to (\beta \to \alpha) \tag{A1}$$

$$(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)) \tag{A2}$$

then the deduction theorem holds, i.e., for all $\Gamma \subseteq$ For, $\alpha, \beta \in$ For:

(DT) $\quad \Gamma \cup \{\alpha\} \vdash_{\mathcal{C}} \beta \quad \text{iff} \quad \Gamma \vdash_{\mathcal{C}} \alpha \to \beta.$

Consequently, using (DT), the following formulas are derivable from (A1) and (A2) using (MP):

$$\alpha \to \alpha \tag{id}$$

$$(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma)) \tag{c}$$

$$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma)) \tag{tr}$$

$$(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta) \tag{w}$$

$$((\beta \to \gamma) \to \alpha) \to ((\beta \to \alpha) \to \alpha) \tag{pc}$$

$$\alpha \to ((\alpha \to \beta) \to \beta) \tag{mp}$$

For the kind of calculi considered, a calculus $\mathcal{C}$ is said to be *paraconsistent* iff the *principle of explosion* does not hold in $\mathcal{C}$, i.e, for some $\alpha, \beta \in$ For we have $\{\alpha, {\sim}\alpha\} \nvdash_{\mathcal{C}} \beta$. Every such paraconsistent calculus is not trivial, i.e., some formula is not its thesis.

We say that calculi $\mathcal{C} = \langle \text{For}, \text{Ax}_{\mathcal{C}}, \vdash_{\mathcal{C}} \rangle$ and $\mathcal{C}' = \langle \text{For}, \text{Ax}_{\mathcal{C}'}, \vdash_{\mathcal{C}'} \rangle$ are *equivalent* iff $\text{Ax}_{\mathcal{C}'} \subseteq \text{Th}(\mathcal{C})$ and $\text{Ax}_{\mathcal{C}} \subseteq \text{Th}(\mathcal{C}')$. In such cases, we have $\vdash_{\mathcal{C}} = \vdash_{\mathcal{C}'}$, and so $\text{Th}(\mathcal{C}) = \text{Th}(\mathcal{C}')$.

A matrix for For is a triple $\mathfrak{M} = \langle \mathbb{V}, \mathbb{D}, \mathbb{I} \rangle$, where

- $\mathbb{V}$ is a non-empty set of *truth values*;
- $\mathbb{D}$ is a non-empty proper subset of $\mathbb{V}$ – the set of *designated truth values*;
- $\mathbb{I} = \{I_{\to}, I_{\sim}\}$ is a set of *interpretation mappings* for used propositional connectives such that $I_{\to} \colon \mathbb{V}^2 \longrightarrow \mathbb{V}$ and $I_{\sim} \colon \mathbb{V} \longrightarrow \mathbb{V}$. These mappings correspond to truth tables.

A *valuation* in $\mathfrak{M}$ is any function $v\colon \mathrm{For} \longrightarrow \mathbb{V}$ such that for all $\alpha, \beta \in \mathrm{For}$:

- $v(\alpha \to \beta) = I_{\to}(v(\alpha), v(\beta))$,
- $v({\sim}\alpha) = I_{\sim}(v(\alpha))$.

For all $\alpha \in \mathrm{For}$ and $\Gamma \subseteq \mathrm{For}$, we say that α is a *semantic consequence* of Γ in $\mathfrak{M}$ (in symbols: $\Gamma \models_{\mathfrak{M}} \alpha$) iff for every valuation v: if $v(\beta) \in \mathbb{D}$ for every $\beta \in \Gamma$, then $v(\alpha) \in \mathbb{D}$. We say that α is *valid* in $\mathfrak{M}$ iff $\emptyset \models_{\mathfrak{M}} \alpha$.

We say that a calculus $\mathcal{C}$ is *sound* with respect to a matrix $\mathfrak{M}$ iff all theses of $\mathcal{C}$ are valid in $\mathfrak{M}$. Moreover, we say that $\mathcal{C}$ is *complete* with respect to the matrix $\mathfrak{M}$ iff all formulas valid in $\mathfrak{M}$ are theses of $\mathcal{C}$.

Non-deterministic bivaluational semantics are also used for the calculi considered in this paper. A $\mathcal{C}$-*bivaluation* is any function $v\colon \mathrm{For} \longrightarrow \{1, 0\}$ satisfying some conditions for $\to$ and $\sim$, which are suitable for $\mathcal{C}$. For all $\alpha \in \mathrm{For}$ and $\Gamma \subseteq \mathrm{For}$, α is a *semantic consequence* of Γ in $\mathcal{C}$ (in symbols: $\Gamma \models_{\mathcal{C}} \alpha$) iff for every $\mathcal{C}$-bivaluation v: if $v(\beta) = 1$ for every $\beta \in \Gamma$, then $v(\alpha) = 1$. A formula α is a $\mathcal{C}$-tautology iff $v(\alpha) = 1$, for every $\mathcal{C}$-bivaluation v.

2. Sette's Calculus

2.1. The original axiomatization

Nearly half a century ago, Sette introduced a propositional calculus, denoted as **P1**, which was paraconsistent only at the level of variables, that is, $\{p, {\sim}p\} \nvdash_{\mathbf{P1}} q$, for any $p, q \in \mathrm{Var}$. At the level of complex formulas, **P1** has all the properties of classical propositional calculus, including $\{\alpha, {\sim}\alpha\} \vdash_{\mathbf{P1}} \beta$, if $\alpha \notin \mathrm{Var}$. Sette's calculus is defined by the following axiom schemas: (A1), (A2) and

$$({\sim}\alpha \to {\sim}\beta) \to (({\sim}\alpha \to {\sim}{\sim}\beta) \to \alpha) \tag{N1}$$

$${\sim}(\alpha \to {\sim}{\sim}\alpha) \to \alpha \tag{N2}$$

$$(\alpha \to \beta) \to {\sim}{\sim}(\alpha \to \beta) \tag{N3}$$

and the rule (MP).

Sette accepts only the symbols $\sim$ and $\to$ as primitive connectives. Conjunction, disjunction, and equivalence are introduced *via* the following definitions:

$$\alpha \wedge \beta := (((\alpha \to \alpha) \to \alpha) \to {\sim}((\beta \to \beta) \to \beta)) \to {\sim}(\alpha \to {\sim}\beta)$$

$$\alpha \vee \beta := (\alpha \to {\sim}{\sim}\alpha) \to ({\sim}\alpha \to \beta)$$

$$\alpha \leftrightarrow \beta := (\alpha \to \beta) \wedge (\beta \to \alpha)$$

Note that disjunction can be defined solely in terms of implication, as it is in Łukasiewicz's three–valued logic: $\alpha \vee \beta := (\alpha \to \beta) \to \beta$. The definition of conjunction can also be simplified: $\alpha \wedge \beta := {\sim}(\alpha \to {\sim}(\alpha \to \beta))$. Alternative definitions have been proposed, e.g., in [11, 17].

2.2. Semantics for P1

It is known that **P1** is sound and complete with respect to the matrix $\mathfrak{M}_{\mathbf{P1}} = \langle \{1, 2, 0\}, \{1, 2\}, \sim, \to \rangle$, where the interpretations of connectives of $\to$ and $\sim$ are defined by the following truth tables:

$\to$	1	2	0
1	1	1	0
2	1	1	0
0	1	1	1

$\sim$	
1	0
2	1
0	1

That is:

Fact 2.1. *A formula is a thesis of* **P1** *iff it is valid in* $\mathfrak{M}_{\mathbf{P1}}$.

Notice that the three-valued semantics is not the only option available. For instance, a modal interpretation for **P1** was discussed in [1, 23], the so-called society semantics was presented in [11, 5], and a non-deterministic bivaluational semantics for **P1** was introduced in [8].

P1-bivaluations satisfy the following conditions for all $\alpha, \beta \in$ For:

if $v({\sim}\alpha) = 0$ then $v(\alpha) = 1$,
if $v({\sim}{\sim}\alpha) = 1$ then $v({\sim}\alpha) = 0$,
if $v({\sim}(\alpha \to \beta)) = 1$, then $v(\alpha \to \beta) = 0$,
$v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$.

Theorem 2.1 (8, pp. 1114–1116). *For all* $\Gamma \subseteq$ For *and* $\alpha \in$ For:

$$\Gamma \vdash_{\mathbf{P1}} \alpha \quad \textit{iff} \quad \Gamma \models_{\mathbf{P1}} \alpha.$$

The application of non-deterministic bivaluational semantics in the field of paraconsistency is not a novel concept. In 1977, da Costa and Alves proposed this type of semantics for the hierarchy of C_n-systems, where $n < \omega$ [see 9, pp. 622–623]. There are also other paraconsistent calculi that possess an adequate and decidable bivaluational semantics [see, e.g., 6, 16, 7].

2.3. An equivalent axiomatization

Sette's calculus, as demonstrated in [8], can be axiomatized by (MP) and the following axioms: (A1), (A2) and

$$\sim\alpha \to (\sim\sim\alpha \to \beta) \tag{A3}$$

$$(\alpha \to \beta) \to (\sim(\alpha \to \beta) \to \gamma) \tag{A4}$$

$$(\alpha \to \beta) \to ((\sim\alpha \to \beta) \to \beta) \tag{A5}$$

Since (A3)–(A5) are valid in $\mathfrak{M}_{\mathbf{P1}}$, they are thesis of **P1**, by Fact 2.1. Let **P1**$'$ be the calculus based on the set of axioms $\{$(A1), . . . , (A5)$\}$ and (MP), as the sole rule of inference. We see that $\mathrm{Ax}_{\mathbf{P1}'} \subseteq \mathrm{Th}(\mathbf{P1})$. We will prove that also $\mathrm{Ax}_{\mathbf{P1}} \subseteq \mathrm{Th}(\mathbf{P1}')$, i.e., **P1** and **P1**$'$ are equivalent.

LEMMA 2.1. 1. *The following formula is derivable from* (A1), (A2), (A5) *and* (id) *using* (MP)

$$(\sim\alpha \to \alpha) \to \alpha \tag{F1}$$

2. *The following formula is derivable from* (A1), (A2), (A4), (c), (tr) *and* (w) *using* (MP)

$$((\alpha \to \beta) \to \beta) \to (\sim(\sim\alpha \to \alpha) \to \beta) \tag{F2}$$

3. *The following formula is derivable from* (A1), (A2), (A5), (c), (tr) *and* (F1) *using* (MP)

$$(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\sim(\sim\alpha \to \alpha) \to \beta) \to \gamma)) \tag{F3}$$

4. *The following formula is derivable from* (A1)–(A3) *and* (c) *using* (MP)

$$\sim\sim\alpha \to \alpha \tag{F4}$$

5. *The following formula is derivable from* (A1), (A2), (tr), (w), (mp), (id), (F2) *and* (F3) *using* (MP)

$$((\alpha \to \beta) \to \alpha) \to \alpha \tag{PL}$$

PROOF. In all cases we will use (DT). (F1):

1.	$\sim\alpha \to \alpha$	hyp.
2.	$(\alpha \to \alpha) \to ((\sim\alpha \to \alpha) \to \alpha)$	(A5)
3.	α	(id), 1, 2, (MP)

(F2):

1.	$(\alpha \to \beta) \to \beta$	hyp.
2.	$\sim(\sim\alpha \to \alpha)$	hyp.
3.	$(\sim\alpha \to \alpha) \to (\alpha \to \beta)$	2, (A4), (c), (MP)

4. $\alpha \to (\sim\alpha \to \alpha)$ (A1)
5. $\alpha \to (\alpha \to \beta)$ 4, 3, (tr), (MP)
6. $\alpha \to \beta$ 5, (w), (MP)
7. β 1, 6, (MP)

(F3):
1. $\alpha \to \gamma$ hyp.
2. $\beta \to \gamma$ hyp.
3. $\sim(\sim\alpha \to \alpha) \to \beta$ hyp.
4. $\sim(\sim\alpha \to \alpha) \to \gamma$ 2, 3, (tr), (MP)
5. $(\sim(\sim\alpha \to \alpha) \to \gamma) \to (((\sim\alpha \to \alpha) \to \gamma) \to \gamma)$ (A5), (c), (MP)
6. $((\sim\alpha \to \alpha) \to \gamma) \to \gamma$ 5, 4, (MP)
7. $(\sim\alpha \to \alpha) \to \gamma$ (F1), 1, (tr), (MP)
8. γ 4, 7, (A5), (MP)

(F4):
1. $\sim\sim\alpha$ hyp.
2. $\sim\sim\alpha \to (\sim\alpha \to \alpha)$ (A3), (c), (MP)
3. $\sim\alpha \to \alpha$ 1, 2, (MP)
4. α (F1), 3, (MP)

(PL):
1. $(\alpha \to \beta) \to \alpha$ hyp.
2. $\beta \to (\alpha \to \beta)$ (A1)
3. $\beta \to \alpha$ (tr), 1, 2, (MP)
4. $\alpha \to ((\alpha \to \beta) \to \beta)$ (mp)
5. $(\alpha \to \beta) \to ((\alpha \to \beta) \to \beta)$ (tr), 1, 4, (MP)
6. $(\alpha \to \beta) \to \beta$ (w), 5, (MP)
7. $\sim(\sim\alpha \to \alpha) \to \beta$ (F2), 6, (MP)
8. $(\sim(\sim\alpha \to \alpha) \to \beta) \to \alpha$ (F3), (id), 3, (MP)
9. α 7, 8, (MP) ⊣

The set of propositional formulas that contains implication alone, without any other connectives, is referred to as the propositional implicational language. It is well-known that $\{$(A1), (A2), (PL), (MP)$\}$ constitutes the implicational fragment of classical propositional calculus ($\mathbf{CPC}^{\to}$, for short).

The above lemma with the lemma below entail that $\mathrm{Ax}_{\mathbf{P1}} \subseteq \mathrm{Th}(\mathbf{P1}')$.

LEMMA 2.2. 1. (N1) *is derivable from* (A1)–(A3), (tr), (F1) *using* (MP).
2. (N2) *is derivable from* (A1), (A2), (A4), (c) *and* (PL) *using* (MP).
3. (N3) *is derivable from* (A1), (A2), (A4), (c), (tr) *and* (F1) *using* (MP).

PROOF. In all cases we will use (DT). (N1):

1.	$\sim\alpha \to \sim\beta$	hyp.
2.	$\sim\alpha \to \sim\sim\beta$	hyp.
3.	$\sim\beta \to (\sim\sim\beta \to \alpha)$	(A3)
4.	$\sim\alpha \to (\sim\sim\beta \to \alpha)$	1, 3, (tr), (MP)
5.	$\sim\alpha \to \alpha$	2, (A2), (MP)
6.	α	5, (F1), (MP)

(N2):

1.	$\sim(\alpha \to \sim\sim\alpha)$	hyp.
2.	$\sim(\alpha \to \sim\sim\alpha) \to ((\alpha \to \sim\sim\alpha) \to \alpha)$	(c), (A4), (MP)
3.	$(\alpha \to \sim\sim\alpha) \to \alpha$	1, 2, (MP)
4.	$((\alpha \to \sim\sim\alpha)) \to \alpha) \to \alpha$	(PL)
5.	α	3, 4, (MP)

(N3):

1.	$\alpha \to \beta$	hyp.
2.	$\sim(\alpha \to \beta) \to \sim\sim(\alpha \to \beta)$	1, (A4), (c), (MP)
3.	$\sim\sim\sim(\alpha \to \beta) \to \sim\sim(\alpha \to \beta)$	2, (F4), (tr), (MP)
4.	$\sim\sim(\alpha \to \beta)$	3, (F1), (MP) $\dashv$

Therefore, we obtain that Sette's calculus can be axiomatizable by (A1)–(A5) using (MP).

THEOREM 2.2. **P1** *and* **P1′** *are equivalent.*

Finally, notice that:

THEOREM 2.3. (A1)–(A5) *and* (MP) *are mutually independent.*

PROOF. Independence of (A1) is established by the matrix $\langle\{1, 2, 0\}, \{1, 2\}, \sim, \to\rangle$, where implication and negation are defined by

$\to$	1	2	0
1	1	1	0
2	0	0	0
0	1	1	1

	$\sim$
1	0
2	0
0	1

It is straightforward to verify that this matrix satisfies (A2)–(A5) and (MP), but it fails to validate (A1) when both α and β are assigned the value 2.

Independence of (A2) is established by $\langle\{1, 2, 0\}, \{1\}, \sim, \to\rangle$, where the connectives are defined by the following truth tables:

$\to$	1	2	0
1	1	0	0
2	1	0	1
0	1	1	1

	$\sim$
1	0
2	1
0	1

(A1), (A3)–(A5) are valid in this matrix and (MP) preserves validity. However, (A2) is invalidated (when α and γ are assigned the value 2, while β is assigned the value 1).

Independence of (A3) is established by $\langle\{1, 2, 0\}, \{1, 2\}, \sim, \to\rangle$, where the connectives are defined by the truth tables:

$\to$	1	2	0
1	1	1	0
2	1	1	0
0	1	1	1

	$\sim$
1	0
2	1
0	2

(A1), (A2), (A4), (A5) are valid in this matrix and (MP) preserves validity. However, (A3) is invalidated (when both α and β is assigned the value 0).

Independence of (A4) is established by $\langle\{1, 2, 0\}, \{1, 2\}, \sim, \to\rangle$, where the connectives are defined by the truth tables:

$\to$	1	2	0
1	1	2	0
2	1	1	0
0	1	1	1

	$\sim$
1	0
2	1
0	1

(A4) is invalid in this matrix (when α is assigned the value 1, β is assigned the value 2 and γ is assigned the value 0).

Independence of (A5) can be easily established by the matrix $\langle\{1, 0\}, \{1\}, \sim, \to\rangle$, where the connectives are defined by the truth tables:

$\to$	1	0
1	1	0
0	1	1

	$\sim$
1	0
0	0

(A5) is invalidated in this matrix (when α and β are assigned the value 0).

Now we prove that (MP) is independent of each of (A1)–(A5). To establish this fact, we apply $\langle\{1, 2, 0\}, \{1, 2\}, \sim, \to\rangle$, where the connectives are defined by the following truth tables [see 12, p. 195]:

$\to$	1	2	0
1	1	0	0
2	1	1	1
0	1	1	1

	$\sim$
1	2
2	1
0	1

All (A1)–(A5) are valid in this matrix. However, (MP) does not preserve validity; for example, $\sim(p \to \sim\sim p)$ and $(\sim(p \to \sim\sim p) \to p$ are valid (since all valuations assign 2 to the first formula) but p is not valid [cf. 12, p. 195]. So (MP) also does not preserve the designated values. ⊣

3. The Literal Paraconsistent Calculus

Sette's calculus is paraconsistent at the level of variables, but not in relation to complex formulas. In this section, we define a calculus that is expected to be paraconsistent at the level of literals. For any $p \in \mathrm{Var}$, let $\sim_0 p = p$ and $\sim_{k+1} p = \sim\sim_k p$ for any $k \geqslant 0$. By *literal* we mean any formula of the form $\sim_k p$. Let $\mathrm{LI} := \{\sim_k p : p \in \mathrm{Var} \text{ and } k \geqslant 0\}$.

Consider the Literal Paraconsistent Calculus, $\mathbf{P_{LI}}$, defined in the language with implication and negation, with axioms (A1), (A2), (A4), (A5) and (MP) as the sole rule of inference. Clearly, $\mathbf{P_{LI}}$ is a proper subsystem of **P1** (cf. Theorems 2.2 and 2.3). $\mathbf{P_{LI}}$ also serves as an example of a calculus in which for any $\alpha \notin \mathrm{LI}$, the pair $\{\alpha, \sim\alpha\}$ yields any β. It is worth mentioning that there are other pairs of formulas from which "anything" follows. For instance, $\{q, \sim(p \to q)\} \vdash_{\mathbf{P_{LI}}} r$ holds for all $p, q, r \in \mathrm{Var}$. So also for all $\alpha, \beta, \gamma \in \mathrm{For}$, $\{\beta, \sim(\alpha \to \beta) \vdash_{\mathbf{P_{LI}}} \gamma$.

From Lemmas 2.1 and 2.2 we obtain:

FACT 3.1. *Formulas* (F1)–(F3), (PL), (N2) *and* (N3) *are theses of* $\mathbf{P_{LI}}$.

$\mathbf{P_{LI}}$-bivaluations satisfy the following conditions for all $\alpha, \beta \in \mathrm{For}$:

(i) if $v(\sim\alpha) = 0$ then $v(\alpha) = 1$,
(ii) if $v(\sim(\alpha \to \beta)) = 1$ then $v(\alpha \to \beta) = 0$,
(iii) $v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$.

By induction on the length of a derivation, we obtain soundness:

THEOREM 3.1. *For all* $\Gamma \subseteq \mathrm{For}$, $\alpha \in \mathrm{For}$*: if* $\Gamma \vdash_{\mathbf{P_{LI}}} \alpha$*, then* $\Gamma \models_{\mathbf{P_{LI}}} \alpha$.

For the proof of completeness, we apply the method based on the notion of maximal non-trivial sets of formulas [see, e.g., 4, Section 2.2]. Before proceeding, we define a few terms and review some results.

Let $\mathcal{C} = \langle \mathrm{For}, \mathrm{Ax}_{\mathcal{C}}, \vdash_{\mathcal{C}} \rangle$ be a calculus satisfying Tarskian properties: reflexivity, transitivity and monotonicity. For any $\Delta \subseteq \mathrm{For}$, we say that Δ is a *closed theory of* $\mathcal{C}$ if and only if for any $\beta \in \mathrm{For}$: $\Delta \vdash_{\mathcal{C}} \beta$ iff $\beta \in \Delta$. Moreover, we say that Δ is *maximal non-trivial with respect to* $\alpha \in \mathrm{For}$

in $\mathcal{C}$ if and only if (1) $\Delta \nvdash_{\mathcal{C}} \alpha$, and (2) for every $\beta \in \mathrm{For}$, if $\beta \notin \Delta$ then $\Delta \cup \{\beta\} \vdash_{\mathcal{C}} \alpha$.

LEMMA 3.1 (4, Lemma 2.2.5). *Every maximal non-trivial set with respect to some formula is a closed theory.*

LEMMA 3.2 (25, Theorem 3.31). *For all $\Gamma \subseteq \mathrm{For}$ and $\alpha \in \mathrm{For}$ such that $\Gamma \nvdash_{\mathcal{C}} \alpha$, there is a maximal non-trivial set Δ with respect to α in $\mathcal{C}$ such that $\Gamma \subseteq \Delta$.*

Since $\mathbf{P_{LI}}$ satisfies Tarskian properties, the above lemmas holds for it. Moreover, we have:

LEMMA 3.3. *For any maximal non-trivial set Δ with respect to $\alpha \in \mathrm{For}$ in $\mathbf{P_{LI}}$ the following mapping $v_\Delta \colon \mathrm{For} \longrightarrow \{1, 0\}$ is a $\mathbf{P_{LI}}$-valuation*:

$$v_\Delta(\delta) = 1 \text{ iff } \delta \in \Delta \tag{$\star$}$$

PROOF. For (i): Suppose that $v_\Delta(\sim\beta) = 0 = v_\Delta(\beta)$. Then, by ($\star$), $\sim\beta \notin \Delta$ and $\beta \notin \Delta$. Since Δ is a maximal non-trivial set with respect to α, then $\Delta \cup \{\beta\} \vdash_{\mathbf{P_{LI}}} \alpha$ and $\Delta \cup \{\sim\beta\} \vdash_{\mathbf{P_{LI}}} \alpha$. By (DT), we get $\Delta \vdash_{\mathbf{P_{LI}}} \beta \to \alpha$ and $\Delta \vdash_{\mathbf{P_{LI}}} \sim\beta \to \alpha$. Hence, by (A5) and (DT), $\{\beta \to \alpha, \sim \beta \to \alpha\} \vdash_{\mathbf{P_{LI}}} \alpha$. So also $\Delta \vdash_{\mathbf{P_{LI}}} \alpha$. Since Δ is a closed theory, $\alpha \in \Delta$. But, from the main assumption, $\alpha \notin \Delta$. This leads to a contradiction.

For (ii): Assume that $v_\Delta(\sim(\beta \to \gamma)) = 1 = v_\Delta(\beta \to \gamma)$. Then, by ($\star$), $\sim(\beta \to \gamma) \in \Delta$ and $\beta \to \gamma \in \Delta$. So $\Delta \vdash_{\mathbf{P_{LI}}} \sim(\beta \to \gamma)$ and $\Delta \vdash_{\mathbf{P_{LI}}} \beta \to \gamma$. Hence, by (A4) and (DT), $\{\sim(\beta \to \gamma), \beta \to \gamma\} \vdash_{\mathbf{P_{LI}}} \alpha$. So also $\Delta \vdash_{\mathbf{P_{LI}}} \alpha$. Since Δ is deductively closed, $\alpha \in \Delta$. However, according to our main assumption, $\alpha \notin \Delta$. This leads to a contradiction.

For (iii): "$\Rightarrow$" Suppose that $v_\Delta(\beta \to \gamma) = 1 = v_\Delta(\beta)$ and $v_\Delta(\gamma) = 0$. Then by ($\star$), we have $\beta \to \gamma \in \Delta$, $\beta \in \Delta$ and $\gamma \notin \Delta$. So $\Delta \vdash_{\mathbf{P_{LI}}} \beta \to \gamma$ and $\Delta \vdash_{\mathbf{P_{LI}}} \beta$. Since (mp) is a thesis of $\mathbf{P_{LI}}$, by (DT), $\{\beta \to \gamma, \beta\} \vdash_{\mathbf{P_{LI}}} \gamma$. So also $\Delta \vdash_{\mathbf{P_{LI}}} \gamma$. Thus, we obtain a contradiction: $\gamma \in \Delta$ and $\gamma \notin \Delta$.

"$\Leftarrow$" Firstly, assume that $v_\Delta(\beta) = 0 = v_\Delta(\beta \to \gamma)$. Then, by ($\star$), $\beta \notin \Delta$ and $\beta \to \gamma \notin \Delta$. Since Δ is a maximal non-trivial set with respect to α, then $\Delta \cup \{\beta\} \vdash_{\mathbf{P_{LI}}} \alpha$ and $\Delta \cup \{\beta \to \gamma\} \vdash_{\mathbf{P_{LI}}} \alpha$. By (DT), we obtain $\Delta \vdash_{\mathbf{P_{LI}}} \beta \to \alpha$ and $\Delta \vdash_{\mathbf{P_{LI}}} (\beta \to \gamma) \to \alpha$. Since (pc) is a thesis of $\mathbf{P_{LI}}$, by (DT), $\{\beta \to \alpha, (\beta \to \gamma) \to \alpha\} \vdash_{\mathbf{P_{LI}}} \alpha$. So also $\Delta \vdash_{\mathbf{P_{LI}}} \alpha$. Since Δ is deductively closed, by the main assumption, we obtain a contradiction: $\alpha \in \Delta$ and $\alpha \notin \Delta$. Secondly, suppose that $v_\Delta(\gamma) = 1$. Then, by ($\star$),

$\gamma \in \Delta$. So $\Delta \vdash_{\mathbf{P_{LI}}} \gamma$. Moreover, by (A1) and (DT), $\{\gamma\} \vdash_{\mathbf{P_{LI}}} \beta \to \gamma$. So also $\Delta \vdash_{\mathbf{P_{LI}}} \beta \to \gamma$. Hence $\beta \to \gamma \in \Delta$. So $v_\Delta(\beta \to \gamma) = 1$. ⊣

Notice that Lindenbaum-Łoś theorem holds for any finitary calculus $\mathcal{C}$. Thus, we obtain completeness:

Theorem 3.2. *For all* $\Gamma \subseteq \mathrm{For}$ *and* $\alpha \in \mathrm{For}$*: if* $\Gamma \models_{\mathbf{P_{LI}}} \alpha$*, then* $\Gamma \vdash_{\mathbf{P_{LI}}} \alpha$*.*

Proof. Assume that $\Gamma \nvdash_{\mathbf{P_{LI}}} \alpha$. Then, by Lemma 3.2, there is a maximal non-trivial set Δ with respect to α in $\mathbf{P_{LI}}$ such that $\Gamma \subseteq \Delta$. For the valuation v_Δ from the proof of Lemma 3.3, we have $v_\Delta(\alpha) = 0$ (since $\alpha \notin \Delta$) and $v_\Delta(\beta) = 1$, for any $\beta \in \Delta$. This implies that $\Delta \not\models_{\mathbf{P_{LI}}} \alpha$ and, in particular, $\Gamma \not\models_{\mathbf{P_{LI}}} \alpha$. ⊣

In $\mathbf{P_{LI}}$, the operation $\sim$ taken as a primitive connective, is not the only form of negation that can be used. A new connective of negation can be introduced *via* the definition: $\neg\alpha := \alpha \to \sim(\alpha \to \alpha)$. An important aspect of this connective is that it possesses some *notorious* properties (at least from the perspective of parconsistency). For instance, for all $\alpha, \beta \in \mathrm{For}$, $\{\alpha, \neg\alpha\} \vdash_{\mathbf{P_{LI}}} \beta$ holds. This is not surprising given that the connective has all the features of classical negation.

In logical literature, there are several references to the concept of literal paraconsistent calculi; however, most have been developed primarily as a secondary aspect of paranormal logics. A few examples will help us to illustrate this point. In 1988, Puga and da Costa presented a calculus to capture "some of Vasiliev's more important intuitions" [26, p. 205]. This calculus, denoted as $\mathbf{V}$, has conjunction, disjunction, implication, and negation as primitive connectives. It comprises the positive fragment of **CPC** along with the following axiom for negation for all $\alpha, \beta \notin \mathrm{Var}$:[1]

$$(\sim\alpha \to \beta) \to ((\sim\alpha \to \sim\beta) \to \alpha) \tag{ne}$$

It is easy to see that (A3) is a thesis of $\mathbf{V}$, indicating that $\mathbf{V}$ is not paraconsistent at the level of literals. However, if one modifies the restriction imposed on (ne): from $\alpha, \beta \notin \mathrm{Var}$ to $\alpha, \beta \notin \mathrm{LI}$, paraconsistency at the level of literals is achieved.

Another example is the paranormal logic $\mathbf{I^1P^1}$. The logic is defined by the matrix $\mathfrak{M}_{\mathbf{I1P1}} = \langle \{1, 2, 3, 0\}, \{1, 2\}, \sim, \wedge, \vee, \to \rangle$, where the interpretations of connectives $\sim$, $\wedge$, $\vee$ and $\to$ are defined in Table 1.[2] Since

[1] The calculus **V** was also discussed, under a different name, in [15, 14].

[2] The logic $\mathbf{I^1P^1}$ was presented, e.g., in [11, 13].

	$\sim$
1	0
2	1
3	0
0	1

$\to$	1	2	3	0
1	1	1	0	0
2	1	1	0	0
3	1	1	1	1
0	1	1	1	1

$\wedge$	1	2	3	0
1	1	1	0	0
2	1	1	0	0
3	0	0	0	0
0	0	0	0	0

$\vee$	1	2	3	0
1	1	1	1	1
2	1	1	1	1
3	1	1	0	0
0	1	1	0	0

Table 1. Truth tables in $\mathfrak{M}_{\mathbf{I1P1}}$

(A3) is valid in $\mathfrak{M}_{\mathbf{I1P1}}$, the same issue arises as in the case of Puga and da Costa's calculus.

4. A Gently Paraconsistent Weakening of $\mathbf{P_{LI}}$

A paraconsistent calculus $\mathcal{C}$ that admits, for all $\alpha, \beta \in \mathrm{For}$, the principle of *gentle explosion*: $\{\alpha, \sim\alpha, \sim\sim\alpha\} \vdash_{\mathcal{C}} \beta$, is called *gently* paraconsistent. The calculus presented below is expected to comply with this principle iff $\alpha \notin \mathrm{LI}$. Let us denote it as $\mathbf{GP_{LI}}$. $\mathbf{GP_{LI}}$ is obtained from $\mathbf{P_{LI}}$ by replacing (A4) with (PL) plus the following formula:

$$(\alpha \to \beta) \to (\sim(\alpha \to \beta) \to (\sim\sim(\alpha \to \beta) \to \gamma)) \qquad \text{(A4}^\star\text{)}$$

FACT 4.1. *(PL) is independent from (A1), (A2), (A4***), (A5) and (MP). So,* $\mathbf{GP_{LI}}$ *is a proper subsystem of* $\mathbf{P_{LI}}$.

PROOF. Independence of (PL) is established by $\langle\{1, 2, 0\}, \{1\}, \sim, \to\rangle$, where the connectives are defined by

$\to$	1	2	0
1	1	2	0
2	1	1	0
0	1	1	1

	$\sim$
1	0
2	1
0	1

(A1), (A2), (A4*) and (A5) are valid in this matrix and (MP) preserves validity. To show that (PL) is invalid in this matrix, it is sufficient to assign the value 2 to α and 0 to β. ⊣

A bivalent semantics for $\mathbf{GP_{LI}}$ can be given through minor modifications to the semantics proposed for $\mathbf{P_{LI}}$. $\mathbf{GP_{LI}}$-bivaluations satisfy the following conditions for all $\alpha, \beta \in \text{For}$:

(i) if $v(\sim\alpha) = 0$ then $v(\alpha) = 1$,
(ii) if $v(\sim\sim(\alpha \to \beta)) = 1$, then $v(\sim(\alpha \to \beta)) = 0$ or $v(\alpha \to \beta) = 0$,
(iii) $v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$.

THEOREM 4.1. *For all $\Gamma \subseteq \text{For}$ and $\alpha \in \text{For}$: $\Gamma \vdash_{\mathbf{GP_{LI}}} \alpha$ iff $\Gamma \models_{\mathbf{GP_{LI}}} \alpha$.*

The proof coincides with the proof of Theorems 3.1 and 3.2. The only notable difference lies in the proof of Lemma 3.3.

LEMMA 4.1. *For any maximal non-trivial set Δ with respect to α in $\mathbf{GP_{LI}}$ the mapping v_Δ defined by ($\star$) is a $\mathbf{GP_{LI}}$-valuation.*

PROOF. For (i) and (iii), proceed as in the proof of Lemma 3.3.

For (ii): Assume that $v_\Delta(\sim\sim(\beta \to \gamma)) = v_\Delta(\sim(\beta \to \gamma)) = v_\Delta(\beta \to \gamma) = 1$. Then, by ($\star$), we have $\sim\sim(\beta \to \gamma) \in \Delta$, $\sim(\beta \to \gamma) \in \Delta$ and $\beta \to \gamma \in \Delta$. So $\Delta \vdash_{\mathbf{GP_{LI}}} \sim\sim(\beta \to \gamma)$, $\Delta \vdash_{\mathbf{GP_{LI}}} \sim(\beta \to \gamma)$ and $\Delta \vdash_{\mathbf{GP_{LI}}} \beta \to \gamma$. Hence, by (A4*) and (DT), $\{\sim\sim(\beta \to \gamma), \sim(\beta \to \gamma), \beta \to \gamma\} \vdash_{\mathbf{GP_{LI}}} \alpha$. So also $\Delta \vdash_{\mathbf{GP_{LI}}} \alpha$. Thus, we obtain a contradiction: $\alpha \in \Delta$ and $\alpha \notin \Delta$. ⊣

Finally, the triple of non-literal formulas: α, $\sim\alpha$ and $\sim\sim\alpha$ entails any β, which results in trivializing $\mathbf{GP_{LI}}$.

5. Controllable Paraconsistency

A propositional logic that contains the positive part of **CPC** and the formula

$$(\alpha \to \sim\alpha) \to \sim\alpha \tag{F1*}$$

as the sole axiom for negation, is well-known as **CLuN** in the literature [see, e.g., 3, p. 229]. We assume that the propositional implication-negation fragment of **CLuN*** serves as the foundational calculus, with negation treated as a subcontrary-forming operator.[3] Let **CLuN*** be the propositional implication-negation fragment of **CLuN**.

FACT 5.1. **CLuN*** *is axiomatizable by* (A1), (A2), (A5), (PL) *using* (MP).

[3] **CLuNs** and the other calculi presented in this section are not functionally complete.

PROOF. For soundness, it suffices to verify whether (A5) is a thesis of **CLuN***. It can be done using semantics adequate for **CLuN**: the bivaluation semantics from [3, p. 231] or the three-values non-deterministic semantics from [2, Section 3.2.1]. For completeness, notice that (F1*) is derivable from (A5) and (id) using (MP) and (DT). ⊣

Now let us consider the extension of **CLuN***, $\mathbf{P_{AB}}$ for short, obtained by adding

$$\sim(\alpha \to \beta) \to \alpha \tag{A4a}$$

$$\sim(\alpha \to \beta) \to \sim\beta \tag{A4b}$$

to {(A1), (A2), (A5), (PL), (MP)}.

FACT 5.2. *Formulas (A4a) and (A4b) are theses of* $\mathbf{P_{LI}}$.

PROOF. (A4a) is derivable from (c), (A4), (PL) using (DT) and (MP).

For (A4b):

1.	$\sim(\alpha \to \beta)$	hyp.
2.	$(\alpha \to \beta) \to \sim\beta$	1, (A4), (c), (MP)
3.	$\beta \to \sim\beta$	(tr), (A1), 2, (MP)
4.	$\sim\beta$	(c), (A5), (id), 3, (MP) ⊣

FACT 5.3. *Formula (A4) is not a thesis of* $\mathbf{P_{AB}}$. *So,* $\mathbf{P_{AB}}$ *is a proper subsystem of* $\mathbf{P_{LI}}$.

PROOF. This follows form applying the matrix $\langle\{1, 2, 0\}, \{1, 2\}, \sim, \to\rangle$, where the connectives are defined by

$\to$	1	2	0
1	1	2	0
2	1	1	0
0	1	1	1

	$\sim$
1	0
2	2
0	1

(A4) is invalid in this matrix (when α is assigned the value 1, β assigned the value 2 and γ assigned the value 0). ⊣

$\mathbf{P_{AB}}$-bivaluations satisfy the following conditions for all $\alpha, \beta \in \mathrm{For}$:

if $v(\sim\alpha) = 0$ then $v(\alpha) = 1$,
if $v(\sim(\alpha \to \beta)) = 1$ then $v(\alpha) = 1$ and $v(\sim\beta) = 1$,
$v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$.

In a similar way as for Theorems 3.1 and 3.2 we obtain:

THEOREM 5.1. *For all* $\Gamma \subseteq \mathrm{For}$ *and* $\alpha \in \mathrm{For}$: $\Gamma \vdash_{\mathbf{P_{AB}}} \alpha$ *iff* $\Gamma \models_{\mathbf{P_{AB}}} \alpha$.

Given that α is of the form $\varphi \to \chi$, we say that a paraconsistent calculus $\mathcal{C}$ is *controllable* iff $\{\alpha, \sim\alpha\} \vdash_{\mathcal{C}} \varphi$ and $\{\alpha, \sim\alpha\} \vdash_{\mathcal{C}} \chi$, for all $\varphi, \chi \in \mathrm{For}$.[4] It can be easily verified that $\{\varphi \to \chi, \sim(\varphi \to \chi)\} \nvdash_{\mathbf{P_{AB}}} \beta$, for some $\varphi, \chi, \beta \in \mathrm{For}$. It suggests that $\mathbf{P_{AB}}$ is paraconsistent not only at the level of literals but also at the level of complex formulas. On the other hand, we have both $\{\varphi \to \chi, \sim(\varphi \to \chi)\} \vdash_{\mathbf{P_{AB}}} \varphi$ and $\{\varphi \to \chi, \sim(\varphi \to \chi)\} \vdash_{\mathbf{P_{AB}}} \chi$, for any $\varphi, \chi \in \mathrm{For}$, which indicates that $\mathbf{P_{AB}}$ is paraconsistent in a *controllable* manner. The former result is not surprising, as (A4a) is an axiom of $\mathbf{P_{AB}}$. For the latter, assume that $\varphi \to \chi$ and $\sim(\varphi \to \chi)$. Then we get φ, by (A4a), $\sim(\varphi \to \chi)$ and (MP); and finally, χ, by $\varphi \to \chi$, φ and (MP). In the next paragraphs, we will discuss three exemplary extensions of $\mathbf{P_{AB}}$, with the last one being considered to be the top extension of all controllable paraconsistent calculi.

The first, denoted by $\mathbf{P_N}$, is obtained by adding (F4) to $\mathbf{P_{AB}}$. We have $\{\alpha \to \sim\beta, \sim(\alpha \to \sim\beta)\} \vdash_{\mathbf{P_N}} \beta$ and $\{\alpha \to \sim\beta, \sim(\alpha \to \sim\beta)\} \vdash_{\mathbf{P_N}} \sim\beta$, for all $\alpha, \beta \in \mathrm{For}$. Note that despite deriving both β and $\sim\beta$ from the set of premises $\{\alpha \to \sim\beta, \sim(\alpha \to \sim\beta)\}$, it is not the case that any γ follows. Roughly speaking, a (sub-)contradiction at the level of complex formulas implies a (sub-)contradiction at any level, but this does not entail any formula.

$\mathbf{P_N}$-bivaluations satisfy the following conditions for all $\alpha, \beta \in \mathrm{For}$:

if $v(\sim\alpha) = 0$ then $v(\alpha) = 1$,
if $v(\sim\sim\alpha) = 1$ then $v(\alpha) = 1$,
if $v(\sim(\alpha \to \beta)) = 1$ then $v(\alpha) = 1$ and $v(\sim\beta) = 1$,
$v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$.

For $\mathbf{P_N}$, we can obtain:

THEOREM 5.2. *For all $\Gamma \subseteq \mathrm{For}$ and $\alpha \in \mathrm{For}$: $\Gamma \vdash_{\mathbf{P_N}} \alpha$ iff $\Gamma \models_{\mathbf{P_N}} \alpha$.*

FACT 5.4. *$\mathbf{P_N}$ is a proper subsystem of* **P1**.

The second calculus, denoted by $\mathbf{P_{AB^\star}}$, is obtained by adding to $\mathbf{P_{AB}}$ the following new axiom:

$$\alpha \to (\sim\beta \to \sim(\alpha \to \beta)) \tag{A4c}$$

Although (A4c) is not a thesis of **P1**, it is a thesis of several well-known paraconsistent logics, including Nelson's logic **N4**, **CLuNs** and **CAR** [see, e.g., 21, 22, 3, 24, 10, 20].

[4] The adjective *controllable* was already used in the context of paraconsistency but with a different restriction [see, e.g., 6, Section 4.1].

$\mathbf{P_{AB^\star}}$-bivaluations satisfy the conditions for all $\alpha, \beta \in \mathrm{For}$:

if $v(\sim\alpha) = 0$ then $v(\alpha) = 1$,
$v(\sim(\alpha \to \beta)) = 1$ iff $v(\alpha) = 1$ and $v(\sim\beta) = 1$,
$v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$.

For $\mathbf{P_{AB^\star}}$, we can obtain:

THEOREM 5.3. *For all* $\Gamma \subseteq \mathrm{For}$ *and* $\alpha \in \mathrm{For}$: $\Gamma \vdash_{\mathbf{P_{AB^\star}}} \alpha$ *iff* $\Gamma \models_{\mathbf{P_{AB^\star}}} \alpha$.

The third calculus, denoted by $\mathbf{CLuNs}^\star$, we are going to discuss is the propositional implication-negation fragment of **CLuNs** without the constant $\bot$ [see, e.g., 3, pp. 229–230]. $\mathbf{CLuNs}^\star$ is an extension of **N4** by adding (PL) and (F1*) (or, alternatively, (A5)). $\mathbf{CLuNs}^\star$ contains the axioms of $\mathbf{CPC}^{\to}$ and the following postulates for negation: (A4a), (A4b), (A4c), (A5), (F4) and

$$\alpha \to \sim\sim\alpha \tag{A6}$$

The sole rule of inference is (MP). Unlike in $\mathbf{P_{AB}}$, $\mathbf{P_{AB^\star}}$ and $\mathbf{P_N}$, the formulas $\sim\sim(\alpha \to \beta) \to (\sim\sim\alpha \to \sim\sim\beta)$ and $(\sim\sim\alpha \to \sim\sim\beta) \to \sim\sim(\alpha \to \beta)$ are theses of $\mathbf{CLuNs}^\star$. Consequently, we have the rules for driving double negation "inwards" (and "outwards") [cf. 3, p. 230].

$\mathbf{CLuNs}^\star$-bivaluations satisfy the conditions for all $\alpha, \beta \in \mathrm{For}$:

(i) if $v(\sim\alpha) = 0$ then $v(\alpha) = 1$,
(ii) $v(\sim\sim\alpha) = 1$ iff $v(\alpha) = 1$,
(iii) $v(\sim(\alpha \to \beta)) = 1$ iff $v(\alpha) = 1$ and $v(\sim\beta) = 1$,
(iv) $v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$.

For $\mathbf{CLuNs}^\star$ we obtain:

THEOREM 5.4. *For all* $\Gamma \subseteq \mathrm{For}$, $\alpha \in \mathrm{For}$: $\Gamma \vdash_{\mathbf{CLuNs}^\star} \alpha$ *iff* $\Gamma \models_{\mathbf{CLuNs}^\star} \alpha$.

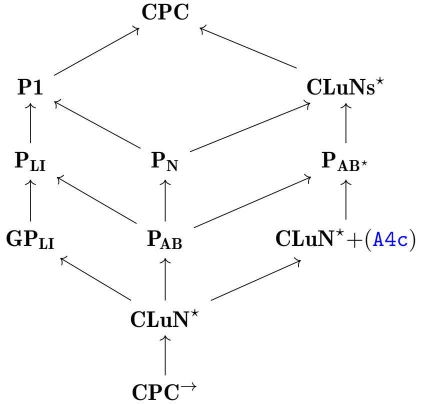
For the most part, the proof proceeds in a similar way to those of Theorems 3.1 and 3.2. We only show that the following lemma holds:

LEMMA 5.1. *For any maximal non-trivial set* Δ *with respect to* α *in* $\mathbf{CLuNs}^\star$, $v_\Delta \colon \mathrm{For} \longrightarrow \{1, 0\}$ *defined by* ($\star$) *is a* $\mathbf{CLuNs}^\star$*-valuation.*

PROOF. For (i) and (iv): Proceed as in Lemma 3.3. For (ii) and (iii): By applying axioms: (F4), (A6), (A4a), (A4b) and (A4c). ⊣

6. Final Remarks

Paraconsistent logics are defined in such a way as to accept contradictions without falling into triviality. The principle of explosion is expected to fail, its universal validity is questioned. However, some exceptions to this principle may be admitted in special cases depending on the shape or form of formulas. We can assume, just like in case of the *literal* and *controllable* paraconsistency, that a pair of the formulas α and $\sim\alpha$ entails a β if and only if α is neither a propositional variable nor is its iterated negation, or that the formula β must be subject to certain restrictions. It was the basic idea behind the logical calculi that have been presented in this paper. Together, they all form the structure shown in the following figure:



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