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Strong Kleene Logics as a Tool for Modelling Formal Epistemic Norms

Abstract. In this paper, we present two ways of modelling every epistemic formal conditional commitment that involves (at most) three key epistemic attitudes: acceptance, rejection and neither acceptance nor rejection. The first one consists of adopting the plurality of every mixed Strong Kleene logic (along with an epistemic reading of the truth-values), and the second one involves the use of a unified system of six-sided inferences, named **6SK**, that recovers the validities of each mixed Strong Kleene logic. We also introduce a sequent calculus that is sound and complete with respect to both approaches. We compare both accounts, and finally, we suggest that the plurality of Strong Kleene logic as well as the general framework **6SK** are linked to formal epistemic norms *via* bridge principles.

Keywords: epistemic attitudes; formal commitments; bridge principles; three-valued logics; substructural logics

1. Formal conditional commitment (for three key epistemic attitudes)

It is common — and reasonable — to think that, in a valid inference, or at least in sound ones, premises provide support for the conclusions (one of them, at a minimum). Or, to put it more clearly, that *accepting* the premises provides reasons to *accept* the conclusions. Nevertheless, this way of understanding validity does not represent all the different kinds of epistemic support that certain propositions can receive in various normative situations faced in everyday life (and also in philosophical contexts). For example, sometimes the *rejection* of certain sentences provides good

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reasons to *accept* others. Or the fact that one *does not accept nor reject* a set of sentences warrants *accepting* or *rejecting* some conclusion.[1]

Rationally accepting and rejecting propositions are two kinds of attitudes that can be deemed as *epistemic*, at least in the minimal sense that we regard them to be somewhat justified, and even based on reasons.[2] There are certain norms that we seem to follow in these normative situations. Some of those norms are purely *formal*, in the sense that they are not about any particular set of propositions, sentences or judgements. For this reason, we shall call them *formal epistemic norms*. In the following, we provide two ways in which to account for all possible formal epistemic norms that refer exclusively to the epistemic attitudes of accepting, rejecting and neither accepting nor rejecting. The first way to account for these kinds of formal commitments involves the use of the plurality of all mixed Strong Kleene logics (accompanied by an appropriate epistemic interpretation of the three truth-values). Thus, it relies on a certain kind of logical pluralism. The second account uses a single logic to accommodate each of these formal epistemic norms. One of the most peculiar things about it is the non-standard way of understanding the notion of inference. *Inferences* will not be ordered pairs of sets of sentences, but ordered pairs of triples of sets of sentences.

The rest of this paper is organized as follows. In Section 2, we introduce the family of all mixed Strong Kleene logics. In Section 3, we present an epistemic reading available for each of these three-valued systems. In Section 4, we propose a unified framework in which one can represent the logics previously introduced, and we present a proof system for it, `SC6SK`. This is a generalization of the result given in [15] to every Strong Kleene mixed logic. In Section 5, we suggest two bridge principle schemas that explain how these logics give rise to norms. In Section 6,

[1] We should better stop now to explain what we mean by *neither accepting nor rejecting*. This is less a name for an specific attitude than a label for a variety of attitudes. Different ways of "neither accepting nor rejecting" include being uncertain (that a certain event in fact holds), suspending belief, or having as much reasons to accept than to reject a certain proposition. One can also take it as tolerantly accepting (that is, neither strictly accepting nor strictly rejecting), using the strict/tolerant terminology found in [7]. They are all species of the same *genus*, as they are all epistemic attitudes that cannot be interpreted neither as forms of accepting nor as ways of rejecting sentences, propositions, etc. This justifies using the same label group them as a third kind of epistemic position towards sentences, besides accepting and rejecting.

[2] When we talk about *accepting* or *rejecting* sentences and propositions in this article, we will understand the terms as *rationally accepting* or *rationally rejecting*.

we lay out a comparison of both accounts, and give arguments in favour of the second approach. And finally, in Section 7, we conclude with some final remarks.

Now we are ready to introduce the first way to account for formal epistemic norms that relate the epistemic attitudes of acceptance, rejection and (the different types of epistemic attitudes of) neither acceptance nor rejection: adopting the plurality of every mixed Strong Kleene logic.

2. First account: the plurality of mixed Strong Kleene logics

The first way to model formal epistemic commitments is through mixed three-valued Strong Kleene logics. Before saying how this can be done, we will start by defining what a (mixed) Strong Kleene logic is, and which are they.

2.1. Mixed Strong Kleene logics

In this paper we will focus on propositional mixed three-valued logics based on the Strong Kleene valuations, but there are other three-valued schemas one can choose to work with: sub- and super-valuations, Weak Kleene valuations, or the Łukasiewicz three-valued schema, are just a few cases worth mentioning.

A *logic* is usually defined as an algebra plus a consequence relation, also understood as a set of inferences or as a standard that determines a set of (valid) inferences. For the sake of simplicity, we will focus on propositional logics. Let $\mathcal{L}$ be a propositional language with the connectives $\wedge$, $\vee$ and $\neg$, of arities 2, 2, and 1 respectively, and intended as conjunction, disjunction and negation. Let Var be a countably infinite set of propositional variables, we denote by $\mathrm{FOR}_{\mathcal{L}}$ the absolutely free algebra of formulas of $\mathcal{L}$, with Var as its generating set.

So far, we have given an informal characterization of consequence relations. Though formally they can be defined in different ways, we focus on what Chemla, Egré and Spector [5] call *mixed* consequence relations.[3]

[3] Though mixed consequence relations were available long before, most notably, in [13, 9]. Further developments on mixed consequence can be found in Chemla and Egré's [4, 3].

Definition 2.1. A *consequence relation* for $\mathcal{L}$ is a subset of $\wp(\text{FOR}_{\mathcal{L}}) \times \wp(\text{FOR}_{\mathcal{L}})$. A consequence relation for $\mathcal{L}$ is *mixed* if and only if for every inference $\langle \Gamma, \Delta \rangle$, where Γ and Δ are sets of sentences, Δ follows from Γ if and only if for every valuation v, if $v(\gamma) \in \mathcal{D}^+$ for every $\gamma \in \Gamma$, then $v(\delta) \in \mathcal{D}^-$ for some $\delta \in \Delta$, for some pair of standards $\mathcal{D}^+$ and $\mathcal{D}^-$.

A *standard* is a set of truth values. And for any standard $\mathcal{D}^{+/-}$, valuation v and formula α, we say that $v(\alpha)$ *meets* or *satisfies* the standard $\mathcal{D}^{+/-}$ if and only if $v(\alpha) \in \mathcal{D}^{+/-}$. *Standards* for premises and conclusions can also be understood as specifying which are the values each formula might have in a *sound* argument or inference.

If $\mathcal{D}^+ \neq \mathcal{D}^-$, then the mixed consequence relation is *impure*; otherwise, it is *pure*. We will introduce the different kinds of three-valued logics that can be characterized through the Strong Kleene schema, i.e., where sentential connectives behave like this:

	$\neg$
1	0
½	½
0	1

$\wedge$	1	½	0
1	1	½	0
½	½	½	0
0	0	0	0

$\vee$	1	½	0
1	1	1	1
½	1	½	½
0	1	½	0

Moreover, $\rightarrow$ and $\leftrightarrow$ can be defined in the usual way.

Mixed Strong Kleene logics belong to one and only one of the following five categories: (1) pure logics, (2) disjoint logics, (3) p-logics, (4) q-logics and (5) overlapping logics. (1) and (2) have been already introduced by Chemla, Egré and Spector in [5] and by Pailos in [15], respectively. (3) and (4) are well-known kinds of substructural logics, though we understand them in a slightly unorthodox way regarding the original presentation given, correspondingly, by Frankowski [9] and Malinowski [13, 14]. Finally, (5) refers to logics whose consequence relation consists in a pair of standards such that neither of them is included in the other, though their intersection is not empty. These logics, as far as we know, have never been studied in the literature.

If the empty set is taken as a *bona fide* standard, there are 64 mixed three-valued logics based on the Strong Kleene schema — i.e., 64 different mixed Strong Kleene logics. If not, there are "just" 49 different mixed Strong Kleene logics. In the sequel, we will present all of them, not only to be exhaustive, but also for philosophical reasons: even though *de facto* some inferences involving certain epistemic attitudes are more prevalent than others in practice, we think all of these possible combinations are, in principle, equally legitimate. Hence why all are presented on equal footing. Finally, it is worth noticing that, since we are working with sets, we have Contraction and Exchange for free in all of these logics.

In order to improve readability and keep track of which logic is being discussed, we will rename the standards we will be talking about — and therefore the logics they help characterize. We will keep the substructural terminology, and use **s** and **t** for $\{1\}$ and $\{1, ½\}$, respectively.[4] We borrow the remaining terminology from [15]. We will use **n** for $\{½\}$ and the $\varnothing$ sign for the empty set. Finally, we will use an operation $\overline{x}$, that provides the complement of x. Here is a list of the new vocabulary introduced:

$$\mathbf{s} = \{1\} \qquad \overline{\mathbf{s}} = \{½, 0\} \qquad \mathbf{t} = \{1, ½\} \qquad \overline{\mathbf{t}} = \{0\}$$
$$\mathbf{n} = \{½\} \qquad \overline{\mathbf{n}} = \{1, 0\} \qquad \varnothing = \varnothing \qquad \overline{\varnothing} = \{1, ½, 0\}$$

Using these new abbreviations, we will introduce in the following subsections a way to characterize every mixed Strong Kleene logic that can be represented with a pair of these eight labels. While the first sign of the pair stands for the set $\mathcal{D}^+$ (i.e., the "premise standard"), the second sign represents the set $\mathcal{D}^-$ (i.e., the "conclusion standard").

2.2. Pure logics

There are eight different mixed pure logics based on the Strong Kleene schema — at least if we take the empty set and its complement as legitimate standards. (If we do not, then there are six of them.) The following two are the most well known three-valued pure logics:

- **ss**, also known as **K3**.
- **tt**, usually referred to as **LP**.

The logics **K3** and **LP** were originally introduced in [11, 1, 17] and have been widely studied in the literature. Instead, we shall focus on the other six lesser-known logics of this group. (For a more thorough presentation of all mixed pure Strong Kleene logics, we defer to [16].) Further, let $p, q \in \mathrm{Var}$ and $p \neq q$.

[4] The **s** and the **t** correspond, respectively, to the notions of being *strictly* and *tolerantly* satisfied. A valuation v *tolerantly* satisfies a formula φ if and only if $v(\varphi) \in \{1, ½\}$, and it *strictly* satisfies φ if and only if $v(\varphi) \in \{1\}$. These notions have been widely used in the literature about (non-transitive and non-reflexive) substructural logics, but were originally introduced by Cobreros et al. in [6].

The logic **nn** does not have tautologies (as every formula can receive a classical truth-value), and it does not validate every inference with a non-empty set of premises, either. For example, it does not validate $p \vDash q$. **nn** is a paraconsistent and a paracomplete logic, because neither Explosion (in the more standard form $p, \neg p \vDash q$) nor Excluded Middle (i.e., $\vDash p \vee \neg p$) are **nn**-valid. A valuation v such that $v(p) = ½$ and $v(q) \in \{1, 0\}$ is a counterexample to Explosion and a valuation v' such that $v'(p) \in \{1, 0\}$ is a counterexample to Excluded Middle. **nn** is not a sublogic of Classical Logic. For example, $p \vDash \neg p$ and $\neg p \vDash p$ are two valid inferences in **nn** that are not valid in Classical Logic. (This also makes it incomparable with **LP** and **K3**.)

The logic $\overline{\textbf{nn}}$ lacks valid inferences with an empty set of conclusions. Nevertheless, $\overline{\textbf{nn}}$ is not trivial. $\overline{\textbf{nn}}$ validates no formula — as the valuation that gives every formula the value ½ can witness. Since it has no tautologies, it invalidates Excluded Middle, making it a paracomplete logic. It is also not explosive — as p and $\neg p$ may receive classical values, but q might receive the non-classical value. Thus, $\overline{\textbf{nn}}$ is also paraconsistent. $p \vDash \neg p$ is an example of an inference valid in $\overline{\textbf{nn}}$ but invalid in Classical Logic, which also makes $\overline{\textbf{nn}}$ contra-classical.

The logic $\overline{\textbf{tt}}$ is such that every inference with a classical tautology as a premise turns out valid. This logic, though, has no tautologies — as the valuation v that gives the non-classical value to every propositional letter, and thus to every formula, invalidates every formula. It is not included neither in $\overline{\textbf{nn}}$ nor in **nn**, since $p \vee \neg p \vDash q$ is valid in this logic but invalid in the other two. And it does not have these as sublogics neither, as $p \vDash \neg p$ is invalid in $\overline{\textbf{tt}}$, but valid in those logics, while $p \vDash \neg p$ is invalid in $\overline{\textbf{tt}}$ but is valid in $\overline{\textbf{nn}}$ and **nn**. $\overline{\textbf{tt}}$ is also paracomplete, as it invalidates Excluded Middle (because it has no tautologies). $\overline{\textbf{tt}}$ is also contra-classical. For instance, $p \vDash p \wedge \neg p$ is valid in it, but not in Classical Logic.

2.3. Disjoint logics

Disjoint logics are impure logics in which $\mathcal{D}^+ \cap \mathcal{D}^- = \varnothing$, i.e., the standard for premises does not share any truth-value with the standard for conclusions. There are twenty six different three-valued disjoint logics based on the Strong Kleene schema — i.e., twenty six different *Strong Kleene disjoint logics*.

We will mention some facts about three of these logics to illustrate how they behave. For a more comprehensive picture of Strong Kleene disjoint logics, see [15].[5]

The logic **sn** is such that any inference with a classical contradiction as a premise is valid, e.g., $p \wedge \neg p \vDash q$. In fact, the same happens with any disjoint three-valued logic with **s** as the standard for premises.

The logic $\mathbf{\bar{t}n}$ is not empty either, as any inference with a classically valid formula as a premise is valid — e.g., $p \vee \neg p \vDash q$. In fact, the same happens with any disjoint three-valued logic with $\mathbf{\bar{t}}$ as the standard for premises.

The logic $\mathbf{n\bar{t}}$ is not empty if, for example, $\top$ or $\bot$ are part of the language. In particular, any inference with $\bot$ or $\top$ as a premise or a conclusion is valid. But if no such constants are available, then, as in previous cases, the valuation that gives value ½ to every propositional letter invalidates every inference (and sentence).

2.4. P-logics

Though the most usual way to define p-logics is through sets of inferential validities that obey closure properties such as reflexivity and monotonicity, we will refer to *p-logics* as a *mixed* and *impure* logics for which $\mathcal{D}^+ \neq \varnothing$ and $\mathcal{D}^+ \subset \mathcal{D}^-$. The first clause intends to exclude every disjoint logic with the empty set as the standard for premises. This, though, does not give us the ordinary extension of (mixed Strong Kleene) p-logics. For example, according to our new definition, neither **CL** nor **LP** count as p-logics. But as we intend to fit every three-value Strong Kleene logic into one and only one category, this terminological adjustment is more suitable for our purposes.

It is pretty straightforward to show that in every p-logic Identity and Weakening hold. It is also not hard to show that Cut (i.e., if $\Gamma \vDash A, \Delta$ and $\Gamma, A \vDash \Delta$, then $\Gamma \vDash \Delta$) is invalid in all of them, just by taking a

[5] Notice that in [15] $\varnothing\varnothing$ is defined as a disjoint logic, whilst here we have categorized it as a pure logic. What defines a logic as pure is the fact that the standard for premises and the one for conclusions are the same. Meanwhile, disjoint logics ask for the corresponding standards not to have any values in common. Since $\varnothing\varnothing$ fits both categories, we choose to follow the intuition that dictates that disjoint logics should be impure. This puts $\varnothing\varnothing$ in a more intuitive place, among the pure logics.

valuation that assigns to the cut-formula, for every p-logic in the list, the only value in $\mathcal{D}^+$ that does not belong to $\mathcal{D}^-$.[6]

Some of these logics are supraclassical, e.g., $\mathbf{s}\overline{\varnothing}$, a trivial logic regarding non-empty premises and conclusions, **st** matches exactly classical logic (at least at the inferential level), and even others are contraclassical — e.g., $\mathbf{n}\overline{\mathbf{s}}$, a logic that invalidates $p \vDash p \vee q$, but validates any inference with a classical contradiction as its only conclusion (or even with a set of classically unsatisfiable formulas as the set of its conclusions). Nevertheless, no p-logic is strictly sub-classical.[7]

2.5. Q-logics

Though the most usual way to define q-consequence relations — that extensionally characterize q-logics — is as sets of inferential validities that obey closure properties such as monotonicity and quasi-closure, we will also employ the term *q-logics* in a slightly modified way. Q-logics, in our sense, are *mixed* and *impure* logics for which $\mathcal{D}^- \neq \varnothing$ and $\mathcal{D}^- \subset \mathcal{D}^+$. In a similar fashion as before, the first clause excludes disjoint logics with the empty set as the standard for conclusions. And again, this way of understanding them does not collapse with what is ordinary referred to as q-logics. For example, according to our new definition, neither **CL** nor **K3** count as q-logics.

There are twelve different Strong Kleene q-logics — which can be obtained from the twelve Strong Kleene p-logics by switching the premises and the conclusion's standards. It is not hard to show that Cut and Weakening hold in every q-logic. Identity is invalid in every q-logic, a fact witness by a valuation that assigns to the formula that appears in

[6] All these facts have been originally proved in [9], for p-logics defined in the original sense. But each of our p-logics is also a p-logic according to the way Frankowski uses the term. The same can be said regarding q-logics in our sense and Malinowski's.

[7] The following is a quick proof of this fact. The ones that have two values in $\mathcal{D}^+$ must have all the three values in $\mathcal{D}^-$, making every formula a tautology of these logics. The ones such that $\mathcal{D}^+ = \{0\}$ are not included in classical logic because every inference with a subset of premises classical satisfiable in every valuation — for example, with a classical tautology as a premise — turn out valid in these logics, but not in **CL**. The ones such that $\mathcal{D}^+ = \{½\}$ are not included in classical logic because they validate the inferential schema $A \vDash \neg A$. Finally, the ones such that $\mathcal{D}^+ = \{1\}$ are not included in classical logic neither. The only logic of this type we have not already talked about is $\mathbf{s}\overline{\mathbf{n}}$, and this logic also validates $A \vDash \neg A$.

both the premise and conclusion side the truth-value that belongs to $\mathcal{D}^+$ but doesn't belong to $\mathcal{D}^-$. (This has been originally proved in [13, 14].)

Some of these logics are strictly sub-classical (e.g., $\overline{\mathbf{s}}\overline{\mathbf{t}}$, an empty logic), others are contra-classical (e.g., $\overline{\varnothing}\mathbf{s}$, a logic that invalidates Identity; like every q-logic), but validates any inference with a classical contradiction as one of its conclusions. (In fact, it validates any inference with a set of classically unsatisfiable formulas as a subset of its conclusions.) But, of course, there is no strictly supra-classical q-logic (i.e., q-logics that strictly include **CL**) because each of these logics invalidates at least one case of Identity.

2.6. Overlapping logics

Not every three-valued Strong Kleene inferential logic falls into one of the four previous categories. Overlapping logics are such that the intersection between $\mathcal{D}^-$ and $\mathcal{D}^+$ is not empty — so they cannot be disjoint logics — , but also none of these sets is included in the other member of the pair of standards — and thus cannot be neither p-logics nor q-logics nor pure logics. For matters of simplicity, we label them as *overlapping logics*. To the best of our knowledge, this particular family of logics has not been previously investigated in the literature.

Each of these logics is non-reflexive. To check this, consider any instance of Identity with a letter $p \in \mathrm{Var}$ for which there is a valuation v such that $v(p) \in \mathcal{D}^+$ but $v(p) \notin \mathcal{D}^-$. Moreover, and for similar reasons, none of these logics validate Cut. Once again, consider a case with the empty-sequent as the end-sequent, and a cut-formula A such that there is a valuation v, $v(A) \in \mathcal{D}^-$ but $v(A) \notin \mathcal{D}^+$.

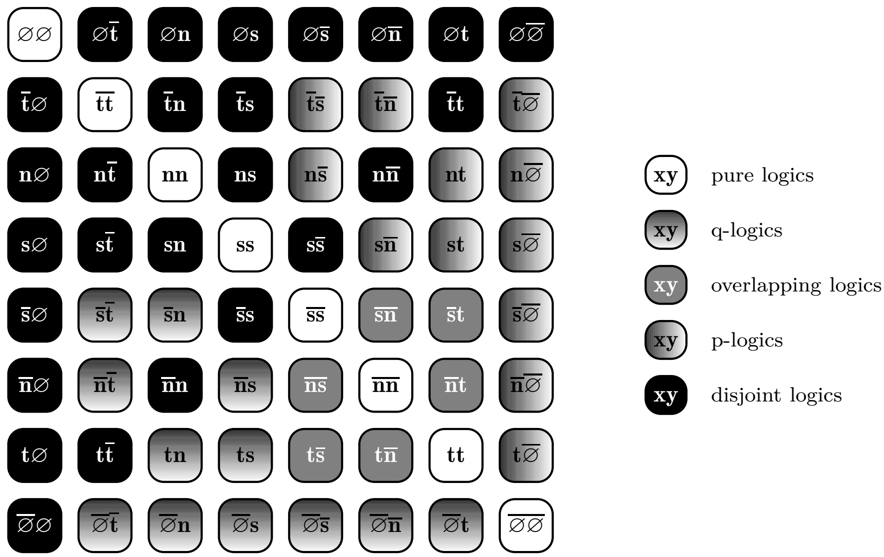
Nevertheless, these logics are monotonic, contractive and validate Exchange, as they are inferential logics, and inferences, in this framework, are pairs of *sets* of formulas. Given their novelty, we will stress some features of three of these logics:

The logic $\mathbf{t}\overline{\mathbf{n}}$ is strictly subclassical. And in fact, it is an empty logic (if no truth-value constants are added to the language).

The logic $\mathbf{t}\overline{\mathbf{s}}$ is not comparable with classical logic, as it, e.g., invalidates Identity but validates $A \vDash \neg A$. (In fact, there are no strictly supraclassical logic — i.e., logics that strictly include **CL** — because each of these logics invalidates Identity.)

The logic $\overline{\mathbf{s}}\mathbf{t}$ is not comparable with classical logic, as it invalidates Identity, like every overlapping-logic, but validates, for example, $\neg A \vDash A$.

We summarize the membership of every mixed Strong Kleene logic in their corresponding category with the next diagram:



Every row represents every mixed logic with the same standard for premises, and each column represents every mixed logic with the same standard for conclusions.

3. An epistemic reading of Strong Kleene logics

As we mentioned in the beginning, it is common to think that, in a valid inference, or at least in sound ones, premises provide reasons to accept the conclusions. Another, and even more clarifying, way of putting it is that *accepting* the premises provides reasons for *accepting* the conclusions. However, this assumption leaves aside other ways in which certain epistemic attitudes towards propositions are able to give epistemic support to others. In that sense, it leaves aside other possible formal commitments that we encounter in everyday, and in philosophical, life.

For example, sometimes the *acceptance* of certain sentences warrants the *rejection* of others. Or the fact that one *does not accept nor reject* some collection of sentences provides good reasons to either *accept* or *reject* some conclusion. These situations cannot be straightforwardly represented by valid or sound inferences of logics that aim at capturing acceptance-preserving or truth-preserving facts, in one way or another. Nevertheless, they can be represented by valid inferences of other logics. We claim that all of them — i.e., every formal epistemic norm related only to the attitudes (or the type of attitudes) of accepting, rejecting, or neither accepting nor rejecting — can be linked to the valid inferences of some mixed Strong Kleene logic (*via* some bridge principle). In this section, we will present a kind of epistemic interpretation available for all of these logics.[8]

It is by now standard to think that each value of a three-valued logic can be suitable related to a specific epistemic attitude, informational condition or metaphysical state. Specifically, value 1 can be related to the metaphysical state of being true, but also to being certain — i.e., to the epistemic state of certainty that a specific agent might be in with respect to a specific proposition — , to the epistemic attitude of accepting that proposition, or to the informational condition of 'being told true'. The value ½, in its turn, can be related to the metaphysical state of being neither true nor false (namely, of some event or object being *undetermined* in some regard, e.g. neither having nor not having some property), the dialetheic situation of being both true and false (that is, of being *overdetermined* in some aspect, e.g. being both P and not P, for some property P; for more about it, see, e.g., [18]), but it can also be understood as a state of uncertainty, of suspension of belief, or one in which the agent has as much reasons to accept the proposition as the ones she has to reject it — which are all forms of neither accepting nor rejecting a proposition — , and even to the informational stated of 'being told neither true nor false' (or being told both of them). Finally, the value 0 is usually related to the metaphysical state of being false,

[8] This type of justification has been previously elaborated by [15, 16] for two different sub-families of mixed Strong Kleene logics. We should stress that, even though we have chosen to deepen in the epistemic way to understand these logics, this does not mean that we are claiming that ours is the only kind of interpretation available for these logics. In fact, it is possible to give either alethic interpretations — i.e., in terms of the metaphysical states of being true, being false and being neither true nor false (or both) — or informational readings — i.e., in which certain pieces of information (sentences or propositions) can be marked as being told true, being told false and neither being told true nor being told false — or being told both. The latter is the interpretation that Belnap develops for **FDE** in [2], that can be easily extended to every mixed Strong Kleene logic. We will consider this informational interpretation later on, though our main focus is on what we have called the *epistemic interpretation*, given our interests.

but also to the epistemic attitude of rejection — or the probably different one of being certain of the falsity of the proposition,[9] but also to the informational condition of 'being told false'.

With these different ways to understand the three truth-values, it is possible to develop philosophical interpretations for all of these logics. In particular, if we attach these values to the (kinds of) epistemic attitudes usually associated with them (i.e., as accepting, neither accepting nor rejecting, or as rejecting certain sentence), we will more or less automatically develop epistemic interpretations for all of these logics.

Before giving a precise definition and some examples, let us make explicit the epistemic attitudes we want to attach to the sets of values under consideration:

$\mathbf{s}$: accepting	$\overline{\mathbf{s}}$: not accepting
$\mathbf{n}$: neither accepting nor rejecting	$\overline{\mathbf{n}}$: accepting or rejecting
$\mathbf{t}$: not rejecting	$\overline{\mathbf{t}}$: rejecting
$\varnothing$: having no attitude	$\overline{\varnothing}$: having any attitude

In principle, every set of values encodes an epistemic attitude, be it basic as in $\mathbf{s}$ or $\overline{\mathbf{t}}$, be it more complex as in $\overline{\mathbf{n}}$ or $\overline{\varnothing}$. Some combinations can be discarded as being problematic, but they should be treated case by case. Now we can define what is an epistemic interpretation for a mixed Strong Kleene logic:

Definition 3.1 (Epistemic interpretation). An inference is valid in a Strong Kleene logic $\mathbf{xy}$ if and only if, if an agent has an attitude related to some values in $\mathbf{x}$ with respect to every premise, then the agent should have an attitude related to one of the values in $\mathbf{y}$ towards at least one conclusion.

We provide some examples that aim to give the reader a general idea of how these interpretations look. The first example we will consider is given by $\overline{\mathbf{t}}\overline{\mathbf{t}}$. This logic can be seen as qualifying as valid inferences of *rejected* sentences from *rejected* sentences — i.e., inferences such that if every premise is rejected, then at least one conclusion is rejected. Both $p \wedge \neg q \vDash p$ and $p \wedge \neg p \vDash \neg p$ are invalid in $\overline{\mathbf{t}}\overline{\mathbf{t}}$, as they should: if someone rejects $p \wedge \neg p$, its because she rejects one (and only one) of its conjuncts.

[9] These attitudes may or may not be identical to the one of accepting the negation of the proposition rejected. As this discussion is not central to the point we want to state here, we will leave it aside for the moment.

But it could be either one. Thus, it would not be reasonable to demand her to reject, say $\neg p$, because if she rejects p, then she should in fact do the opposite: accept its negation. Yet $p \vDash p \wedge \neg p$ is valid in it, despite not being valid in Classical Logic. Once again, this seems right: if someone rejects a sentence p, then she should reject the conjunction of p with any other formula.

Another example is given by logic **sn**. This is a disjoint logic that can be interpreted as qualifying as valid inferences of *non accepted nor rejected* sentences from *accepted* sentences — i.e., inferences such that if every premise is accepted, then at least one conclusion is neither accepted nor rejected. And this is why $A \nvDash \neg A$: if A is accepted, then $\neg A$ is rejected, i.e., it is not neither accepted nor rejected.

We will provide one final example. The logic $\bar{\mathbf{s}}\mathbf{t}$ is an overlapping-logic such that $\neg A \nvDash A$, but it invalidates Identity. We can justify the first fact by pointing out that if $\neg A$ is either rejected, or neither accepted or rejected, A should be either accepted or neither accepted or rejected. The second fact can be justified by saying that if a sentence is rejected, it cannot be accepted in the same moment.

These logics, interpreted in this way, can be applied to a specific case, i.e., to the epistemic situation of some particular person. Take, for example, $\mathbf{n}\bar{\mathbf{t}}$. This logic validates the inferences such that if every premise is neither accepted nor rejected, then the set of conclusions contains at least one rejected sentence. Thus, in order to obtain what sentences must be rejected by a particular agent in some specific situation, given the set of sentences she neither accepts nor rejects, the valuations must be restricted to the ones that give the value ½ to the sentences (that represent the ones) that are in fact neither accepted nor rejected by the agent. The set of conclusions of valid inferences (restricted to the set of valuations that give the value ½ to every premise), will be the ones that contain at least one sentence that must be rejected (i.e., whose rejection the subject is committed to, given the set of sentences she neither accept nor reject).

Thus, philosophical interpretations for all of these logics are available, or at least for most of them. In particular, we are unsure whether this kind of interpretation can be extended to mixed logics involving $\overline{\varnothing}$ or $\varnothing$ as standards for premises or conclusions. In the case where $\varnothing$ acts as the standard for premises, we get a trivial logic with respect to non-empty premise inferences. Moreover, it is not clear what the empty set might represent as an attitude, or as a mix of attitudes. There seems to be an analogous problem with logics that have $\overline{\varnothing}$ as the standard for conclusions. In each of these cases we get a trivial logic regarding non-empty conclusion inferences.[10] Besides, what kind of commitments would a logic that is trivial for inferences with non-empty premises (resp. conclusions) represent?

Having said that, in the case of logics that have $\overline{\varnothing}$ as the standard for premises, there seems to be a clear philosophical reading, of the kind we have already proposed. These logics validate just the inferences that have at least one conclusion that can be accepted/rejected/neither accepted nor rejected, or a combination of them, depending on the case. If our focus is on sentences rather than on inferences, these logics seem to be useful. Nevertheless, if we reject some subset of every possible mixed Strong Kleene logic, it will not be the case that, according to this approach, *everything goes* regarding mixed Strong Kleene logics.

The epistemic interpretation of these logics — in terms of acceptance, rejection and a third kind of attitude different from them — is not new. However, it has been restricted to logics defined through standards that are upsets.[11] We have generalize this kind of interpretation for every mixed Strong Kleene logic, regardless of whether it is defined through standards that are upsets or not. And what we gained from this is the possibility to represent, with the valid inferences of the logics that are usually left aside (or not even considered as logics), every possible formal commitment that relates these three types of epistemic attitudes.

Nevertheless, we are not committed to the claim that this is the only way to capture these kinds of epistemic commitments — but just that it is a fairly reasonable way to do it. Specifically, it is in principle possible to capture all formal epistemic norms (for the three epistemic attitudes previously listed) in a single unified theory, or logic, that does not require one to commit to any kind of logical pluralism. Thus, this is one way to go if a monist approach is preferred. The next section is devoted to it.

4. Second account: a unified logic for every three-valued formal epistemic commitment

To achieve a unified framework that encapsulates all formal commitments possible, given this three epistemic attitudes, we first need to

[10] For more about how bad logics that are trivial with respect to a particular set of sentences are, see [21].

[11] Following [5], a standard is an *upset* iff if $x \leq y$ and $x \in \mathcal{D}$, then $y \in \mathcal{D}$.

modify the way in which we understand inferences, or better yet, to give up the traditional way in which we conceive them: instead of seeing them as *pairs* of sets (or multisets, or sequences) of formulas, we embrace a *six-sided* understanding of inferences — or a two-sided type of inference such that both the premises and conclusions are neither sets nor multisets, but sequences of three sets of sentences. These type of inferences are a version of the sequents that have been introduced by Francez in [8] under the name of *poly-sequents*.[12] We will introduce the logic **6SK** that deals with this form of inferences, but first, we need to introduce some technical machinery.

Let $\mathcal{L}$ be a sentential language with the connectives $\wedge$, $\vee$, and $\neg$, of arities 2, 2, and 1, and intended as conjunction, disjunction and negation, respectively. Let Var be a countable set of sentential variables $\{p, q, r, ...\}$. By $\mathrm{FOR}_{\mathcal{L}}$ we denote the set of well formed formulas of $\mathcal{L}$, defined as usual. We define an expression of the form $\phi \to \psi$ as $\neg\phi \vee \psi$.

Definition 4.1. A six-sided inference on $\mathcal{L}$ is any element of $\wp(\mathrm{FOR}_{\mathcal{L}}) \times \wp(\mathrm{FOR}_{\mathcal{L}}) \times \wp(\mathrm{FOR}_{\mathcal{L}}) \times \wp(\mathrm{FOR}_{\mathcal{L}}) \times \wp(\mathrm{FOR}_{\mathcal{L}}) \times \wp(\mathrm{FOR}_{\mathcal{L}})$.

So, a six-sided inference is any tuple $\Gamma_1 \mid \Gamma_{½} \mid \Gamma_0 \vDash \Delta_1 \mid \Delta_{½} \mid \Delta_0$ such that $\Gamma_1, \Gamma_{½}, \Gamma_0, \Delta_1, \Delta_{½}, \Delta_0 \subseteq \mathrm{FOR}_{\mathcal{L}}$. We let $\mathrm{INF}(\mathcal{L})$ denote the set of all six-sided inferences.

Definition 4.2. A six-sided logic $\boldsymbol{L}$ for $\mathcal{L}$ is a pair $\langle \mathcal{L}, \vDash_{\boldsymbol{L}} \rangle$, where $\vDash_{\boldsymbol{L}} \subseteq \wp(\mathrm{FOR}_{\mathcal{L}}) \times \wp(\mathrm{FOR}_{\mathcal{L}}) \times \wp(\mathrm{FOR}_{\mathcal{L}}) \times \wp(\mathrm{FOR}_{\mathcal{L}}) \times \wp(\mathrm{FOR}_{\mathcal{L}}) \times \wp(\mathrm{FOR}_{\mathcal{L}})$

With this definition of a six-sided logic, we can specify now how our target logic, **6SK**, works.

Definition 4.3. For any six-sided inference $\Gamma_1 \mid \Gamma_{½} \mid \Gamma_0 \vDash \Delta_1 \mid \Delta_{½} \mid \Delta_0$, it is valid in the six-sided logic for Strong Kleene matrices **6SK** if and only if, for every valuation v, if for every $\gamma_1 \in \Gamma_1$, $v(\gamma_1) = 1$, for every $\gamma_{½} \in \Gamma_{½}$, $v(\gamma_{½}) = ½$ and for every $\gamma_0 \in \Gamma_0$, $v(\gamma_0) = 0$, then either

[12] Francez introduces them with the aim of obtaining a proof-system such that acceptance and rejection can play a role both in the premises and the conclusions, and not as in Restall's [19] understanding a traditional two-sided inferences and sequents, where premises are related to the things an agent accepts, while the conclusions are linked with what that agent rejects. Moreover, the way Francez reads poly-sequents does not depend on a specific semantics for them, thus — according to him — being better suited for a proof-theoretic semantics explanation of the meaning of logical constants.

for some $\delta_1 \in \varDelta_1$, $v(\delta_1) = 1$, or for some $\delta_{½} \in \varDelta_{½}$, $v(\delta_{½}) = ½$ or for some $\delta_0 \in \varDelta_0$, $v(\delta_0) = 0$.

This logic is designed to validate six-sided inferences that correspond to valid inferences in some mixed Strong Kleene logic, for any mixed Strong Kleene logic.

Fact 4.1. *Let $\mathcal{V} = \{1, ½, 0\}$. For every mixed Strong Kleene logic* **xy**, *$\varGamma \vDash_{\mathbf{xy}} \varDelta$ if and only if for all $\varGamma_1$, $\varGamma_{½}$ and $\varGamma_0$ such that $\varGamma = \bigcup_{i\in\mathbf{x}} \varGamma_i$ and $\varnothing = \bigcup_{i\in\mathcal{V}-\mathbf{x}} \varGamma_i$ there are some $\varDelta_1$, $\varDelta_{½}$ and $\varDelta_0$ such that $\varDelta = \bigcup_{i\in\mathbf{y}} \varDelta_i$ and $\varnothing = \bigcup_{i\in\mathcal{V}-\mathbf{y}} \varDelta_i$ for which $\varGamma_1 \mid \varGamma_{½} \mid \varGamma_0 \vDash \varDelta_1 \mid \varDelta_{½} \mid \varDelta_0$ is valid in* **6SK**.

This fact claims that the validity of an inference in a Strong Kleene logic **xy** can be established by checking the validity of a set of inferences in **6SK**, given some constraints. These constraints demand for a distribution of the premises and conclusions of the original inference in subsets of propositions, used for building inferences to be checked in **6SK**. Before proving Fact 4.1, it is worth presenting some examples (i.e., some instances of this fact), with the hope that it makes it easier to understand how — and in what sense — **6SK** captures the different mixed Strong Kleene logic's validities.

- $\varGamma \vDash_{\mathbf{s\bar{t}}} \varDelta$ is valid if and only if $\varGamma \mid \varnothing \mid \varnothing \vDash_{\mathbf{6SK}} \varnothing \mid \varnothing \mid \varDelta$ is valid.
- $\varGamma \vDash_{\mathbf{st}} \varDelta$ is valid if and only if $\varGamma \mid \varnothing \mid \varnothing \vDash_{\mathbf{6SK}} \varDelta_1 \mid \varDelta_{½} \mid \varnothing$ is valid, for some $\varDelta_1$ and $\varDelta_{½}$ such that $\varDelta = \varDelta_1 \cup \varDelta_{½}$.
- $\varGamma \vDash_{\mathbf{n\bar{n}}} \varDelta$ is valid if and only if $\varnothing \mid \varGamma \mid \varnothing \vDash_{\mathbf{6SK}} \varDelta_1 \mid \varnothing \mid \varDelta_0$ is valid, for some $\varDelta_1$ and $\varDelta_0$ such that $\varDelta = \varDelta_1 \cup \varDelta_0$.
- $\varGamma \vDash_{\mathbf{ts}} \varDelta$ is valid if and only if $\varGamma_1 \mid \varGamma_{½} \mid \varnothing \vDash_{\mathbf{6SK}} \varDelta \mid \varnothing \mid \varnothing$ is valid, for every $\varGamma_1$ and $\varGamma_{½}$ such that $\varGamma = \varGamma_1 \cup \varGamma_{½}$.
- $\varGamma \vDash_{\mathbf{t\bar{s}}} \varDelta$ is valid if and only if $\varGamma_1 \mid \varGamma_{½} \mid \varnothing \vDash_{\mathbf{6SK}} \varnothing \mid \varDelta_{½} \mid \varDelta_0$ is valid, for every $\varGamma_1$ and $\varGamma_{½}$ and some $\varDelta_{½}$ and $\varDelta_0$ such that $\varGamma = \varGamma_1 \cup \varGamma_{½}$ and $\varDelta = \varDelta_{½} \cup \varDelta_0$.

Proof. (From left to right): Suppose $\varGamma \vDash_{\mathbf{xy}} \varDelta$ is valid. Then, for every valuation v, if for every $\gamma \in \varGamma$, $v(\gamma) \in \mathbf{x}$, then, for some $\delta \in \varDelta$, $v(\delta) \in \mathbf{y}$. We need to check whether for every $i, j, k \in \{0, ½, 1\}$, such that $\varGamma = \varGamma_i \cup \varGamma_j \cup \varGamma_k$, the corresponding six-sided inference is valid. But then, either for some i such that $i \in \mathbf{x}$, it is not the case that for every $\gamma_i \in \varGamma_i$, $v(\gamma_i) = i$, or for every i such that $i \in \mathbf{x}$ and for every $\gamma_i \in \varGamma_i$, $v(\gamma_i) = i$. If the former is the case, then the valuation v satisfies the six-sided inference. If the latter is the case, then we need to check whether, for some $j \in \mathbf{y}$, it is the case that some $\delta_j \in \varDelta_j$ $v(\delta_j) = j$. But, as $\varGamma \vDash_{\mathbf{xy}} \varDelta$, it is true that for some $\delta \in \varDelta$, $v(\delta) \in \mathbf{y}$. Thus, it is also true that for some $\delta_j \in \varDelta_j$ $v(\delta_j) = j$.

(From right to left): Suppose that for all $\varGamma_1$, $\varGamma_{½}$ and $\varGamma_0$ such that $\varGamma = \bigcup_{i \in \mathbf{x}} \varGamma_i$, and, for every i in $\{1, ½, 0\}$, if $i \notin \mathbf{x}$, then $\varGamma_i = \varnothing$, there are $\varDelta_1$, $\varDelta_{½}$ and $\varDelta_0$ such that $\varDelta = \bigcup_{i \in \mathbf{y}} \varDelta_i$ and for every i in $\{1, ½, 0\}$, if $i \notin \mathbf{y}$, then $\varDelta_i = \varnothing$, for which $\varGamma_1 \mid \varGamma_{½} \mid \varGamma_0 \vDash \varDelta_1 \mid \varDelta_{½} \mid \varDelta_0$ is valid in **6SK**. Thus, in each of these six-sided inferences, if all premises satisfy their positions, then at least one conclusion satisfies its position. This can happen only if not every premise in $\varGamma$ receives a value in $\mathbf{x}$ or some conclusion in $\varDelta$ a value in $\mathbf{y}$. Therefore, $\varGamma \vDash \varDelta$ is valid in $\mathbf{xy}$. □

As we stressed before, Fact 4.1 demands for a set of inferences some constraints, so that each inference corresponds to a different distribution of premises and conclusions in the six-sided inferences. But these constraints make us check some combinations that are redundant.

Take, for example, a logic $\mathbf{xy}$ such that $\mathbf{x} = \{1, 0\}$. Then the set of premises $\varGamma$ should be distributed along the sets $\varGamma_1$ and $\varGamma_0$. This means that inferences for which, for instance, $\varGamma_1$ and $\varGamma_0$ share some formula will be checked in **6SK**, but inferences with this feature will be trivially valid: every valuation v fails in giving value 1 to every formula in $\varGamma_1$ and giving value 0 to every formula in $\varGamma_0$, since there is at least one formula in both $\varGamma_1$ and $\varGamma_0$ that receives one value but not both. As a result, checking inferences with subsets of $\varGamma$ that do not overlap is enough.

On the other hand, consider a logic $\mathbf{xy}$ such that $\mathbf{y} = \{1, 0\}$. Then the set of conclusions $\varDelta$ should be distributed along the sets $\varDelta_1$ and $\varDelta_0$. For the possible distributions of $\varDelta$, we need that, at least one of them satisfies Definition 4.3 by having, for every valuation, a formula that is satisfied in its corresponding place. We could check every possible distribution of $\varDelta$ in $\varDelta_1$ and $\varDelta_0$, but if we take $\varDelta_1$ and $\varDelta_0$ to be the same as $\varDelta$ it would be enough. This is the case where every conclusion is checked for every value in $\mathbf{y}$, so if there is a valuation with no formula assigned to the corresponding value, every other distribution of conclusions would also result in an invalid six-sided inference; and if one of the these other distributions result in a valid inference, the highlighted distribution would also result in a valid inference. Therefore, it is enough to checked inferences where every subset of conclusions coincides with $\varDelta$.

FACT 4.2. *Let $\mathcal{V} = \{1, ½, 0\}$ and let $\mathbf{x}, \mathbf{y} \subseteq \mathcal{V}$. Let $\varGamma$ and $\varDelta$ be any two sets of formulas. For all $\varGamma_1$, $\varGamma_{½}$ and $\varGamma_0$ such that $\varGamma = \bigcup_{i \in \mathbf{x}} \varGamma_i$ and*

$\varnothing = \bigcup_{i \in \mathcal{V}-\mathbf{x}} \Gamma_i$, there are Δ_1, $\Delta_{½}$ and Δ_0 such that $\Delta = \bigcup_{i \in \mathbf{y}} \Delta_i$ and $\varnothing = \bigcup_{i \in \mathcal{V}-\mathbf{y}} \Delta_i$, for which $\Gamma_1 \mid \Gamma_{½} \mid \Gamma_0 \vDash \Delta_1 \mid \Delta_{½} \mid \Delta_0$ is valid in **6SK** *if and only if for all Γ_1, $\Gamma_{½}$ and Γ_0 such that $\Gamma = \bigcup_{i \in \mathbf{x}} \Gamma_i$, $\varnothing = \bigcup_{i \in \mathcal{V}-\mathbf{x}} \Gamma_i$, and for all $i, j \in \mathbf{x}$, if $i \neq j$ then $\varnothing = \Gamma_i \cap \Gamma_j$, $\Gamma_1 \mid \Gamma_{½} \mid \Gamma_0 \vDash \Delta_1 \mid \Delta_{½} \mid \Delta_0$ is valid in* **6SK**, *with $\Delta_i = \Delta$ if $i \in \mathbf{y}$ and $\Delta_i = \varnothing$ otherwise.*

Proof. In the first place, we have to prove that checking any inference $\Gamma_1 \mid \Gamma_{½} \mid \Gamma_0 \vDash \Delta_1 \mid \Delta_{½} \mid \Delta_0$ such that there are $i, j \in \mathbf{x}$, with $i \neq j$ and $\varnothing \neq \Gamma_i \cap \Gamma_j$ is redundant, by showing that it is trivially valid, so these are not the instances of Fact 4.1 that matter. If there are Γ_i and Γj (with $i \neq j$) such that they share a formula, say σ, for every valuation v, $v(\sigma)$ cannot be equal to both i and j, since $i \neq j$. Therefore, for every valuation v is not the case that for every $\gamma_i \in \Gamma_i$, $v(\gamma_i) = i$ and that for every $\gamma_j \in \Gamma_j$, $v(\gamma_j) = j$. So, such six-sided sequent is trivially valid by Definition 4.3.

In the second place, we have to prove that checking any inference besides the ones where $\Delta_i = \Delta$ if $i \in \mathbf{y}$ and $\Delta_i = \varnothing$ if $i \notin \mathbf{y}$ is redundant. If an inference with this distribution of conclusions (i.e., where every conclusion-side is either Δ or $\varnothing$, depending on $\mathbf{y}$) is valid, then for every valuation v, there is a $\delta_i \in \Delta_i$ such that $v(\delta_i) = i$, for some i. If that is the case, we need not check any other conclusions distribution for a given premises distribution, by Definition 4.3. If that initial distribution is not valid, there is a valuation v such that every $\delta_i \in \Delta_i$ is such that $v(\delta_i) \neq i$, for every i. Removing any formula from any of those sets wouldn't make any difference, since any other distribution of conclusions would also form invalid inferences. □

There is an obvious relation between six-sided inferences and three-sided disjunctive sequents. In [20] we find the ways in which valid inferences in some mixed Strong Kleene logics correspond to valid three-sided sequents. Before making this explicit, and see how it (positively) affects us, we will introduce the kind of disjoint three-sided sequents we will be talking about.

Definition 4.4. A disjunctive sequent $\Gamma \mid \Sigma \mid \Delta$ is satisfied by a valuation v iff $v(\gamma) = 0$ for some $\gamma \in \Gamma$, or $v(\sigma) = ½$ for some $\sigma \in \Sigma$, or $v(\delta) = 1$ for some $\delta \in \Delta$. A sequent is valid if and only if it is satisfied by every valuation. A valuation is a counterexample to a sequent if the valuation does not satisfy the sequent.

There is a close relation between validity in a given mixed Strong Kleene logic and validity of *a certain kind* of disjunctive three-sided sequents. The following are some examples:

- $\Gamma \vDash_{\mathbf{n\bar{n}}} \Delta$ is valid if and only if $\Gamma, \Delta \mid \varnothing \mid \Gamma, \Delta$ is valid.
- $\Gamma \vDash_{\mathbf{s\bar{t}}} \Delta$ is valid if and only if $\Gamma, \Delta \mid \Gamma \mid \varnothing$ is valid.
- $\Gamma \vDash_{\mathbf{st}} \Delta$ is valid if and only if $\Gamma \mid \Gamma, \Delta \mid \Delta$ is valid.
- $\Gamma \vDash_{\mathbf{ts}} \Delta$ is valid if and only if $\Gamma \mid \varnothing \mid \Delta$ is valid.

The following fact follows straightforward from both the definitions of mixed Strong Kleene logic and three-sided disjunctive sequent.

Fact 4.3. *For every mixed Strong Kleene logic* $\mathbf{xy}$, $\Gamma \vDash_{\mathbf{xy}} \Delta$ *if and only if* $\Gamma_0, \Delta_0 \mid \Gamma_{½}, \Delta_{½} \mid \Gamma_1, \Delta_1$ *is valid, where for every* $i, j \in \{0, ½, 1\}$, *if* $i \in \mathbf{x}, \Gamma_i = \varnothing$, *if* $i \notin \mathbf{x}, \Gamma_i = \Gamma$, *if* $j \in \mathbf{y}, \Delta_j = \Delta$ *and if* $j \notin \mathbf{y}, \Delta_j = \varnothing$.

The fact below follows immediately from the definition of validity of a six-sided inference and the definition of validity of a disjunctive three-sided sequent:

Fact 4.4. *A six-sided inference* $\Gamma_1 \mid \Gamma_{½} \mid \Gamma_0 \vDash \Delta_1 \mid \Delta_{½} \mid \Delta_0$ *is valid in* **6SK** *if and only if* $\Gamma_1, \Gamma_{½}, \Delta_0 \mid \Gamma_1, \Gamma_0, \Delta_{½} \mid \Gamma_0, \Gamma_{½}, \Delta_1$ *is a valid three-sided disjunctive sequent.*

One important thing that remains to be settled is whether it is possible to design proof systems for this logic. In the sequel, we will present a calculus that is both sound and complete with respect to **6SK**. As the previous result shows the correspondence of every six-sided inference with a disjunctive three-sided sequent, it will be enough to give a proof-system for the latter. Our target proof theory, then, is the three-sided disjunctive sequent system `SC6SK` (for "Sequent Calculus for **6SK**").[13]

As we said, `SC6SK` can be used as a proof system for **6SK**. `SC6SK` includes, as usual, some axioms and rules. A sequent is provable if and only if it follows from the axioms by some finite number (possibly zero) of applications of the rules. As we are working with sets, the effects of the structural rules of Exchange and Contraction are built in. Moreover, Weakening is built into the Identity axiom. Still, to make things easier, we will include the structural rule of Weakening as an explicit rule. We

[13] This system strongly resembles — and is obviously based on — the one David Ripley present in [20] for the truth theories based on **ST**, **TS**, **LP** and **K3**. It was first introduce in [15] for different purposes. The only difference between these two systems is that it was originally called **DL** because it was designed to capture validities for what the author called "disjoint logics".

will have three versions of a three-sided Cut rule. Identity (*Id*) is the only axiom-schema of SC6SK. *Weak*, *Cut1*, *Cut2* and *Cut3* are structural rules. The remaining are SC6SK's operational rules.

$$Id\ \frac{}{A, \Gamma \mid A, \Sigma \mid A, \Delta} \qquad Cut1\ \frac{\Gamma, A \mid \Sigma \mid \Delta \qquad \Gamma \mid \Sigma, A \mid \Delta}{\Gamma \mid \Sigma \mid \Delta}$$

$$Weak\ \frac{\Gamma \mid \Sigma \mid \Delta}{\Gamma, \Gamma' \mid \Sigma, \Sigma' \mid \Delta, \Delta'} \qquad Cut2\ \frac{\Gamma \mid \Sigma \mid \Delta, A \qquad \Gamma \mid \Sigma, A \mid \Delta}{\Gamma \mid \Sigma \mid \Delta}$$

$$Cut3\ \frac{\Gamma, A \mid \Sigma \mid \Delta \qquad \Gamma \mid \Sigma \mid \Delta, A}{\Gamma \mid \Sigma \mid \Delta}$$

$$\mathrm{L}\neg\ \frac{\Gamma \mid \Sigma \mid \Delta, A}{\Gamma, \neg A \mid \Sigma \mid \Delta} \qquad \mathrm{L}\wedge\ \frac{\Gamma, A, B \mid \Sigma \mid \Delta}{\Gamma, A \wedge B \mid \Sigma \mid \Delta}$$

$$\mathrm{M}\neg\ \frac{\Gamma \mid \Sigma, A \mid \Delta}{\Gamma \mid \Sigma, \neg A \mid \Delta} \qquad \mathrm{R}\wedge\ \frac{\Gamma \mid \Sigma \mid \Delta, A \qquad \Gamma \mid \Sigma \mid \Delta, B}{\Gamma \mid \Sigma \mid \Delta, A \wedge B}$$

$$\mathrm{R}\neg\ \frac{\Gamma, A \mid \Sigma \mid \Delta}{\Gamma \mid \Sigma \mid \Delta, \neg A}$$

$$\mathrm{M}\wedge\ \frac{\Gamma \mid \Sigma, A \mid \Delta, A \qquad \Gamma \mid \Sigma, B \mid \Delta, B \qquad \Gamma \mid \Sigma, A, B \mid \Delta}{\Gamma \mid \Sigma, A \wedge B \mid \Delta}$$

As $\vee$ and $\rightarrow$ can be defined in terms of the former, we will not specify rules for them. To prove Completeness, we make use of the *Derived Cut* rule (that can be inferred from the three basic rules of Cut in pretty much the way Ripley did it for his system in [20]):

$$Derived\ Cut\ \frac{\Gamma, A \mid \Sigma, A \mid \Delta \qquad \Gamma \mid \Sigma, A \mid \Delta, A \qquad \Gamma, A \mid \Sigma \mid \Delta, A}{\Gamma \mid \Sigma \ \mid \Delta}$$

We now present the main results regarding SC6SK:

Theorem 4.1 (Soundness). *If a sequent $\Gamma \mid \Sigma \mid \Delta$ is provable in* SC6SK, *then it is a valid three-sided disjunctive sequent.*

Proof. The axioms are valid, and validity is preserved by the rules, as can be checked without too much trouble. □

The system SC6SK is also complete. The following theorem is proven in [15], though the three-sided disjunctive sequents in [15] are related only to disjoint logics, and here we can relate them to any mixed logic, as done in Fact 4.3:

THEOREM 4.2 (Completeness). *If a sequent $\Gamma \mid \Sigma \mid \Delta$ is valid, then it is provable in* SC6SK*.*

Due to the fact that every valid six-sided inference corresponds to a valid disjunctive three-sided sequent, as Fact 4.4 shows, then the fact that SC6SK is sound and complete with respect to three-sided sequents validity implies that, it can be used for proving every six-sided inference in **6SK**, though not every provable sequent corresponds to a six-sided inference in **6SK**.

Finally, we would like to make some comments about some of the different forms of metainferences that might be reasonably labelled as "Cut" in SC6SK. First, notice that not only Derived Cut is derivable from the three basic forms of Cut, but it is possible (and not hard) to prove that each of those basic forms of Cut can be derived using Derived Cut and none of the three basic forms of Cut of SC6SK—in fact, each proof only used the premises of each Derived Cut (one of them should be used twice), Weakening, and Derived Cut at the final step (we leave the details of this proof to the reader). Moreover, the translation of both traditional forms of Cut, one context-free and the other context-sharing, as they should be interpreted in **ST**, are both not derivable in SC6SK, but the translation of these two metainferential schemas/rules according to either **K3**, **LP** and **TS** are both derivable. We show this for the context-sharing versions in both **ST** and **LP**, and leave the other proofs as exercises to the reader.

$$\textbf{ST}\text{-Cut}\ \frac{\Gamma, A \mid \Gamma, \Delta, A \mid \Delta \qquad \Gamma \mid \Gamma, \Delta, A \mid \Delta, A}{\Gamma \mid \Gamma, \Delta \mid \Delta}$$

$$\textbf{LP}\text{-Cut}\ \frac{\Gamma, A \mid \Delta \mid \Delta \qquad \Gamma \mid \Delta, A \mid \Delta, A}{\Gamma \mid \Delta \mid \Delta}$$

The following is a proof of **LP**-Cut's derivability:

$$\mathit{Cut3}\ \dfrac{\mathit{Cut1}\ \dfrac{\mathit{Weak}\ \dfrac{\Gamma, A \mid \Delta \mid \Delta}{\Gamma, A \mid \Delta \mid \Delta, A} \qquad \Gamma \mid \Delta, A \mid \Delta, A}{\Gamma \mid \Delta \mid \Delta, A} \qquad \Gamma, A \mid \Delta \mid \Delta}{\Gamma \mid \Delta \mid \Delta}$$

And the next is a failed proof attempt for **ST**-Cut's derivability:

$$???\ \dfrac{\mathit{Cut3}\ \dfrac{\Gamma, A \mid \Gamma, \Delta, A \mid \Delta \qquad \Gamma \mid \Gamma, \Delta, A \mid \Delta, A}{\Gamma \mid \Gamma, \Delta, A \mid \Delta} \qquad \Gamma, A \mid \Gamma, \Delta, A \mid \Delta}{\Gamma \mid \Gamma, \Delta \mid \Delta}$$

The failure of the last route to prove that the translation of Cut in the system according to how to translate **ST**-inferences in three-sided disjunctive sequents, does not prove that that metainferential schema/rule is not derivable. But any other strategy will fail, because it is not possible to obtain, from the two premises, plus all the axioms and rules of SC6SK a sequent like the conclusion of *Cut3* in the failed proof, but with the cut-formula obtaining just in the left side, or just in the right side — which is what we need to get the conclusion of the translation of Cut in **ST** (i.e., that special sequent without occurrences of the cut-formula), together with the actual conclusion of *Cut3* in the proof.

These results seems like an asset of SC6SK, as they reproduce the relationship of each of these four Strong Kleene mixed logics with the two forms of Cut (i.e., context-sharing and context-free), i.e., that Cut is a metainferential schema that is locally valid in all of them except for **ST**.

But probably the most important result is the admissibility (i.e., eliminability) of the three *basic* forms of Cut SC6SK, i.e., *Cut1*, *Cut2* and *Cut3*, thus proving the theory is consistent. In the appendix, the proof of the following theorem is given:

Theorem 4.3 (Cut-elimination for SC6SK). *If a three-sided sequent $\Gamma \mid \Sigma \mid \Delta$ has a proof in* SC6SK*, then there is cut-free proof of $\Gamma \mid \Sigma \mid \Delta$ in* SC6SK *(i.e., a proof without occurrences of either Cut1, Cut2 or Cut3).*

5. Bridge principles for both accounts

In and of itself, claims about logical validity don't seem to be explicitly normative. Following [12], we say it is by way of some *bridge principles* that they give rise to norms. As the name indicates, these principles bridge facts about logical consequence with norms that govern epistemic attitudes towards propositions.[14]

A bridge principle takes the form of a material conditional: the antecedent states a fact about logical validity (i.e., that an inference is valid or a formula is a logical truth, or maybe that a given agent believes this is so), while its consequent contains a normative claim concerning the

[14] An anonymous referee points out that [7] presents a related view about the interpretation of the consequence relation. Though it is not explicitly stressed out as bridge principles, we thank her for establishing the connection of their work with [12].

agent's epistemic attitudes towards the relevant propositions. These are the formal commitments we have called *formal epistemic norms*.

Here, we propose two new schemas for generating bridge principles, one for each account presented. In the first one, the antecedent is a validity of a Strong Kleene logic **xy**, and the consequence is a formal epistemic norm connecting the propositional epistemic attitudes related to both **x** and **y**:

DEFINITION 5.1 ((First) Bridge Principle Schema). If $\Gamma \vDash_{\mathbf{xy}} \Delta$, then if you have (any combination of) the epistemic attitude(s) related to (the members of) **x** with respect to every γ in Γ, you must have[15] (one of) the epistemic attitude(s) related to **y** with respect to some $\delta \in \Delta$.

That is: either you have (any of the) epistemic attitude(s) related to $\overline{\mathbf{x}}$ towards some $\gamma \in \Gamma$, or you have (some of the) epistemic attitude(s) related to **y** towards some δ in Δ. Or stated negatively: if $\Gamma \vDash_{\mathbf{xy}} \Delta$, then do not have (any combination of) the epistemic attitude(s) related to **x** with respect to every $\gamma \in \Gamma$, while having (any of) the epistemic attitude(s) related to $\overline{\mathbf{y}}$ towards every $\delta \in \Delta$.

What these bridge principles tell us is that the valid inferences of a Strong Kleene logic are a way of expressing the *coherence* of an agent's epistemic attitudes towards certain propositions in a given normative situation. For that reason, violating a bridge principle equates to an agent holding incoherent epistemic attitudes in that situation.

For example, in the case of $\mathbf{n\bar{t}}$, a logic in which neither accepting nor rejecting all of the premises entails rejecting some conclusion, we can construct the following equivalent bridge principles:

Bridge principles for $\mathbf{n\bar{t}}$. If $\Gamma \vDash_{\mathbf{n\bar{t}}} \Delta$, then if you neither accept nor reject every γ in Γ, you must reject some δ in Δ. Equivalently: if $\Gamma \vDash_{\mathbf{n\bar{t}}} \Delta$, then either you accept or reject some γ in Γ, or you reject some δ in Δ. Or in negative form: if $\Gamma \vDash_{\mathbf{n\bar{t}}} \Delta$, then don't withhold your judgment over every γ in Γ while accepting or neither accepting nor rejecting every δ in Δ (since this principle states it is incoherent for an agent to neither accept nor reject every premise whilst accepting or neither accepting nor rejecting every conclusion).

[15] In the literature on the normativity of logic, we can find different variations w.r.t. the type of deontic operator (permission, obligation, etc.) and its scope (over the whole conditional, just the consequent, etc.). Here we decided to use the obligation operator with a narrow scope with respect to the consequent, but nothing hangs on this particular choice.

Now, let us go on to review the second bridge principle schema. In this case, the antecedent states a validity of **6SK**, and the consequence is a formal epistemic norm connecting the propositional epistemic attitudes related to Γ_i and Δ_j:

Definition 5.2 ((Second) Bridge Principle Schema). If $\Gamma_1 \mid \Gamma_{½} \mid \Gamma_0 \vDash \Delta_1 \mid \Delta_{½} \mid \Delta_0$, is valid in **6SK**, then if you accept every member of Γ_1, neither accept nor reject every member of $\Gamma_{½}$, and reject every member of Γ_0, then you must either accept some member of Δ_1, or neither accept nor reject some member of $\Delta_{½}$, or reject some member of Δ_0.

Notice that the second bridge principle schema is more general, since it implies every standard determination of the first one. In other words, each choice of **x** and **y** for the first schema is already present in the second one.

Now that we presented both ways of accounting for all possible formal epistemic norms referred to acceptance, rejection, and neither acceptance nor rejection, let us compare both approaches.

6. Comparison between the two accounts

One feature of **6SK** that we deem as an advantage over the first approach is its *expressive power*. In particular, the general framework provided by **6SK** can be considered more expressive than the plurality of mixed Strong Kleene logics, as it allows to represent in a clear and direct way some *material* commitments an agent might have, i.e., some commitments an agent assumed despite not being *formally* valid in **6SK** (nor in any mixed Strong Kleene logic).

For instance, suppose that $\Gamma_1 \mid \Gamma_{½} \mid \Gamma_0 \vDash \Delta_1 \mid \Delta_{½} \mid \Delta_0$ is not valid in **6SK**, but each Γ_i is different than the others Γ_j (for $i \neq j$) and is also different than each Δ_k. Assume something similar happens with each Δ_k. In that case, an agent embracing $\Gamma_1 \mid \Gamma_{½} \mid \Gamma_0 \vDash \Delta_1 \mid \Delta_{½} \mid \Delta_0$ means that if she accepts each formula in Γ_1, neither accepts nor reject each formula in $\Gamma_{½}$ and rejects each formula in Γ_0, then she is committed to either accepting some formula in Δ_1, or to neither accepting nor rejecting some formula in $\Delta_{½}$ or to rejecting some formula in Δ_0. From our viewpoint, this is an asset of the logic: even though six-sided sequents can have an equivalent formulation in some mixed Strong Kleene logic, they also might not, and it is not obvious how these kinds of material commitments could be expressed nor which mixed Strong Kleene logic is better suited for expressing them.

This is closely connected to the fact that **6SK** is *stronger* than the sum of each mixed Strong Kleene logic, i.e., it includes every inference valid in some mixed Strong Kleene logic, but also has validities that do not correspond to any valid inference in any mixed Strong Kleene logic. For instance, the six-sided inference $\varnothing \vDash q, s \mid q \mid q, r$ is **6SK**-valid, as every valuation v satisfies it. Nevertheless, it cannot be represented by any valid inference, in any of the mixed Strong Kleene logics.[16]

7. Conclusions and future work

In this article, we have presented two different ways of modelling every epistemic formal conditional commitment that involves (at most) three key epistemic attitudes towards propositions: acceptance, rejection and neither acceptance nor rejection. The first one is a pluralistic proposal, and uses the collection of every mixed Strong Kleene logic — *including some new logics that have so far been unexplored in the literature* — , accompanied by an epistemic reading of the truth-values. The second one employs six-sided inferences, and recovers the validities of each mixed Strong Kleene logic in one single framework. We have also introduced a sequent calculus that is sound and complete with respect to both approaches. Finally, we suggested that both the plurality of Strong Kleene logics and the unified system **6SK** gain normative status and model the intended formal epistemic norms thanks to bridge principles.

However, some lines of inquiry remain open. We briefly mentioned that there are several other schemas that one can choose to work with, instead of the ones given by the Strong Kleene valuations. So, exploring these other alternatives and the different mixed logics they would yield (that, in turn, might model a different set of formal epistemic norms, and subsequently give rise to new norms *via* some other bridge principles) is one possible route in which to continue researching. Thus,

[16] The proof is not hard, but quite long. In light of Fact 4.4, either $0 \in \mathbf{x}$ (i.e., the premise-standard) or not. In both cases, either $0 \in \mathbf{y}$ (i.e., the conclusion-standard) or not. If $0 \in \mathbf{x}$, then Γ_0 is empty. If not, $\Gamma_0 = \Gamma$. And if $0 \in \mathbf{y}$, then Δ_0 is empty. If not, $\Delta_0 = \Delta$. The proof then moves, in each of these four branches, to consider whether $\frac{1}{2} \in \mathbf{x}$ or not, and whether $\frac{1}{2} \in \mathbf{y}$ or not, and then to whether $1 \in \mathbf{x}$ or not, and whether $1 \in \mathbf{y}$ or not. Contradiction emerges sooner or later in any of these branches.

we leave as pending work the possibility of employing a different three-valued schema, such as the Weak Kleene or supervaluational one, or switching to a four-valued setting altogether. The latter would allow us to make more fine-grained distinctions among the set of attitudes we have grouped together under the label '*neither acceptance nor rejection*'. But, on the downside, working with extra truth-values (or with non-compositional schemas, for that matter) would increase the complexity of the investigation considerably.

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Appendix: Cut-Elimination

We demonstrate that the three rules of Cut are eliminable in SC6SK. The proof strategy is adapted from [10, §1.8.2]. We start by defining the height of a derivation:

Definition 7.1. Let $\mathcal{D}$ denote a derivation (proof), and S a sequent. Then we define the *height* of a derivation recursively as follows:

- Let $\mathcal{D}$ be a proof of an axiom. Then $h\mathcal{D} = 1$.
- Let $\mathcal{D}$ be a proof of S, S be immediately deduced from S', and $\mathcal{D}'$ be a proof of S'. Then $h\mathcal{D} = h\mathcal{D}' + 1$;
- Let $\mathcal{D}$ be a proof of S, S be immediately deduced from S' and S'', with $\mathcal{D}'$ and $\mathcal{D}''$ being their proofs, respectively. Then $h\mathcal{D} = \max(h\mathcal{D}', h\mathcal{D}'') + 1$.

Writing $\vdash_n S$ means that the sequent S has height n at most.

We provide a proof by means of a double induction: (i) on the sum of the heights of proofs of both premises, and (ii) on the complexity of the cut-formula A. This will allow us to prove, in the first place, that Cut on axiomatic sequents is eliminable. Then, the inductive step will establish that it is also eliminable for sequents of any height.

1. **Base case**. We show the proof for $Cut1$, the other two cases are completely analogous. Assume $\mathcal{D}$ is a derivation in which $Cut1$ is used to derive $\Gamma \mid \Sigma \mid \Delta$ from two instances of *Id*.

1.1. *Base case.* The cut-formula is a propositional variable, $A := p$. We have two subcases to consider, one where the cut-formula is active in *Id* and one where it is not:

- Subcase (i). If $\Gamma, A \mid \Sigma \mid \Delta := \Gamma, p \mid \Sigma', p \mid \Delta', p$, then considering the context-sharing requirement, the other premise is of the form $\Gamma \mid \Sigma', p, p \mid \Delta', p$ whereas the conclusion is $\Gamma \mid \Sigma', p \mid \Delta', p$. Given that we are working with sets, both the conclusion and the right premise are identical.
- Subcase (ii). If a formula B, which is not the cut-formula, appears in Γ, Σ and Δ, then the conclusion of *Cut1* is $\Gamma \mid \Sigma \mid \Delta := \Gamma', B \mid \Sigma', B \mid \Delta', B$ (an axiom). Therefore, it can be proven in a one step derivation without using *Cut1*.

1.2. *Inductive step.* If Cut on cut-formulas of complexity $< k$ is eliminable, then Cut on a cut-formula of complexity k is eliminable. Notice that the basis above covers the cases where the cut-formula is a negation or a conjunction, since nothing in the proof depends on A being atomic. Hence, the proof is the same.

Conclusion of 1.1 and 1.2: Cut on axiomatic sequents is eliminable.

2. **Inductive step**. If Cut with the left (right) height $< n$ is eliminable, then Cut with height n is eliminable too. We have to consider the following (exhaustive) cases: either the cut-formula is principal in both premises or is not principal in at least one premise.

2.1. *Base case.* First, consider the case where the cut-formula is an atom and is principal in both premises. If this is the case, then $A := p$ and it was obtained from *Id* or *Weak.* If at least one premise of Cut is an axiom, we proceed in a similar fashion as in 1. Otherwise, the cut-formula is an atom, is principal in both premises and it was obtained from *Weak*:

$$\dfrac{\textit{Weak}\;\dfrac{\begin{matrix}\vdots\\ \Gamma \mid \Sigma \mid \Delta\end{matrix}}{\Gamma, p \mid \Sigma \mid \Delta} \qquad \textit{Weak}\;\dfrac{\begin{matrix}\vdots\\ \Gamma \mid \Sigma \mid \Delta\end{matrix}}{\Gamma \mid \Sigma, p \mid \Delta}}{\Gamma \mid \Sigma \mid \Delta}\;\textit{Cut1}$$

Hence, there is also a derivation of $\Gamma \mid \Sigma \mid \Delta$ of lesser height that, by hypothesis, uses no Cuts.

Second, suppose the cut-formula is an atom and is not principal in at least one of the premises. Let it be the right premise. Then there are several subcases to consider: one where the right premise $\varGamma \mid \varSigma, p \mid \varDelta$ was obtained by one of the rules for negation, the rules for conjunction or *Weak*. We see what happens when it is deduced from a single-premise and a multipremise rule.

Subcase (i). The right premise deduced by L¬:

$$
Cut1\ \dfrac{\begin{matrix}\vdots\\ \varGamma', \neg B, p \mid \varSigma \mid \varDelta\end{matrix} \qquad \mathrm{L}\neg\ \dfrac{\begin{matrix}\vdots\\ \vdash_{m-1} \varGamma' \mid \varSigma, p \mid \varDelta, B\end{matrix}}{\vdash_m \varGamma', \neg B \mid \varSigma, p \mid \varDelta}}{\varGamma', \neg B \mid \varSigma \mid \varDelta}
$$

Take the derivation of the left premise and replace each $\varDelta$ with $\varDelta, B$. All such replacements keep the application of rules correct, since B appears as context in every axiomatic sequent used to derive $\varGamma', \neg B, p \mid \varSigma \mid \varDelta$ and also in each subsequent step. Hence, we obtain a derivation of $\varGamma', \neg B, p \mid \varSigma \mid \varDelta, B$.

Take the derivation of the right premise and replace each $\varGamma'$ with $\varGamma', \neg B$. Again, these replacements also keep the application of rules correct. We omit the L¬ step, since it is no longer necessary at this stage, thus reducing the height of the right premise:

$$
\mathrm{L}\neg\ \dfrac{Cut1\ \dfrac{\begin{matrix}\vdots\\ \varGamma', \neg B, p \mid \varSigma \mid \varDelta, B\end{matrix} \qquad \begin{matrix}\vdots\\ \vdash_{m-1} \varGamma', \neg B \mid \varSigma, p \mid \varDelta, B\end{matrix}}{\varGamma', \neg B \mid \varSigma \mid \varDelta, B}}{\varGamma', \neg B, \neg B \mid \varSigma \mid \varDelta}
$$

We repeat $\neg B$ for clarity, but given that we are working with sets, we know we have arrived at the same sequent. Notice that we did not reduce the complexity of the cut-formula, but we made a permutation of the application of rules after which the new Cut is of lesser height, so it follows (by the inductive hypothesis) that it is eliminable.

Subcase (ii). The right premise deduced by R∧:

$$
Cut1\ \dfrac{\begin{matrix}\vdots\\ \vdash_n \varGamma, p \mid \varSigma \mid \varDelta', B \wedge C\end{matrix} \qquad \mathrm{R}\wedge\ \dfrac{\begin{matrix}\vdots\\ \varGamma \mid \varSigma, p \mid \varDelta', B\end{matrix} \qquad \begin{matrix}\vdots\\ \varGamma \mid \varSigma, p \mid \varDelta', C\end{matrix}}{\vdash_m \varGamma \mid \varSigma, p \mid \varDelta', B \wedge C}}{\vdash_j \varGamma \mid \varSigma \mid \varDelta', B \wedge C}
$$

Where $j = \max(n, m) + 1$. After performing similar replacements, using *Cut1*, we arrive at:

$$\mathrm{R}\wedge\dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots\\ \Gamma, p \mid \Sigma \mid \Delta', B\wedge C, B & \Gamma \mid \Sigma, p \mid \Delta', B\wedge C, B\end{array}}{\vdash_{j'<j} \Gamma \mid \Sigma \mid \Delta', B\wedge C, B} \quad \dfrac{\begin{array}{cc}\vdots & \vdots\\ \Gamma, p \mid \Sigma \mid \Delta', B\wedge C, C & \Gamma \mid \Sigma, p \mid \Delta', B\wedge C, C\end{array}}{\vdash_{j'<j} \Gamma \mid \Sigma \mid \Delta', B\wedge C, C}}{\Gamma \mid \Sigma \mid \Delta', B\wedge C, B\wedge C}$$

Now we have introduced two cuts instead of one but both of lower height, and thus, eliminable by inductive hypothesis.

2.2. *Inductive step.* If Cut on cut-formulas of complexity $< k$ is eliminable, then Cut on a cut-formula of complexity k is eliminable. First, consider the case where the cut-formula is principal in both premises.

Subcase (i). The cut-formula is principal in both premises and it is a negation. Then this

$$\mathit{Cut1}\ \dfrac{\mathrm{L}\neg\dfrac{\begin{array}{c}\vdots\\ \Gamma \mid \Sigma \mid \Delta, A\end{array}}{\Gamma, \neg A \mid \Sigma \mid \Delta} \quad \mathrm{M}\neg\dfrac{\begin{array}{c}\vdots\\ \Gamma \mid \Sigma, A \mid \Delta\end{array}}{\Gamma \mid \Sigma, \neg A \mid \Delta}}{\Gamma \mid \Sigma \mid \Delta}$$

is replaced with:

$$\mathit{Cut2}\ \dfrac{\begin{array}{cc}\vdots & \vdots\\ \Gamma \mid \Sigma \mid \Delta, A & \Gamma \mid \Sigma, A \mid \Delta\end{array}}{\Gamma \mid \Sigma \mid \Delta}$$

The original derivation is not replaced with a proof of the conclusion without Cut. However, in the new proof, Cut is performed on a formula of lesser complexity, hence it is eliminable by the (subsidiary) inductive hypothesis.

Subcase (ii). The cut-formula is principal in both premises and it is a conjunction. Then this

$$\mathit{Cut1}\ \dfrac{\mathrm{L}\wedge\dfrac{\begin{array}{c}\vdots\\ \Gamma, A, B \mid \Sigma \mid \Delta\end{array}}{\Gamma, A\wedge B \mid \Sigma \mid \Delta} \quad \mathrm{M}\wedge\dfrac{\begin{array}{ccc}\vdots & \vdots & \vdots\\ \Gamma \mid \Sigma, A \mid \Delta, A & \Gamma \mid \Sigma, B \mid \Delta, B & \Gamma \mid \Sigma, A, B \mid \Delta\end{array}}{\Gamma \mid \Sigma, A\wedge B \mid \Delta}}{\Gamma \mid \Sigma \mid \Delta}$$

is replaced with:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \mid \Sigma,B \mid \Delta,B}}{\Gamma,A \mid \Sigma,B \mid \Delta,B}\,W \quad \dfrac{\dfrac{\vdots}{\Gamma,A,B \mid \Sigma \mid \Delta}}{\Gamma,A,B \mid \Sigma,B \mid \Delta}\,W}{\Gamma,A \mid \Sigma,B \mid \Delta}\,Cut3 \quad \dfrac{\vdots}{\Gamma,A,B \mid \Sigma \mid \Delta}}{\Gamma, A \mid \Sigma \mid \Delta}\,Cut1 \quad \dfrac{\dfrac{\mathfrak{B}_1}{\Gamma,B \mid \Sigma,A \mid \Delta} \quad \dfrac{\vdots}{\Gamma \mid \Sigma,A,B \mid \Delta}}{\Gamma \mid \Sigma,A \mid \Delta}\,Cut1}{\Gamma \mid \Sigma \mid \Delta}\,Cut1$$

Where $\mathfrak{B}_1$ is:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma,A,B \mid \Sigma \mid \Delta}}{\Gamma,A,B \mid \Sigma,A \mid \Delta}\,Weak \quad \dfrac{\dfrac{\vdots}{\Gamma \mid \Sigma,A \mid \Delta,A}}{\Gamma,B \mid \Sigma,A \mid \Delta,A}\,Weak}{\Gamma,B \mid \Sigma,A \mid \Delta}\,Cut3$$

Notice that, even though we used more Cuts than before, all are eliminable by the inductive hypothesis because they were performed on formulas of lesser complexity.

Now we move on to the case where the cut-formula is not principal in at least one premise. If it is not principal in the left premise, then some formula B in either Γ, Σ or Δ was introduced by *Weak*, or by the corresponding negation rule, or by the corresponding conjunction rule. This leaves us with 9 subcases to consider. We prove two, and leave the remaining cases to the reader.

Subcase (i). The left premise is deduced from M$\wedge$:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma,A \mid \Sigma, B \mid \Delta, B} \quad \dfrac{\vdots}{\Gamma,A \mid \Sigma, C \mid \Delta, C} \quad \dfrac{\vdots}{\Gamma,A \mid \Sigma, B, C \mid \Delta}}{\Gamma,A \mid \Sigma, B \wedge C \mid \Delta}\,\mathrm{M}\wedge \quad \dfrac{\vdots}{\Gamma \mid \Sigma,A, B \wedge C \mid \Delta}}{\Gamma \mid \Sigma, B \wedge C \mid \Delta}\,Cut1$$

Take the derivation of each premise of M$\wedge$ and replace each Σ with $\Sigma, B \wedge C$. As previously stated, these replacements keep the application of rules correct, given that $B \wedge C$ appears as context in every axiomatic sequent and also in each subsequent step.

First, we take the right premise of *Cut1* and perform similar replacements, in order to obtain a derivation of $\Gamma \mid \Sigma, A, B \wedge C, B \mid \Delta, B$. Then, using *Cut1*, we obtain that:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, A \mid \Sigma, B \wedge C, B \mid \Delta,B} \quad \dfrac{\vdots}{\Gamma \mid \Sigma, A, B \wedge C, B \mid \Delta, B}}{\Gamma \mid \Sigma, B \wedge C, B \mid \Delta, B} \quad \dfrac{\mathfrak{B}_1}{\Gamma \mid \Sigma, B \wedge C, C \mid \Delta, C} \quad \dfrac{\mathfrak{B}_2}{\Gamma \mid \Sigma, B \wedge C, B, C \mid \Delta}}{\Gamma \mid \Sigma, B \wedge C, B \wedge C \mid \Delta}\,\mathrm{M}\wedge$$

Second, we adjust contexts in the derivation of the right premise of *Cut1* once more, so as to get $\Gamma \mid \Sigma, A, B \wedge C, C \mid \Delta, C$. Thus, we get that $\mathfrak{B}_1$ is:

$$\dfrac{\begin{matrix}\vdots \\ \Gamma, A \mid \Sigma, B \wedge C, C \mid \Delta, C\end{matrix} \qquad \begin{matrix}\vdots \\ \Gamma \mid \Sigma, A, B \wedge C, C \mid \Delta, C\end{matrix}}{\Gamma \mid \Sigma, B \wedge C, C \mid \Delta, C}\ \mathit{Cut1}$$

Third, we make the pertinent changes in the right premise of *Cut1* one last time, in order to get a derivation of $\Gamma \mid \Sigma, A, B \wedge C, B, C \mid \Delta$. From here, we get that $\mathfrak{B}_2$ is:

$$\dfrac{\begin{matrix}\vdots \\ \Gamma, A \mid \Sigma, B \wedge C, B, C \mid \Delta\end{matrix} \qquad \begin{matrix}\vdots \\ \Gamma \mid \Sigma, A, B \wedge C, B, C \mid \Delta\end{matrix}}{\Gamma \mid \Sigma, B \wedge C, B, C \mid \Delta}\ \mathit{Cut1}$$

Subcase (ii). The left premise is deduced from M¬:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \vdash_{n-1} \Gamma, A \mid \Sigma, B \mid \Delta\end{matrix}}{\vdash_n \Gamma, A \mid \Sigma, \neg B \mid \Delta}\ \mathrm{M}\neg \qquad \begin{matrix}\vdots \\ \vdash_m \Gamma \mid \Sigma, A, \neg B \mid \Delta\end{matrix}}{\Gamma \mid \Sigma, \neg B \mid \Delta}\ \mathit{Cut1}$$

Take the derivation of the left sequent and change each Σ for $\Sigma, \neg B$, and take the derivation of the right sequent and change each Σ for Σ, B. With these changes, we replace the previous proof with:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \vdash_{n-1} \Gamma, A \mid \Sigma, B, \neg B \mid \Delta\end{matrix} \qquad \begin{matrix}\vdots \\ \vdash_m \Gamma \mid \Sigma, A, B, \neg B \mid \Delta\end{matrix}}{\Gamma \mid \Sigma, B, \neg B \mid \Delta}\ \mathit{Cut1}}{\Gamma \mid \Sigma, \neg B, \neg B \mid \Delta}\ \mathrm{M}\neg$$

Given that Contraction is worked into the system since we are using sets, this permutation gives us a proof of the same sequent.

CONCLUSION OF 2.1 AND 2.2: *Cut on sequents of any height is eliminable.*

CONCLUSION OF 1 AND 2: *Cut is eliminable.*

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