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Some Remarks on the Logic of Probabilistic Relevance

Abstract. In this paper we deepen some aspects of the statistical approach to relevance by providing logics for the syntactical treatment of probabilistic relevance relations. Specifically, we define conservative expansions of Classical Logic endowed with a ternary connective $\rightsquigarrow$ — indeed, a constrained material implication — whose intuitive reading is "x materially implies y and it is relevant to y under the evidence z". In turn, this ensures the definability of a formula in three-variables $\mathbb{R}(x, z, y)$ which is the representative of relevance in the object language. We outline the algebraic semantics of such logics, and we apply the acquired machinery to investigate some term-defined weakly connexive implications with some intuitive appeal. As a consequence, a further motivation of (weakly) connexive principles in terms of relevance and background assumptions obtains.

Keywords: relevance; statistical relevance; relevant implication; relating logic; connexive logic

1. Introduction

Detecting *relevant* facts is essential in reasoning. It allows us to reduce the search space of valuable information when dealing with propositions whose truth value cannot be deduced by our acquired body of knowledge.

Depending on its specific applications, the concept of relevance has been treated within different disciplines, from different perspectives, and by means of different methodologies. Such a heterogeneity becomes even more striking if one considers that, over time, "to be relevant" has assumed, implicitly or explicitly, different meanings, even within the same field of inquiry (think, e.g., to information theory [11, 28, 40]), depending on the specific functions it has been meant for.

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Over the past years, the theory of *statistical relevance* (SR) has received great attention due to its ubiquitous applications in philosophy of science and Artificial Intelligence. An *event* A is said to be statistically relevant to an event C under an evidence B with respect to a probability measure P, whenever it holds that

$$P(C|B) \neq P(C|A\&B).$$

Put another way, A is relevant to C under B if the probable occurrence of A provides a supplement of information to the "background assumption" B in order to estimate the probability of C.

From a philosophical perspective, SR has been investigated in the framework of the theory of scientific explanation. Making good use of *relevance measures*, R. Carnap [3] examines positive/negative relevance and its properties within the framework of logical foundations of probability. W. C. Salmon [38, 39] applies SR to explain the relationship between the *explanans* and the *explanandum* in his theory of scientific explanation, with the aim of overcoming difficulties arising in Hempelian *inductive-statistical* and *deductive-nomological* models [see, e.g., 17]. Amongst others, an important application of SR is in defining *(in)dependency* of random variables, a key ingredient to modeling reasoning under uncertainty in Artificial Intelligence (see p. 113 below and, e.g., [33, 34]).

Although recognized as one of the most important concepts in epistemology, some counter-intuitive aspects of SR led some authors to question its actual adequacy in capturing the notion of relevance, as assumed in concrete argumentation. Indeed, the conjunction of two events A and B which are positively (negatively) relevant to a given hypothesis C on the light of a certain piece of evidence D need not be itself positively (negatively) relevant: it can be either negatively (positively) relevant or even irrelevant [see, e.g., 3, §65, §69]. Therefore, upon denoting by R_C the relation of being statistically relevant under the evidence that C, one has that the condition

$$A \,\mathrm{R}_C\, B,\ D \,\mathrm{R}_C\, B \text{ and } A \cap D \cap C \neq \emptyset \text{ imply } (A \cap D) \,\mathrm{R}_C\, B, \tag{1}$$

need not hold in general. However, as observed by J. M. Keynes, if one considers A relevant to B if it *increases* the "weight" of an argument supporting or rejecting B, then it is reasonable, and theoretically preferable, to assume that a proposition part of which is for and part against

B must be considered, taken as a whole, as relevant even if it leaves the probability of B unchanged [cf. 23]. Therefore, SR seems not to cope well with information which is not relevant *per se*, but might be so in the light of new evidence. To overcome such difficulties, Keynes provides a notion of relevance to the effect that A is relevant to B under an evidence C if it entails jointly with C an event D, not itself entailed by C alone, which is statistically relevant to B under C. In other words, A is relevant to B under C if, together with C produces some information, not already contained in C, which affects the probability of B. Unfortunately, such a definition was later shown to have a *trivializing* effect [see 3].

Starting from Keynes' perspective on SR, P. Gärdenfors [13] aims to provide a "logic of relevance" which, on the one hand, preserves important features of SR and, on the other hand, is able to partially cope with the aforementioned theoretical difficulties. First, it is important to observe that [13] treats relevance relations and probability measures as relations and probabilities on *formulas* in the language of Classical Logic **CPL**, rather than on events. In the sequel, we will use $\alpha \, \mathrm{R}_\gamma \, \beta$ as a shortcut for "α is relevant to β under the evidence γ", while $\alpha \, \mathrm{I}_\gamma \, \beta$ will stand for "α is irrelevant to β under γ". Indeed, as Gärdenfors points out, SR satisfies the following conditions, for any $\alpha, \beta, \gamma, \delta$:

(R0) $\vdash_{\mathbf{CPL}} \gamma \to (\alpha \leftrightarrow \delta)$ implies $\alpha \, \mathrm{R}_\gamma \, \beta$ iff $\delta \, \mathrm{R}_\gamma \, \beta$;
(R1) $\alpha \, \mathrm{R}_\gamma \, \beta$ iff not $\alpha \, \mathrm{I}_\gamma \, \beta$;
(R2) $\alpha \, \mathrm{R}_\gamma \, \beta$ iff $\neg\alpha \, \mathrm{R}_\gamma \, \beta$;
(R3) $(\alpha \vee \neg\alpha) \, \mathrm{I}_\gamma \, \beta$;
(R4) If α is *contingent* on γ, then $\alpha \, \mathrm{R}_\gamma \, \alpha$,

where 'α is contingent on γ' here means that $\nvdash_{\mathbf{CPL}} \gamma \to \neg\alpha$ and $\nvdash_{\mathbf{CPL}} \gamma \to \alpha$. To satisfy Keynes' requirements, the ideal would be extending the above set of axioms by the condition

$$\alpha \, \mathrm{R}_\gamma \, \beta \text{ and } \nvdash_{\mathbf{CPL}} \neg(\alpha \wedge \delta \wedge \gamma) \text{ imply } (\alpha \wedge \delta) \, \mathrm{R}_\gamma \, \beta. \tag{R5}$$

Unfortunately, this is not possible on pain of trivialization. In fact, adding (R5) to (R0)–(R4) entails that, for any α, β, γ, if α and β are contingent on γ, then $\alpha \, \mathrm{R}_\gamma \, \beta$ [see 13, Theorem 1]. Nevertheless, if one interprets Keynes' intuition as asserting that $\alpha \, \mathrm{R}_\gamma \, \beta$ holds if either α is statistically relevant to β under γ *or* there exists δ which is not *per se* relevant to β under γ, but it does under the evidence that $\alpha \wedge \gamma$, then one obtains that for this new notion of relevance (R0)–(R4) hold. Moreover, although it still does not satisfy, in general, condition (1) [by Theorem 1, Theorem 2 13], the following *conjunction criterion for irrelevance* holds.

$$\text{If } \alpha \,\mathrm{I}_\gamma\, \beta, \text{and } \delta \,\mathrm{I}_\gamma\, \beta \text{ then } (\alpha \wedge \delta) \,\mathrm{I}_\gamma\, \beta.$$

It is not difficult to see that the above accounts of the logic of relevance based on probability theory, henceforth called probabilistic relevance, namely [3, 13], and those who criticize it [see, e.g., 41] have always provided "logics" of relevance in terms of sets of conditions/postulates that the relevance relation should satisfy. However, a natural question arises. Is it possible to define a logic of probabilistic relevance to be meant as a *structural consequence relation* [in the sense of 12]? Put another way, is it possible to define a logic $\vdash$ in which a relation of probabilistic relevance (or some not too abstract generalization thereof) can be treated syntactically? What should a suitable semantics for it look like? The present paper attempts to provide answers to these questions.

Relating logics [RLs, see, e.g., 19, 21] rest on the idea that the semantical evaluation of complex formulas in a model might depend not only on the "truth value" of their subformulas, but also on the intensional relation among them. Interestingly enough, such investigations have provided logical systems which are expansions of **CPL** by means of *relating implications* obtained from material implications by imposing constraints depending on certain binary (*relating*) relations between antecedents and consequents: the so-called *Boolean Logics with Relating Implication* (BLRI). For the purposes of the present work, one of the most interesting features of these logics is the fact that, in many cases, relating implications allow us to represent relating relations in the object language [20].

Inspired by relating logics' machinery, we axiomatize three logics, **D**, **G**, and **SR**, which are expansions of **CPL** with a ternary connective $\rightsquigarrow$ such that $\rightsquigarrow(\alpha, \gamma, \beta)$ has the following reading: "α *materially* implies β and α is relevant to β under the evidence γ", where 'relevant' can be interpreted in terms of a common generalization of SR and Gärdenfors' relevance, Gärdenfors relevance, and SR, respectively. In turn, $\rightsquigarrow$ allows us to define a three-variable term $\mathbb{R}(x, y, z)$ which, depending on the specific axioms of the system under exam, satisfies salient features of SR, Gärdenfors' relevance, or both.

To define such logics, we follow the *reverse algebraisation* approach, as advocated, e.g., in [4]: we introduce suitable quasi-varieties of algebras — indeed, expansions of Boolean algebras — which, on the basis of general results on the algebraizability of logics, turn out to be the equivalent algebraic semantics of **D**, **G**, or **SR**. Interestingly enough, these algebras find their “concrete” representatives in Boolean algebras with a ternary operation which coincides with material implication, if the antecedent is statistically/Gärdenfors-like relevant to the consequent with respect to a strictly positive state, and bottom otherwise.

The idea of a consequent which is entailed by the antecedent modulo some “background information” is clearly not new in the literature. For example, this is in some way codified, as M. J. Dunn observes in [6], by truth conditions for relevant implications axiomatized by logics R and E [1] *via* their Routley-Meyer semantics. However, in that work, relevance is interpreted according to Sperber and Wilson’s theory [45]. The idea of expanding **CPL** with relevant implications can be found, e.g., in [27].

It can be seen that our approach allows us to treat probabilistic relevance in a qualitative shape. This fact is compatible with J. Pearl’s recommendations [34]. As Pearl writes,

> In a commonsense reasoning system, [...] the language used for representing information should allow assertions about dependency relationships to be expressed qualitatively, directly, and explicitly.

Therefore,

> it would be interesting to explore how assertions about relevance can be inferred qualitatively, and whether assertions equivalent to those made about probabilistic dependencies can be derived *logically* without references to numerical quantities. [34, pp. 79–81]

Moreover, our approach allows us to overcome some difficulties arising in alternative theories of relevance based on logical consequence. Indeed, in some of the latest philosophical investigations on the subject, many authors have agreed that relevance should be better defined at a *metalogical* level [see, e.g., 5]. Although capable of capturing the intuitive concept of relevance at a great level of generality, these accounts make it difficult to deal with the reiteration or self-embedding of ‘relevant’, like, e.g., ‘α is relevant to β under δ is relevant to γ’. Moreover, some metalogical approaches to relevance do not address the problem of providing formal systems to treat relevance relations syntactically (proof-theoretically) or, from an algebraic perspective, arithmetically [see, e.g., 5]. Although with a lower level of generality, our approach allows to address both challenges with ease.

Connexive logic [43] is a well established stream of research aimed at investigating formal systems which provide accounts of *connections*, or *compatibilities*, between antecedents and consequents of sound conditionals. Such connections are expressed in a language containing a unary (negation) connective $\sim$ and a binary (implication) connective $\Rightarrow$ by means of the following formulas:

(AT1) $\sim(\alpha \Rightarrow \sim\alpha)$;
(AT2) $\sim(\sim\alpha \Rightarrow \alpha)$;
(BT1) $(\alpha \Rightarrow \beta) \Rightarrow \sim(\alpha \Rightarrow \sim\beta)$;
(BT2) $(\alpha \Rightarrow \sim\beta) \Rightarrow \sim(\alpha \Rightarrow \beta)$.

(AT1) and (AT2) are commonly known as Aristotle's Theses, while (BT1) and (BT2) are known as Boethius' Theses. A connexive logic is nothing but a logic $\vdash$ having the above formulas as theorems with respect to a negation $\neg$ and a non-symmetric implication $\rightarrow$, where "non-symmetric" here means that the following inference rule does not hold:[1]

$$(\alpha \rightarrow \beta) \vdash (\beta \rightarrow \alpha).$$

The latter requirement, which we call *the principle of non-symmetry*, is essential, since $\rightarrow$ must be understood as a genuine implication rather than as an equivalence. Apparently, (AT1), (AT2), (BT1), and (BT2) are falsified in **CPL** whenever implications with *false* antecedents are considered.

The concept of relevance has been advocated to clarify the kind of connection between antecedents and consequents of connexive implications. Indeed, taking his cue from the influential [31], R. Routley has argued that such a connection should be meant as an *intentional* relevance relation between contents [36]. However, against that intuition, relevance logics [1] and connexive principles turn out to be mutually "incompatible", in the sense that their combination leads to logics which are contradictory, trivial, or witness the failure of the Variable Sharing Principle or the Effective Use in Proof [9], the trademarks of relevance logic.

In the last part of this work we put in good use the acquired algebraic machinery to prove that, indeed, something like Routley's intuition can be obtained once relevance is understood as probabilistic. In fact, we will show that the ternary operation $\rightsquigarrow$ allows to define binary connectives

[1] In the literature non-symmetry of implication is often assumed just by requiring that $(A \rightarrow B) \rightarrow (B \rightarrow A)$ is not a theorem [see, e.g., 43].

with some intuitive meaning which can be meant as implications (so they are non-symmetric), and satisfy Aristotle's Theses as well as weak forms of Boethius' Theses with respect to classical negation.

The paper is organized as follows. In Section 2 we dispatch basic notions that will be expedient for the development our arguments. Section 3 presents the notion of statistical and weak relevance in the framework of Boolean algebras and (strictly positive) states over them. Section 4 introduces d-algebras, G-algebras, and SR-algebras, which will be shown (Section 4.1) to be the algebraic semantics of logics **D**, **G** and **SR**, respectively. In Section 4.2 we prove that some term-definable operations in d-algebras, G-algebras, or SR-algebras, induce weakly connexive implications in the corresponding logics. We close in Section 5.

2. Preliminaries

In this section we recall basic notions that will be useful in the sequel. We assume the reader is familiar with basic concepts of universal algebra and Boolean algebras. We refer to [2] and [15] for details.

2.1. Some algebraic logic

Let $\mathrm{Fm}_{\mathcal{L}}$ be the absolutely free algebra (the formula-algebra) of a fixed type $\mathcal{L}$ built up over a countably infinite set Var of propositional variables. A *consequence relation* on $\mathrm{Fm}_{\mathcal{L}}$ is a relation $\vdash\, \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}) \times \mathrm{Fm}_{\mathcal{L}}$ s.t. for all $\Gamma, \Delta \in \mathcal{P}(\mathrm{Fm}_{\mathcal{L}})$ and $\alpha \in \mathrm{Fm}_{\mathcal{L}}$,

(R) If $\alpha \in \Gamma$, then $\Gamma \vdash \alpha$.
(M) If $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \alpha$.
(C) If $\Gamma \vdash \alpha$ and $\Delta \vdash \beta$ for all $\beta \in \Gamma$, then $\Delta \vdash \alpha$.

A *logic* $\boldsymbol{L}$ [see 12, Definition 1.5] is a consequence relation $\vdash_{\boldsymbol{L}}\, \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}) \times \mathrm{Fm}_{\mathcal{L}}$, which is *substitution-invariant* in the sense that for every substitution (i.e. endomorphism) $\sigma\colon \mathrm{Fm}_{\mathcal{L}} \to \mathrm{Fm}_{\mathcal{L}}$, if $\Gamma \vdash_{\boldsymbol{L}} \alpha$, then $\sigma\Gamma \vdash_{\boldsymbol{L}} \sigma\alpha$. Whenever the reference to $\boldsymbol{L}$ will be clear from the context, we will write $\vdash$ in place of $\vdash_{\boldsymbol{L}}$.

Given a fixed type $\mathcal{L}$, we will denote by $\mathrm{Eq}_{\mathcal{L}}$ the set of *equations* on $\mathrm{Fm}_{\mathcal{L}}$, i.e., the set of pairs of the form (φ, ψ), where $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$. Using a customary notation, whenever no danger of confusion will result, we will denote an equation (φ, ψ) by $\varphi \approx \psi$. We call a *transformer of* *equations in (a set of) formulas* a mapping $\rho\colon \mathrm{Eq}_{\mathcal{L}} \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}})$. Similarly we define a *transformer τ of formulas in a set of equations.* A transformer is *structural* if it commutes with substitutions. It is easily seen that a transformer ρ from equations to formulas is structural if and only if there exists $\Delta(x, y) \subseteq \mathrm{Fm}_{\mathcal{L}}$ such that for any $\epsilon, \delta \in \mathrm{Fm}_{\mathcal{L}}$, $\rho(\epsilon = \delta) := \Delta(\epsilon, \delta)$. Similarly, a transformer τ from formulas to equations is structural if and only if there exists $E(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ such that for any $\alpha \in \mathrm{Fm}_{\mathcal{L}}$, $\tau(\varphi) := E(\varphi)$.

Definition 2.1. A logic $\boldsymbol{L}$ is *algebraizable* when there is a class $\mathcal{K}$ of algebras (of the same type) and there are structural transformers τ, ρ (from formulas to equations and from equations to formulas, respectively) such that for all $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ and all $\Theta \cup \{\epsilon = \delta\} \subseteq \mathrm{Eq}$, the following hold:

(ALG1) $\Gamma \vdash_{\boldsymbol{L}} \alpha$ if and only if $\tau(\Gamma) \models_{\mathcal{K}} \tau(\alpha)$;
(ALG2) $\Theta \models_{\mathcal{K}} \epsilon = \delta$ if and only if $\rho(\Theta) \vdash_{\boldsymbol{L}} \rho(\epsilon \approx \delta)$;
(ALG3) $x \dashv\vdash_{\boldsymbol{L}} \rho\tau(x)$;
(ALG4) $x \approx y \Dashv\models_{\mathcal{K}} \tau\rho(x \approx y)$.

By [12, Proposition 3.12], a logic $\boldsymbol{L}$ satisfies (ALG1) and (ALG4) w.r.t. structural transformers τ, ρ if and only if it satisfies (ALG2) and (ALG3). Therefore, $\boldsymbol{L}$ is algebraizable by means of ρ, τ iff it satisfies one of the above equivalent sets of conditions. In this case, we call $\Delta(x, y)$ and $E(x)$ the *equivalence formulas* and the *defining equations*, respectively.

Remark 2.1 (cf. 12, Corollary 3.18). A logic $\boldsymbol{L}$ is algebraizable iff it is algebraizable (with the same transformers) with respect to a *largest* (indeed the only) *generalized quasivariety* $\mathcal{K}^*$ (of course, of the same type) called the *equivalent algebraic semantics* of $\boldsymbol{L}$ [see 12, Definition 1.72].

Proposition 2.1 (cf. 12, Proposition 3.13). *Let $\boldsymbol{L}$ be an algebraizable logic with equivalence formulas $\Delta(x, y) \subseteq \mathrm{Fm}_{\mathcal{L}}$. The following conditions are satisfied*:

(R) $\vdash_{\boldsymbol{L}} \Delta(x, x)$
(Sy) $\Delta(x, y) \vdash_{\boldsymbol{L}} \Delta(y, x)$
(Tr) $\Delta(x, y) \cup \Delta(y, z) \vdash_{\boldsymbol{L}} \Delta(x, z)$
(Re) $\bigcup_{i=1}^{n} \Delta(x_i, y_i) \vdash_{\boldsymbol{L}} \Delta(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n))$, *for any $f \in \mathcal{L}$*
(MP) $x, \Delta(x, y) \vdash_{\boldsymbol{L}} y$

The labels (R), (Sy), (Tr), (Re), (MP) stand for 'Reflexivity', 'Symmetry', 'Transitivity', 'Replacement' and 'Modus Ponens', respectively.

The following proposition ensures that any extension as well as (under minimal requirements) any expansion of an algebraizable logic is still algebraizable.

PROPOSITION 2.2 (cf. 12, Proposition 3.31). *Let $\boldsymbol{L}$ be an algebraizable logic with respect to a class $\mathcal{K}$ of similar algebras and with transformers τ, ρ.*

1. *Every extension $\boldsymbol{L}'$ of $\boldsymbol{L}$ is algebraizable as well, with respect to a subclass $\mathcal{K}'$ of $\mathcal{K}$ and with the same transformers;*
2. *If $\boldsymbol{L}'$ is an expansion of $\boldsymbol{L}$ such that $\vdash_{\boldsymbol{L}'}$ satisfies condition (Re) for the additional connectives, then $\boldsymbol{L}'$ is algebraizable with the same transformers with respect to a class $\mathcal{K}'$ (of the same expanded type).*

Let $\boldsymbol{L}$ be an algebraizable logic. Then $\boldsymbol{L}$ is algebraizable *in the sense of Blok and Pigozzi* (*BP-algebraizable*) if it is both finitary and has a finite set of equivalence formulas [cf. 12, Definition 3.39].

PROPOSITION 2.3 (12, Proposition 3.44). *Let $\boldsymbol{L}$ be a BP-algebraizable logic, with equivalence formulas $\Delta(x, y)$ and defining equations $E(x)$, and let it be axiomatized by a set Ax of axioms and a set InfR of proper inference rules. Its equivalent algebraic semantics is given by the sets of equations*

(τ-Ax) $E(\alpha)$, *for each* $\alpha \in Ax$

and the sets of quasi-equations

(τ-*InfR*) $(\bigwedge \bigcup_{i=1}^{n} E(\alpha_i)) \curvearrowright \epsilon = \delta$, *for each* $\alpha_1, \ldots, \alpha_n \vdash \beta \in InfR$ *and* $\epsilon \approx \delta \in E(\beta)$;

(Red) $(\bigwedge E(\Delta(x, y)) \curvearrowright x = y$.[2]

Let $\mathrm{Fm}_{\mathbf{CPL}}$ be the absolutely free algebra generated by an infinite countable set of variables Var over the language $\{\wedge, \vee, \neg, \top, \bot\}$. The following result is well known.

PROPOSITION 2.4. *Classical Logic* **CPL** *is algebraizable with respect to the variety $\mathcal{BA}$ of Boolean algebras as its equivalent algebraic semantics by means of transformers $\tau\colon \mathrm{Fm}_{\mathbf{CPL}} \to \mathcal{P}(\mathrm{Fm}^2_{\mathbf{CPL}})$ and $\rho\colon \mathrm{Fm}^2_{\mathbf{CPL}} \to \mathcal{P}(\mathrm{Fm}_{\mathbf{CPL}})$ such that for any $\alpha \in Fm_{\mathbf{CPL}}$ and $\epsilon \approx \delta \in Fm^2_{\mathbf{CPL}}$, $\tau(\alpha) := \{\alpha \approx \top\}$ and $\rho(\epsilon \approx \delta) := \{\epsilon \leftrightarrow \delta\}$, where, for any $\alpha, \beta \in Fm_{\mathbf{CPL}}$, $\alpha \leftrightarrow \beta$ is defined in the customary way.*

[2] Note that here $\bigwedge$ stands for *metatheoretical* conjunction.

Recall that given a Boolean algebra $\boldsymbol{B} = (B, \wedge, \vee, ', \bot, \top)$[3], a (finitely additive) *state* over $\boldsymbol{B}$ is a mapping $m\colon B \to [0,1]$, where $[0,1]$ is the real unit interval, satisfying:

1. $m(\top) = 1$;
2. $m(x \vee y) = m(x) + m(y)$, for any $x, y \in A$ such that $x \leq y'$.

Of course, for any $\boldsymbol{B} \in \mathcal{BA}$, any homomorphism $h\colon \boldsymbol{B} \to \boldsymbol{B}_2$, where $\boldsymbol{B}_2$ is the two-elements Boolean algebra regarded as a sub-algebra of the *standard MV-algebra* over the real unit interval [see, e.g., 30], can be regarded as a state over $\boldsymbol{B}$.

A state over a Boolean algebra $\boldsymbol{B}$ is said to be *strictly positive* provided that $m(a) = 0$ implies $a = \bot$. In other words, a state is strictly positive if it assigns a degree of probability strictly greater than 0 to any proposition which is not a contradiction. Of course, not any Boolean algebra admits a strictly positive state. However, it does if and only if it satisfies Kelley's condition [22], namely it is a *countable union of subsets with positive intersection number* [cf. 7, Definition 0.1]. In the sequel, any state considered is meant to be strictly positive. Therefore, when dealing with a Boolean algebra and a state over it, we will always mean a Boolean algebra satisfying Kelley's condition.

Let us recall some well known properties of (strictly positive) states [see, e.g., 22, 25].

Lemma 2.5. *Let $\boldsymbol{B}$ be a Boolean algebra and let $m\colon \boldsymbol{B} \to [0,1]_{\mathbb{R}}$ be a state over $\boldsymbol{B}$. For any $x, y \in B$, the following hold*:

1. *$x < y$ implies $m(x) < m(y)$;*
2. *$m(x') = 1 - m(x)$;*
3. *$m(x \vee y) = m(x) + m(y) - m(x \wedge y)$;*

Given a Boolean algebra $\boldsymbol{B}$, let us denote by $\mathcal{S}^{\boldsymbol{B}}$ the set of states over $\boldsymbol{B}$. Moreover, for any $m \in \mathcal{S}^{\boldsymbol{B}}$ and $x, y \in B$ such that $m(x) \neq 0$, we define the *conditional probability* of y *given* x by

$$m(y|x) = \frac{m(x \wedge y)}{m(x)}.$$

[3] In order to keep our notation as unambiguous as possible, here and in the sequel the least (greatest) element of a Boolean algebra $\boldsymbol{B}$ will be denoted by $\bot$ ($\top$).

2.2. (Weakly) connexive theses in algebraizable expansions of CPL

Let $\boldsymbol{L}$ be an expansion of **CPL** over a language $\mathcal{L}'$ such that $\vdash_{\boldsymbol{L}}$ satisfies (Re) of Proposition 2.1 for any new connective from $\mathcal{L}'$. Then, by Proposition 2.2, $\boldsymbol{L}$ is algebraizable with the same transformers with respect to a generalized quasivariety $\mathcal{K}$ of expansions of Boolean algebras. Let $t(x,y), n(x) \in \mathrm{Fm}_{\mathcal{L}'}$ be a binary and a unary term, respectively. We say that $t(x,y)$ satisfies (AT1), (AT2), (BT1) or (BT2) in $\boldsymbol{L}$ w.r.t. $n(x)$ provided that the corresponding theorem obtained by replacing $\Rightarrow$ by $t(x,y)$, and $\neg$ by $n(x)$ holds in $\boldsymbol{L}$. In turn, this is equivalent to say that, for any $\boldsymbol{A} \in \mathcal{K}$, the following equational renderings of connexive theses hold:

(AT1) $\quad n(t(x,n(x))) \approx \top;$
(AT2) $\quad n(t(n(x),x)) \approx \top;$
(BT1) $\quad t(t(x,y),n(t(x,n(y)))) \approx \top;$
(BT2) $\quad t(t(x,n(y)),n(t(x,y))) \approx \top.$

In particular, if $n(x) = x'$, since $'$ is an involution, one has that connexive theses reduce, e.g., to (AT1) and (BT1). Moreover, it is easily seen that $t(x,y)$ is not symmetric, provided that there is an $\boldsymbol{A} \in \mathcal{K}$ in which the quasi-equation $t(x,y) = \top \curvearrowright t(y,x) = \top$ does not hold.

In [44], a weaker notion of connexivity is formulated. A logic $\boldsymbol{L}$ endowed with a binary and a unary connective $\Rightarrow$ and $\sim$, respectively, is called *weakly connexive* if it satisfies (AT1), (AT2), and the following two *weak* versions of Boethius theses:

(WB1) $\quad \alpha \Rightarrow \beta \vdash {\sim}(\alpha \Rightarrow {\sim}\beta)$
(WB2) $\quad \alpha \Rightarrow {\sim}\beta \vdash {\sim}(\alpha \Rightarrow \beta)$

In the light of the above discourse, given an expansion $\boldsymbol{L}$ of **CPL** which is algebraizable w.r.t. a generalized quasi-variety $\mathcal{K}$ by means of the same transformers, we say that a binary term $t(x,y)$ satisfies the weak connexive theses w.r.t. a unary term $n(x)$, if it is non-symmetric in the above sense and, moreover, the equational (AT1) and (AT2), as well as the following quasi-equational renderings of (WB1) and (WB2) hold in $\mathcal{K}$:

(WB1) $\quad t(x,y) = \top \curvearrowright n(t(x,n(y))) = \top;$
(WB2) $\quad t(x,n(y)) = \top \curvearrowright n(t(x,y)) = \top.$

Of course, if $n(x)$ is assumed to be x', weak connexivity reduces to satisfying, e.g., (AT1) and (WB1).

3. Statistical relevance and Gärdenfors' account

In this section we summarize probabilistic accounts of relevance [3, 13] that will be expedient for the development of our discourse, namely statistical relevance, and Gärdenfors' notion, henceforth called *weak relevance*. These concepts will be treated within the framework of Boolean algebras.

Definition 3.1. Let $\boldsymbol{B} \in \mathcal{BA}$, $a, b, c \in B$, and $m \in \mathcal{S}^{\boldsymbol{B}}$.

1. a is *m-statistically relevant* to b with respect to c, written $a \, \mathrm{R}_c^m \, b$, provided that $m(c \wedge a) \neq 0$ and $m(b|a \wedge c) \neq m(b|c)$;
2. a is *m-statistically irrelevant* to b with respect to c, written $a \, \mathrm{I}_c^m \, b$, provided that it does not hold that $a \, \mathrm{R}_c^m \, b$.
3. a is *m-weakly relevant* to b with respect to c, written $a\mathcal{R}_c^m b$, provided that $m(c \wedge a) \neq 0$, and at least one of the following hold:
   (a) $a \, \mathrm{R}_c^m \, b$;
   (b) There exists $d \in B$ such that $d \mathrm{I}_c^m \, b$, $m(a \wedge c \wedge d) \neq 0$, and $d \, \mathrm{R}_{a \wedge c} \, b$.
4. a is *strongly irrelevant* to b with respect to c, written $a\mathcal{I}_c^m b$, iff it does not hold that $a\mathcal{R}_c^m b$.

Of course, by definition, one has that $\mathrm{R} \subseteq \mathcal{R}$, for any Boolean algebra $\boldsymbol{B}$ and any $m \in \mathcal{S}^{\boldsymbol{B}}$. Therefore, weak relevance extends statistical relevance to the effect that a proposition a might be relevant to b in the light of a piece of evidence c even if a does not impact the probability of b *directly*, but it makes a previously irrelevant proposition d relevant, once added as a further piece of evidence to c.

If $\boldsymbol{B} \in \mathcal{BA}$, whenever the reference to a given $m \in \mathcal{S}^{\boldsymbol{B}}$ will be clear, we will write simply $a \, \mathrm{R}_c \, a$ $(a\mathcal{R}_c b, a \, \mathrm{I}_c \, b, a\mathcal{I}_c b)$ in place of $a \, \mathrm{R}_c^m \, b$ $(a\mathcal{R}_c^m b, a \, \mathrm{I}_c^m \, b, a\mathcal{I}_c^m b)$.

The following lemma provides a (of course, not exhaustive) summary of some interesting properties of R establishing relationships between Boolean operations and statistical relevance. It will be clear that, although R does not satisfy (1) nor (R5), cf. Example 4.1, weaker versions of the latter hold.

Lemma 3.1. *Let $\boldsymbol{B} \in \mathcal{BA}$, $m \in \mathcal{S}^{\boldsymbol{B}}$. Then, for any $x, y, z, u \in B$:*

1. $x \, \mathrm{R}_z \, y$ *iff* $y \, \mathrm{R}_z \, x$;
2. $x \, \mathrm{R}_z \, y$ *iff* $x \, \mathrm{R}_z \, y'$;
3. $x \, \mathrm{R}_z \, y$ *iff* $x' \, \mathrm{R}_z \, y$;
4. $x \, \mathrm{I}_z \top$ *and* $x \, \mathrm{I}_z \perp$;

5. *If* $u\,\mathrm{I}_z\,y$, *then* $x\,\mathrm{R}_{z\wedge u}\,y$ *iff* $(x \wedge u)\,\mathrm{R}_z\,y$;
6. $x\,\mathrm{R}_z\,y$, $u\,\mathrm{I}_z\,y$, *and* $x \wedge u \wedge z \neq \bot$ *imply* $(x \wedge u)\,\mathrm{R}_z\,y$;
7. $x\,\mathrm{R}_z\,y$ *iff* $(x \wedge z)\,\mathrm{R}_z\,y$;
8. $x\,\mathrm{R}_z\,y$ *iff* $(z \to x)\,\mathrm{R}_z\,y$;
9. $z \nleq x, x'$ *iff* $x\,\mathrm{R}_z\,x$.

Proof. The statements can be proven by means of simple probability theory and so they are left to the reader. □

It is worth highlighting that the notion of statistical irrelevance I is substantially different from the concept of conditional independence of discrete random variables given, e.g., in [33, pp. 82–87]. Indeed, while the former is about events/propositions, the latter concerns (sets of) discrete random variables. To make the differences even more apparent, we might exemplify the notion of conditional independence using the concepts we introduced so far as follows.

Let $U := \{x_1, \dots, x_n\}$ $(n \geqslant 1)$ be a finite set of variables that we call *discrete random variables*. For any $1 \leqslant i \leqslant n$, let $D_i \subseteq \mathbb{R}$ be the *range* of x_i, namely the set of values that x_i may take. Therefore, we consider random variables with real values only. Now, let

$$K := \{t \in \mathbb{R}^U : t(x_i) \in D_i,\ \text{for any } 1 \leqslant i \leqslant n\}.$$

Let us consider the Boolean algebra $\mathcal{P}(K)$. For any $1 \leqslant i \leqslant n$ and $r \in D_i$, let

$$A_{x_i:=r} = \{t \in K : t(x_i) = r\}.$$

Now, let m be a state over $\mathcal{P}(K)$. We call m a *joint probability distribution* over U. For any $X = \{x_{i_1}, \dots, x_{i_k}\} \subseteq U$ and $R = \{r_{i_1}, \dots, r_{i_k}\} \subseteq \mathbb{R}$ with $k \geqslant 1$ and $r_{i_j} \in D_{i_j}$ we set

$$\boldsymbol{x} = \textstyle\bigcap_{j=1}^{k} A_{x_{i_j}:=r_{i_j}}.$$

We call $\boldsymbol{x}$ a *configuration* of X. Let $X, Y, Z \subseteq U$. We say that X *is independent on* Y *given* Z if, *for any* configuration $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ of X, Y and Z, respectively, one has [cf., e.g., 33, p. 83]: $\boldsymbol{x}\mathrm{I}_{\boldsymbol{z}}{}^{m}\boldsymbol{y}$. It can be noticed that, while statistical irrelevance concerns triples of events, conditional independence is a relation between *sets* of events.

Let us now turn our attention to $\mathcal{R}$. In what follows, we provide a characterization of weak relevance simplifying Definition 3.1. The next remark is almost immediate.

Remark 3.1. For any Boolean algebra $\boldsymbol{B}$, $b, c \in B$, $m \in \mathcal{S}^{\boldsymbol{B}}$, one has that $\top\mathcal{I}_c^m b$.

Lemma 3.2. *Let* $\boldsymbol{B} \in \mathcal{BA}$ *and* $m \in \mathcal{S}^{\boldsymbol{B}}$. *Then, for any* $x, y, z \in B$, *the following are equivalent*:

1. $x \mathcal{R}_z^m y$;
2. $x \,\mathrm{R}_z\, y$ *or there exists* $u \in B$ *such that* $u \,\mathrm{I}_z\, y$ *and* $(u \wedge x) \,\mathrm{R}_z\, y$;
3. *There exists* $u \in B$ *such that* $u \,{\mathrm{I}_z}^m\, y$ *and* $(x \wedge u) \,{\mathrm{R}_z}^m\, y$.

Proof. "1 $\Rightarrow$ 2" Let us assume w.l.o.g. that $x \,\mathrm{I}_z\, y$. Therefore, there exists $u \in B$ such that $u\mathrm{I}_z y$, $m(u \wedge x \wedge z) \neq 0$ and $u\mathrm{R}_{x\wedge z} y$. Assume by way of contradiction that $(u \wedge x) \,\mathrm{I}_z\, y$. By Definition 3.1 and $m(u \wedge x \wedge z) \neq 0$, one must have that

$$\frac{m(u \wedge x \wedge z \wedge y)}{m(u \wedge x \wedge z)} = \frac{m(z \wedge y)}{m(z)}.$$

Since $u \,\mathrm{R}_{x\wedge z}\, y$, we have that

$$\frac{m(u \wedge x \wedge z \wedge y)}{m(u \wedge x \wedge z)} \neq \frac{m(x \wedge z \wedge y)}{m(x \wedge z)}$$

and so

$$\frac{m(x \wedge z \wedge y)}{m(x \wedge z)} \neq \frac{m(z \wedge y)}{m(z)},$$

against our hypothesis.

"2 $\Rightarrow$ 1" Assume without loss of generality that $x \,\mathrm{I}_z\, y$. Therefore, by hypothesis, there exists $u \in B$ such that $u \,\mathrm{I}_z\, y$ and $(u \wedge x) \,\mathrm{R}_z\, y$. In particular, this means that $m(u \wedge x \wedge z) \neq 0$, i.e., $u \wedge x \wedge z \neq \bot$ (by strict positivity) and so (again by strict positivity) $m(x \wedge z) \neq 0$, since $x \wedge z \neq \bot$. But then, in view of our working hypotheses:

$$\frac{m(u \wedge x \wedge z \wedge y)}{m(u \wedge x \wedge z)} \neq \frac{m(z \wedge y)}{m(z)} = \frac{m(x \wedge z \wedge y)}{m(x \wedge z)}.$$

"1 $\Rightarrow$ 3" If $x \,\mathrm{R}_z\, y$, then, by Remark 3.1, it is sufficient to set $u = \top$. Otherwise, set $u = e$, where e witnesses the weak relevance of x w.r.t. y under z. By hypothesis, one has $m(x \wedge z \wedge u) \neq 0$ and so $x \wedge z \wedge u \neq \bot$. This means that $x \wedge z \neq \bot$, i.e., $m(x \wedge z) \neq 0$. Since $x \,\mathrm{I}_z\, y$, one must have

$$\frac{m(x \wedge z \wedge y)}{m(x \wedge z)} = \frac{m(y \wedge z)}{m(z)}.$$

Therefore, we have also

$$\frac{m(u \wedge x \wedge z \wedge y)}{m(u \wedge x \wedge z)} \neq \frac{m(x \wedge z \wedge y)}{m(x \wedge z)} = \frac{m(z \wedge y)}{m(z)},$$

i.e., $(x \wedge u) \,\mathrm{R}_z\, y$.

"3 ⇒ 1" If there exists u such that $u\, \mathrm{I}_z\, y$ and $(u \wedge x)\, \mathrm{R}_z\, y$, one has $m(x \wedge u \wedge z) \neq 0$, and so $x \wedge u \wedge z \neq \bot$. In particular, it follows that $x \wedge z \neq \bot$ and so $m(x \wedge z) \neq \bot$. Now, if $x\, \mathrm{R}_z\, y$, then we are done. Otherwise, one must have

$$\frac{m(x \wedge z \wedge y)}{m(x \wedge z)} = \frac{m(y \wedge z)}{m(z)}.$$

So, we conclude that

$$\frac{m(u \wedge x \wedge z \wedge y)}{m(u \wedge x \wedge z)} \neq \frac{m(z \wedge y)}{m(z)} = \frac{m(x \wedge z \wedge y)}{m(x \wedge z)}. \qquad \square$$

In the sequel we will make free use of Lemma 3.2 without further reference.

For any Boolean algebra $\boldsymbol{B}$, $a, b \in B$, using customary notations, we set $a \to b := a' \vee b$ and $a \leftrightarrow b := (a \to b) \wedge (b \to a)$. The proof of the next result can be adapted to the framework of Boolean algebras from [13]. However, for the reader's convenience we recall it in the sequel.

Lemma 3.3 (cf. 13, Theorem 3). *Let $\boldsymbol{B} \in \mathcal{BA}$, $m \in \mathcal{S}^{\boldsymbol{B}}$. For any $x, y, z, u \in B$, the following hold*:

(R1) $z \leq x \leftrightarrow y$ *implies* $x\mathcal{R}_z u$ *iff* $y\mathcal{R}_z u$.
(R2) $x\mathcal{R}_z y$ *iff not* $x\mathcal{I}_z y$;
(R3) $x\mathcal{R}_z y$ *iff* $x'\mathcal{R}_z y$;
(R4) $\top\mathcal{I}_z y$;
(R5) *If* $z \not\leq x$ *and* $z \not\leq x'$, *then* $x\mathcal{R}_z x$;
(R6) *If* $x\mathcal{I}_z u$ *and* $y\mathcal{I}_z u$, *then* $(x \wedge y)\mathcal{I}_z u$.

Proof. Let $a, b, c, d \in B$. (R1) Suppose that $a\mathcal{R}_c d$. Then there exists $e \in B$ such that $e\, \mathrm{I}_c\, d$ but $(a \wedge e)\, \mathrm{R}_c\, d$. Therefore,

$$\frac{m(a \wedge e \wedge c \wedge d)}{m(e \wedge a \wedge c)} \neq \frac{m(c \wedge d)}{m(c)}.$$

our conclusion follows upon noticing that $c \leq a \leftrightarrow b$ entails $c \wedge a = c \wedge b$. (R2) follows trivially by definition. (R4) follows from Remark 3.1. Concerning (R5), suppose that $c \not\leq a$ and $c \not\leq a'$ but $a\mathcal{I}_c a$. In particular, one has $a\, \mathrm{I}_c\, a$. Now, if $m(a \wedge c) = 0$, then by the strict positivity of m, one has $a \wedge c = 0$, i.e., $a \leq c'$, a contradiction. So one has $m(a \wedge c) \neq 0$ and also

$$1 = \frac{m(a \wedge c)}{m(a \wedge c)} = \frac{m(a \wedge c)}{m(c)},$$

i.e., $m(c) = m(a \wedge c)$. Hence, $0 = m(a \wedge c) - m(c) = m(a \wedge c) - m(c) + 1 - 1$. We conclude $1 = m((a \wedge c) \vee c') = m(a \vee c')$. But then $m(a' \wedge c) = 0$, i.e., $a \leq c$, again a contradiction. Concerning (R6), suppose that $a \wedge b \mathcal{R}_c d$. This means that there exists $e \in B$ such that $e \, \mathrm{I}_c \, d$ and $(a \wedge b \wedge e) \, \mathrm{R}_c \, d$. If $(b \wedge e) \mathrm{R}_c d$, then $b \mathcal{R}_c d$ and we are done. Otherwise, suppose that $(b \wedge e) \mathrm{I}_c d$. But then $a \mathcal{R}_c d$, as desired. Finally, let us prove (R3). Suppose that $a \mathcal{R}_c d$ and assume towards a contradiction that $a' \mathcal{I}_c d$. In view of Lemma 3.1, one has that $a \mathrm{I}_c d$ and $a' \mathrm{I}_c d$. Let $e \in B$ be such that $e \mathrm{I}_c d$ and $(a \wedge e) \mathrm{R}_c d$. Note that $m(d|c \wedge a' \wedge e) = m(d|c) = m(d|c \wedge a') = m(d|c \wedge a)$. One has

$$\frac{m(e \wedge c \wedge d)}{m(e \wedge c)} = \frac{m(e \wedge c \wedge d \wedge a)}{m(c \wedge e \wedge a)} \frac{m(c \wedge e \wedge a)}{m(e \wedge c)} + \frac{m(e \wedge c \wedge d \wedge a')}{m(c \wedge e \wedge a')} \frac{m(c \wedge e \wedge a')}{m(e \wedge c)}.$$

In turn, this implies

$$\frac{m(e \wedge c \wedge d)}{m(e \wedge c)} \frac{m(c \wedge e \wedge a)}{m(e \wedge c)} = \frac{m(e \wedge c \wedge d)}{m(e \wedge c)} (1 - \frac{m(c \wedge e \wedge a')}{m(e \wedge c)}) = \frac{m(e \wedge c \wedge d \wedge a)}{m(c \wedge e \wedge a)} \frac{m(c \wedge e \wedge a)}{m(e \wedge c)}.$$

So we have

$$\frac{m(c \wedge d)}{m(c)} = \frac{m(c \wedge e \wedge d)}{m(e \wedge c)} = \frac{m(e \wedge c \wedge d \wedge a)}{m(c \wedge e \wedge a)},$$

a contradiction. □

It is worth remarking that, given $\boldsymbol{B} \in \mathcal{BA}$, $m \in \mathcal{S}^{\boldsymbol{B}}$, and $a \in B$, while R_a is symmetric by Lemma 3.1, $\mathcal{R}_a$ need not be so, as witnessed by [13, pp. 362–363] (cf. Example 4.1). Moreover, $\mathcal{R}_a$ need not be reflexive unless the antecedent of condition R5 from Lemma 3.3 holds.

Lemma 3.4. *Let $\boldsymbol{B} \in \mathcal{BA}$, $m \in \mathcal{S}^{\boldsymbol{B}}$. Then, for any $x, y, z \in B$:*

1. *$x \mathcal{R}_y x$ iff $y \nleq x$, $y \nleq x'$;*
2. *$x \mathcal{R}_z y$ iff $(x \wedge z) \mathcal{R}_z y$;*
3. *$x \mathcal{I}_z \top$ and $x \mathcal{I}_z \bot$.*

Proof. *Ad* 1. The right-to-left direction follows by Lemma 3.3. Conversely, if $a \mathcal{R}_b a$, then one has that $a \neq \bot$, by Lemma 3.3(R3,R4). Moreover, if $b \leq a$, then for any $d \in B$ such that $d \, \mathrm{I}_b \, a$, one has

$m(a|(a \wedge d) \wedge b) = m(a|b \wedge d) = m(a|b)$. Since d is arbitrary, one has that $a\mathcal{I}_b a$. Similarly, if $b \leq a'$, one has $a'\mathcal{I}_b a$, since for any $d \in B$ such that $d \, \mathrm{I}_b \, a$, one has $m(a|(a' \wedge d) \wedge b) = m(a|b \wedge d) = m(a|b)$. Therefore, by (R4) of Lemma 3.3, one has $a\mathcal{I}_b a$. *Ad* 2. It follows upon noticing that, if there exists $d \in B$ such that $d \, \mathrm{I}_c \, b$ and $(a \wedge d) \, \mathrm{R}_c \, b$, then one has also $m(b|(a \wedge c) \wedge d \wedge c) = m(b|a \wedge d \wedge c) \neq m(b|c)$. The converse follows similarly. *Ad* 3. It is a consequence of Lemma 3.1(4). □

4. Algebras of relevance

In this section we define classes of algebras with the aim of providing an equivalent algebraic semantics for expansions of **CPL** by means of a new connective which allows us to represent relevance in the object language. To achieve this goal, we outline a class of expansions of Boolean algebras with a new ternary operation embodying the idea of letting material implications be defined provided that the antecedent is (statistically or weakly) relevant to the consequent under a given piece of evidence. More precisely, we introduce structures which are abstract counterparts of Boolean algebras expanded with a ternary implication operation $\rightsquigarrow$ such that the intended reading of $\rightsquigarrow (x, z, y)$ is "x materially implies y and x is weakly/statistically relevant (w.r.t. a given state m) to y under the evidence z". On the one hand, this new connective allows us to explore the "logical consequences" of statistical and weak relevance. On the other hand, it can be used to introduce a notion of relevance in the object language. Therefore, this approach grants the possibility of making (generalizations of) probabilistic relevance relations amenable to a smooth algebraic/arithmetical treatment. At the same time, it yields the possibility of expressing sentences involving "nestings" and reiterations of relevance relations with ease.

Let $\boldsymbol{B} \in \mathcal{BA}$. Given a mapping $\rightsquigarrow : B^3 \to B$, let us denote $\rightsquigarrow(x, z, y)$ by $x \rightsquigarrow_z y$. Moreover, in what follows we set

$$x \, \mathbb{R}_y \, z := (x \rightsquigarrow_y z) \vee (x' \rightsquigarrow_y z),$$

as well as $x \, \mathbb{I}_z \, y := (x \, \mathbb{R}_z \, y)'$.

Definition 4.1. A *d-algebra* is a structure $\boldsymbol{A} = (A, \rightsquigarrow, \wedge \vee, ', \bot, \top)$ of the type $(3, 2, 2, 1, 0, 0)$ satisfying the following conditions:

1. $(A, \wedge \vee, ', \bot, \top)$ is a Boolean algebra;

2. $\rightsquigarrow$ is a ternary operation satisfying the following conditions:
   (D1) $x \rightsquigarrow_z \top \approx \bot$;
   (D2) $(x \rightsquigarrow_z y) \wedge x \leq y$;
   (D3) $(x' \rightsquigarrow_z y) \wedge y \leq x \rightsquigarrow_z y$;
   (D4) $x \rightsquigarrow_z y \leq (x \wedge z) \rightsquigarrow_z y$;
   (D5) $(z \to x) \rightsquigarrow_z y \leq x \rightsquigarrow_z y$;
   (D6) $x \rightsquigarrow_z x \approx x' \rightsquigarrow_z x'$:
   (D7) $x \rightsquigarrow_z y \leq (x \rightsquigarrow_z x) \wedge (y \rightsquigarrow_z y)$;
   (D8) $(x \wedge z) \rightsquigarrow_z y \leq x \,\mathbb{R}_z\, y$.

Let us denote by $\mathcal{D}$ the variety of d-algebras.

The next lemma shows that, indeed, d-algebras are nothing but abstract counterpart of Boolean algebras endowed with a ternary connective embodying the constraint of material implication by statistical/weak relevance with respect to a given state.

Lemma 4.1. *Let $\boldsymbol{B} \in \mathcal{BA}$ and $m \in \mathcal{S}^{\boldsymbol{B}}$. Upon defining $\rightsquigarrow^* \colon B^3 \to B$ such that for any $x, y, z \in B$:*

$$x \rightsquigarrow_z^* y = \begin{cases} x \to y & \text{if } x\mathcal{R}_z^m y \\ \bot & \text{otherwise} \end{cases}$$

then $(B, \rightsquigarrow^*, \wedge, \vee, ', \bot, \top)$ is a d-algebra and, for any $x, y, z \in B$, $x \mathrm{R}_z y = \top$ iff $x\mathcal{R}_z^m y$. Moreover, the same holds replacing $\mathcal{R}^m$ by R^m.

Proof. (D1) follows by Lemma 3.4(3) while (D2) holds by definition. (D3) follows by Lemma 3.3(R3). (D4) is a consequence of Lemma 3.4(2), while (D5) holds by 3.3(R3) and again Lemma 3.4(2). (D6) follows by Lemma 3.3(R5). (D7) follows upon noticing that if $x \rightsquigarrow_z^* y \neq \bot$, then one must have $x\mathcal{R}_z y$. By Lemma 3.2, there exists $u \in B$ such that $u \,\mathrm{I}_z\, y$ but $(u \wedge x) \,\mathrm{R}_z\, y$. This means that

$$\frac{m(u \wedge x \wedge z \wedge y)}{m(u \wedge x \wedge z)} \neq \frac{m(z \wedge y)}{m(z)}. \tag{2}$$

However, if $z \leq x$, then (2) implies that $u \,\mathrm{R}_z\, y$, a contradiction. $z \leq x'$ is impossible since otherwise both sides would be 0. Similarly, if $z \leq y$ or $z \leq y'$, then both sides of (2) would be 1 or 0, respectively. We conclude that $x\mathcal{R}_z x$ and $y\mathcal{R}_z y$, and so we have $x \rightsquigarrow_z^* x = y \rightsquigarrow_z^* y = \top$. That $x \,\mathbb{R}_z\, y = \top$ iff $x\mathcal{R}_z y$ is clear.

Concerning the moreover part, (D1) follows by Lemma 3.1(4) while (D2) holds by definition. (D3) is a consequence of Lemma 3.1(3), (D4) and (D8) hold because of Lemma 3.1(7). (D5) follows by Lemma 3.1(8) while (D6) is a consequences of Lemma 3.1(1)-(3), (9). (D7) holds upon noticing that $x \rightsquigarrow_z y \neq \bot$ implies that $x \,\mathrm{R}_z{}^m\, y$ and, in turn, this implies $y \wedge z \neq \bot \neq y \wedge z'$. Therefore, by Lemma 3.1(9), we have $y \,\mathrm{R}_z\, y$ and $y \rightsquigarrow_z^* y = 1$. $x \rightsquigarrow_z x = 1$, follows similarly upon noticing that $x \,\mathrm{R}_z{}^m\, y$ iff $y \,\mathrm{R}_z{}^m\, x$, and so $x \wedge z \neq \bot \neq x \wedge z'$. □

In what follows we show that, for any d-algebra $\boldsymbol{A}$, once we consider the term $x \,\mathbb{R}_z\, y$ as embodying the notion of relevance, $x \,\mathbb{R}_z\, y$ fulfills some conditions which are satisfied both from $\mathcal{R}$ and R.

Lemma 4.2. *Let $\boldsymbol{B}$ be a d-algebra. The following hold*:

1. $\bot \rightsquigarrow_z y \approx x \rightsquigarrow_z \bot \approx \bot \approx x \rightsquigarrow_y y \approx x \rightsquigarrow_y y' \approx \top \rightsquigarrow_x y \approx x \rightsquigarrow_x y \approx x \rightsquigarrow_{x'} y$;
2. $\top \,\mathbb{R}_z\, y \approx \bot \,\mathrm{R}_z\, y \approx x \,\mathrm{R}_z\, \bot \approx x \,\mathrm{R}_z\, \top \approx x \,\mathrm{R}_x\, y \approx x \,\mathrm{R}_{x'}\, y \approx x \,\mathrm{R}_{y'}\, y$;
3. $(x \wedge z) \,\mathrm{R}_z\, y \leq x \,\mathrm{R}_z\, y$;
4. $x \rightsquigarrow_z y \leq (z \to x) \,\mathrm{R}_z\, y$;
5. $(z \to x)' \rightsquigarrow_z y \leq x \,\mathrm{R}_z\, y$;
6. $(z \to x) \,\mathrm{R}_z\, y = x \,\mathrm{R}_z\, y$;
7. $x \rightsquigarrow_z x \neq \bot$ *implies* $z \nleq x$ *and* $z \nleq x'$.

Proof. *Ad* 1. The identities can be proven by means of (D1), (D6) and (D4). For example, we have $\top \mathrm{R}_z y = (\top \rightsquigarrow_z y) \vee (\bot \rightsquigarrow_z y) \leq \top \rightsquigarrow_z \top = \bot$ and so also $x \rightsquigarrow_{x'} y, x' \rightsquigarrow_{x'} y \leq x \,\mathrm{R}_{x'}\, y = (x \rightsquigarrow_{x'} y) \vee (x' \rightsquigarrow_{x'} y) \leq x \rightsquigarrow_{x'} x \leq (x \wedge x') \rightsquigarrow_{x'} y = \bot \rightsquigarrow_{x'} y = \bot$, by (D4), (D6), and (D7). The other equalities can be proven similarly. *Ad* 2. It is a direct consequence of item 1. *Ad* 3. It follows by $(x \wedge z)' \rightsquigarrow_z y = (z \to x') \rightsquigarrow_z y \leq x' \rightsquigarrow_z y \leq x \,\mathbb{R}_z\, y$, by (D5), and by (D8). *Ad* 4. One has $x \rightsquigarrow_z y \leq (x \wedge z) \rightsquigarrow_z y \leq (x \wedge z) \mathrm{R}_z y = ((z \to x) \wedge z) \mathrm{R}_z y \leq (z \to x) \mathrm{R}_z y$, by item 3. *Ad* 5. Note that $(z \to x)' \rightsquigarrow_z y \leq (z \wedge x') \mathrm{R}_z y \leq x' \mathrm{R}_z y = x \mathrm{R}_z y$, by item 3. *Ad* 6. By (D5) and item 5, one has $(z \to x) \mathrm{R}_z y \leq x \mathrm{R}_z y$. Conversely, note that by (D4), one has $x' \rightsquigarrow_z y \leq (x' \wedge z) \rightsquigarrow_z y = (z \to x)' \rightsquigarrow_z y \leq (z \to x) \,\mathrm{R}_z\, y$. Therefore, by item 4, the desired result obtains. *Ad* 7. It follow by (D4) and item 1. □

From now on, given $\boldsymbol{A} \in \mathcal{D}$, we set, for any $\{x_1, \ldots, x_n, z\} \subseteq A$ $(n \geqslant 1)$:

$$\mathrm{C}_z(x_1, \ldots, x_n) := \textstyle\bigwedge_{i=1}^n x_i \rightsquigarrow_z \bigwedge_{i=1}^n x_i.$$

In view of Lemma 4.2(7), we will read $\mathrm{C}_z(x_1, \ldots, x_n)$ as "$x_1, \ldots, x_n$ are *contingent* on evidence z". Moreover, for any $\{x_1, \ldots, x_n\} \subseteq A$,

$\mathrm{C}_\top(x_1,\ldots,x_n) \neq \bot$ implies $\bigwedge_{i=1}^n x_i \neq \bot, \top$. In view of this fact, $\mathrm{C}_\top(x_1,\ldots,x_n)$, that we shorten by $\mathrm{C}(x_1,\ldots,x_n)$, will be read as '$\{x_1, \ldots, x_n\}$ is *consistent* and *non-trivially* satisfiable'.

Perhaps a few words on the intuitive reading of (D1)–(D8) deserve to be made. D1 asserts that no proposition a implies the "absolute truth" $\top$ under any evidence b, since a cannot affect the truth of $\top$. (D2) is clear. The meaning of (D3) becomes apparent once it is replaced by the provably equivalent identity

$$y \leq (x \rightsquigarrow_z y) \leftrightarrow (x' \rightsquigarrow_z y).$$

Indeed, (D3) suggests that, if a proposition c turns out to be true, then the truth value of any other proposition a does not carry much information on the truth of c under any evidence b. (D4) codifies the idea that, if a relevantly implies c under the evidence b, then "a and b" is still relevant for and implies c knowing that b is true. (D8) suggests that if "a and b" relevantly implies c under the evidence b, then necessarily a must be relevant to c under b. (D5) asserts that knowing that "if b, then a" relevantly implies the truth of c under the evidence that b entails that a relevantly implies c under b. (D6) is clear, while (D7) just states that if a relevantly implies c under an evidence b then both a and c are contingent w.r.t. b.

The next lemma provides a further insight on properties of the relation $\mathbb{R}$.

Lemma 4.3. *Let $\boldsymbol{B}$ be a d-algebra. Then, for any $x, y, z, u \in B$, the following hold*:

1. $x \,\mathrm{R}_z\, y \approx x' \,\mathrm{R}_z\, y$;
2. $(x \to y) \wedge (x \,\mathrm{R}_z\, y) = (x \rightsquigarrow_z y)$;
3. $(x \wedge z) \,\mathrm{R}_z\, y \approx x \,\mathrm{R}_z\, y$;
4. $z \leq (x \leftrightarrow u)$ *implies* $x \,\mathrm{R}_z\, y = u \,\mathrm{R}_z\, y$.

Proof. *Ad* 1. It is trivial. *Ad* 2. Let us compute using distributivity

$$\begin{aligned}
(x \to y) \wedge x \,\mathrm{R}_z\, y &= (x \to y) \wedge [(x \rightsquigarrow_z y) \vee (x' \rightsquigarrow_z y)] \\
&= (x \rightsquigarrow_z y) \vee [(x \to y) \wedge (x' \rightsquigarrow_z y)] \quad \text{by (D2)} \\
&= (x \rightsquigarrow_z y) \vee [(x' \wedge (x' \rightsquigarrow_z y)) \vee (y \wedge (x' \rightsquigarrow_z y))] \\
&= (x \rightsquigarrow_z y) \vee (y \wedge (x' \rightsquigarrow_z y)) = (x \rightsquigarrow_z y) \quad \text{by (D2), (D3)}
\end{aligned}$$

Ad 3. By Lemma 4.2 and item 1, one has $x \,\mathrm{R}_z\, y = x' \,\mathrm{R}_z\, y = (z \to x') \,\mathrm{R}_z\, y = (z \wedge x)' \,\mathrm{R}_z\, y = (z \wedge x) \,\mathrm{R}_z\, y$. *Ad* 4. It follows upon noticing that

$z \le x \leftrightarrow u$ implies $x \wedge z = u \wedge z$. So, by item 3, $x \,\mathrm{R}_z\, y = (z \wedge x) \,\mathrm{R}_z\, y = (z \wedge u) \,\mathrm{R}_z\, y = u \,\mathrm{R}_z\, y$. □

Theorem 4.4. *Let $\boldsymbol{A} = (A, \rightsquigarrow, \wedge, \vee, ', \bot, \top)$ be an algebra of the type $(3, 2, 2, 1, 0, 0)$ whose $\{\wedge, \vee, ', \bot, \top\}$-reduct is a Boolean algebra. Then $\boldsymbol{A} \in \mathcal{D}$ if and only if the following hold in $\boldsymbol{A}$*:

1. $x \,\mathrm{R}_z \top \approx \bot$
2. $(x \to y) \wedge (x \,\mathrm{R}_z\, y) = (x \rightsquigarrow_z y)$;
3. $(x \wedge z) \,\mathrm{R}_z\, y \approx x \,\mathrm{R}_z\, y$;
4. $x \,\mathbb{R}_z\, y \le (x \,\mathbb{R}_z\, x) \wedge (y \,\mathbb{R}_z\, y)$.

Proof. One direction follows by Lemma 4.2 and Lemma 4.3. Conversely, (D1) follows directly from item 1. (D2) follows by item 2 as $x \rightsquigarrow_z y \le x \to y$ implies $(x \rightsquigarrow_z y) \wedge x \le y$. (D3) follows by item 2, distributivity, and (D2) since

$$\begin{aligned}
x \rightsquigarrow_z y &= (x' \vee y) \wedge x \,\mathbb{R}_z\, y \\
&= (x' \vee y) \wedge ((x \rightsquigarrow_z y) \vee (x' \rightsquigarrow_z y)) \\
&= (x \rightsquigarrow_z y) \vee (x' \wedge (x' \rightsquigarrow_z y)) \vee (y \wedge (x' \rightsquigarrow_z y)) \\
&= (x \rightsquigarrow_z y) \vee (y \wedge (x' \rightsquigarrow_z y)).
\end{aligned}$$

Let us prove (D4). First, by (D1), (D3) and distributivity, one can easily see that $(*)$ $(x' \rightsquigarrow_z y) \wedge (x \to y) \le x \rightsquigarrow_z y$. Now, we compute

$$\begin{aligned}
x \rightsquigarrow_z y &= (x \to y) \wedge x \,\mathbb{R}_z\, y \\
&= (x \to y) \wedge (x \wedge z) \,\mathbb{R}_z\, y \\
&= ((x \to y) \wedge (x \wedge z) \rightsquigarrow_z y) \vee ((x \to y) \wedge (x \wedge z)' \rightsquigarrow_z y).
\end{aligned}$$

The latter expression reduces to $(x \to y) \wedge (x \wedge z) \rightsquigarrow_z y$ since, by $(*)$, $((x \wedge z)' \rightsquigarrow_z y) \wedge (x \to y) \le ((x \wedge z)' \rightsquigarrow_z y) \wedge ((x \wedge z) \to y) \le (x \wedge z) \rightsquigarrow_z y$.

Let us prove (D5). We have

$$\begin{aligned}
(z \to x) \rightsquigarrow_z y &= ((z \to x) \to y) \wedge ((z \to x) \,\mathbb{R}_z\, y) \\
&= ((z \wedge x') \vee y) \wedge ((z \to x) \,\mathbb{R}_z\, y) \\
&= [(z \vee y) \wedge (x' \vee y))] \wedge ((z \to x) \,\mathbb{R}_z\, y) \\
&\le (x' \vee y) \wedge ((z \to x)' \,\mathbb{R}_z\, y) = (x' \vee y) \wedge ((z \wedge x') \,\mathbb{R}_z\, y) \\
&= (x' \vee y) \wedge (x' \,\mathbb{R}_z\, y) = (x' \vee y) \wedge (x \,\mathbb{R}_z\, y) = x \rightsquigarrow_z y.
\end{aligned}$$

We prove (D6). $x \,\mathbb{R}_z\, y = x' \,\mathbb{R}_z\, y \le x' \,\mathbb{R}_z\, x' = x \,\mathbb{R}_z\, x'$. Therefore, $x \,\mathbb{R}_z\, x \le x \,\mathbb{R}_z\, x' \le x \,\mathbb{R}_z\, x$. This means that, by item 2, $x \rightsquigarrow_z x = x \,\mathbb{R}_z\, x'$

and so $x' \rightsquigarrow_z x' \leq x \rightsquigarrow_z x$ as well as $x' \rightsquigarrow_z x' \leq x \rightsquigarrow_z x$. As regards (D7), note that $x \rightsquigarrow_z x = x \to x \wedge x \,\mathbb{R}_z\, x = x \,\mathbb{R}_z\, x$. This means that $x' \rightsquigarrow_x \leq x \rightsquigarrow_z x$ and so $x \,\mathbb{R}_z\, x \leq x \rightsquigarrow_z x$. We conclude that $x \rightsquigarrow_z y \leq x \,\mathbb{R}_z\, y \leq (x \,\mathbb{R}_z\, x) \wedge (y \,\mathbb{R}_z\, y) \leq (x \rightsquigarrow_z x) \wedge (y \rightsquigarrow_z y)$. (D8) is a direct consequence of item 3. $\square$

Theorem 4.5. *The subvariety $\mathcal{V}(\mathbf{D}_2)$ of $\mathcal{D}$ generated by the two-element d-algebra $\mathbf{D}_2$ is axiomatized by the identity*

$$x \,\mathbb{R}_z\, y \approx \bot. \tag{Ir}$$

Proof. Note that $\mathbf{D}_2 \models$ (Ir), and so $\mathcal{V}(\mathbf{D}_2) \models$ (Ir). Concerning the converse, we show that any $\boldsymbol{A} \in \mathcal{V} \subseteq \mathcal{D}$, where $\mathcal{V}$ is axiomatized by (Ir), is a subdirect product of $\mathbf{D}_2$. To see this, first note that congruences over members of $\boldsymbol{A}$ coincide with congruences over its Boolean reduct, since (Ir) yields that $x \rightsquigarrow_z y = \bot$, for any $x, y, z \in A$. Since a Boolean algebra $\boldsymbol{B}$ is subdirectly irreducible iff $|B| = 2$, we conclude that the only subdirectly irreducible member of $\mathcal{V}$ is $\mathbf{D}_2$. Therefore, we conclude $\mathcal{V} = \mathcal{V}(\mathbf{D}_2)$. $\square$

In other words, the algebraic semantics of **CPL** is regained as a “limit case”, once propositions are considered irrelevant to each other.

Note that, for any d-algebra $\boldsymbol{A}$, $x, y, z \in A$, one has that $x \,\mathbb{R}_z\, y = \top$ ($x \,\mathbb{I}_z\, y = \top$) iff $x \rightsquigarrow_z y = x \to y$ and $x' \rightsquigarrow_z y = x' \to y$ ($x \rightsquigarrow_z y = \bot = x' \rightsquigarrow_z y$). Of course, a natural *desideratum* would be to have that $x \,\mathbb{R}_z\, y$ can assume only two values, namely $\top$ or $\bot$. In turn, this is equivalent (by Lemma 4.3(2) and basic properties of d-algebras) to the fact that $x \rightsquigarrow_z y$ equals either material implication or $\bot$. Therefore, one might ask if d-algebras enjoy such a property, or if this condition can be expressed by means of a set of generalized quasi-equations. The latter requirement is of interest, since we have aimed to provide the *equivalent* algebraic semantics of a logic which, by general results, is always a generalized quasivariety [by 12, Corollary 3.18]. Unfortunately, the answer to both questions is no. In fact, such a condition would not be preserved by *direct products* (cf., e.g., Lemma 4.15 below, [12, Definition 1.72]). Consequently, there is no subset $\mathcal{A} \subseteq \mathcal{D}$, in whose members $\rightsquigarrow$ has the smooth behavior outlined above, that may serve as the equivalent algebraic semantics of a logic.

We close this section by showing how to extend d-algebras in order to obtain structures in which the term-defined relation $\mathbb{R}$ satisfies some characteristic conditions of $\mathcal{R}$ and R. Interestingly enough, in some cases, it turns out that a property of $\mathbb{R}$ like, e.g., symmetry is equivalent to the fact that $\rightsquigarrow$ satisfies *unrestrictedly* some Boolean identities.

Definition 4.2. A *Gärdenfors'algebra* (G-algebra) is a d-algebra satisfying the further condition:

$$(x \wedge y) \, \mathrm{R}_z \, u \leq (x \, \mathrm{R}_z \, u) \vee (y \, \mathrm{R}_z \, u). \tag{G}$$

From now on, we will denote by $\mathcal{G}$ the variety of G-algebras. It can be seen that (G) fails in $\mathcal{D}$. Therefore $\mathcal{D} \subsetneq \mathcal{G}$.

Example 4.1. Following [13, p. 362], consider the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and the Boolean algebra $(\mathcal{P}(A), \cap, \cup, ^c, \emptyset, A)$. Let $m \colon \mathcal{P}(A) \to [0, 1]$ be defined according to the following probabilities assigned to atoms in $\mathcal{P}(A)$: $m(\{1\}) = 0,04$; $m(\{2\}) = 0,18$; $m(\{3\}) = 0,12$; $m(\{4\}) = 0,11$; $m(\{5\}) = 0,13$; $m(\{6\}) = 0,06$; $m(\{7\}) = 0,08$; $m(\{8\}) = 0,07$; $m(\{9\}) = 0,21$. Let us define $\rightsquigarrow^* \colon \mathcal{P}(A)^3 \to \mathcal{P}(A)$ upon setting $X \rightsquigarrow^*_Z Y = X^c \cup Y$, if $X \, \mathrm{R}_Z{}^m \, Y$, and $X \rightsquigarrow^*_Z Y = \emptyset$, otherwise. A routine check shows that $(\mathcal{P}(A), \rightsquigarrow^*, \cap, \cup, ^c, \emptyset, A)$ is a d-algebra. Now, upon setting $X := \{1, 3, 4, 5\}$, $Y := \{1, 3, 2, 6\}$ and $Z := \{1, 6, 8, 7\}$, it can be seen that $(X \cap Z) \, \mathrm{R}_A \, Y$ but $X \, \mathrm{I}_A \, Y$ and $Z \, \mathrm{I}_A \, Y$. This clearly entails that $A = (X \cap Z) \, \mathrm{R}_A \, Y \not\subseteq X \, \mathrm{R}_A \, Y \cup Z \, \mathrm{R}_A \, Y = \emptyset$.

Example 4.2. Let us consider the algebra $\boldsymbol{A} = (\{\bot, \top, a, a'\}, \rightsquigarrow, \wedge, \vee, ', \bot, \top)$ such that $(\{\bot, \top, a, a'\}, \wedge, \vee, ', \bot, \top)$ is the four-element Boolean algebra and, $x \rightsquigarrow_\top x = \top$ and $x \rightsquigarrow_\top x' = x'$, for any $x \in \{a, a'\}$ and $x \rightsquigarrow_y z = \bot$ in any other case. It is easily seen that $\boldsymbol{A}$ is a G-algebra.

Recall the definition of $\rightsquigarrow^*$ from Lemma 4.1. From Lemmas 3.3(R6) and 4.1 we obtain:

Proposition 4.6. *Let $\boldsymbol{B} \in \mathcal{BA}$ and $m \in \mathcal{S}^{\boldsymbol{B}}$. Then $(B, \rightsquigarrow^*, \wedge, \vee, ', \bot, \top)$ is a G-algebra.*

As recalled by Example 4.1, in general, statistical and weak relevance do not coincide. Therefore, it naturally raises the question of whether, as for a generalized version of Gärdenfors' weak relevance, an algebraic treatment of some generalization of strong relevance may be given. In other words, we are after expansions of Boolean algebras in which the relevance relation $\mathbb{R}$ defined in terms of $\rightsquigarrow$ satisfies conditions from Lemma 3.1.

Definition 4.3. A d-algebra $\boldsymbol{A} = (A, \rightsquigarrow, \wedge, \vee, ', \bot, \top)$ is called an *SR-algebra* provided that the following conditions are satisfied:

(SR1) $x \rightsquigarrow_z y \approx y' \rightsquigarrow_z x'$;
(SR2) $(x \rightsquigarrow_z y) \wedge x \approx (y \rightsquigarrow_z x) \wedge y$;
(SR3) $(y \,\mathbb{I}_z\, u) \wedge (x \rightsquigarrow_{z\wedge u} y) \leq ((x \wedge u) \rightsquigarrow_z y)$;
(SR4) $(y \,\mathbb{I}_z\, u) \wedge ((x \wedge u) \rightsquigarrow_z y) \leq x \,\mathbb{R}_{z\wedge u}\, y$;
(SR5) $(x \rightsquigarrow_z y) \wedge (y \,\mathbb{I}_z\, u) \wedge \mathrm{C}(x, z, u) \leq (x \wedge u) \rightsquigarrow_z y$, where C is defined as in p. 120.

Let us denote by $\mathcal{SR}$ the variety of SR-algebras.

Note that (D2) and (D6) can be easily proven by means of the remaining axioms of SR-algebras.

Proposition 4.7. *Let* $\boldsymbol{B} \in \mathcal{BA}$ *and* $m \in \mathcal{S}^{\boldsymbol{B}}$. *Then* $(B, \rightsquigarrow^*, \wedge, \vee, ', \bot, \top)$ *is an SR-algebra.*

Proof. $(B, \rightsquigarrow^*, \wedge, \vee, ', \bot, \top)$ is a d-algebra by Lemma 4.1. (SR1) and (SR2) follow by the commutativity of ${\mathrm{R}_z}^m$ (for any $z \in B$). (SR3) and (SR4) are direct consequences of Lemma 3.1(5) and (6), respectively. (SR5) follows from Lemma 3.1(6). □

Lemma 4.8. *Any SR-algebra* $\boldsymbol{A}$ *satisfies the following conditions*:

1. $x \,\mathbb{R}_z\, y = y \,\mathbb{R}_z\, x$;
2. $(u \,\mathbb{I}_z\, y) \wedge (x \,\mathbb{R}_{z\wedge u}\, y) \approx (u \,\mathbb{I}_z\, y) \wedge ((x \wedge u) \,\mathbb{R}_z\, y)$;
3. $x \,\mathbb{R}_z\, y \wedge u \,\mathbb{I}_z\, y \wedge \mathrm{C}(x, z, u) \leq (x \wedge u) \,\mathbb{R}_z\, y$.

Proof. *Ad* 1. By (SR1) $x' \rightsquigarrow_z y = y' \rightsquigarrow_z x \leq y \,\mathbb{R}_z\, x$. Now, one has that $(x \rightsquigarrow_z y) \wedge y \leq x' \rightsquigarrow_z y \leq y' \rightsquigarrow_z x \leq y \mathbb{R}_z y$ and also $(x \rightsquigarrow_z y) \wedge y' = (y' \rightsquigarrow_z x') \wedge y' = (x' \rightsquigarrow_z y') \wedge x' \leq x' \rightsquigarrow_z y' = y \rightsquigarrow_z x \leq y \mathbb{R}_z x$. Therefore we have $x \rightsquigarrow_z y = [(x \rightsquigarrow_z y) \wedge y] \vee [(x \rightsquigarrow_z y) \wedge y'] \leq y \,\mathbb{R}_z\, x$. We conclude that $x \mathbb{R}_z y \leq y \mathbb{R}_z x$. Similarly we have $y \mathbb{R}_z x \leq x \mathbb{R}_z y$ and so the desired result obtains. *Ad* 2. By (SR3) one has that $(y \,\mathbb{I}_z\, u) \wedge (x \rightsquigarrow_{z\wedge u} y) \leq (x \wedge u) \rightsquigarrow_z y$. Moreover, $(y \,\mathbb{I}_z\, u) \wedge (x' \rightsquigarrow_{z\wedge u} y) \wedge y \leq (y \,\mathbb{I}_z\, u) \wedge (x \rightsquigarrow_{z\wedge u} y) \leq (x \wedge u) \rightsquigarrow_z y \leq (x \wedge u) \,\mathbb{R}_z\, y$ and

$$\begin{aligned}
(y \,\mathbb{I}_z\, u) \wedge (x' \rightsquigarrow_{z\wedge u} y) \wedge y' &= (y \,\mathbb{I}_z\, u) \wedge (y' \rightsquigarrow_{z\wedge u} x) \wedge y' \\
&= (y \,\mathbb{I}_z\, u) \wedge (x \rightsquigarrow_{z\wedge u} y') \wedge x \\
&= (y' \,\mathbb{I}_z\, u) \wedge (x \rightsquigarrow_{z\wedge u} y') \wedge x \\
&\leq (y' \,\mathbb{I}_z\, u) \wedge ((x \wedge u) \rightsquigarrow_z y') \leq (x \wedge u) \,\mathbb{R}_z\, y' \\
&= (x \wedge u) \,\mathbb{R}_z\, y
\end{aligned}$$

We conclude $(y \mathbb{I}_z u) \wedge (x' \rightsquigarrow_{z\wedge u} y) \leq (x \wedge u) \mathbb{R}_z y$; so $(y \mathbb{I}_z u) \wedge (x \mathbb{R}_{z\wedge u} y) \leq (x \wedge u) \,\mathbb{R}_z\, y$. The converse can be proven similarly by means of (SR4).

Ad 3. $(x' \rightsquigarrow_z y) \wedge y \wedge (y \,\mathbb{I}_z\, u) \wedge \mathrm{C}(x,z,u) \leq (x \rightsquigarrow_z y) \wedge y \wedge (y \,\mathbb{I}_z\, u) \wedge \mathrm{C}(x,z,u) \leq (x \wedge u) \rightsquigarrow_z y \leq (x \wedge u) \,\mathbb{R}_z\, y$. Also,

$$\begin{aligned}(x' \rightsquigarrow_z y) \wedge y' \wedge (y \,\mathbb{I}_z\, u) \wedge \mathrm{C}(x,z,u) &= (y' \rightsquigarrow_z x) \wedge y' \wedge (y' \,\mathbb{I}_z\, u) \wedge \mathrm{C}(x,z,u) \\ &= (x \rightsquigarrow_z y') \wedge x \wedge (y' \,\mathbb{I}_z\, u) \wedge \mathrm{C}(x,z,u) \\ &\leq (x \wedge u) \rightsquigarrow_z y \leq (x \wedge u) \,\mathbb{R}_z\, y.\end{aligned}$$

Thus, by (SR5), we have $(x \,\mathbb{R}_z\, y) \wedge (y \,\mathbb{I}_z\, u) \wedge \mathrm{C}(x,z,u) \leq (x \wedge u) \,\mathbb{R}_z\, y$. □

Theorem 4.9. *A d-algebra $\boldsymbol{A} \in \mathcal{SR}$ if and only if it satisfies* (1)–(3) *of Lemma 4.8.*

Proof. On direction follows from Lemma 4.8. Let us prove the converse.

(SR1) Note that $x \rightsquigarrow_z y = (x \to y) \wedge (x \mathbb{R}_z y) = (y' \to x') \wedge (y' \mathbb{R}_z x') = y' \rightsquigarrow_z x'$, upon combining basic properties of d-algebras and item 1. Similarly, (SR2) follows upon noticing that $x \wedge (x \rightsquigarrow_z y) = x \wedge (x \to y) \wedge (x \,\mathbb{R}_z\, y) = (x \wedge y) \wedge (x \,\mathbb{R}_z\, y) = y \wedge (y \to x) \wedge (y \,\mathbb{R}_z\, x) = y \wedge (y \rightsquigarrow_z x)$. (SR3) follows directly from item 2. Finally, as regards (SR4), by item 3, it follows that $(x \rightsquigarrow_z y) \wedge (y \,\mathbb{I}_z\, u) \wedge \mathrm{C}(x,z,u) \leq (x \wedge u) \,\mathbb{R}_z\, y$. Therefore, we conclude

$$\begin{aligned}(x \wedge u) \rightsquigarrow_z y &\geq (x \rightsquigarrow_z y) \wedge (y \,\mathbb{I}_z\, u) \wedge \mathrm{C}(x,z,u) \wedge ((x \wedge u) \to y) \\ &= (x \rightsquigarrow_z y) \wedge (y \,\mathbb{I}_z\, u) \wedge \mathrm{C}(x,z,u) \wedge ((x \to y) \vee (u \to y)) \\ &\geq (x \rightsquigarrow_z y) \wedge (y \,\mathbb{I}_z\, u) \wedge \mathrm{C}(x,z,u) \wedge (x \to y) \\ &= (x \rightsquigarrow_z y) \wedge (y \,\mathbb{I}_z\, u) \wedge \mathrm{C}(x,z,u).\end{aligned}$$ □

4.1. Turning algebras into logics

As a natural follow-up of the above considerations, in this section we provide three logics, here presented in the form of Hilbert-style calculi, which are algebraizable in the sense of J. W. Blok and D. Pigozzi [cf. 12] w.r.t. d-algebras, Gärdenfors' algebras, and SR-algebras, respectively. As it will be clear soon, this task can be achieved with ease.

Let us consider the absolutely free algebra $\mathrm{Fm}_{\mathcal{L}'}$ in the algebraic language $\mathcal{L}' = \{\wedge, \vee, \rightsquigarrow, ', \bot, \top\}$ generated by an infinite countable set of variables Var. We let $\alpha \to \beta$, $\alpha \leftrightarrow \beta$, and $\alpha \rightsquigarrow_\gamma \beta$, $\alpha \,\mathbb{R}_\gamma\, \beta$, and $\alpha \,\mathbb{I}_\gamma\, \beta$ be defined as customary, for any $\alpha, \beta, \gamma \in \mathrm{Fm}_{\mathcal{L}'}$.

The logic **D** is the smallest structural consequence relation $\vdash_{\mathbf{D}} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}'}) \times \mathrm{Fm}_{\mathcal{L}'}$ closed under instances of the following axioms/inference rules. Let $\mathrm{Taut}_{\mathbf{CPL}}$ be the set of tautologies of **CPL**.

(A1) $\vdash_{\mathbf{D}} \delta$, for any $\delta \in \mathrm{Taut}_{\mathbf{CPL}}$ in the language $\{\wedge, \vee, \neg, \top, \bot\}$;
(A2) $\vdash_{\mathbf{D}} \neg(\alpha \rightsquigarrow_{\gamma} \top)$;
(A3) $\vdash_{\mathbf{D}} (\alpha \rightsquigarrow_{\gamma} \beta) \to (\alpha \to \beta)$;
(A4) $\vdash_{\mathbf{D}} (\neg\alpha \rightsquigarrow_{\gamma} \beta) \to (\beta \to (\alpha \rightsquigarrow_{\gamma} \beta))$;
(A5) $\vdash_{\mathbf{D}} (\alpha \rightsquigarrow_{\gamma} \beta) \to ((\alpha \wedge \gamma) \rightsquigarrow_{\gamma} \beta)$;
(A6) $\vdash_{\mathbf{D}} ((\gamma \to \alpha) \rightsquigarrow_{\gamma} \beta) \to (\alpha \rightsquigarrow_{\gamma} \beta)$;
(A7) $\vdash_{\mathbf{D}} (\alpha \rightsquigarrow_{\gamma} \alpha) \leftrightarrow (\neg\alpha \rightsquigarrow_{\gamma} \neg\alpha)$;
(A8) $\vdash_{\mathbf{D}} (\alpha \rightsquigarrow_{\gamma} \beta) \to (\alpha \rightsquigarrow_{\gamma} \alpha) \wedge (\beta \rightsquigarrow_{\gamma} \beta)$;
(A9) $\vdash_{\mathbf{D}} ((\alpha \wedge \gamma) \rightsquigarrow_{\gamma} \beta) \to \alpha \, \mathrm{R}_{\gamma} \, \beta$;
(A10) $\alpha_1 \leftrightarrow \alpha_2, \gamma_1 \leftrightarrow \gamma_2, \delta_1 \leftrightarrow \delta_2 \vdash_{\mathbf{D}} (\alpha_1 \rightsquigarrow_{\gamma_1} \delta_1) \leftrightarrow (\alpha_2 \rightsquigarrow_{\gamma_2} \delta_2)$;
(A11) Modus ponens: $\alpha \to \beta, \alpha \vdash_{\mathbf{D}} \beta$.

It is easily seen that, for any $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}'}$, $\Gamma \vdash_{\mathbf{D}} \varphi$ if and only if there exists $\gamma_1, \ldots, \gamma_n$ ($n \geqslant 1$) such that $\gamma_n = \varphi$ and, for any $1 \leqslant i \leqslant n$:

1. $\gamma_i \in \Gamma$, or
2. γ_i is the instance of an axiom, or
3. there exist $j, k < i$ such that $\gamma_k = \gamma_j \to \gamma_i$.
4. γ_i has been obtained by (A10) from $\gamma_j = \alpha_1 \leftrightarrow \alpha_2$, $\gamma_k = \beta_1 \leftrightarrow \beta_2$ and $\gamma_l = \delta_1 \leftrightarrow \delta_2$, $j, k, l < i$.

Of course, a formula α will be said *provable* in (or a *theorem* of) **D** provided that $\vdash_{\mathbf{D}} \alpha$.

In the light of Lemma 4.1, in order to provide a logic for $\mathcal{R}$, we consider the following axiomatic extensions of **D**.

Definition 4.4. The logic $\vdash_{\mathbf{G}}$ is the extension of **D** by the axiom scheme

$$(\alpha \wedge \delta) \, \mathrm{R}_{\gamma} \, \beta \to (\alpha \, \mathrm{R}_{\gamma} \, \beta) \vee (\delta \, \mathrm{R}_{\gamma} \, \beta).$$

To obtain a logic for R we just have to add axioms according to Definition 4.3.

Definition 4.5. The logic **SR** is the extension of **D** by the following inference rules:

1. $\vdash_{\mathbf{SR}} (\alpha \rightsquigarrow_{\gamma} \beta) \leftrightarrow (\neg\beta \rightsquigarrow_{\gamma} \neg\alpha)$;
2. $\vdash_{\mathbf{SR}} (\alpha \rightsquigarrow_{\gamma} \beta) \wedge \alpha \leftrightarrow (\beta \rightsquigarrow_{\gamma} \alpha) \wedge \beta$;
3. $\vdash_{\mathbf{SR}} [(\beta \, \mathbb{I}_{\gamma} \, \delta) \wedge (\alpha \rightsquigarrow_{\gamma \wedge \delta} \beta)] \to (\alpha \wedge \delta) \rightsquigarrow_{\gamma} \beta$;
4. $\vdash_{\mathbf{SR}} [(\beta \, \mathbb{I}_{\gamma} \, \delta) \wedge ((\alpha \wedge \delta) \rightsquigarrow_{\gamma} \beta)] \to \alpha \, \mathbb{R}_{\gamma \wedge \delta} \, \beta$;
5. $\vdash_{\mathbf{SR}} (\alpha \rightsquigarrow_{\gamma} \beta) \wedge (\beta \, \mathbb{I}_{\gamma} \, \delta) \wedge \mathrm{C}(\alpha, \gamma, \delta) \to [(\alpha \wedge \delta) \rightsquigarrow_{\gamma} \beta]$.

In view of (A10), **D**, **G**, and **SR** are expansions of **CPL** satisfying (Re) from Proposition 2.1. Therefore, by Proposition 2.4, Proposition 2.2(2), and Remark 2.1, **D**, **G** and **SR** are algebraizable by means of transformers $\tau(\alpha) \mapsto \{\alpha \approx \top\}$ and $\rho(\epsilon \approx \delta) \mapsto \{\epsilon \leftrightarrow \delta\}$ with respect to classes of algebras $\mathcal{Q}$, $\mathcal{Q}'$ and $\mathcal{Q}''$, respectively, as their equivalent algebraic semantics. Moreover, **D**, **G**, and **SR** are finitary (by definition) and they are algebraizable by means of a finite set of equivalence formulas. Therefore, they are BP-algebraizable (see p. 109). So, by applying Proposition 2.3, we have an explicit axiomatization of $\mathcal{Q}$, $\mathcal{Q}'$ and $\mathcal{Q}''$, respectively.

In view of the above discussion, in order to show that **D**, **G** and **SR** are algebraizable with equivalent algebraic semantics $\mathcal{D}$, $\mathcal{G}$ and $\mathcal{SR}$, respectively, it suffices to show that $\mathcal{Q} = \mathcal{D}$, $\mathcal{Q}' = \mathcal{G}$, and $\mathcal{Q}'' = \mathcal{SR}$. In the sequel we focus on **D**, since **G** and **SR** can be treated in the same way.

By Proposition 2.3, we have that $\mathcal{Q}$ is the quasi-variety axiomatized by the equations

(τ-Ax) $\alpha \approx \top$, for each $\alpha \in \mathrm{Taut}_{\mathbf{CPL}} \cup \{\mathrm{A}i : 1 \leqslant i \leqslant 9\}$;

and sets of quasi-equations (τ-InfR)

$$x \approx \top \text{ and } x \to y \approx \top \curvearrowright y \approx \top;$$
$$\textstyle\bigwedge_{i=1}^{3} x_i \leftrightarrow y_i \approx \top \curvearrowright (x_1 \rightsquigarrow_{x_2} x_3) \leftrightarrow (y_1 \rightsquigarrow_{y_2} y_3) \approx \top,$$

and (Red) $x \leftrightarrow y \approx \top \curvearrowright x \approx y$.

To show that $\mathcal{Q} \subseteq \mathcal{D}$, we prove that any $\boldsymbol{A} = (A, \rightsquigarrow, \wedge, \vee, ', \bot, \top) \in \mathcal{Q}$ satisfies equations of d-algebras. We sketch the proof leaving further details as a simple exercise for the reader.

(a) $(A, \wedge, \vee, ', \bot, \top) \in \mathcal{BA}$. To see this, one proves that $\boldsymbol{A}$ satisfies axioms of Boolean algebras. For example, we show that $\boldsymbol{A} \models x \wedge (x \vee y) \approx x$. By ($\tau$-Ax), one has that $\boldsymbol{A} \models ((x \wedge (x \vee y)) \leftrightarrow x) \approx \top$ (since $(x \wedge (x \vee y)) \leftrightarrow x \in \mathrm{Taut}_{\mathbf{CPL}}$). Therefore, the desired conclusion follows by (Red).

(b) (D1)–(D8) hold. We prove, e.g., (D4). By (τ-Ax), one has that $\boldsymbol{A} \models (x \rightsquigarrow_z y) \to (y \to (x' \rightsquigarrow_z y)) \approx \top$. By (a), one has that $\boldsymbol{A} \models (x \rightsquigarrow_z y) \wedge y \leq x' \rightsquigarrow_z y$.

To prove that $\mathcal{D} \subseteq \mathcal{Q}$, one proceeds exactly as above by showing that (τ-Ax), (τ-InfR) and (Red) hold in $\mathcal{D}$. For example, to show that any $\boldsymbol{A} \in \mathcal{D}$ satisfies (τ-Ax), it suffices to observe that the reduct $(A, \rightsquigarrow, \wedge, \vee, ', \bot, \top)$ is a Boolean algebra and so, for any $\alpha \in \mathrm{Taut}_{\mathbf{CPL}}$, $\alpha \approx \top$

holds. Moreover, for any $\alpha \in \{\mathrm{A}i : 1 \leq i \leq 9\}$, $\alpha \approx \top$ is ensured by axioms of d-algebras and residuation (of $\rightarrow$ w.r.t. $\wedge$). (τ-InfR) and (Red) are straightforward. Note that, as regards tautologies of **CPL**, one can confine oneself to taking just one of the many finite axiomatizations of Classical Logic available in the literature.

In view of the above reasoning, we have just proven the following result.

Theorem 4.10. *The logics* **D**, **G** *and* **SR** *are algebraizable with* $\mathcal{D}$, $\mathcal{G}$ *and* $\mathcal{SR}$, *respectively, as their equivalent variety semantics with transformers* $\tau\colon \alpha \mapsto \{\alpha \approx \top\}$ *and* $\rho\colon \epsilon \approx \delta \mapsto \{\epsilon \leftrightarrow \delta\}$.

In the light of Theorem 4.10, we have indeed a stronger result. Recall that a logic $\boldsymbol{L}$ is *regularly* (BP-)algebraizable if it is (BP-)algebraizable and, moreover, it satisfies

$$x, y \vdash_{\boldsymbol{L}} \Delta(x, y), \tag{G}$$

for some set $\Delta(x, y)$ of equivalence formulas [cf. 12, Definition 3.49]. By Theorem 4.10, **D**, **G** and **SR** are BP-algebraizable. Moreover, since any $\mathcal{V} \in \{\mathcal{D}, \mathcal{G}, \mathcal{SR}\}$ satisfy

$$x \approx \top, y \approx \top \models_{\mathcal{V}} x \leftrightarrow y \approx \top,$$

one has **D**, **G** and **SR** satisfy (G). Therefore, **D**, **G** and **SR** are regularly algebraizable. So, by [12, Proposition 4.58], they are complete with respect to classes of reduced matrices $\langle \boldsymbol{A}, \{\top^{\boldsymbol{A}}\}\rangle : \boldsymbol{A} \in \mathcal{K}\}$, where $\mathcal{K}$ stands for $\mathcal{D}$, $\mathcal{G}$ and $\mathcal{SR}$, respectively [see 12, Definition 4.37].

Despite its customary proof, Theorem 4.10 has the following important consequences. Recall that a logic $\boldsymbol{L}$ over a language $\mathcal{L}$ has the uni-term Deduction Detachment property w.r.t. a binary term $t(x, y)$ if it satisfies, for any $\Gamma \cup \{\alpha, \beta\} \subseteq \mathrm{Fm}_{\mathcal{L}}$:

$$\Gamma, \alpha \vdash_{\boldsymbol{L}} \beta \text{ iff } \Gamma \vdash_{\boldsymbol{L}} t(\alpha, \beta).$$

Proposition 4.11. *The uni-term Deduction Detachment property w.r.t.* $\rightarrow$ *fails for* **D**, **G** *and* **SR**.

Proof. First, we prove that $x, x \rightsquigarrow_z y \vdash_{\mathbf{D}} \bot$ ($x, y, z \in \mathrm{Var}$). In fact, let $\boldsymbol{A} \in \mathcal{D}$ and $h : \mathrm{Fm}_{\mathcal{L}'} \to \boldsymbol{A}$ be an arbitrary homomorphism. If $h(x) = \top$, then $h(x \rightsquigarrow_z y) = h(x) \rightsquigarrow_{h(z)} h(y) = \top \rightsquigarrow_{h(z)} h(y) = \bot$, by Lemma 4.2(1). So, $\mathcal{D}$ vacuously satisfies

$$\tau(x), \tau(x \rightsquigarrow_z y) \models_{\mathcal{D}} \tau(\bot).$$

Therefore, by Theorem 4.10, the desired conclusion obtains. Now, suppose towards a contradiction that the uni-term Deduction Detachment property w.r.t. $\to$ holds for $\mathbf{D}$. One has $\vdash_{\mathbf{D}} x \to \neg(x \rightsquigarrow_z y)$. So, any d-algebra satisfies the identity $x \leq (x \rightsquigarrow_z y)'$. But this is impossible, as in the d-algebra from Example 4.2 one has, e.g., that $a \not\leq (a \rightsquigarrow_\top a)' = \bot$. $\square$

However, since, by Proposition 2.1 and [12, Theorem 6.7], any algebraizable logic is *protoalgebraic* [see 12, Definition 6.1], $\mathbf{D}$ ($\mathbf{G}$, $\mathbf{SR}$) satisfies the *Parametrised Local Deduction-Detachment Theorem* with respect to a family Φ of *Deduction-Detachment sets with parameters* [see 12, Definition 6.21, Theorem .22]. In order to keep the present manuscript within a reasonable length, we postpone the explicit characterization of Φ to a future work.

Remark 4.1. As a consequence of Proposition 4.11, in view of [12, Corollary 3.73], it can be seen that the (proper) inference rule (A10) is *not* admissible in $\mathbf{D}$ ($\mathbf{G}$, $\mathbf{SR}$).

It is worth noticing that, due to Theorem 4.10, $\mathbf{D}$ ($\mathbf{G}$, $\mathbf{SR}$) provides inference schemes with a rather counter-intuitive flavour. For example, beside the already mentioned

$$\alpha, \alpha \rightsquigarrow_\gamma \beta \vdash_{\mathbf{D}} \bot, \tag{3}$$

the following inference scheme is derivable in $\mathbf{D}$ ($\mathbf{G}$, $\mathbf{SR}$):

$$\alpha \vdash_{\mathbf{D}} \alpha \, \mathbb{I}_\gamma \, \beta. \tag{4}$$

In other words, $\mathbf{D}$ ($\mathbf{G}$, $\mathbf{SR}$) allows us to prove that no “definitely”, or *a priori*, true statement, even if not tautological, can be relevant to any other statement under an arbitrary evidence. Such a drawback has been already pointed out by [14]. Indeed, the above inference might look like rather counter-intuitive at first sight. For example, it seems difficult to deny that a statement like “The first 982976262 even numbers are sums of two prime numbers” could be relevant for the statement “Goldbach’s conjecture is true”. However, these “pathological” inference schemes have a quite natural explanation. Indeed, R and $\mathcal{R}$ are intrinsically probabilistic. A proposition a is statistically/weakly relevant to b with respect to an evidence c if the probability of a increases or decreases directly/indirectly the probability that b occurs or does not occur given that c happens, and a necessary condition of this fact is that a must be *contingent* w.r.t. c. Therefore, if a is *a priori* true, then it can be considered without loss of generality as a part of c and so absolutely non influential on the evaluation of the probability of b, *given* c. This last fact is witnessed, e.g., by the inference rule:

$$\alpha \vdash_{\mathbf{D}} \beta \, \mathbb{R}_{\gamma} \, \delta \leftrightarrow \beta \, \mathbb{R}_{\gamma \wedge \alpha} \, \delta.$$

However, if one wants to avoid the above phenomena, another route is possible. In fact, instead of considering the $\top$-assertional logic of $\mathcal{D}$ one may take the order-logic $\vdash_{\mathbf{D}^{\leq}}$ over the signature $\mathcal{L}'$ defined as follows, for any $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}'}$:

$$\Gamma \vdash_{\mathbf{D}^{\leq}} \alpha \text{ iff } \exists \Gamma' \subseteq \Gamma,\ |\Gamma'| < \omega,\ \mathcal{D} \models \textstyle\bigwedge_{\gamma \in \Gamma'} \gamma \leq \alpha. \tag{OL}$$

We can obtain the logics $\vdash_{\mathbf{G}^{\leq}}$ and $\vdash_{\mathbf{SR}^{\leq}}$ as above by replacing $\mathcal{D}$ by $\mathcal{G}$ and $\mathcal{SR}$, respectively, in (OL). A little thought shows that, in this case, inference schemes (3) and (4) are no longer valid, since, for example, $x \wedge (x \rightsquigarrow_z y) \approx \bot$ fails in at least a G-algebra and an SR-algebra. The interested reader can easily find a counterexample in the structures from Example 4.2 and Example 4.1, respectively. Moreover, in the light of general facts on logics induced by *equationally orderable quasivarieties* [18], $\mathbf{D}^{\leq}$, $\mathbf{G}^{\leq}$ and $\mathbf{SR}^{\leq}$ enjoy interesting properties. For example, by [18, Proposition 2.16, Theorem 2.13], $\mathbf{D}^{\leq}$, $\mathbf{G}^{\leq}$ and $\mathbf{SR}^{\leq}$ are *fully self-extensional*. Moreover, while, by Proposition 4.11, $\mathbf{D}$, $\mathbf{G}$ and $\mathbf{SR}$ do not satisfy the uni-term Deduction Detachment property w.r.t. $\to$, $\mathbf{D}^{\leq}$, $\mathbf{G}^{\leq}$ and $\mathbf{SR}^{\leq}$ do.

Proposition 4.12. *$\mathbf{D}^{\leq}$ ($\mathbf{G}^{\leq}$, $\mathbf{SR}^{\leq}$) satisfies the uni-term Deduction Detachment Property w.r.t. $\to$.*

Proof. We prove that, for any $\Gamma \cup \{\alpha, \beta\} \subseteq \mathrm{Fm}_{\mathcal{L}'}$, we have $\Gamma, \alpha \vdash_{\mathbf{D}^{\leq}} \beta$ iff $\Gamma \vdash_{\mathbf{D}^{\leq}} \alpha \to \beta$. The right-to-left direction follows upon noticing that if $\mathcal{D} \models \bigwedge_{i=1}^{n} \gamma_i \leq \alpha \to \beta$, then $\mathcal{D} \models \bigwedge_{i=1}^{n} \gamma_i \wedge \alpha \leq (\alpha \to \beta) \wedge \alpha \leq \beta$. Conversely, let us assume that $\Gamma, \alpha \vdash_{\mathbf{D}^{\leq}} \beta$. Then we have that there exists $\Gamma' \subseteq \Gamma \cup \{\alpha\}$, where $|\Gamma'| < \omega$, such that $\mathcal{D} \models \bigwedge_{\gamma' \in \Gamma'} \gamma' \wedge \alpha \leq \beta$. Therefore, the desired conclusion follows by residuation. $\square$

It is clear that d(G,SR)-algebras are obtained from Boolean algebras by expanding their signature with a new ternary connective obtained by constraining material implications by means of a ternary relation involving their antecedents and consequents. As already mentioned in

Section 1, this idea finds its source of inspiration in relating logics' machinery. Now, it can be seen that the framework we deal with in this paper allows to define many "relating implications" $\rightsquigarrow_z$, rather than a single one, depending on how we parameterize $\rightsquigarrow$. Therefore, logics $\mathbf{D}, \mathbf{G}$ and $\mathbf{SR}$ might be regarded as expansions of $\mathbf{CPL}$ with a countable infinity of implication connectives $\rightsquigarrow_\gamma$ which can be obtained from material implications by imposing constraints depending on that the antecedents and consequents are in a relation R_γ or not. Consequently, rather than mono-relating logics [see, e.g., 21], our logics might be considered as "multi-relating logics". Now, it naturally rises the question if a multi-relating semantics for $\mathbf{D}, \mathbf{G}$ and $\mathbf{SR}$ may be provided. Let us consider $\boldsymbol{L} \in \{\mathbf{D}, \mathbf{G}, \mathbf{SR}\}$. One might ask if there exists a class $\mathfrak{L}$ of structures of the form $\langle v, \{R_\gamma\}_{\gamma \in \mathrm{Fm}_{\mathcal{L}'}} \rangle$ where $v\colon \mathrm{Var} \to \{\top, \bot\}$ is an interpretation of propositional variables and $\{R_\gamma\}_{\gamma \in \mathrm{Fm}_{\mathcal{L}'}}$ is such that $R_\gamma \subseteq \mathrm{Fm}^2_{\mathcal{L}'}$, for any $\gamma \in \mathrm{Fm}_{\mathcal{L}'}$, which define a relating semantics for $\boldsymbol{L}$ as follows. Each $M = \langle v, \{R_\gamma\}_{\gamma \in \mathrm{Fm}_{\mathcal{L}'}} \rangle \in \mathfrak{L}$ should induce a valuation over $\mathrm{Fm}_{\mathcal{L}'}$ to be defined inductively as follows:

1. $M \models x$ iff $v(x) = \top$, for any $x \in \mathrm{Var}$;
2. $M \models \top$ and $M \not\models \bot$;
3. $M \models \neg\alpha$ iff $M \not\models \alpha$;
4. $M \models \alpha \wedge \beta$ iff $M \models \alpha$ and $M \models \beta$;
5. $M \models \alpha \vee \beta$ iff $M \models \alpha$ or $M \models \beta$;
6. $M \models \alpha \rightsquigarrow_\gamma \beta$ iff $M \not\models \alpha$ or $M \models \beta$, and $(\alpha, \beta) \in R_\gamma$.

Now, upon setting, for any $M \in \mathfrak{L}$ and $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}'}$

1. $\Gamma \models_M \alpha$ provided that, if $M \models \gamma$, for any $\gamma \in \Gamma$, then $M \models \alpha$, and
2. $\Gamma \models_{\mathfrak{L}} \alpha$ iff $\Gamma \models_M \alpha$, for any $M \in \mathfrak{L}$,

the following result should be provable, for any $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}'}$:

$$\Gamma \vdash_{\boldsymbol{L}} \alpha \text{ iff } \Gamma \models_{\mathfrak{L}} \alpha. \tag{5}$$

However, this is not the case. Let us consider by way of example the case $\boldsymbol{L} = \mathbf{D}$. Assume that there exists a class $\mathfrak{D}$ of multi-relating models satisfying (5). In the light of Lemma 4.2, and by the algebraization Theorem 4.10, one should have that, for any $\alpha, \gamma \in \mathrm{Fm}_{\mathcal{L}'}$, $(\bot, \alpha), (\alpha, \top) \notin R_\gamma$. Now, let $M = \langle v, \{R_\gamma\}_{\gamma \in \mathrm{Fm}_{\mathcal{L}'}} \rangle \in \mathfrak{D}$, $x, y, z \in \mathrm{Var}$. One has that $M \models x \rightsquigarrow_z y$ iff $M \not\models x$ or $M \models y$, and $(x, y) \in R_z$. If $M \not\models x$, then $M \models x \leftrightarrow \bot$. So, by (A10), one has $M \models \bot \rightsquigarrow_z y$. But this means that $(\bot, y) \in R_z$, which is impossible. Similarly, the case $M \models y$

leads to $M \models x \rightsquigarrow_z \top$ and so $(x, \top) \in R_z$, again a contradiction. So, one should have $M \not\models x \rightsquigarrow_z y$. Since M is arbitrary, one should have also $\models_{\mathfrak{D}} \neg(x \rightsquigarrow_z y)$ and, by hypothesis, $\vdash_{\mathbf{D}} \neg(x \rightsquigarrow_z y)$. Hence, by Theorem 4.10, $x \rightsquigarrow_z y \approx \bot$ should hold in any d-algebra. But this is not the case, since there exists $\boldsymbol{A} \in \mathcal{D}$ such that $|A| > 2$ satisfying, for any $a \in A \smallsetminus \{\top, \bot\}$, $a \rightsquigarrow_\top a = \top$. See Example 4.2. Moreover, the same problems occur if one replaces **D** by the stronger **G** or **SR**.

It is clear that a counterexample for **D** to have a multi-relating semantics in the above sense rests substantially on the use of (A10). However, it seems reasonable to ask if, although one is not able to obtain a strong completeness theorem, a weaker result is still available. In fact, it naturally raises the question of whether, for any $\boldsymbol{L} \in \{\mathbf{D}, \mathbf{G}, \mathbf{SR}\}$, there exists a class of multi-relating models $\mathfrak{L}$ such that for any $\alpha \in \mathrm{Fm}_{\mathcal{L}'}$,

$$\vdash_{\boldsymbol{L}} \alpha \text{ iff } \models_{\mathfrak{L}} \alpha.$$

To this end, we can confine ourselves to consider the logics $\mathbf{D}^*$, $\mathbf{G}^*$, and $\mathbf{SR}^*$ obtained from **D**, **G**, and **SR**, respectively, by replacing (A10) by the less demanding inference rule $(A10^*)$

$$\vdash \alpha_1 \leftrightarrow \alpha_2, \vdash \gamma_1 \leftrightarrow \gamma_2, \vdash \beta_1 \leftrightarrow \beta_2 \text{ implies } \vdash \alpha_1 \rightsquigarrow_{\gamma_1} \beta_1 \leftrightarrow \alpha_2 \rightsquigarrow_{\gamma_2} \beta_2.$$

Of course, routine Lindenbaum-Tarski arguments yield the next theorem.

Theorem 4.13. *For any $\alpha \in \mathrm{Fm}_{\mathcal{L}'}$, the following hold*:

1. $\vdash_{\mathbf{D}^*} \alpha$ *iff* $\models_{\mathcal{D}} \alpha$;
2. $\vdash_{\mathbf{G}^*} \alpha$ *iff* $\models_{\mathcal{G}} \alpha$;
3. $\vdash_{\mathbf{SR}^*} \alpha$ *iff* $\models_{\mathcal{SR}} \alpha$.

Therefore, we leave the following problem to future investigation.

Problem 4.1. Show that, for any $\boldsymbol{L} \in \{\mathbf{D}^*, \mathbf{G}^*, \mathbf{SR}^*\}$, there exists a family $\mathfrak{L}$ of multi-relating models such that for any $\alpha \in \mathrm{Fm}_{\mathcal{L}'}$, $\vdash_{\boldsymbol{L}} \alpha$ iff $\models_{\mathfrak{L}} \alpha$.

4.2. Weakly connexive implications and relevance

The intuition linking together connexive principles and content relationships between antecedents and consequents of sound conditionals dates back to the dawn of logic, in particular to Chrysippus' definition of sound conditionals:

> And those who introduce the notion of *connexion* say that a conditional is sound when the contradictory of its consequent is incompatible with its antecedent. [Sextus Empiricus, 24, p. 129]

Such a perspective was later revived in E. J. Nelson's work on intentional relations. Indeed, [31] observes that, once entailment is defined in terms of inconsistency of the antecedent with the proper contradictory of the consequent, namely as encoding " a necessary connexion between meaning" [see 31, p. 445] Aristotle's as well as Boethius' Theses follow. In [36], R. Routley pushes forward Nelson's intuition by identifying such a "necessary connexion" in the notion of relevance. Quoting Routley:

> This requirement [*the connexion*] coincides with the broad requirement of relevance: for if antecedent and consequent enjoy a meaning connexion then they are relevant in meaning to one another, and if they are relevant in meaning to one another then they have through the relevance relation a connexion in meaning. Thus the general classes of connexive and relevant logics are one and the same. [36, p. 393]

However, as recalled by [9] (see also [29]), connexive and relevance logics are mutually incompatible, in the sense that their combination leads to contradictions, or to triviality, or to the failure of distinctive nice properties of relevant implications. Therefore, although Routley's claim seems to have a certain plausibility and intuitive appeal, it seems that connexive principles and relevance, at least to the extent that a suitable notion of relevance is captured by relevance logics, are somehow unrelated. Such a conclusion becomes even more relevant for our discourse when one takes into account M. J. Dunn's interpretation of relevant implication yielded by Routley-Meyer' semantics. Indeed, Dunn [6] interprets the ternary relation R on which Routley-Meyer frames [see, e.g., 6, p. 15] are based as a relevance relation between states of information in a given context. To clarify the notion of relevance he is referring to, Dunn cites D. Sperber and D. Wilson: "an input is relevant to an individual when its processing in a context of available assumptions yields a positive cognitive effect" [6, 45].

In the sequel we show that, once probabilistic (weak or statistical) relevance is taken into account, our framework somehow gives substance to Routley's intuitions. In fact, we will show that d(G, SR)-algebras' framework allow to define weakly connexive implications with a rather intuitive meaning which encode the relevancy of the antecedent w.r.t. the consequent under some reasonable background assumptions. Furthermore, in the last part of the section we will argue that a notion of implication inspired by Chrysippus' account of conditionals can be formalized and shown to be "almost" weakly connexive, once the concept of consistency borrowed from Section 4 is taken into account. Of course, such a notion of "consistent" is different from Nelson's. Indeed, it is closer to C.I. Lewis' account of consistency [cf. 26] than to the spirit of [31].

In the sequel we will consider the following binary term-operations in the language of d-algebras:

1. $x \Rightarrow_1 y := x \rightsquigarrow_{\mathrm{C}(x,y)} y$;
2. $x \Rightarrow_2 y := x \rightsquigarrow_{x\mathrm{R}_y x} y$;
3. $x \Rightarrow_3 y := x \rightsquigarrow_{x' \rightsquigarrow_\top y'} y$;
4. $x \Rightarrow_4 y := x \rightsquigarrow_{y \to x} y$.

A few words on the reading of the above connectives are in order. $\Rightarrow_1$ reflects the idea of a conditional whose antecedent is relevant to the consequent under the evidence that both are non-trivially satisfiable and, moreover, they are consistent, i.e., they do not contradict each other. Interestingly enough, these conditions closely resemble, although from a different theoretical perspective, the definition of G. Priest's weakly connexive implication defined within the account of negation as cancellation [35, p. 145].

$\Rightarrow_2$ subsumes the idea that the antecedent of a sound conditional relevantly implies the consequent under the evidence that the former is contingent on the latter (cf. p. 119). Such a connective is reminiscent of *contingency conditionals* as outlined in [42], in which it is argued that natural language conditionals have antecedents which are possible, namely either *neutrally indeterminate* or *neutrally contingent*, where a statement is neutrally contingent "if it is neither necessary – or necessarily true — nor impossible – or necessarily false" [42, p. 301]. Note that $\Rightarrow_2$ relativises the contingency of the antecedent to the truth of the consequent.

As regards $\Rightarrow_3$, this implication embodies the idea of a kind of conditional in which the antecedent relevantly implies the consequent given that the falsity of the latter is dependent on, and a consequence of, the falsity of the former. A suggestive concrete example of conditional of the kind encoded by $\Rightarrow_2$ might be the following: "x is the only reason justifying y". Note that a connexive implication with a similar reading has been proposed by [10] within the framework of Semi-Heyting algebras.

Finally, it is clear that $\Rightarrow_4$ encodes the idea of a conditional in which the antecedent relevantly implies the consequent under the assumption that the latter materially implies the former. Interestingly enough, $\Rightarrow_4$ can be somehow regarded as an implication encoding the kind of connection argued by [38] between a hypothesis H and (the conjunction of) its empirical consequences C which *confirm* it, whenever the *entailment condition* holds, as outlined by Hempel's theory of confirmation [17, 32].[4]

Theorem 4.14. *For any $i \in \{1,2,3,4\}$, $\boldsymbol{A} \in \mathcal{D}$:*

1. $\boldsymbol{A} \models x \Rightarrow_i x' \approx \bot$;
2. $\boldsymbol{A} \models x \Rightarrow_i y = \top$ *implies* $\boldsymbol{A} \models (x \Rightarrow_i y')' = \top$.

Moreover, for any $i \in \{1,2,3,4\}$, there exists $\boldsymbol{A} \in \mathcal{D}$ and $a, b \in A$ such that $a \Rightarrow_i b = \top$ but $b \Rightarrow_i a \neq \top$.

Proof. $i = 1$. Note that $x \Rightarrow_1 x' = x \rightsquigarrow_{\mathrm{C}(x,x')} x' = x \rightsquigarrow_\bot x' = \bot$, by Lemma 4.2(1). Also, assume that $x \Rightarrow_1 y = \top$. This means that $x \leq y$. Moreover, $x \Rightarrow_1 y' = (x \wedge y) \Rightarrow_1 y' = (x \wedge y) \rightsquigarrow_{\mathrm{C}(x \wedge y, y')} y' = (x \wedge y) \rightsquigarrow_\bot y' = \bot$. So $(x \Rightarrow_1 y')' = \top$.

$i = 2$. Note that $x \,\mathrm{R}_{x'}\, x = \bot$ by Lemma 4.2(2). Therefore, $x \Rightarrow_2 x' = x \rightsquigarrow_\bot x' = \bot$. Now, if $x \Rightarrow_2 y = \top$, then $x \leq y$ and so $x \,\mathrm{R}_{y'}\, x = \bot$. This implies $x \Rightarrow_2 y' = \bot$.

Let us consider the case $i = 3$. One has that $x \Rightarrow_3 x' = x \rightsquigarrow_{(x' \rightsquigarrow_\top x)} x'$. Now, upon observing that $x = x \vee (x' \wedge x' \rightsquigarrow_y x) = x \vee (x' \rightsquigarrow_y x)$, we have $x \rightsquigarrow_{x' \rightsquigarrow_y x} x' \leq (x \wedge (x' \rightsquigarrow_y x)) \rightsquigarrow_{x' \rightsquigarrow_y x} x' \leq (x' \rightsquigarrow_y x) \rightsquigarrow_{x' \rightsquigarrow_y x} x' = \bot$. Our conclusion follows by setting $y := \top$. Assume $x \Rightarrow_3 y = \top$. So $x \leq y$. Now, one has that $y = y \vee (x' \wedge (x' \rightsquigarrow_z y)) = y \vee (x' \rightsquigarrow_z y)$. Consequently, we have $x' \rightsquigarrow_z y \leq y$ and also $y \rightsquigarrow_{x' \rightsquigarrow_z y} u = \bot$. In particular, we have $y \rightsquigarrow_{x' \rightsquigarrow_z y} y = \bot$. Now, observe that, for any $z, u \in A$, we have $z \rightsquigarrow_u z = z' \rightsquigarrow_u z'$. This is an easy consequence of (D6). So we have that $y' \rightsquigarrow_{x' \rightsquigarrow_z y} y' = y \rightsquigarrow_{x' \rightsquigarrow_z y} y = \bot$. Therefore, again by (D6), we have $x \Rightarrow_3 y' = x \rightsquigarrow_{x' \rightsquigarrow_\top y} y' \leq y' \rightsquigarrow_{x' \rightsquigarrow_\top y} y' = \bot$. Finally, as regards $i = 4$, first we have that $x \Rightarrow_4 x' = x \rightsquigarrow_{x' \to x} x' = x \rightsquigarrow_x x' = \bot$. Now, assume that $x \Rightarrow_4 y = \top$. this means that $x \rightsquigarrow_{y \to x} y = \top$ and so $x \leq y$. But then $x \rightsquigarrow_{y' \to x} y' = x \rightsquigarrow_{y \vee x} y' = x \rightsquigarrow_y y' = \bot$.

To prove that none of the $\Rightarrow_i'$ s is symmetric we will make use of the SR-algebra $(\mathcal{P}(A), \rightsquigarrow^*, \cap, \cup, {}^c, \emptyset, A)$ from Example 4.1 and fix $X := \{1,2\}$, $Y := \{1,2,3\}$. For $i = 1$, one has $\mathrm{C}(X,Y) = X \rightsquigarrow_A X = A$ and $m(Y) \neq 1$. It is easily seen that this entails $X \,\mathrm{R}_A\, Y$ and so $X \Rightarrow_1 Y = A$. However, $Y \Rightarrow_1 X \neq A$ for obvious reasons. Concerning the case $i = 2$, considering X and Y as above, easy computations show that $X \mathrm{R}_Y X = A$ and so, reasoning as above, we have $X \Rightarrow_2 Y = A$ although the converse

[4] Quoting Hempel: "Any sentence which is entailed by an observation report is confirmed by it" [17, p. 31].

does not hold. Concerning $i = 3$, just note that $X \,\mathrm{R}_A\, Y$ entails that $Y^c \,\mathrm{R}_A\, X^c$, by Lemma 3.1. So $Y^c \rightsquigarrow_A X^c = X \to Y = A$. And so $X \rightsquigarrow_{Y^c \rightsquigarrow_A X^c} Y = X \rightsquigarrow_A Y = A$, namely also in this case our claim obtains. Let us prove the case $i = 4$. Note that $Y \to X := \{1, 2, 4, 5, 6, 7, 8, 9\}$. It can be seen that $m(Y \to X) \neq m(X)$ and so $X \,\mathrm{R}_{Y\to X}{}^m\, Y$. We conclude that $X \Rightarrow_4 Y = A \neq Y \Rightarrow_4 X$. □

In the light of Definition 2.1, Theorem 4.10, Theorem 4.14, and remarks from Section 2.2, we have that, for any $\boldsymbol{L} \in \{\mathbf{D}, \mathbf{G}, \mathbf{SR}\}$, and any $i \in \{1, 2, 3, 4\}$, the following inference schemes hold:

$$\vdash_{\boldsymbol{L}} \neg(\varphi \Rightarrow_i \neg\varphi) \quad \text{and} \quad \varphi \Rightarrow_i \psi \vdash_{\boldsymbol{L}} \neg(\varphi \Rightarrow_i \neg\psi).$$

Moreover, by the last part of the proof of Theorem 4.14, it follows that, for any $\varphi \in \mathrm{Fm}_{\mathcal{L}'}$ and $1 \leqslant i \leqslant 4$: $\varphi \Rightarrow_i \psi \nvdash_{\boldsymbol{L}} \psi \Rightarrow_i \varphi$,[5] namely $\Rightarrow_i$ is not symmetric. Therefore, $\Rightarrow_i$ $(1 \leqslant i \leqslant 4)$ can be regarded as a full fledged weakly connexive implications in $\mathbf{D}$, $\mathbf{G}$, or $\mathbf{SR}$.

Remark 4.2. In general, for any $i, j \in \{1, 2, 3, 4\}$, $i \neq j$ implies $\Rightarrow_i \neq \Rightarrow_j$. In fact, let us consider the SR-algebra $(\mathcal{P}(A), \rightsquigarrow^*, \cap, \cup, {}^c, \emptyset, A)$ from Example 4.1. One has that $\{1\} \Rightarrow_1 \{1\} = \{1\} \rightsquigarrow^*_{\mathrm{C}(\{1\})} \{1\} = \{1\} \rightsquigarrow^*_A \{1\} = A$, but $\{1\} \Rightarrow_2 \{1\} = \{1\} \rightsquigarrow_{\{1\}\mathrm{R}_{\{1\}}\{1\}} \{1\} = \emptyset$. Moreover, for $X := \{2, 6, 7, 8, 9\}$ and $Y := \{4, 5, 7, 8, 9\}$, one can prove that $X \Rightarrow_1 Y = X \to Y$ but $X \Rightarrow_3 Y = \emptyset$, since $X^c \rightsquigarrow_A Y^c = \emptyset$. Moreover, one has also that $X \Rightarrow_2 Y \neq \emptyset$. In fact, it is easily seen that Lemma 3.4(1) still holds if one replaces $x\mathcal{R}_y x$ by $x \,\mathrm{R}_y\, x$. Therefore, we conclude $X \,\mathrm{R}_Y{}^m\, X$ upon noticing that $Y \not\subseteq X, X^c$. Furthermore, we have that $X^c \Rightarrow_1 Y^c = \emptyset$, while $X^c \Rightarrow_4 Y^c = X^c \rightsquigarrow^*_{Y^c \to X^c} Y^c \neq \emptyset$, since $X^c \mathrm{R}_{Y^c \to X^c}{}^m Y^c$. Now, to show that $\Rightarrow_2 \neq \Rightarrow_4$, just consider $U := \{1, 2, 3\}$ and $Z := \{1, 2\}$. One has that $U \Rightarrow_2 Z = \emptyset$, since $Z \subseteq U$. However, we have $U \Rightarrow_4 Z = U \to Z$, since $U \,\mathrm{R}_A\, Z$. Finally, one can see that $X \Rightarrow_3 Y = \emptyset$, since $X \,\mathrm{I}_A\, Y$ implies $Y^c \,\mathrm{I}_A\, X^c$, by Lemma 3.1. However, $X \Rightarrow_4 Y = X \rightsquigarrow_{Y \to X} Y = X \to Y$, since a direct calculation shows that $m(Y|X) = m(Y|(Y \to X) \cap X) = 0,6 \neq 0,47 = m(Y|Y \to X)$.

It can be seen that the notion of consistency (or compatibility), if expressed by C, allows to define the connective $x \Rightarrow_c y := x \rightsquigarrow_{\mathrm{C}(x,y')'} y$ which may be regarded as an implication encoding, *mutatis mutandis*, the kind of entailment recommended by Chrysippus.

[5] Note that, in turn, this implies that $\nvdash_{\boldsymbol{L}} (\varphi \Rightarrow_i \psi) \to (\psi \Rightarrow_i \varphi)$.

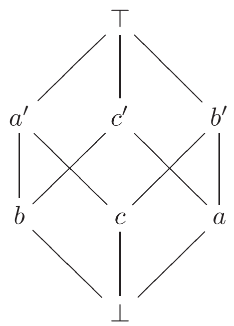

Figure 1. The algebra $\boldsymbol{A}$

$\rightsquigarrow_{b'}$	$\bot$	$\top$	a	a'	b	b'	c'	c
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\top$
a'	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\top$
b	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
b'	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
c'	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\top$
c	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\top$

Table 1. Cayley table of $\rightsquigarrow_{b'}$

It can be observed that $\Rightarrow_c$ satisfies weak Boethius' Thesis in $\mathcal{D}$. In fact, let $\boldsymbol{A} \in \mathcal{D}$. Assume that $x \rightsquigarrow_{\mathrm{C}(x,y')'} y = x \Rightarrow_c y = \top$. This means that $x \leq y$. Therefore $\mathrm{C}(x,y')' = \bot' = \top$. Now, it can be seen that $\mathrm{C}(x,y)' = \mathrm{C}(x)' = (x \rightsquigarrow_\top x)' = \top' = \bot$, since $\top = x \rightsquigarrow_\top y \leq x \rightsquigarrow_\top x$. Therefore, $x \Rightarrow_c y' = x \rightsquigarrow_{\mathrm{C}(x,y)'} y' = x \rightsquigarrow_\bot y' = \bot$. Consequently, $(x \Rightarrow_c y')' = \top$.

However, in general, $\Rightarrow_c$ does not satisfy Aristotle's Thesis. In fact, let us consider the eight-element d-algebra $\boldsymbol{A} = (\{\bot, \top, a, b, c, a', b', c'\}, \rightsquigarrow, \wedge, \vee, ', \bot, \top)$ depicted in Fig. 1, in which the implication $\rightsquigarrow$ is defined as follows. For any $x \in \{a, c, b, a', c', \bot\}$, $\rightsquigarrow_x$ is the constant function $\bot$. Moreover, $\rightsquigarrow_\top$ is such that $a \rightsquigarrow_\top a = a \rightsquigarrow_\top a' = a' \rightsquigarrow_\top a' = b$ and $x \rightsquigarrow_\top y = \bot$ in any other case, while $\rightsquigarrow_{b'}$ is defined according to Table 1.

It is easily checked that $(a \Rightarrow_c a')' = (a \rightsquigarrow_{\mathrm{C}(a)'} a')' = (a \rightsquigarrow_{b'} a')' = \top' = \bot$. Now, although Aristotle's Thesis fails in general, it can be seen that it holds in the "concrete" d-algebra $\boldsymbol{B}^* = (B, \rightsquigarrow^*, \wedge, \vee, ', \bot, \top)$ obtained extending a Boolean algebra $\boldsymbol{B}$ by means of $\rightsquigarrow^*$ to be defined as in Lemma 4.1 with respect to a given $m \in \mathcal{S}^{\boldsymbol{B}}$ and a relation among R^m and $\mathcal{R}^m$. In fact, we have the following

Remark 4.3. Let $\boldsymbol{A} \in \mathcal{D}$. Then $\boldsymbol{A} \models (x \Rightarrow_c x')'$, if $\boldsymbol{A}$ satisfies the following condition, for any $x \in A$:

$$\text{if } x \notin \{\bot, \top\} \text{ then } x \rightsquigarrow_\top x = \top. \tag{6}$$

In fact, assuming w.l.o.g. $x \notin \{\bot, \top\}$, one has $x \Rightarrow_c x' = x \rightsquigarrow_{\mathrm{C}(x,x'')'} x' = x \rightsquigarrow_{(x \rightsquigarrow_\top x)'} x' = x \rightsquigarrow_\bot x' = \bot$, i.e. $(x \Rightarrow_c x')' = \top$.

Lemma 4.15. *Let $\boldsymbol{A} \in \mathcal{D}$. If $\boldsymbol{A}$ satisfies (6), then $\boldsymbol{A}$ is simple.*

Proof. Let θ be a congruence over $\boldsymbol{A}$, suppose that $\theta \neq \Delta$, and consider $(a, b) \in \theta$ such that $a \neq b$. We can assume w.l.o.g. that $a \nleq b$. One has that $a \wedge b' \neq \bot$ and $(a \wedge b', \bot) \in \theta$. In turn, this implies that $(a \wedge b' \rightsquigarrow_\top a \wedge b', \bot \rightsquigarrow_\top a \wedge b') = (\top, \bot) \in \theta$, i.e. $\theta = \nabla$. This means that $\boldsymbol{A}$ is simple. $\square$

Remark 4.4. We observe that the converse of Lemma 4.15 does not hold, in general. In fact, let us consider the d-algebra $\boldsymbol{A}$ whose Boolean reduct is the algebra depicted in Figure 1, and such that $\rightsquigarrow$ is defined as follows. We set $x \rightsquigarrow_z y = \bot$, for any $z \in A \setminus \{\top\}$ and we let $\rightsquigarrow_\top$ be defined according to the next Cayley table:

$\rightsquigarrow_\top$	$\bot$	$\top$	a	b	c	a'	b'	c'
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	a'	$\top$	$\bot$	a'	$\bot$	$\bot$
b	$\bot$	$\bot$	$\bot$	b'	$\bot$	$\bot$	b'	$\bot$
c	$\bot$	$\bot$	$\bot$	$\bot$	c'	$\bot$	$\bot$	c'
a'	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	a'	$\bot$	$\bot$
b'	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	b'	$\bot$
c'	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	c'

Of course, if one asks if condition (6) can be expressed by means of a set Q of generalized quasi-equations axiomatizing a generalized quasivariety $\mathcal{Q} \subseteq \mathcal{D}$ which is the equivalent algebraic semantics of a logic $\boldsymbol{L}$ in which $\Rightarrow_c$ is fully weakly connexive, then the answer is negative. In fact, Lemma 4.15 shows that (6) is not preserved by direct products. As simple as it is, this fact has another important consequence. In fact, it shows that there does not exist a generalized quasivariety $\mathcal{V} \subseteq \mathcal{D}$ such that for any $\boldsymbol{A} = (A, \rightsquigarrow, \wedge, \vee, ', \bot, \top) \in \mathcal{V}$, there exists a state m over the subreduct $(A, \wedge, \vee, ', \bot, \top)$ such that $\rightsquigarrow = \rightsquigarrow^*$, where $\rightsquigarrow^*$ is defined as in Lemma 4.1 w.r.t. R^m or $\mathcal{R}^m$. In other words, there is no hope of providing a logic which is algebrizable, and its equivalent algebraic semantics is given by all and only the "concrete" models we started from in our investigation.

5. Conclusion and future research

In this paper we deepened the probabilistic account of relevance as outlined in [3, 13]. Specifically, we showed how to exploit the concept of statistical/weak relevance in order to obtain a formal treatment of conditionals in which the probability of the antecedent affects directly or indirectly the probability of the consequent under a given piece of evidence. More precisely, we dealt with the problem of providing an expansion of **CPL** with a ternary implication connective encoding conditionals in which the antecedent materially implies the consequent, and is relevant to it, under a given piece of evidence. To this aim, we have applied Blok and Pigozzi's theory of algebraizable logics in order to obtain logics which are algebraizable w.r.t. varieties of expansions of Boolean algebras with a ternary operations $\rightsquigarrow$. Such and operation generalizes the ternary operation $\rightsquigarrow^*$ which can be defined on a Boolean algebra by demanding that antecedents of "sound" material implications are statistically (weakly) relevant to consequents given some evidence w.r.t. a given (strictly positive) state. However, although d-algebras, G-algebras, and SR-algebras share some important features with the "concrete" models they arise from, algebras we dealt with fall short of capturing properties of such structures in their entirety. Therefore, the following problem rises.

Problem 5.1. Find sufficient and necessary conditions under which a G(SR)-algebra $\boldsymbol{A} = (A, \rightsquigarrow, \wedge, \vee, ', \bot, \top)$ is such that $\rightsquigarrow = \rightsquigarrow^*$, where $\rightsquigarrow^*$ is defined by means of $\mathcal{R}^m(\mathrm{R}^m)$, for some suitable strictly positive state m over the sub-reduct $(A, \wedge, \vee, ', \bot, \top)$.

In Section 4.1 we observed that logics **D**, **G** and **SR** might be regarded as expansions of **CPL** with a ternary connective which can be defined in a similar manner as relating implications are obtained in the framework of Boolean Logics with Relating Implication [20]. Against such an intuition, we showed that a multi-relating semantics for **D**, **G** and **SR**, at least if defined as we do at p. 131, does not exist on pain of contradiction. However, if one considers the logic $\mathbf{D}^*$ ($\mathbf{G}^*$, $\mathbf{SR}^*$) axiomatized by replacing the non-admissible inference rule (A10) in **D** (**G**, **SR**) by the weaker ($A10^{*}$), then the problem of providing a relating semantics with respect to which $\mathbf{D}^{*}$ ($\mathbf{G}^{*}$, $\mathbf{SR}^{*}$) is weakly complete might be reduced to showing that ($A10^{*}$) is admissible. Let us consider the case of $\mathbf{D}^{*}$.

DEFINITION 5.1. Let $\mathfrak{D}$ be the class of multi-relating models of the form $\langle v, \{R_\gamma\}_{\gamma\in \mathrm{Fm}_{\mathcal{L}'}}\rangle$ such that for any $\{\alpha,\beta,\gamma\}\subseteq \mathrm{Fm}_{\mathcal{L}'}$, the following conditions hold:

1. $(\alpha,\top)\notin R_\gamma$;
2. The following are equivalent:
   (a) $(\alpha,\beta)\in R_\gamma$
   (b) $(\neg\alpha,\beta)\in R_\gamma$
   (c) $(\alpha\wedge\gamma,\beta)\in R_\gamma$
   (d) $(\gamma\to\alpha,\beta)\in R_\gamma$
3. $(\alpha,\beta)\in R_\gamma$ implies $(\alpha,\alpha),(\beta,\beta)\in R_\gamma$.

A customary argument by induction on the length of proofs yields that, if ($A10^{*}$) is admissible in $\mathbf{D}^{*}$, then $\vdash_{\mathbf{D}^{*}}\subseteq\, \models_{\mathfrak{D}}$. Moreover, suppose that $\nvdash_{\mathbf{D}^{*}} \alpha$. Applying a routine Lindenbaum argument, one can find a *maximally consistent* [cf. 20, p. 18] set of formulas Γ such that $\alpha\notin\Gamma$. Now, let us consider the pair $\langle v_\Gamma, \{R^{\Gamma}_\gamma\}_{\gamma\in\mathrm{Fm}_{\mathcal{L}'}}\rangle$ such that for any $x\in \mathrm{Var}$, $v_\Gamma(x)=\top$ iff $x\in\Gamma$ (note that, in particular, v_Γ is well defined) and, for any $\beta,\gamma,\delta\in\mathrm{Fm}_{\mathcal{L}'}$,

$$(\beta,\delta)\in R^{\Gamma}_\gamma \text{ iff } \beta\,\mathbb{R}_\gamma\,\delta\in\Gamma.$$

By virtue of Theorem 4.13 and Theorem 4.4, it can be seen that, for any $\gamma\in\mathrm{Fm}_{\mathcal{L}'}$, R^{Γ}_γ satisfies conditions (1)–(3) of Definition 5.1. In other words, we have $\langle v_\Gamma, \{R^{\Gamma}_\gamma\}_{\gamma\in\mathrm{Fm}_{\mathcal{L}'}}\rangle\in\mathfrak{D}$. Now, if one considers the evaluation over $\mathrm{Fm}_{\mathcal{L}'}$ induced by $\langle v_\Gamma, \{R^{\Gamma}_\gamma\}_{\gamma\in\mathrm{Fm}_{\mathcal{L}'}}\rangle$ as at p. 131, it can be seen that for any $\beta\in\mathrm{Fm}_{\mathcal{L}'}$,

$$\langle v_\Gamma, \{R^{\Gamma}_\gamma\}_{\gamma\in\mathrm{Fm}_{\mathcal{L}'}}\rangle\models\beta \text{ iff } \beta\in\Gamma.$$

Therefore, we conclude that $\langle v_\Gamma, \{R^{\Gamma}_\gamma\}_{\gamma\in\mathrm{Fm}_{\mathcal{L}'}}\rangle\not\models\alpha$, and so $\not\models_{\mathfrak{D}}\alpha$. The above reasoning boils down to the following

THEOREM 5.1. *If ($A10^{*}$) is admissible in $\mathbf{D}^{*}$, then, for any $\alpha\in\mathrm{Fm}_{\mathcal{L}'}$:*

$$\vdash_{\mathbf{D}^{*}}\alpha \textit{ iff } \models_{\mathfrak{D}}\alpha.$$

Under the same hypothesis, in order to obtain analogous results for **G** and **SR**, it suffices to consider the class $\mathfrak{G}$, whose definition extends Definition 5.1 with the further condition

$$(\alpha \wedge \delta, \beta) \in R_\gamma \text{ implies } (\alpha, \beta) \in R_\gamma \text{ or } (\delta, \beta) \in R_\gamma,$$

and $\mathfrak{S}$, which is obtained from Definition 5.1 by adding conditions which mirrors items from Lemma 4.8, i.e.,

1. $(\alpha, \beta) \in R_\gamma$ iff $(\beta, \alpha) \in R_\gamma$;
2. If $(\delta, \beta) \notin R_\gamma$ then $(\alpha, \beta) \in R_{\gamma \wedge \delta}$ iff $(\alpha \wedge \delta, \beta) \in R_\gamma$;
3. If $(\alpha, \beta) \in R_\gamma$, $(\delta, \beta) \notin R_\gamma$, and $((\alpha \wedge \gamma) \wedge \delta, (\alpha \wedge \gamma) \wedge \delta) \in R_\top$ imply $(\alpha \wedge \delta, \beta) \in R_\gamma$,

respectively. Therefore, Problem 4.1 is reduced to

Problem 5.2. Show that $(A10^*)$ is admissible in $\mathbf{D}^*$, $\mathbf{G}^*$, and $\mathbf{SR}^*$.

In general, we believe that the following problem is worth of some attention.

Problem 5.3. Find the least extension $\boldsymbol{L}$ of $\mathbf{D}^*$ which is weakly complete w.r.t. a multi-relating semantics.

Finally, as shown in Section 4.2, the framework of d(G,SR)-algebras allows us to provide term-defined weakly connexive implications with a rather transparent interpretation. However, our work does not address the question if some fully connexive implications are definable.

Problem 5.4. Provide a non-symmetric term-definable binary operation $\Rightarrow$ on d-algebras (or G-, or SR-algebras) satisfying connexive theses w.r.t. $'$.

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