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Weak Kleene Logic and Topic-Sensitive Logics

Abstract. This paper makes first steps toward a systematic investigation of how pertinence to topic contributes to determine deductively valid reasoning along with preservation of designated values. I start from the interpretation of Weak Kleene Logic **WKL** as a reasoning tool that preserves truth and topic pertinence, which is offered by Jc Beall. I keep Beall's motivations and I argue that **WKL** cannot meet them in a satisfying way. In light of this, I propose an informal definition of a *topic-sensitive logic* and I proceed to turn it into a formal one by deploying the topic-algebraic framework from the *Topic-Sensitive Intentional Modalities* project by Francesco Berto. I apply the framework in order to define semantically a 'classical topic-sensitive logic' CTSL that meets Beall's motivations and proves topic-sensitive. Then, I prove results that connect CTSL and any possible topic-sensitive logic to the tradition of containment logics and provide a unification tool for a wide range of recent proposals in philosophical logic. A topic-theoretical interpretation of **WKL** is then offered without prejudice to the fact that the logic is not topic-sensitive. Finally, the paper discusses some conceptual issues and research perspectives.

Keywords: topic sensitivity; containment logics; **WKL**; topic algebra; classical topic-sensitive logic; topic-algebraic semantics; logical consequence

1. Introduction

Beall (2016) introduces a number of points in order to motivate an interpretation of three-valued logic **WKL** as an inferential machinery which preserves pertinence to the topic of discourse along with truth. The motivations proposed by Beall sound natural and convincing, but one point seems to fly in the face of the interpretation he proposes for **WKL**: the logic has *Explosion* among its valid rules — that is: $A, \neg A \vdash_{\mathbf{WKL}} B$ — and of course, since *Explosion* allows us to conclude *any* formula from

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inconsistent premises, it also apparently allows us to conclude formulas that have nothing to do with the topic of the premises.

In this paper, I start from this problem and I pursue some achievements that are relevant in relation to the growing literature in philosophical logic on the role played by topic pertinence in reasoning or propositional attitudes (see, e.g., Beall, 2016; Berto, 2018, 2019a,b, 2022; Berto and Hawke, 2021; Ferguson, 2023; and, indirectly, Hawke, 2018; Yablo, 2014). First, I look for an informal definition of what it is for a logic to be *topic sensitive*. This criterion will single out rules such as *Explosion* and *Addition* as incompatible with topic sensitivity. Second, I apply a simple extension of the topic-algebraic framework introduced by Berto (2018, 2019a,b) — in turn inspired by Fine (1986) — in order to give a rigorous formal definition of a *topic-sensitive logic*. Third, I build a formal system which meets Beall's motivations better than **WKL**. This will be defined through a *topic-theoretical semantics* that exploits the topic-algebraic framework presented and includes a *topic containment* condition. I call the resulting logic CTSL — short for 'classical topic-sensitive logic'. Fourth, I establish some connections between *topic-sensitive logics* as I define them and a family of *containment logics* which is extensively explored by Ferguson (2015) and which stems from a tradition dating back to Parry (1932, 1968). In particular, I prove that a logic is *topic sensitive* in the sense of the present paper *if and only if* it satisfies the so-called $\vdash$-*proscriptive principle*. This principle imposes that all valid inferences are such that all the atomic sentences contained in the conclusion are among those contained in the premises. This result is presented as a corollary of a more general result, which shows that the topic containment condition we use in defining a topic-sensitive logic is actually equivalent with the syntactic requirement from the $\vdash$-proscriptive principle.

The two results show that there are systematic connections between topic-sensitive logics and a family of containment logics, on the one hand, and even more general connections with wider families of logics, on the other. For instance, extensional containment logics $\mathbf{S}_{\mathsf{fde}}$ and $\mathbf{S}^{\star}_{\mathsf{fde}}$, which are interpreted by Ciuni et al. (2018) as paraconsistent logics of meaninglessness, immediately qualify as topic-sensitive logics. Also, logics from the tradition of *Topic-Sensitive Intentional Modalities* by Berto (2018, 2019a,b) and Berto and Hawke (2021) actually belong to a further family of containment logics. An immediate consequence of the two main results of the paper is that: $\Gamma \vdash_{\mathrm{CTSL}} B \iff \Gamma \vdash_{\mathbf{CL}} B$ and $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)$, where for every set Γ of formulas from standard propositional language, $\mathsf{at}(\Gamma)$ is the set of atomic sentences occurring in some formulas in Γ—notice that I write $\mathsf{at}(A)$ instead of $\mathsf{at}(\{A\})$. That is, CTSL is the greatest fragment of classical logic satisfying the condition $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)$—whence the label 'classical' for the logic. This reveals CTSL as the containment logic by Johnson (1977), Parks-Clifford (1989), Zinov'ev (1973). None of these papers, however, define a topic-theoretical semantics for CTSL, and none of these papers relates the logic to the issue of topic sensitivity in reasoning. The paper also shows how to provide a topic-theoretical interpretation of **WKL** and discusses some perspectives for future research.

Structure of the paper. The paper proceeds as follows. In the remainder of the Introduction, I discuss the relevance of the paper and its methodology, and I provide some background by introducing **WKL**. In Section 2, I present Beall's motivations for a topic-theoretical interpretation of **WKL** and I claim that such motivations are better met by a logic that, contrary to **WKL**, does not satisfy *Explosion*. I discuss and reject an argument to the effect that *Explosion* and topic sensitivity are compatible with one another, which I call 'the compatibility argument', and I give an informal condition at which a logic can qualify as *topic sensitive*. In Section 3, I introduce the intended extension of the topic-algebraic framework by (Berto, 2019a,b), and I give a formal definition of a *topic-sensitive logic*. Against this background, I introduce CTSL and define its topic-theoretical semantics. The logic is topic sensitive and meets the motivations by Beall. In Section 4, CTSL is revealed to be a containment logic. This follows from two more general results. The first proves that the distinctive topic containment condition of topic-sensitive logics and the distinctive atomic containment condition from $\vdash$-containment logics are actually equivalent. The second concludes that every topic-sensitive logic is a $\vdash$-containment logic, and vice versa. The wider implications of the first result are also discussed. In Section 5, I define a topic-theoretical semantics for **WKL**, thus showing that the logic can receive a topic-theoretical interpretation, as Beall suggests, without prejudice to the fact that **WKL** fails to be a topic-sensitive logic. Section 6 briefly discusses open problems and possible future research. Finally, Section 7 summarizes the paper and presents some conclusions.

Relevance and methodology of the paper. There is increasing interest today toward the role topic pertinence plays in reasoning, propositional attitudes and in the theory of meaning. The topic sensitivity of propositional attitudes is the main focus of the *Topic-Sensitive Intentional Modalities* project by Berto (2018, 2019a,b, 2022) and Berto and Hawke (2021), a theory of topic and the role of *aboutness* in determining the meaning of a proposition are the focus of Hawke (2018) and Yablo (2014). However, to this day only Beall (2016) touches upon the issue of the role topic sensitivity would play in deductive reasoning — that is, how considerations concerning topic would contribute to determine valid inference schemes. Beall (2016), however, does not attempt at a systematic investigation of the issue.

This is an unsatisfactory gap. The present paper fills it by making first steps toward a systematic investigation and elaboration of topic-sensitive logics. A general understanding of topic sensitivity and what it means for a logic to be topic sensitive are crucial in this. The relevance of the proposal from Section 3 — and particularly of Definition 3.3 — follows from the pivotal role it plays in this enterprise. The relevance of Theorem 4.1 and Corollary 4.1 from Section 4 — the main results from the paper — does not lie in their technical import — they do not require challenging or novel proof techniques — but in their ability to connect a galaxy of different traditions in logic, such as the many-valued, extensional containment logics interpreted as *paraconsistent logics of meaninglessness* by Ciuni et al. (2018), the *Topic-Sensitive Intentional Modalities* project mentioned above, and Parry systems where conditionals satisfy their corresponding proscriptive principle (see Section 4 for this). Although a connection between topic algebras and Parry systems is already outlined by Fine (1986) (the content algebra there is *de facto* a topic algebra), this is the first paper to show the connection in its full generality, and to discuss the implications the connection has for a wide research area in philosophical logic. Also, the two results may provide helpful methodological instructions that allow one for working with the semantics of $\vdash$-containment logics or with the topic-theoretical semantics presented here, while achieving exactly the same end, and they secure that any logical enterprise satisfying the appropriate proscriptive principle will output a logic satisfying some form of topic sensitivity. In a nutshell, the relevance of the results is connected to the role they can play as a unification tool in a research area that is expanding fast.

This paper deploys a semantical methodology. As is standard, I consider a logic $\boldsymbol{L}$ to be a pair consisting of a given language (here, standard propositional language) and a consequence relation $\vdash$. When convenient, I write $\vdash_{\boldsymbol{L}}$ for the consequence relation intended for logic $\boldsymbol{L}$, and I talk about $\boldsymbol{L}$-consequence. All the consequence relations defined here are matrix-based (and hence include the obvious component of 'preservation of designated values'), but the ones I define from Section 3 on also include a topic-algebraic component. I present matrices just once by the generic notation $\langle \mathcal{T}, \neg, \vee, \wedge, \supset, \mathcal{D} \rangle$, with $\mathcal{T}$ being the universe (in our standard logic-oriented interpretation, the set of truth values), $\mathcal{D}$ being a privileged subset of the universe (the set of designated truth values), $\neg, \vee, \wedge, \supset$ being the standard operations on an algebra — and by abusing notation a bit, I also use the same symbols for the connectives from propositional language. In all other occasions, I will refer to the sets of truth values and of designated truth values just via the appropriate lists of elements.

Background: Weak Kleene Logic. Weak Kleene Logic (**WKL**) results once we interpret the standard notion of logical consequence as preservation of designated values on the matrix $\mathcal{M}_{\mathbf{WKL}} = \{\mathcal{T}, \neg, \vee, \wedge, \supset, \mathcal{D}\}$, where $\mathcal{D} = \{\mathbf{t}\}$ is the set of designated values, and $\{\mathcal{T}, \neg, \vee, \wedge, \supset\}$ is a *Weak Kleene algebra*, with $\mathcal{T} = \{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$, and $\neg, \vee, \wedge, \supset$ behaving as follows:[1]

	$\neg$
t	**f**
f	**t**
n	**n**

$\vee$	**t**	**f**	**n**
t	**t**	**t**	**n**
f	**t**	**f**	**n**
n	**n**	**n**	**n**

$\wedge$	**t**	**f**	**n**
t	**t**	**f**	**n**
f	**f**	**f**	**n**
n	**n**	**n**	**n**

$\supset$	**t**	**f**	**n**
t	**t**	**f**	**n**
f	**t**	**t**	**n**
n	**n**	**n**	**n**

Standard propositional language is interpreted by the usual valuation functions which, in this case, associate each formula of the language to a value among $\{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$ in conformity with the above table. I call $\mathcal{V}_{\mathbf{WKL}}$ the class of valuations that satisfy the requirement. **WKL**-consequence is defined as truth-preservation. That is:

$$\Gamma \vdash_{\mathbf{WKL}} B \iff v(B) = \mathbf{t} \text{ for every } v \in \mathcal{V}_{\mathbf{WKL}} \text{ such that } v[\Gamma] \subseteq \{\mathbf{t}\}.$$

The following characterization of $\vdash_{\mathbf{WKL}}$ — which immediately follows from (Ciuni and Carrara, 2019, Theorem 4.3) — will be useful in many points of this paper:[2]

[1] From now on, when introducing a matrix I represent the set of truth values and that of the designated values by the list of their elements, keeping them fix in first and third position, respectively.

[2] Fact 1.1 appears as Corollary 4.7 of (Ciuni and Carrara, 2019) and as Theorem 2.3.1 of (Urquhart, 2002). I use $\perp$ as an abbreviation for $p \vee \neg p$.

FACT 1.1 (Characterization of $\vdash_{\mathbf{WKL}}$).

$$\Gamma \vdash_{\mathbf{WKL}} B \Leftrightarrow \begin{cases} \Gamma \vdash_{\mathbf{CL}} \bot & \text{or} \\ \Gamma \vdash_{\mathbf{CL}} B, \text{ and} & \mathsf{at}(B) \subseteq \mathsf{at}(\Gamma) \end{cases}$$

Notice also that the semantical machinery of **WKL** satisfies the so-called *contamination property*. In particular:

DEFINITION 1.1 (Contaminating value). Given an algebra $\{\mathcal{T}, \neg, \vee, \wedge, \supset\}$, a $\boldsymbol{v} \in \mathcal{T}$, a set Form of wff, and a set $\mathcal{V}$ of valuation functions, I say that:

$\boldsymbol{v} \in \mathcal{T}$ is *contaminating* (relative to Form and $\mathcal{V}$) if and only if: For every $v \in \mathcal{V}$ and formula $A \in \mathsf{Form}$, $v(A) = \boldsymbol{v}$ iff $v(p) = \boldsymbol{v}$ for some $p \in \mathsf{at}(A)$.

Out of symbols, a value $\boldsymbol{v}$ is contaminating if and only if one atomic sentence p taking value $\boldsymbol{v}$ suffices for any formula A in which p occurs to have value $\boldsymbol{v}$, regardless of the values of all other atomic sentences occurring in A.[3] I say that the semantics of a logic satisfies the *contamination property* if and only if it includes a contaminating value. It is easy to see that $\mathbf{n}$ from the Weak Kleene algebra is contaminating relative to standard propositional language and $\mathcal{V}_{\mathbf{WKL}}$, to the effect that the semantics of **WKL** satisfies the contamination property.

The first (and to this day best known) philosophical interpretation of **WKL** comes from (Bochvar, 1938), the very first paper in which **WKL** was introduced. In particular, Bochvar takes **WKL** as an adequate logic to reason in presence of meaningless expressions.[4] Indeed, Bochvar reads the third value $\mathbf{n}$ from the Weak Kleene Algebra as 'meaningless', since the contaminating behavior of $\mathbf{n}$ would make it fit representing the equally contaminating behavior of meaninglessness.[5] Since the introduction of new atomic sentences in the conclusion of a reasoning exposes one to the risk of meaninglessness, meeting condition $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)$ is necessary for an inference $\Gamma \vdash B$ to be valid. Otherwise, I could go

[3] Contamination is not to be confused with the property stating that, for every model v and formula A, *if* $v(A) = \mathbf{n}$, *then* $v(p) = \mathbf{n}$ for some $p \in \mathsf{at}(A)$. This weaker principle is common to all subclassical paracomplete logics having a matrix-based semantics.

[4] That is, syntactic constructs that are (well-formed) sentences and yet fail to convey a meaning.

[5] That meaninglessness is contaminating sounds natural: if a syntactic construct is meaningless, every syntactic construct involving it will also be meaningless.

from true premises to meaningless conclusions. One important point to notice is that, in **WKL**, this atomic containment condition is satisfied (and required) only if Γ is classically consistent ($\Gamma \nvdash_{\mathbf{WKL}} \bot$). No classical inconsistency can be true in any $v \in \mathcal{V}_{\mathbf{WKL}}$, to the effect that, for any inconsistent Γ, there is *no* model in which all Γs are true *and* B is false *or* meaningless, regardless of the inclusion relations between $\mathsf{at}(\Gamma)$ and $\mathsf{at}(B)$.[6]

2. Beall's interpretation of WKL, and its motivations

Beall (2016) proposes a novel interpretation of **WKL** as a reasoning machinery which preserves pertinence to the topic of discourse along with truth. Under this reading, the behavior of $\vdash_{\mathbf{WKL}}$ is interpreted as a guarantee that the conclusion of a valid inference does not add any more topics to those from the premises. In presence of this property, Beall talks about 'topic preservation'. I prefer to talk about 'topic sensitivity', and I will do so in this paper. Beall's interpretation is

> [...] motivated by the following ideas, all of which I take to be prima facie plausible (and offer here without argument).
> 1. A theory is about all and only what its elements — that is, the claims in the theory — are about.
> 2. Conjunctions, disjunctions and negations are about exactly whatever their respective subsentences are about:
>   (a) Conjunction $A \wedge B$ is about exactly whatever A and B are about.
>   (b) Disjunction $A \vee B$ is about exactly whatever A and B are about.
>   (c) Negation $\neg A$ is about exactly whatever A is about.
> 3. Theories in English are rarely about every topic expressible in English. (Beall, 2016, p. 139)

Notice that, by *a theory*, Beall actually means a *deductively closed theory* (see Beall, 2016, p. 139). I keep Beall's motivations in this paper: I believe they are a natural starting point for any research in topic-sensitive logics, and exactly as Beall, I will take them on board with no

[6] One worry about Bochvar's interpretation is that logical operations would apply to propositions (that is, sentences that can have a meaning), to the effect that they would not apply to meaningless syntactical construct and the latter would have no possible role to play in our reasoning, contrary to what Bochvar apparently implies. I do not go through an analysis of this worry, because that would take us too far from our topic. Let us just notice that, exactly as Beall, I do not take this worry as decisive against Bochvar's interpretation. See (Beall, 2016, §3) for this.

argument. Also, they are as easy as needing no comment. Let me just notice that motivation 3 implies that theories are, usually, not *maximal* — that is, they do not contain either A or $\neg A$ for some formula A defined in the language. I does not interpret this in strictly quantitative terms, but as meaning that the theories we tend to direct our attention are not 'theories of everything' — that is, theories encompassing every topic. Beall goes on by saying:

> Putting these simple ideas together tells against using mere truth-preserving consequence as a relation under which all theories in English are closed. After all, Addition is truth preserving; however, it isn't topic preserving given (2b) and (3). As per (2b) the disjunction of *Grass is green* and *Brexit happened* is about grass and Brexit; but, by way of a witness for (3), our true theory of grass is not about Brexit. Addition can take theories off-topic. (Beall, 2016, pp. 139–140)

In short: suppose we are discussing grass. Ideally, our (deductively closed) theory of grass must contain the (true) sentence *Grass is green*, but not the disjunction *Grass is green or Brexit happened*, because the second disjunct is off-topic w.r.t. the topic 'the color of grass'. Thus, *Addition* is incompatible with topic sensitivity, since *it might make us conclude sentences that are off-topic w.r.t. the premises*.

A natural move if one endorses motivations 1–3 and the related remark by Beall (2016, pp. 139–140) is to go for a formal machinery that is topic sensitive and in which, therefore, *Addition* fails. Beall (2016, p. 140) singles out **WKL** to this purpose. In particular:

> The foregoing ideas motivate a very simple but attractive interpretation of **WKL** as a logic that concerns not simply truth-preservation but truth-and-topic preservation [. . .] The proposal: read the value **n** not simply as true but rather as true and on-topic, and similarly **f** as false and on-topic. Finally, read the third value **n** as off-topic. This interpretation, I claim, motivates the **WKL** treatment of connectives, at least if the background motivating ideas [. . .] are held fixed.
> (Beall, 2016, p. 140)

Remark 2.1 (Classical logic as the basic topic-insensitive logic). Choosing **WKL** as one's tool for topic-sensitive reasoning immediately qualifies **CL** as the basic tool for topic-*in*sensitive reasoning — this is immediate if one looks at Fact 1.1. I follow Beall (2016) in this, and I will take **CL** as our provider of topic-insensitive inference schemes. This will lead us to look for a topic-sensitive version of **CL**. However, neither Beall nor I commit with the philosophical view that **CL** is the best possible reasoning tool (or 'the one and only right logic'), to be later constrained in order to make it topic sensitive. Beall's favored reasoning tool (net of topicality issues), after all, is **FDE**. In (Ciuni et al., 2018), a number of logics are explored that satisfy the requirements for topic sensitivity (in view of Theorem 4.1 below) and are based on *paraconsistent* logics. In that paper, such logics are seen as viable *paraconsistent logics of meaninglessness*.

I find Beall's motivations convincing, but I believe that the pack of ideas and remarks provided by (Beall, 2016, pp. 139–140) better motivates some logic *different* from **WKL**. I believe this because doubts analogous to those raised against *Addition* apply to *Explosion*. Why so? Well, suppose I state the contradiction that *Grass is green and grass is not green*. Then, *Explosion* makes me conclude anything — including that *Brexit happened*. And with this sentence we get off-topic w.r.t. the premises — the very same problem we had with *Addition*. Therefore, any topic-sensitive logic worth its name should fail *Explosion*. Since **WKL** validates the rule, it proves not adequate as a topic-sensitive logic, contrary to Beall's purposes.

So far so good, but before we go to CTSL as a (non-explosive) tool for topic-sensitive reasoning, let us come back to the claim that, with Explosion, we may '*get off-topic w.r.t. the premises*.' I go through this in more detail since, intuitive as it sounds, there is an argument against it.

2.1. The compatibility argument presented and dismissed

In order to see the argument for the *compatibility* between *Explosion* and *topic sensitivity*, remember that the motivations by Beall are actually intended for *deductively closed* theories. Hence, one could start from a special case of motivation 1 and state: 'A (deductively closed) inconsistent theory is about all and only what its elements — that is, the claims in the theory — are about.' One could then go on and say: 'Okay, but a (deductively closed) inconsistent theory is a theory that contains *any* sentence, and hence is among those few theories that are about *any* topic. No problem at all with *Explosion*, then!'

Let's call this 'the compatibility argument'. It implies that the argument against *Addition* does not extend to (deductively closed) inconsistent theories, since those are theories about *anything*. Hence, one cannot possibly go off-topic w.r.t. any inconsistent theory, which makes *Explosion* unproblematic.

Remark 2.2 (Addition). Notice that the compatibility argument does not turn into an argument in favor of *Addition*. This is important, because, of course, one could claim: 'Ok, a deductively closed theory whatever is a theory which includes $A \vee B$, if it includes A (or B). The topic expressed by B (or A) is *already there* anyway! So if you condone *Explosion*, you must also condone *Addition*.' This does not work, however. Indeed, *Addition* would force *every* theory T to be about *everything* (by just the presence of disjunctions $A \vee B$ such that $A \in T$ but $B, \neg B \notin T$), in sheer violation of motivation 3. By contrast, *Explosion* does this just to inconsistent theories, with no patent violation of motivation 3 or of any other motivation provided by Beall. Also, rejection of *Addition* is not arbitrary (that is, done just because 'otherwise I screw motivation 3 up'). I can provide a criterion under which $A \vee B$ can be included in a deductively closed theory T without going off-topic w.r.t. T: $A \vee B \in T$ *if and only if* $A, B \vee \neg B \in T$, or $A \vee \neg A, B \in T$. Notice that this just fits $A, B \vee \neg B \vdash_{\mathbf{WKL}} A \vee B$.

Going back to the compatibility argument, I believe that it proves flawed, at least under a very natural reading of deductive closure. Indeed, we can naturally see *deductive closure* as a way to generate *theories* from *theories*. This just suggests itself if we think of the *consequence operator* $\mathcal{C}\colon \mathfrak{T} \to \mathfrak{T}$, where $\mathfrak{T} = 2^{\mathsf{Form}}$ for Form the formulas defined in a given language. Of course:

$$\mathcal{C}(T) = \{B \mid \mathit{\Gamma} \vdash B \text{ for some } \mathit{\Gamma} \in 2^{T}\}$$

and of course nothing implies $T = \mathcal{C}(T)$. Thus, the consequence operator — which is in fact a *deductive closure operator* — can be reasonably seen as a way to generate deductively closed theories out of (possibly non-closed) theories. If we go with this very natural reading and go for a topic-sensitive reasoning tool, what we might want is a consequence operation that, in generating a deductively closed theory out of T, does not introduce topics that are novel w.r.t. T. That is, we might want that:

$\mathcal{C}(T)$ does not contain any sentence which is off-topic w.r.t. T.

Now take the (non-closed) theory $\{g \wedge \neg g\}$, with g being *Grass is green*. If *Explosion* is a valid rule, we will have $b \in \mathcal{C}(\{g \wedge \neg g\})$ — b is *Brexit happened*, but could stay for any other atomic sentence. But if this was the case, $\mathcal{C}(\{g \wedge \neg g\})$ would not satisfy the requirement we have just presented.

In light of this, we may reject the compatibility argument and conclude that *Explosion* and *topic sensitivity* are incompatible, that is:

$$\text{if } \vdash \text{ is topic sensitive, then } A, \neg A \nvdash B.$$

Of course, this implies that topic-sensitive logics are *paraconsistent*, albeit it does not imply that topic-sensitive logics need to satisfy classically inconsistent formulas — as we shall see in Section 3.

The argument by Beall against *Addition* and the argument against *Explosion* are motivated by the same presupposition: if your reasoning tool is to be topic sensitive, then the conclusion of a valid inference must not bring in topics that are not comprised in the topic of the premises. Indeed, this is the only way to prevent that the conclusion proves off-topic w.r.t. the premises.

Given the characterization of topic sensitivity from the beginning of this section, this allows for an informal definition of a topic-sensitive logic:

Definition 2.1 (Topic-sensitive logic (informal)). Be $\boldsymbol{L}$ a logic defined by a given language and a given consequence relation $\vdash_{\boldsymbol{L}}$. I say that $\boldsymbol{L}$ is *topic sensitive* if and only if, for all $\Gamma \in 2^{\mathsf{Form}_{\boldsymbol{L}}}$ and $B \in \mathsf{Form}_{\boldsymbol{L}}$:

$$\text{if } \Gamma \vdash_{\boldsymbol{L}} B, \text{ then the topic associated with } B \text{ is comprised in the topic associated with } \Gamma.$$

In the next section, we will equip ourselves with tools that will turn the informal definition above into a rigorous, formal one, and we will deploy the same tool in order to build one topic-sensitive logic.

2.2. A brief sum-up of this section

Summing up the content of this section: the need, for a topic-sensitive logic, to get rid of inference schemes that allow for concluding off-topic sentences follows from motivations 1–3 by Beall (2016, p. 139) and seems to exclude *Addition* and *Explosion* alike. In consequence of this, the explosion-validating **WKL** cannot be a topic-sensitive logic. The argument leading to this conclusion also allows us for an informal definition of a topic-sensitive logic.

3. Topic-Sensitive Logic

In order to turn Definition 2.1 into a formal definition, we first need a framework that allows for a formal treatment of topics. One natural candidate for this is the topic-algebraic framework by Berto (2019a,b) — in turn inspired by the content-algebraic framework by Fine (1986). The version we deploy here extends the original framework by Berto by assigning topics to *sets* of formulas, beside assigning topics to single formulas. First, let's define a *topic algebra* — the definition is by Berto (2019a,b):

Definition 3.1 (Topic Algebra). A topic algebra is a triple $\langle \mathcal{K}, \oplus, \leq \rangle$ such that:

- $\mathcal{K}$ is a non-empty set of *topics*;
- $\oplus$ is a *topic fusion* operation on $\mathcal{K}$ making topics part of larger topics, and satisfying, for every $c, c', c'' \in \mathcal{K}$:
  (a) $c \oplus c = c$
  (b) $c \oplus c' = c' \oplus c$
  (c) $(c \oplus c') \oplus c'' = c \oplus (c' \oplus c'')$
  with $c \oplus c'$ being *the fusion* of the topics c and c'.
- $\leq \subseteq \mathcal{K} \times \mathcal{K}$ is a *topic parthood* relation satisfying, for every c, c':

$$c \leq c' \Leftrightarrow c \oplus c' = c'$$

  with $c \leq c'$ reading 'topic c is *contained* in topic c''.

It is easy to notice that $\leq$ is a *partial order* on $\mathcal{K}$. That is, it is *reflexive* ($c \leq c$), *anti-symmetric* (if $c \leq c'$ and $c' \leq c$, then $c = c'$) and *transitive* (if $c \leq c'$ and $c' \leq c''$, then $c \leq c''$). It is clear by this and the above definition that a topic algebra is a join semilattice. Our next step is defining a way to assign *topics* to *sentences*.

Definition 3.2 (Topic Assignment). Given a topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$ and a set Form of wffs, a *topic assignment* function is $\gamma\colon \mathsf{Atoms} \to \mathcal{K}$ (with Atoms the set of atomic sentences in Form). The function is generalized to arbitrary formulas by setting: $\gamma(A) = \oplus_{p \in \mathsf{at}(A)} \gamma(p)$.

That is: the topic of a sentence A is the fusion of the topics of all the atomic sentences occurring in A. Finally, I generalize the function further to sets of formulas by setting $\gamma(\Gamma) = \oplus_{A \in \Gamma} \gamma(A)$, and imposing:

$$\gamma(\Gamma) \text{ is } \textit{undefined} \text{ if and only if } \Gamma = \varnothing.$$

The two generalizations together imply that $\gamma(\Gamma) = \oplus_{p\in\mathsf{at}(\Gamma)}\gamma(p)$. Notice that, since $(\mathsf{at}(A) \cup \mathsf{at}(B)) = \mathsf{at}(\circ(A, B))$ for every $\circ \in \{\vee, \wedge, \supset\}$, we have $\gamma(A \vee B) = \gamma(A \wedge B) = \gamma(A \supset B) = \gamma(A) \oplus \gamma(B)$. Also, since $\mathsf{at}(A) = \mathsf{at}(\neg A)$, we have $\gamma(A) = \gamma(\neg A)$. These equalities sound natural when it comes to topics. After all, a complex sentence is about what its subsentences (immediate or not) are about, and hence it is ultimately about what its atomic sentences are about. What the topic of a sentence is, in sum, is determined by the 'logically brute stuff' in the sentence, not by the logical operations featured in it. Similar remarks apply to the topic of a *set* of sentences.[7]

The topic-algebraic framework gives us a natural way to interpret motivations 1–3 from (Beall, 2016, p. 139). In particular, we can rephrase them formally as follows:

1. $\gamma(T) = \oplus_{A\in T}\gamma(A)$ for every deductively closed theory $T \in 2^{\mathsf{Form}}$.
2. $\gamma(A \vee B) = \gamma(A \wedge B) = \gamma(A \supset B) = \gamma(A) \oplus \gamma(B)$, $\gamma(\neg A) = \gamma(A)$.
3. $\gamma(T) < \gamma(\mathsf{Form})$ for the deductively closed theories $T \in 2^{\mathsf{Form}}$ we tend to use.

The first two points actually follows from Definition 3.2. The third point states that the topic of the deductively closed theories that we tend to use are *properly* contained in the 'maximal topic' assigned to Form itself. Notice that 3 cannot be satisfied if our consequence relation satisfies *Addition*.[8] On the other hand, *non-maximal* (deductively closed) theories which fail to satisfy *Addition* will satisfy 3.

3.1. Defining topic-sensitive logics

With the topic-algebraic framework at hand, we can provide a formal definition of the notion of a *topic-sensitive logic*:

Definition 3.3 (Topic-sensitive logic (formal)). Let $\boldsymbol{L}$ be a logic defined by a given language and a given consequence relation $\vdash_{\boldsymbol{L}}$. I say that $\boldsymbol{L}$ is *topic sensitive* if and only if, for all $\Gamma \in 2^{\mathsf{Form}_{\boldsymbol{L}}}$ and $B \in \mathsf{Form}_{\boldsymbol{L}}$:

$$\text{if } \Gamma \vdash_{\boldsymbol{L}} B, \text{ then } \gamma(B) \leq \gamma(\Gamma) \text{ for every topic assignment } \gamma \text{ and topic algebra } \langle \mathcal{K}, \oplus, \leq \rangle.$$

[7] For a short defense of the property in question (see Berto, 2019a,b). The property has also been defended by Yablo (2014) in his treatment of *abountness*, and by Fine (2016) in his treatment of *content*.

[8] Indeed, in that case I would have *any* disjunction $A \vee B$ for every $A \in T$, the effect that $\gamma(B) \leq \gamma(T)$ for any $B \in 2^{\mathsf{Form}}$. But this of course implies $\gamma(\mathsf{Form}) \leq \gamma(T)$.

That is: $\boldsymbol{L}$ is topic sensitive *if and only if* all its valid inferences are such that the topic of the conclusion is contained in the topic of the premises. I call the condition '$\gamma(B) \leq \gamma(\Gamma)$ for every topic assignment γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$' a *topic containment condition*. It is clear that Definition 3.3 is a formal rendering of Definition 2.1 in the terms of the topic-algebraic framework presented here.

3.2. A 'classical' topic-sensitive logic

If we take **CL** to be our 'basic' topic-insensitive tool for reasoning, as we are doing in this paper, then we wish that our logic satisfies the following beside Definition 3.3:

$$\text{if } \Gamma \vdash_{\boldsymbol{L}} B, \text{ then } \Gamma \vdash_{\mathbf{CL}} B.$$

That is, we wish our logic to be subclassical. One natural idea, then, is to look for a logic that satisfies *all and only* those classical inference that secures the topic of the conclusions to be contained in that of the premises. In order to do this, I take **CL** and its matrix-based semantics, and I simply impose a further requirement on logical consequence. By doing this, we obtain what I will call *classical topic-sensitive logic* — or, in short, CTSL.[9] In particular, we interpret CTSL on the usual matrix $\mathcal{M}_{\mathbf{CL}} = \{\{\mathbf{t}, \mathbf{f}\}, \neg, \vee, \wedge, \supset, \{\mathbf{t}\}\}$ of **CL**, where $\neg, \vee, \wedge, \supset$ behave as follows:

	$\neg$
t	**f**
f	**t**

$\vee$	**t**	**f**
t	**t**	**t**
f	**t**	**f**

$\wedge$	**t**	**f**
t	**t**	**f**
f	**f**	**f**

$\supset$	**t**	**f**
t	**t**	**f**
f	**t**	**t**

We then define the class $\mathcal{V}_{\text{CTSL}}$ of valuations v that associate each formula of propositional language to a value among $\{\mathbf{t}, \mathbf{f}\}$ in conformity with the above table. Of course, $\mathcal{V}_{\text{CTSL}}$ is nothing but $\mathcal{V}_{\mathbf{CL}}$. Finally, we define CTSL-consequence:

$$\begin{aligned} \Gamma \vdash_{\text{CTSL}} B \iff & v(B) = \mathbf{t} \text{ for every } v \in \mathcal{V}_{\text{CTSL}} \text{ such that } v[\Gamma] \subseteq \{\mathbf{t}\} \\ & \text{and } \gamma(B) \leq \gamma(\Gamma) \text{ for every topic assignment } \gamma \\ & \text{and topic algebra } \langle \mathcal{K}, \oplus, \leq \rangle. \end{aligned}$$

[9] As the reader expects at this point, CTSL is a *paraconsistent* logic — in that it fails *Explosion*. Thus, one could ask in which sense I deem CTSL a 'classical topic-sensitive logic'. In short: the qualification is, albeit a bit loose, justified by the fact that CTSL is the greatest topic-sensitive fragment of **CL**, as we shall see below.

Thus, we have gone from a matrix-based consequence relation for **CL** to a consequence relation that is based on a matrix *and* on topic algebras. It is easy to notice that, by the very definition of $\vdash_{\mathrm{CTSL}}$:

Fact 3.1. 1. *If $\Gamma \vdash_{\mathrm{CTSL}} B$, then $\Gamma \vdash_{\mathbf{CL}} B$.*
2. *If $\Gamma \vdash_{\mathrm{CTSL}} B$, then $\gamma(B) \leq \gamma(\Gamma)$ for every topic assignment γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$.*
3. *$\Gamma \vdash_{\mathrm{CTSL}} B$ iff $\Gamma \vdash_{\mathbf{CL}} B$ and $\gamma(B) \leq \gamma(\Gamma)$ for every topic assignment γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$.*

Fact 3.1.1 tells us that **CL** is subclassical. Fact 3.1.2 tells us that CTSL is a *topic-sensitive logic* as per Definition 3.3. Finally, Fact 3.1.3, tells us that CTSL is not just subclassical: it is the *greatest topic-sensitive fragment* of **CL**.[10] Finally, I wish to draw the reader's attention to the following three notable features of CTSL:

Fact 3.2. 1. $A \nvdash_{\mathrm{CTSL}} A \vee B$
2. $A, \neg A \nvdash_{\mathrm{CTSL}} B$
3. *If $\Gamma \vdash_{\mathbf{CL}} \bot$, then $\Gamma \vdash_{\mathrm{CTSL}} B \Leftrightarrow \gamma(B) \leq \gamma(\Gamma)$ for every topic assignment γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$;*
4. $\varnothing \nvdash_{\mathrm{CTSL}} A$

Fact 3.2.1, Fact 3.2.2 and Fact 3.2.3 immediately follow from the definition of CTSL-consequence. Fact 3.2.1 and Fact 3.2.2 tell us that *Addition* and *Explosion* fail in CTSL, respectively. Fact 3.2.3 adds that only those cases of *Explosion* are valid in CTSL in which the topic of the conclusion is contained in the topic of the premises (for every possible topic assignment and topic algebra). Finally, Fact 3.2.4 tells us that CTSL is *non-tautological*. In order to see this, consider that, for every $A \in \mathsf{Form}$, γ and $p \in \mathsf{at}(A)$, we have $\gamma(p) \neq \gamma(\varnothing)$, to the effect that $(\gamma(A) \oplus \gamma(\varnothing)) \neq \gamma(\varnothing)$, and hence $\gamma(A) \nleq \gamma(\varnothing)$. From this and the definition of CTSL-consequence, we conclude $\varnothing \nvdash_{\mathrm{CTSL}} A$.

Notice that Fact 3.2.2 implies that $A \wedge (\neg A \vee B) \vdash_{\mathrm{CTSL}} B$ and $(A \wedge \neg A) \vee (B \wedge \neg B) \vdash_{\mathrm{CTSL}} (A \wedge \neg A) \wedge (B \wedge \neg B)$. Also, failure of *Explosion* qualifies CTSL as a *paraconsistent* logic. Notice, however, that $\{v \in \mathcal{V}_{\mathrm{CTSL}} \mid v(A) = \mathbf{t}\} = \varnothing$ for any formula A which is classically inconsistent. That is: CTSL does not satisfy any classical inconsistent formula, as we have anticipated in Section 2.

[10] Indeed, the fact equates with having $\mathrm{CTSL} = \{(\Gamma, B) \mid \Gamma \vdash_{\mathbf{CL}} B$ and $\gamma(B) \leq \gamma(\Gamma)$ for every γ for every topic assignment γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle\}$.

Finally, theories that are deductively closed under CTSL satisfy the three motivations by Beall. Indeed, we have $\gamma(T) = \oplus_{A \in T}\gamma(A)$ for every theory T, and in particular, for every theory T such that $T = \{B \in \mathsf{Form} \mid T \vdash_{\mathrm{CTSL}} B\}$. Also, $\gamma(A \vee B) = \gamma(A \wedge B) = \gamma(A \supset B) = \gamma(A) \oplus \gamma(B)$ and $\gamma(\neg A) = \gamma(A)$ are secured by the very generalization of γ to arbitrary formulas. Under two reasonable assumption, we also have $\gamma(T) < \gamma(\mathsf{Form})$ for the theories T that are deductively closed under CTSL and we tend to use. The first assumption is that such theories are non-maximal (we do not usually theorize about everything, not even in our most speculative moments). The second assumption is that we focus on 'non-degenerate' topic assignment functions[11] γ such that $\gamma(T) < \gamma(T')$ for every theory $T, T' \in 2^{\mathsf{Form}}$ such that $\mathsf{at}(T) \subset \mathsf{at}(T')$. Under this assumption, of course, we will have that $\gamma(T) < \gamma(\mathsf{Form})$ for the theories T that are deductively closed under CTSL and we tend to use (indeed, these will not be maximal, to the effect that $\mathsf{at}(T) \subset \mathsf{Atoms}$). Also, the 'non-degenerate' γs secure that no non-maximal CTSL-closed inconsistent theories will be a 'theory of everything', in line with the objection to the compatibility argument in Section 2.

3.3. A brief sum-up of this section

Summing up the content of this section: a slight adjustment of the topic-algebraic framework by Berto (2019a,b) helps us provide a rigorous formal treatment of topics, give a formal definition of a topic-sensitive logic, and semantically define the logic CTSL (for 'classical topic-sensitive logic'). The consequence relation determining its valid inference crucially adds a topic containment condition to the usual condition of truth preservation from premises to conclusion. The logic fails *Explosion* and fits motivations 1–3 by Beall (2016) better than **WKL** can.

4. Topic-sensitive logics and containment logics

In this section, I establish a result that connects *topic-sensitive logics* and the tradition of *containment logics* originating by Parry (1932, 1968) and thoroughly investigated by Ferguson (2015). In particular, the result shows that all topic-sensitive logics are containment logics of a particular kind (the ones satisfying the $\vdash$-proscriptive principle by Ferguson

[11] Not a great label, but nothing pithier springs to mind.

(2015)), and vice versa, to the effect that CTSL turns to be a containment logic.

Containment logics satisfy one of two versions of the *proscriptive principle* (as Parry calls it). One version of the principle — which is the main focus of Ferguson (2015) and of the present section — is:

$$\text{if } \Gamma \vdash B, \text{ then } \mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)$$

which Ferguson calls 'the $\vdash$-proscriptive principle'. It states, informally, that any valid inference from Γ to B must satisfy the requirement $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)$ — call this condition 'atomic containment'. Another version of the principle — which Ferguson calls 'the $\rightarrow$-proscriptive principle' — applies to valid conditionals, and it imposes that, if $\vdash \bigwedge_{A\in\Gamma} A \rightarrow B$,[12] then $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)$. The two versions of the principle give rise to two distinct families of containment logics. I call '$\vdash$-containment logics' those satisfying the $\vdash$-proscriptive principle, and '$\rightarrow$-containment logics' those satisfying the $\rightarrow$-proscriptive principle. In this paper, though, I deal just with the former.

Theorem 4.1 below establishes the equivalence between the topic containment condition which is key in Definition 2.1 and what I have just called 'atomic containment':

Theorem 4.1 (Topic containment and atomic containment). *Given any standard propositional language, the following statements are equivalent:*

1. $\gamma(B) \leq \gamma(\Gamma)$ *for every topic assignment* γ *and topic algebra* $\langle \mathcal{K}, \oplus, \leq \rangle$*;*
2. $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)$.

Proof. The direction from 2 to 1 is immediate and follows from the generalization of γ to sets of formulas. From this, we have that $\gamma(\Gamma) = \oplus_{p\in\mathsf{at}(\Gamma)}\gamma(p)$, which implies that, for every $\gamma : \mathsf{Atoms} \rightarrow \mathcal{K}$, $\gamma(B) \oplus \gamma(\Gamma) = \gamma(\Gamma)$ for every B, Γ such that $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)$. From this and $c \leq c' \Leftrightarrow c \oplus c' = c'$, we conclude $\gamma(B) \leq \gamma(\Gamma)$ for every $\gamma\colon \mathsf{Atoms} \rightarrow \mathcal{K}$ and $B \in \mathsf{Form}, \Gamma \in 2^{\mathsf{Form}}$ such that $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)$. As for the direction from 1 to 2, I prove it by contraposition. Suppose that $\mathsf{at}(B) \nsubseteq \mathsf{at}(\Gamma)$ and take the set $\mathsf{at}(B) \setminus \mathsf{at}(\Gamma)$ of those atomic sentences which are in B, but not in Γ. There can then be a function γ such that $\gamma(p) = c$ for some $p \in \mathsf{at}(B)\setminus\mathsf{at}(\Gamma)$ and some c such that $c \neq \gamma(q)$ for every $q \in \mathsf{at}(\Gamma)$. Together

[12] It is implicitly assumed that the set Γ in question contains finitely many formulas.

with $\gamma(\Gamma) = \oplus_{p\in\mathsf{at}(\Gamma)}\gamma(p)$, this implies $\gamma(B) \oplus \gamma(\Gamma) \neq \gamma(\Gamma)$. From this and $\gamma(B) \leq \gamma(\Gamma') \Leftrightarrow \gamma(B) \oplus \gamma(\Gamma) = \gamma(\Gamma)$, we get $\gamma(B) \nleq \gamma(\Gamma)$. Therefore, if $\mathsf{at}(B) \nsubseteq \mathsf{at}(\Gamma)$, then there are at least a topic assignment function γ and a topic algebra $\langle \mathcal{K}, \oplus, \leq\rangle$ such that $\gamma(B) \nleq \gamma(\Gamma)$. □

An immediate consequence of the fact is, of course:

Corollary 4.1 (Topic-sensitive logics and $\vdash$-containment logics). *Given any logic $\boldsymbol{L}$, the following two statements are equivalent*:

1. *If $\Gamma \vdash_{\boldsymbol{L}} B$, then $\gamma(B) \leq \gamma(\Gamma)$ for every topic assignment γ and topic algebra $\langle \mathcal{K}, \oplus, \leq\rangle$.*
2. *If $\Gamma \vdash_{\boldsymbol{L}} B$, then $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)$.*

That is, a logic satisfies the topic containment condition from Definition 3.3 if and only if it satisfies the atomic containment condition from the $\vdash$-proscriptive principle. Hence, every $\vdash$-containment logic is a *topic-sensitive* logic in the sense specified by Definition 3.3, and vice versa. Thus, for instance extensional containment logics $\mathbf{S}_{\mathsf{fde}}$ and $\mathbf{S}^{\star}_{\mathsf{fde}}$, which are interpreted by Ciuni et al. (2018) as paraconsistent logics of meaninglessness, turn to be topic-sensitive logics. The import of Corollary 4.1, of course, goes beyond this particular case: it shows that the project of building and investigating topic-sensitive reasoning tools will output $\vdash$-containment logics, and that each $\vdash$-containment logic is, *per se*, a reasoning tool that serves the project in question.

4.1. Discussion of Theorem 4.1

Corollary 4.1 is the result that has immediate import for the present paper, and it follows from Theorem 4.1. The relevance of the latter goes far beyond the instrumental function it has in this paper, and I believe it is worth discussing it here.

Theorem 4.1 establishes the equivalence between conditions that have been deployed in two traditions in logic that have distinct and unrelated goals, and that have thus far run unconnected. One tradition is that of *Topic-Sensitive Intentional Modalities* due to (Berto, 2018, 2019a,b; Berto and Hawke, 2021) and given a systematic view in (Berto, 2022). Key to this project are dyadic (epistemic) operators $\Box^{C}A$ which are (variably) strict conditionals.[13] If a formula of the form $\Box^{C}A$ is valid, then it satisfies the topic containment condition, having the topic of A

[13] See (Lewis, 1973) for the notion of a *variable strict conditional.*

contained into that of C. The other tradition is that of matrix-based containment logics (see especially Ciuni et al., 2018; Ferguson, 2015). Theorem 4.1 reveals that these two traditions talk languages that are way closer than expected. Indeed, it implies that what the topic-algebraic framework by Berto (2019a,b) achieves is indeed a family of conditionals that must satisfy atomic containment in order to be valid. Also, it implies that what the matrix-based containment logics from (Ciuni et al., 2018; Ferguson, 2015) achieve is a family of consequence relations that, if reinterpreted in topic-theoretical terms, satisfy the topic containment condition. The true difference in the extent of their proposals lies in their different focuses (conditionals as opposed to consequence relations), not in the additional non-alethic conditions they impose or secure.

Theorem 4.1 also let us understand Berto's logics as *containment logics*. More precisely, Theorem 4.1 reveals the logics from the *Topic-Sensitive Intentional Modalities* projects to be $\rightarrow$-containment logics. Connections between the $\rightarrow$-containment logics and the logics of *Topic-Sensitive Intentional Modalities* are already hinted at by Ferguson (2023), and Fine (1986) presents a topic-algebraic semantics for a known $\rightarrow$-containment logic (Parry's **AI**),[14] but a systematic investigation of the connections between topic containment condition and atomic containment condition is unprecedented, and Theorem 4.1 does more than the two papers hint at: by establishing the equivalence of the two conditions, it secures that every $\rightarrow$-containment logic is *per se* a logic with topic-sensitive valid conditionals, and vice versa, and that every $\vdash$-containment logic is *per se* a topic-sensitive inference generator, and vice versa. This gives a flavor of the potential of Theorem 4.1 as a powerful *unification* tool for logics traditionally placed (or built) under different labels and with the most different purposes.

Finally, Theorem 4.1 opens interesting research methodologies: it tells us that whatever semantics will secure atomic containment (for valid inference or valid conditionals) will also secure topic containment (for valid inference and valid conditionals), and vice versa. Thus, one can transfer the results of any investigation of semantics of the first kind to suitably defined logics designed in topic-theoretical terms, and vice versa. All this reveals the importance of Theorem 4.1 as a bridge

[14] Fine focused on the notion of 'content' rather than that of 'topic', and we should talk about 'content-algebraic framework' in his case — in turn, whether contents coincide with topics is a matter that I will not discuss in this paper. Net of this, Fine's content algebra and Berto's topic algebra are basically the same.

between different (and possibly far or previously unrelated) traditions in philosophical logics.

4.2. CTSL and containment logics

An immediate consequence of Corollary 4.1 is that CTSL is a $\vdash$-containment logic. Also, Corollary 4.1 allows for a different yet equivalent characterization of the logic, that is:

FACT 4.1. *$\mathit{\Gamma} \vdash_{\mathrm{CTSL}} B$ iff $\mathit{\Gamma} \vdash_{\mathbf{CL}} B$ and* $\mathsf{at}(B) \subseteq \mathsf{at}(\mathit{\Gamma})$.

Which is the characterization highlighted in the Introduction. Fact 4.1 reveals that CTSL is the logic by Johnson (1977), Parks-Clifford (1989), Zinov'ev (1973), and that, in consequence of this, CTSL is the fde-fragment of the logic **AI** by Parry (1932, 1968), which plays a prominent role in the history of *containment logics* — for this, see (Ferguson, 2015; Fine, 1986), with the latter being the first to provide a semantics for **AI**.[15]

Finally, notice that Fact 3.2.3 and Fact 4.1 together imply that:

$$\text{if } \mathit{\Gamma} \vdash_{\mathbf{CL}} \bot, \text{ then } \mathit{\Gamma} \vdash_{\mathrm{CTSL}} B \Leftrightarrow \mathsf{at}(B) \subseteq \mathsf{at}(\mathit{\Gamma}).$$

That is: the explosive impact of inconsistent sets $\mathit{\Gamma}$ of formulas is restricted to those formulas whose atomic sentences cannot but be 'on-topic' w.r.t. $\mathit{\Gamma}$. Notice that the CTSL-valid inferences $A \wedge (\neg A \vee B) \vdash_{\mathrm{CTSL}} B$ and $(A \wedge \neg A) \vee (B \wedge \neg B) \vdash_{\mathrm{CTSL}} (A \wedge \neg A) \wedge (B \wedge \neg B)$ are all special cases of that.

4.3. Topic-sensitive logics, containment logics, and right variable inclusion logics

It is worth mentioning a family of logics which enjoys a wide overlap with containment logics and hence, indirectly, with topics-sensitive logics. This is the family of right variable inclusion logics which has been extensively investigated in a number of works including those by Bonzio et al. (2022), Paoli et al. (2021), Pra Baldi (2020). In order to be a right

[15] Notice that **AI** is a $\rightarrow$-containment logic, not a $\vdash$-containment logic. See (Ferguson, 2015) for this. Also, notice that the proposal by Fine (1986) already hints at a connection between the topic containment condition and the $\vdash$-proscriptive principle, although that connection is not actually explored in that paper.

variable inclusion logic, a logic $\boldsymbol{L}$ must satisfy a principle which is *weaker* than the $\vdash$-proscriptive principle, namely:

$$\text{If } \Gamma \vdash_{\boldsymbol{L}} B, \text{ then } \Gamma \vdash_{\boldsymbol{L}} \bot \text{ or } \mathsf{at}(B) \subseteq \mathsf{at}(\Gamma).$$

The above condition and Fact 1.1 together imply that **WKL** is a right variable inclusion logic. Also, the above conditions implies that every $\vdash$-containment logic is a right variable inclusion logic, while the converse does not hold, of course — **WKL** is again an example of this.[16] The fact that the label 'right variable inclusion' is traditionally associated to a family of logics wider than that of $\vdash$-containment logic is the reason why I refer to condition 'If $\Gamma \vdash B$, then $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)$' as to 'the $\vdash$-proscriptive principle', following Ferguson, rather than as to 'right variable inclusion requirement'.

4.4. A brief sum-up of this section

Summing up the content of this section: Theorem 4.1 and Corollary 4.1 connect $\vdash$-containment logics and topic-sensitive logics. An immediate consequence of Corollary 4.1 is that CTSL is a containment logic. More precisely, CTSL is a logic that has appeared, though not with the topic-theoretical semantics presented in this paper, in (Johnson, 1977; Parks-Clifford, 1989; Zinov'ev, 1973). Also, Theorem 4.1 proves to be of great interest in connecting different traditions in philosophical logic, with a great potential as a unification tool. Just to make an example: it gives us an immediate understanding of the logics in the *Topic-Sensitive Intentional Modalities* project by Berto as $\rightarrow$-containment logics.

5. Again on the topic-theoretical interpretation of Weak Kleene Logic

In Section 2 I have argued that motivations 1–3 are better met by a logic different from **WKL**, and in Section 3 I have presented such a logic. **WKL**'s failure in meeting Beall's motivations, however, does not imply

[16] We refer the reader to (Bonzio et al., 2021, 2022) also for 'left variable inclusion logics'. In turn, these are related to a family of systems that somehow reverse the atomic containment required for a system to be a containment logic, and that have been investigated by Ciuni et al. (2018).

that it is impossible to provide a topic-theoretical interpretation of **WKL** along lines that are, basically, those of (Beall, 2016, p. 140):

> [...] read the value $\mathbf{t}$ not simply as true but rather as true and on-topic, and similarly $\mathbf{f}$ as false and on-topic. Finally, read the third value $\mathbf{n}$ as off-topic. (Beall, 2016, p. 140)

Here, I use the topic-algebraic framework from Section 3 in order to provide a rigorous formal rendering of this interpretation. In order to do so, I need first to introduce the auxiliary notion $T_v = \{A \in \mathsf{Form} \mid v(A) = \mathbf{t}\}$ of *the theory built on valuation* v. Notice that the identity and formal properties of v crucially depends on the family of valuations to which v belongs. For instance, if v is classical ($v \in \mathcal{V}_{\mathbf{CL}}$), then T_v will be *maximal* — that is, $A \in T_v$ or $\neg A \in T_v$ for every $A \in \mathsf{Form}$. If $v \in \mathcal{V}_{\mathbf{WKL}}$, by contrast, this needs not be the case — in both cases, however, T_v will meet the disjunctive property — if $A \vee B \in T_v$, then either $A \in T_v$ or $B \in T_v$, and of course they also have other properties in common. We get a topic-theoretical interpretation of **WKL** by taking the semantics of CTSL and exchanging the definition of CTSL-consequence with:

$$\Gamma \vdash B \iff \begin{array}{l} \text{for all } v \in \mathcal{V}_{\mathbf{CL}}\text{, topic assignment function } \gamma \text{ and topic} \\ \text{algebra } \langle \mathcal{K}, \oplus, \leq \rangle\text{: if } \gamma(\Gamma) \leq \gamma(T_v) \text{ and } v[\Gamma] \subseteq \{\mathbf{t}\}, \\ \text{then } \gamma(B) \leq \gamma(T_v) \text{ and } v(B) = \mathbf{t}. \end{array}$$

Clearly, this definition fails the requirements for topic sensitivity when it comes to classically inconsistent premises, exactly as happens with **WKL**. Indeed, if $\Gamma \vdash_{\mathbf{CL}} \bot$, then condition '$\gamma(\Gamma) \leq \gamma(T_v)$ and $v[\Gamma] \subseteq \{\mathbf{t}\}$' can be just *vacuously satisfied* (remember that our valuations are classical), with the usual effects in establishing the consequences of inconsistent premises.

I now prove that the consequence relation $\vdash$ thus defined is actually $\vdash_{\mathbf{WKL}}$. I do this by establishing:

Theorem 5.1. *The following two conditions are equivalent*:

1. *For any* $v \in \mathcal{V}_{\mathbf{CL}}$*, topic assignment function* γ *and topic algebra* $\langle \mathcal{K}, \oplus, \leq \rangle$*: if* $\gamma(\Gamma) \leq \gamma(T_v)$ *and* $v[\Gamma] \subseteq \{\mathbf{t}\}$*, then* $\gamma(B) \leq \gamma(T_v)$ *and* $v(B) = \mathbf{t}$.
2. *Either* $\Gamma \vdash_{\mathbf{CL}} \bot$*, or both* $\Gamma \vdash_{\mathbf{CL}} B$ *and* $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)$.

Proof. As for the direction from 1 to 2: assume that for any $v \in \mathcal{V}_{\mathbf{CL}}$, topic assignment function γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$: if $\gamma(\Gamma) \leq \gamma(T_v)$

and $v[\Gamma] \subseteq \{\mathbf{t}\}$, then $\gamma(B) \leq \gamma(T_v)$ and $v(B) = \mathbf{t}$. There are two possible cases:
Case 1: $v[\Gamma] \cap \{\mathbf{f}\} \neq \emptyset$ for every $v \in \mathcal{V}_{\mathbf{WKL}}$.
Case 2: $v[\Gamma] \subseteq \{\mathbf{t}\}$ for some $v \in \mathcal{V}_{\mathbf{WKL}}$.

As for case 1, it implies that Γ has no classical model — since all classical valuations are **WKL**-valuations, and hence that $\Gamma \vdash_{\mathbf{CL}} \bot$. Obviously, if Γ has no classical model, then condition 1 implies condition 2. As for case 2, consider our initial assumption. Since the set of classical valuations satisfying Γ is nonempty, we have $\Gamma \in 2^{T_v}$ for every $v \in \mathcal{V}_{\mathbf{CL}}$ such that $v[\Gamma] \subseteq \{\mathbf{t}\}$ and, hence, $\gamma(\Gamma) \leq \gamma(T_v)$ for every $v \in \mathcal{V}_{\mathbf{CL}}$ such that $v[\Gamma] \subseteq \{\mathbf{t}\}$ *and* topic assignment function γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$. From this and our initial assumption, we have that for every $v \in \mathcal{V}_{\mathbf{CL}}$, topic assignment function γ, and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$: if $v[\Gamma] \subseteq \{\mathbf{t}\}$, then $v(B) = \mathbf{t}$ and $\gamma(B) \leq \gamma(T_v)$. Thus, I have $\gamma(B) \leq \gamma(T_v)$, for every $v \in \mathcal{V}_{\mathbf{CL}}$ such that $v[\Gamma] \subseteq \{\mathbf{t}\}$, topic assignment function γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$. In turn, this implies that $\gamma(B) \leq \gamma(\Gamma)$ for every topic assignment function γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$. Indeed, suppose that: (a) $\gamma(B) \nleq \gamma(\Gamma)$ for some topic assignment function γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$ (for a classical valuation v satisfying Γ), and consider that there are at least a topic assignment function γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$ such that: (b) for every $p \in \mathsf{at}(T_v) \setminus \mathsf{at}(\Gamma)$, $\gamma(p) = \gamma(q)$ for some $q \in \mathsf{at}(\Gamma)$ — for our given choice of a classical v satisfying Γ. If (a) holds, then there also are at least a topic assignment function γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$ such that they satisfy both (b) (for our choice of v) and (c): for some $r \in \mathsf{at}(B)$ and every $q \in \mathsf{at}(\Gamma)$, $\gamma(r) \neq \gamma(q)$. (c) implies that $\gamma(B) \nleq \gamma(\Gamma)$, and ($b$) implies that $\gamma(T_v) = \gamma(\Gamma)$. Together, they imply $\gamma(B) \nleq \gamma(T_v)$ for some topic assignment function γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$ and valuation v satisfying Γ, thus contradicting an immediate consequence of our initial assumption and, hence, the assumption itself. Thus, I can dismiss (a) and conclude that $\gamma(B) \leq \gamma(\Gamma)$ for every topic assignment function γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$. By Theorem 4.1, this implies that, if condition 1 holds, then $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)$. But of course, condition 1 also implies that $\Gamma \vdash_{\mathbf{CL}} B$. Hence, under case 1, condition 1 implies $\Gamma \vdash_{\mathbf{CL}} \bot$, and under case 2, condition 1 implies $\Gamma \vdash_{\mathbf{CL}} B$ and $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)$. Since case 1 and case 2 together exhaust all the possibilities, I can conclude that condition 1 implies that *either* $\Gamma \vdash_{\mathbf{CL}} \bot$, *or* $\Gamma \vdash_{\mathbf{CL}} B$ and $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)$. That is, condition 1 implies condition 2.

As for the direction from 2 to 1, assume that $\Gamma \vdash_{\mathbf{CL}} \bot$. This implies that the condition 'for every $v \in \mathcal{V}_{\mathbf{CL}}$: if $v[\Gamma] \subseteq \{\mathbf{t}\}$, then $v(B) = \mathbf{t}$' is *vac-*

uously satisfied. In consequence, the same holds for condition 1. Thus, $\Gamma \vdash_{\mathbf{CL}} \bot$ implies condition 1. Now assume $\Gamma \vdash_{\mathbf{CL}} B$ and $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)$. If $\Gamma \vdash_{\mathbf{CL}} \bot$, then the implication of condition 1 is secured by the argument above. If $\Gamma \nvdash_{\mathbf{CL}} \bot$, then there is at least a valuation $v \in \mathcal{V}_{\mathbf{CL}}$ such that $v[\Gamma] \subseteq \{\mathbf{t}\}$. Together with Theorem 4.1, this implies (a) $\Gamma \subseteq T_v$ for any such v, and (b) $\gamma(B) \leq \gamma(\Gamma)$ for every topic assignment function γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$. From (a), we get (c): $\gamma(\Gamma) \leq \gamma(T_v)$ for every topic assignment function γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$. From (b) and (c), we conclude $\gamma(B) \leq \gamma(T_v)$ for every topic assignment function γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$. This in turn implies that, if $\Gamma \vdash_{\mathbf{CL}} B$ and $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)$, then it is the case that, for every $v \in \mathcal{V}_{\mathbf{CL}}$, topic assignment function γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$: if $\gamma(\Gamma) \leq \gamma(T_v)$ and $v[\Gamma] \subseteq \{\mathbf{t}\}$, then $\gamma(B) \leq \gamma(T_v)$ and $v(B) = \mathbf{t}$. But this equates with saying that $[\Gamma \vdash_{\mathbf{CL}} B$ and $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)]$ implies condition 1. Since $\Gamma \vdash_{\mathbf{CL}} \bot$ and $[\Gamma \vdash_{\mathbf{CL}} B$ and $\mathsf{at}(B) \subseteq \mathsf{at}(\Gamma)]$ separately imply condition 1, we conclude that condition 2 implies condition 1. □

Theorem 5.1 and Fact 1.1 together imply that the consequence relation $\vdash$ just defined in this section is nothing but **WKL**'s consequence:

Corollary 5.1. *$\Gamma \vdash_{\mathbf{WKL}} B$ iff for every $v \in \mathcal{V}_{\mathbf{CL}}$, $T \in 2^{\mathsf{Form}}$, topic assignment function γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$: if $\gamma(\Gamma) \leq \gamma(T)$ and $v[\Gamma] \subseteq \{\mathbf{t}\}$, then $\gamma(B) \leq \gamma(T)$ and $v(B) = \mathbf{t}$.*

This proves that a rigorous interpretation of **WKL** along topic-theoretical lines is possible. Indeed, **WKL** is to be seen, in Beall's words, as 'a logic that concerns not simply truth-preservation, but *truth-and-topic preservation* (Beall, 2016, p. 140). Corollary 5.1 implies that **WKL** does exactly this — consider that the 'truth-and-topic preservation' of inconsistent premises is trivial, because no inconsistent premises can be *true and about a given topic*, albeit it can be *about a given topic*. Together with the remarks from Section 2, Corollary 5.1 helps us get a more articulated view on Beall's proposal. In particular, Beall's interpretation is perfectly legitimate and well thought off; just, even under this topic-theoretical reading, **WKL** does not prove a suitable choice against the background of the motivations presented. **WKL** is no option when looking for a topic-sensitive logic as per Definition 3.3.

Finally, the fact below shows that the topic-theoretical interpretation I have envisaged here to **WKL** actually matches the reading of **WKL**'s truth values offered by (Beall, 2016, p. 140). Indeed:

Fact 5.1 (Truth values for **WKL** and topic containment).
For any $v \in \mathcal{V}_{\mathbf{WKL}}$ *and* $A \in \mathsf{Form}$, *we have*:

1. *If* $v(A) = \mathbf{t}$, *then there is a* $v' \in \mathcal{V}_{\mathbf{CL}}$ *such that* $v'(A) = \mathbf{t}$ *and* $\gamma(A) \leq \gamma(T_v)$ *for any topic assignment function* γ *and topic algebra* $\langle \mathcal{K}, \oplus, \leq \rangle$.
2. *If* $v(A) = \mathbf{f}$, *then there is a* $v' \in \mathcal{V}_{\mathbf{CL}}$ *such that* $v'(A) = \mathbf{f}$ *and* $\gamma(A) \leq \gamma(T_v)$ *for any topic assignment function* γ *and topic algebra* $\langle \mathcal{K}, \oplus, \leq \rangle$.
3. *If* $v(A) = \mathbf{n}$, *then there is a* $v' \in \mathcal{V}_{\mathbf{CL}}$ *such that* $v'(A) \in \{\mathbf{t}, \mathbf{f}\}$ *and* $\gamma(A) \nleq \gamma(T_v)$ *for* some *topic assignment function* γ *and topic algebra* $\langle \mathcal{K}, \oplus, \leq \rangle$.

Proof. We start by proving 1. Assume that $v(A) = \mathbf{t}$. Given that $\mathbf{n}$ is a contaminating value in the sense of Definition 1.1, we have that $v(p) \neq \mathbf{n}$ for every $p \in \mathsf{at}(A)$. This implies that $v_{|\mathsf{at}(A)}$ can be extended to a valuation $v' \in \mathcal{V}_{\mathbf{CL}}$ such that $v'(p) = v(p)$ if $p \in \mathsf{at}(A)$ and $v'(q) \in \{\mathbf{t}, \mathbf{f}\}$ if $q \notin \mathsf{at}(A)$. Of course, we have $v'(A) = v(A) = \mathbf{t}$. This implies that, if there is a $v \in \mathcal{V}_{\mathbf{WKL}}$ s.t. $v(A) = \mathbf{t}$, then there is a $v \in \mathcal{V}_{\mathbf{CL}}$ s.t. $v(A) = \mathbf{t}$. Given that $T_v = \{A \in \mathsf{Form} \mid v(A) = \mathbf{t}\}$, we have that $\mathsf{at}(A) \subseteq \mathsf{at}(T_v)$. From this and Theorem 4.1, it follows that $\gamma(A) \leq \gamma(T_v)$ for every topic assignment function γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$.

As for 2, it follows from 1 and the fact that $V(A) = \mathbf{f} \Leftrightarrow V(\neg A) = \mathbf{t}$ for every $v \in \mathcal{V}_{\mathbf{WKL}} \cup \mathcal{V}_{\mathbf{CL}}$ and $\gamma(A) = \gamma(\neg A)$ for every topic assignment function γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$.

As for 3, assume $v \in \mathcal{V}_{\mathbf{WKL}}$ s.t. $v(A) = \mathbf{n}$. This implies that there is at least a $p \in \mathsf{at}(A)$ s.t. $v(p) = \mathbf{n}$. We can turn v into a classical valuation v' by imposing that $v'(p) = v(p)$ for every $p \in \mathsf{Atoms}$ s.t. $v(p) \in \{\mathbf{t}, \mathbf{f}\}$, and $v'(q) \in \{\mathbf{t}, \mathbf{f}\}$ for every $q \in \mathsf{Atoms}$ s.t. $v(q) = \mathbf{n}$. Clearly, I have $v'(A) \in \{\mathbf{t}, \mathbf{f}\}$. Also, since $\mathbf{n}$ is a contaminating value, I have $\mathsf{at}(A) \nsubseteq \mathsf{at}(T_v)$. Otherwise, by our initial assumption that $v(A) = \mathbf{n}$ and the consequent fact that $v(p) = \mathbf{n}$ for some $p \in \mathsf{at}(A)$, I would have $v[T_v] \nsubseteq \{\mathbf{t}\}$, thus contradicting the definition of T_v. Hence, $\mathsf{at}(A) \nsubseteq \mathsf{at}(T_v)$. From this and Theorem 4.1, the latter equates with having $\gamma(A) \nleq \gamma(T_v)$ for some topic assignment function γ and topic algebra $\langle \mathcal{K}, \oplus, \leq \rangle$. □

5.1. A brief sum-up of this section

Summing up the content of this section: although **WKL** is not a topic-sensitive logic — thus proving not to be the best candidate to fit motivations 1–3 from Beall (2016) — it is possible to provide it with a topic-theoretical interpretation by the topic-algebraic framework introduced in Section 3. Also, such an interpretation fits Beall's reading of **WKL**'s truth values from (Beall, 2016, p. 140).

6. Discussion and perspectives for future research

In spotting a flaw in Beall's motivations for a topic-sensitive reading of **WKL**, this paper makes first steps toward a systematic investigation and elaboration of *topic-sensitive logics*. In this section, I briefly discuss a philosophical problem which is connected to the topic-algebraic semantics presented in Section 3 and to Theorem 4.1. Also, I briefly discuss one research perspective that naturally arises from this paper. I plan to devote future papers to both the philosophical problem and the open research problem.

***The filter objection, containment logics, and* CTSL**. Criticisms of the containment logic project from (Parry, 1932, 1968) has been advanced by different sources (see Anderson and Belnap, 1975; Kielkopf, 1975; Routley et al., 1982). The most far-reaching is, likely, the so-called 'filter objection' by Routley et al. (1982). Routley and colleagues actually present many other criticisms against Parry systems (especially against $\rightarrow$-containment logics), but while these are essentially connected to Parry systems' ability to provide paradox-free conditionals,[17] the 'filter objection' is independent from the possible aims or applications of Parry systems. This is how the objection goes:

> The Proscriptive Principle is an attempt to reduce to syntactical form an essentially semantical matter, interrelation of concepts, and suffers from most of the difficulties of such attempts [...] The semantical analysis of Fine's and Dunn's systems reveals that they are effectively strict and material systems, with a variable or contents inclusion requirement thrown on top. The oddities emerging help to show that the trouble with strict and material systems, is not merely, but only incidentally, their variable-sharing failure. The real troubles go deeper and are not repaired simply by throwing on a variable inclusion filter.
>
> (Routley et al., 1982, p. 100)

This is not the place to go through the objection in detail, but there are two points we can take home. First, the proscriptive principle (Routley

[17] By 'paradox-free' here I mean 'free from the paradoxes of material or strict conditionals'.

and colleagues actually focus on the $\rightarrow$-proscriptive principle) is flawed from the start, since it mistakes a semantical problem ('Which insightful interpretation machinery should we envisage for a relevance-sensitive conditional?') for a syntactical one ('We need to impose an atomic containment condition in order to get a suitable conditional'). Second, the semantics for ($\rightarrow$-)containment logics do nothing but *confirming* that: they integrate an otherwise independent insight into the satisfaction of a formula with an additional condition that 'cuts out the deadwood', so to speak. This 'double-barreled analysis' amounts to nothing more than throwing on a syntactic filter on top of a notion of satisfaction that has no intrinsic connection with the filter itself. Summing this up: the semantics proposed for ($\rightarrow$-)containment logics are *ad hoc* and fail to carry any real conceptual insight on why a number of conditional formulas should not be valid.

To be sure, not all containment logics fall victim to the filter objection. Ciuni et al. (2018) present a number of extensional $\vdash$-containment logics whose semantics is not double-barreled, is purely truth-functional and is doubtlessly not *ad hoc*. From this and Theorem 4.1, we have that there are topic-sensitive logics that can be reinterpreted without any recourse to a two-component semantics. However, in light of the results from (Paoli et al., 2021), it is likely that the same does not apply to CTSL, which does not fall in the lot of logics covered by the results from (Ciuni et al., 2018). The reason why this leaves CTSL exposed to the filter objection is clear: Definition 3.3 shapes a notion of logical consequence that is, indeed, double-barreled — unsurprisingly: a two-component analysis is what comes, inevitably, with the topic-algebraic semantics dating back (ultimately) to (Fine, 1986), no matter whether it is applied to conditionals or to the consequence relation.

The questions arising in relation to the filter objection are then: '*Is it possible to provide an insightful alternative semantics for* CTSL*, which does not implement in turn a double-barreled analysis?*' and '*In case the answer is for the negative, is there a way to justify our use of* CTSL *as the limit-case of some reasoning procedure that aims as being, beside topic sensitive, as classical as possible?*'. I wish to answer the two questions in future papers.

Topic-sensitive conditionals. In this paper I have focused on the issue of securing topic-sensitive valid inferences — and then, more generally, topic-sensitive deductive reasoning. Current research on topic pertinence and logic also presents logics that are *not* topic sensitive, according to Definition 3.3, and yet are endowed with topic-sensitive conditionals. Prominent examples of this are the *variably strict conditionals* from the *Topic-Sensitive Intentional Modalities* project by Berto, which I have already mentioned a number of times, and the strict conditional by Fine (1986), which qualifies as a topic-sensitive conditional if we reinterpret Fine's content algebra as a topic algebra. Theorem 4.1 tells us that every logic with a topic-sensitive conditional will be a $\to$-containment logic, and that every $\to$-containment logic is *de facto* a topic-sensitive logic. The result seals a common fate for the two traditions, and yet this is of little comfort. Indeed, none of the semantics provided for $\to$-containment logics have struck researchers as a conceptually insightful interpretation tool, and one of them (the topic-algebraic one) is beset by the filter objection. Then, $\to$-containment logics and topic-sensitive conditionals apparently face the fate of either *lacking real semantical insight*, or being shamelessly *ad hoc* (or both). It follows that, in order to get $\to$-containment logics and topic-sensitive conditionals back on their feet, we must find a semantics for them which looks insightful and genuine.

The question arising in relation to the filter objection are then: '*What semantics, if any, can make conditionals topic sensitive while avoiding a double-barreled analysis and the filter objection?*' which in turn equates with asking '*Is it possible to find a semantics for topic-sensitive conditionals that comes with a genuine conceptual insight?*'. I plan to deal with this open problem in a dedicated paper.

7. Conclusions

In this paper, I have presented results and achievements that are relevant in relation to the growing literature in philosophical logic on the role played by topic pertinence in reasoning, propositional attitudes, or the theory of meaning (see, e.g., Beall, 2016; Berto, 2018, 2019a,b, 2022; Berto and Hawke, 2021; Ferguson, 2023; Hawke, 2018; Yablo, 2014). I have spotted and discussed a drawback in the topic-theoretical interpretation of **WKL** provided by Beall (2016) — namely: the fact that **WKL** validates *Explosion* flies in the face of the claim that the logic is topic sensitive, that is, able to shield our inferences from concluding sentences that are off-topic w.r.t. the premises. Starting from this, I have proposed an informal definition of a topic-sensitive logic (Definition 2.1), I have deployed a simple extension of the topic-algebraic framework by Berto (2019a,b) in order to provide a rigorous formal treatment of topics and a formal definition of a topic-sensitive logic (Definition 3.3), and I have applied the framework in order to define semantically the logic which I label 'classical topic-sensitive logic' (CTSL). The logic is the greatest topic-sensitive fragment of classical logic **CL**, fits Beall's motivations, and turns to be, under an original and unprecedented semantical interpretation, the $\vdash$-containment logic by Johnson (1977), Parks-Clifford (1989), Zinov'ev (1973). Theorem 4.1 proves that the topic containment condition defining topic-sensitive logics is equivalent with the atomic containment condition from the $\vdash$-proscriptive principle from containment logic, and Corollary 4.1 draws the immediate conclusion that every topic-sensitive logic is a $\vdash$-containment logic, and vice versa. The latter result is of interest in that it bridges the topic-sensitive logics we have defined here and a family of containment logics. Just to give an example: extensional containment logics $\mathbf{S}_{\mathsf{fde}}$ and $\mathbf{S}^{\star}_{\mathsf{fde}}$, which are interpreted by Ciuni et al. (2018) as paraconsistent logics of meaninglessness, are revealed to be topic-sensitive logics. The former result opens interesting connections between traditions of philosophical logic that have thus far developed independently, such as the logics of *Topic-Sensitive Intentional Modalities* by Berto (2018, 2019a,b, 2022), Berto and Hawke (2021) and the $\rightarrow$-containment logics by Parry (1932, 1968). The potential of the result as a unification tool of previously unrelated systems, however, is not confined to this particular case. A topic-theoretical interpretation of **WKL** is given by the topic-algebraic framework presented in the paper, thus giving a concrete grasp of the fact that failure to meet the criterion for topic sensitivity does not imply the impossibility of applying a topic-theoretical interpretation. Open problems and research perspectives are briefly discussed and identified as topics for future papers.

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