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Meet-Combination of Consequence Systems

Abstract. We extend meet-combination of logics for capturing the consequences that are common to both logics. With this purpose in mind we define meet-combination of consequence systems. This notion has the advantage of accommodating different ways of presenting the semantics and the deductive calculi. We consider consequence systems generated by a matrix semantics and consequence systems generated by Hilbert calculi. The meet-combination of consequence systems generated by matrix semantics is the consequence system generated by their product. On the other hand, the meet-combination of consequence systems generated by Hilbert calculi is the consequence system generated by their interconnection. We investigate preservation of several properties. Capitalizing on these results we show that interconnection provides an axiomatization for the product. Illustrations are given for intuitionistic and modal logics, Łukasiewicz logic and some paraconsistent logics.

Keywords: combination of logics, meet-combination, consequence systems, product of matrix semantics

Introduction

Combining logics is nowadays an important topic of research. The initial motivation came from the applications, where the need for using different logics (in a "combined" way) became compulsory. A well known example is provided by applications where different operators are relevant for expressing time and space. The first kind of combination was fusion of modal logics investigated in (Thomason, 1980). Another form of combination of modal logics is product (Gabbay et al., 2003(@; Gabbay and Shehtman, 1998). In both cases the set of constructors of the

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combination is obtained by sharing the propositional constructors and adding the modalities of each logic. The semantics is provided by Kripke structures. Starting with a Kripke structure for each logic with the same set of worlds, a Kripke structure for fusion keeps that set of worlds and adds the two accessibility relations. The set of worlds of each Kripke structure for the product is the cartesian product of the sets of worlds in the given structures while the relations are defined component-wise.

As applications became more sophisticated other logics had to be considered besides modal logics. Fibring of logics was an answer to this challenge (Gabbay, 1996, 1999). There are two facets of fibring: unconstrained fibring where no constructors are shared and constrained fibring where some constructors can be shared.

The essence of fibring is that each shared constructor inherits the logical properties of each of its components. Suppose that we share negation in the fibring of classical propositional and intuitionistic logics. For instance the *tertium non datur* would be a property of the shared negation. So classical propositional and intuitionistic logics collapse in the fibring as recognized in (del Cerro and Herzig, 1996). In (Carnielli et al., 2002) modulated fibring was introduced for dealing with this problem.

A new form of combining logics, called meet-combination, was proposed in (Sernadas et al., 2012) for capturing the common logical properties of the constructors of both logics. The linguistic setting for the meet-combination is different from the ones above. The set of formulas is generated by constructors of the form $\langle c_1 c_2 \rangle$ over a set of propositional variables where c_1 and c_2 are constructors with the same arity of the given logics. As an illustration observe that in the meet-combination of classical propositional logic CP and intuitionistic logic J, the commutativity property of constructor $\langle \wedge_{\mathsf{CP}} \vee_{\mathsf{J}} \rangle$ should hold.

Herein we extend meet-combination of logics for capturing also the consequences that are common to both logics. For example the hypothetical syllogism should hold in the meet-combination of classical propositional and intuitionistic logics since it holds in both logics. In order to cope with this goal we work with consequence systems and introduce their meet-combination. This perspective is general enough to allow different semantic domains and calculi for presenting the logics to be combined.

We concentrate on logics endowed with a matrix semantics since it is general enough to accommodate a wide class of logics ranging from intuitionistic and modal logics to multi-valued logics and even some paraconsistent logics. We establish that the meet-combination of the consequence systems generated by the given matrix semantics is the consequence system generated by the product of the argument matrix semantics.

From a deductive point of view we assume that the given logics are described by Hilbert calculi. In this case, we show that the meet-combination of the consequence systems generated by the argument Hilbert calculi is the consequence system generated by their interconnection.

Combination of logics in general raises some challenging theoretical questions: assuming that the given logics have a logical property, is it the case that their combination also has that property? In (Kracht and Wolter, 1991) several preservation results for fusion were proved. Similarly with respect to fibring usually under some conditions (see, e.g., Carnielli et al., 2002, 2008a,b; Marcelino and Caleiro, 2017; Zanardo et al., 2001).

Capitalizing on the definition of meet-combination of consequence systems, we analyze whether properties such as finitariness, structurality, paraconsistency and formal inconsistency are preserved. Moreover, we establish preservation of soundness and completeness when the consequence systems are generated by matrix semantics and Hilbert calculi. This result extends the work in (Sernadas et al., 2012) where preservation of completeness only holds for formulas without schema variables. In the presence of soundness and completeness, the consequence system generated by the product of matrix semantics is precisely the consequence system generated by the interconnection of compatible Hilbert calculi. Hence the interconnection of compatible Hilbert calculi is the right axiomatization of the product of matrix semantics.

The paper is organized as follows. In Section 1 we introduce meet-combination of consequence systems and establish preservation of several properties. In Section 2 we consider the particular case of consequence systems generated by matrix semantics. Then we characterize their meet-combination in terms of the product of the given matrix semantics. We end the section with the preservation of the finite model property. In Section 3 we concentrate on consequence systems generated by Hilbert calculi. We show that their meet-combination corresponds to the interconnection of the calculi. Moreover, we establish preservation of some deductive properties. Finally, in Section 4 we investigate the preservation of soundness and completeness by meet-combination. As a corollary we conclude that product coincides with interconnection. We end the paper drawing some conclusions and outlining future work in Section 5.

1. Meet-combination of consequence systems

We start by discussing meet-combination at the level of consequence systems. Consequence systems were firstly introduced by Alfred Tarski in (1956) and followed by many others including in (Blok and Pigozzi, 1989) with the aim of associating to each set of formulas Γ the set of formulas that are consequences of Γ under some requirements.

Let Ξ be a set of schema or propositional variables. A *consequence system* $\mathcal{C}$ is a pair $(C, \rhd)$, where $C = \{C_n\}_{n \in \mathbb{N}}$ is a family of sets where each C_n is the *set of constructors of arity* n and $\cdot^{\rhd} : \wp L(\Xi, C) \to \wp L(\Xi, C)$ is a map where $L(\Xi, C)$ is the *set of formulas* inductively generated by C over $\Xi \cup C_0$ satisfying the following properties:

- $\Gamma \subseteq \Gamma^{\rhd}$ (extensitivity)
- $(\Gamma^{\rhd})^{\rhd} \subseteq \Gamma^{\rhd}$ (idempotence)
- $\Gamma_1^{\rhd} \subseteq \Gamma_2^{\rhd}$ whenever $\Gamma_1 \subseteq \Gamma_2$ (monotonicity)

We say that $\cdot^{\rhd}$ is a *consequence operator* associating to each set Γ the set of all consequences $\Gamma^{\rhd}$ of Γ. We may use $\Gamma \rhd \varphi$ whenever $\varphi \in \Gamma^{\rhd}$ and $\Gamma \not\rhd \varphi$ whenever $\varphi \notin \Gamma^{\rhd}$. Furthermore, we may write $\rhd \varphi$ whenever $\emptyset \rhd \varphi$. For simplification, when no confusion arises, we may write $L(\Xi)$ for $L(\Xi, C)$ (that is, the absolutely free algebra generated by C over Ξ).When presenting the family C we only define the non-empty sets of constructors. We can say that we consider *Tarskian operators* (see Wójcicki, 1984) due to the choice of properties. Observe that we also have that $(\Gamma^{\rhd})^{\rhd} = \Gamma^{\rhd}$ since $\Gamma \subseteq \Gamma^{\rhd}$ and so, by monotonicity, $\Gamma^{\rhd} \subseteq (\Gamma^{\rhd \mathrm{x}})^{\rhd}$.

Example 1.1. Consider intuitionistic logic J and modal logic K (see Rybakov, 1997). The family of constructors for J is $C_{\mathsf{J},1} = \{\neg\}$ and $C_{\mathsf{J},2} = \{\supset, \wedge, \vee\}$ and the family of constructors for K is $C_{\mathsf{K},1} = \{\neg, \Box\}$ and $C_{\mathsf{K},2} = \{\supset\}$. $\dashv$

A consequence system is *finitary* whenever

$$\Gamma^{\rhd} = \bigcup_{\Psi \in \wp_{\text{fin}} \Gamma} \Psi^{\rhd},$$

where $\wp_{\text{fin}} \Gamma$ is the set of all finite subsets of Γ.

A useful property of consequence systems is structurality, that is, closure for substitution. A *substitution* is a map $\sigma : \Xi \to L(\Xi)$. We extend σ to $\overline{\sigma} : L(\Xi) \to L(\Xi)$ as follows: $\overline{\sigma}(\xi) = \sigma(\xi)$ and $\overline{\sigma}(c(\varphi_1, \ldots, \varphi_n)) = c(\overline{\sigma}(\varphi_1), \ldots, \overline{\sigma}(\varphi_n))$. Furthermore, we denote the set $\{\overline{\sigma}(\psi) : \psi \in \Psi\}$ by $\overline{\sigma}(\Psi)$. As a simplification, we can write σ instead of $\overline{\sigma}$. A consequence system is *structural* or *closed for substitution* whenever

$$\text{if } \Gamma \rhd \varphi \text{ then } \sigma(\Gamma) \rhd \sigma(\varphi) \text{ for every substitution } \sigma.$$

In (Carnielli and Coniglio, 2016; Wójcicki, 1984) a structural and finitary Tarskian logic is called a *standard logic*.

A consequence system is *explosive* whenever there are $\supset \in C_2$ and $\neg \in C_1$ such that $\xi \supset ((\neg \xi) \supset \xi_1) \in \emptyset^{\rhd}$ and it is *paraconsistent* whenever there are $\supset \in C_2$ and $\neg \in C_1$ such that $\xi \supset ((\neg \xi) \supset \xi_1) \notin \emptyset^{\rhd}$, that is explosion is not a consequence of the emptyset.

A paraconsistent consequence system is a *consequence system of formal inconsistency* when there is $\circ \in C_1$ such that $(\circ\xi)\supset(\xi\supset((\neg\xi)\supset\xi_1)) \in \emptyset^{\rhd}$ called *gentle explosion* (for more details see Carnielli and Coniglio, 2016; Carnielli et al., 2007(@). The formula $\circ\xi$ states that ξ is explosive.

We say that $\Gamma \subseteq L(\Xi)$ is *inconsistent* if $\{\varphi \in L(\Xi) : \Gamma \rhd \varphi\} = L(\Xi)$. Observe that if Γ is inconsistent then Γ' is also inconsistent for every $\Gamma' \subseteq L(\Xi)$ such that $\Gamma \subseteq \Gamma'$ which is a consequence of monotonicity.

A consequence system $(C, \rhd)$ is *suitable* if there is a constructor $\mathtt{tt}^n \in C_n$ such that $\mathtt{tt}^n(\varphi_1, \ldots, \varphi_n) \in \emptyset^{\rhd}$ for every $n \in \mathbb{N}$ and formulas $\varphi_1, \ldots, \varphi_n$.

We will explain latter on how to obtain a suitable consequence system out of a consequence system generated from either a matrix semantics (see Remark 2.1) or a Hilbert calculus (see Remark 3.1).

Remark 1.1. Below, we also use C for referring to the family of constructors of a logic enriched with a constructor $\mathtt{tt}^n$ for every $n \in \mathbb{N}$ when such constructors are not present. Moreover, we omit the reference to $\mathtt{tt}^n$ when presenting a family of constructors C.

Meet-combination. The objective now is to define the meet-combination $\mathcal{C} = (C_{12}, \rhd_{12})$ of two suitable consequence systems $\mathcal{C}_k = (C_k, \rhd_k)$ for $k = 1, 2$. Before defining C_{12} there are some points that should be made. The first one is that we should have in the meet as constructors pairs composed of a constructor of C_1 and a constructor of C_2 both of the same arity. The second one consists in saying that we would like to have the elements of C_1 and C_2 as constructors in the meet-combination as well. Finally, it seems natural to say that every consequence in the meet-combination can be projected into the components and every consequence in both projections should also be reflected in the meet-combination. The second point means that we want to see C_{12} as an enrichment of C_1 and C_2 in the sense that we want to recognize in C_{12} the constructors of C_1 and C_2. This objective is attainable by assuming that each component consequence system is suitable which is an assumption from now on.

Definition 1.1. The *family of constructors of the meet-combination* $C_{12} = \{C_{12,n}\}_{n \in \mathbb{N}}$ is such that

$$C_{12,n}=\{\langle c_1 c_2\rangle | c_1 \in C_{1,n}, c_2 \in C_{2,n}\} \cup \{\langle c_1 \mathsf{tt}_2^n\rangle | c_1 \in C_{1,n}\} \cup \{\langle \mathsf{tt}_1^n c_2\rangle | c_2 \in C_{2,n}\}.$$

We assume that Ξ is the set of schema variables that is shared by the two consequence systems. Hence there will be some interaction between $\mathcal{C}_1$ and $\mathcal{C}_2$. For simplicity we use $L_k(\Xi)$ for $L(\Xi, C_k)$ for $k = 1, 2$ and $L_{12}(\Xi)$ for $L(\Xi, C_{12})$. We may also omit the reference to the arity of tt if no confusion arises.

We look at C_{12} as an enrichment of C_1 and C_2 via the embeddings $\eta_1 : c_1 \mapsto \langle c_1 \mathsf{tt}_2\rangle$ for $c_1 \in C_{1,n}$ and $\eta_2 : c_2 \mapsto \langle \mathsf{tt}_1 c_2\rangle$ for $c_2 \in C_{2,n}$. We also denote by η_k the extension of η_k to formulas in $L_k(\Xi)$ such that $\eta_1(\xi) = \eta_2(\xi) = \xi$.

Example 1.2. We define the set of constructors for the meet-combination of J and K. In order to distinguish the constructors with the same symbol we indicate as a subscript the corresponding logic. For instance, we use $\neg_{\mathsf{J}}$ for the negation symbol in intuitionistic logic and $\neg_{\mathsf{K}}$ for the negation symbol in modal logic K. Thus, the family of constructors in the meet-combination of C_{J} and C_{K} is the family $C_{\mathsf{JK}} = \{C_{\mathsf{JK},1}, C_{\mathsf{JK},2}\}$ defined as follows

$$\begin{cases} C_{\mathsf{JK},1}=\{\langle \neg_{\mathsf{J}}\, \mathsf{tt}_{\mathsf{K}}\rangle, \langle \mathsf{tt}_{\mathsf{J}}\, \neg_{\mathsf{K}}\rangle, \langle \mathsf{tt}_{\mathsf{J}} \Box_{\mathsf{K}}\rangle, \langle \neg_{\mathsf{J}}\, \Box_{\mathsf{K}}\rangle, \neg_{\mathsf{JK}}, \langle \mathsf{tt}_{\mathsf{J}} \mathsf{tt}_{\mathsf{K}}\rangle\} \\ C_{\mathsf{JK},2}=\{\langle \supset_{\mathsf{J}} \mathsf{tt}_{\mathsf{K}}\rangle, \langle \wedge_{\mathsf{J}} \mathsf{tt}_{\mathsf{K}}\rangle, \langle \vee_{\mathsf{J}} \mathsf{tt}_{\mathsf{K}}\rangle, \langle \mathsf{tt}_{\mathsf{J}} \supset_{\mathsf{K}}\rangle, \supset_{\mathsf{JK}}, \langle \wedge_{\mathsf{J}} \supset_{\mathsf{K}}\rangle, \langle \vee_{\mathsf{J}} \supset_{\mathsf{K}}\rangle, \langle \mathsf{tt}_{\mathsf{J}} \mathsf{tt}_{\mathsf{K}}\rangle\} \end{cases}$$

where $\neg_{\mathsf{JK}}$ and $\supset_{\mathsf{JK}}$ are used as abbreviations for $\langle \neg_{\mathsf{J}}\, \neg_{\mathsf{K}}\rangle$ and $\langle \supset_{\mathsf{J}} \supset_{\mathsf{K}}\rangle$, respectively. $\dashv$

It is useful to consider the projections for $k = 1, 2$.

Definition 1.2. The k-th *projection* is a map

$$\cdot|^k \colon L_{12}(\Xi) \to L_k(\Xi)$$

such that $\psi|^k$ is inductively defined as follows:

- $\psi|^k$ is ξ when ψ is ξ
- $\psi|^k$ is $c_k(\psi_1|^k, \ldots, \psi_n|^k)$, when ψ is $\langle c_1 c_2\rangle(\psi_1, \ldots, \psi_n)$.

Example 1.3. Recall Example 1.2. Then note that $(\xi_1 \supset_{\mathsf{JK}} (\xi_2 \supset_{\mathsf{JK}} \xi_1))|^{\mathsf{J}}$ is $\xi_1 \supset_{\mathsf{J}} (\xi_2 \supset_{\mathsf{J}} \xi_1)$ and $(\xi_1 \supset_{\mathsf{JK}} (\xi_2 \supset_{\mathsf{JK}} \xi_1))|^{\mathsf{K}}$ is $\xi_1 \supset_{\mathsf{K}} (\xi_2 \supset_{\mathsf{K}} \xi_1)$. Moreover $(\langle \neg_{\mathsf{J}} \Box_{\mathsf{K}}\rangle\xi)|^{\mathsf{J}}$ is $\neg_{\mathsf{J}}\, \xi$, $(\langle \neg_{\mathsf{J}} \Box_{\mathsf{K}}\rangle\xi)|^{\mathsf{K}}$ is $\Box_{\mathsf{K}}\xi$, $(\xi_1 \langle \supset_{\mathsf{J}} \mathsf{tt}_{\mathsf{K}}\rangle\ \xi_2)|^{\mathsf{J}}$ is $\xi_1 \supset_{\mathsf{J}} \xi_2$ and $(\xi_1 \langle \supset_{\mathsf{J}} \mathsf{tt}_{\mathsf{K}}\rangle\ \xi_2)|^{\mathsf{K}}$ is $\mathsf{tt}_{\mathsf{K}}(\xi_1, \xi_2)$. ⊣

We can extend the notion of projection to sets of formulas. The *kth-projection* of Γ is $\Gamma|^k = \{\gamma|^k : \gamma \in \Gamma\}$ for $k = 1, 2$.

Definition 1.3. The *meet-combination* of consequence systems $\mathcal{C}_1$ and $\mathcal{C}_2$ denoted by

$$\mathcal{C}_1 \nabla \mathcal{C}_2$$

is the consequence system $(C_{12}, \rhd_{12})$ such that

$$\Gamma \rhd_{12} \varphi \ \text{ if and only if } \ \Gamma|^k \rhd_k \varphi|^k \ \text{ for each } k = 1, 2$$

for every $\Gamma \cup \{\varphi\} \subseteq L_{12}(\Xi)$.

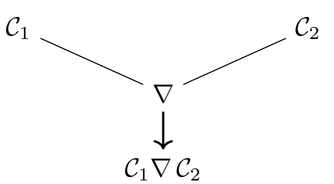

Figure 1. Meet-combination of consequence systems

Proposition 1.1. *The meet-combination $\mathcal{C}_1 \nabla \mathcal{C}_2$ of suitable consequence systems $\mathcal{C}_1$ and $\mathcal{C}_2$ is a suitable consequence system.*

Proof. (1) $\mathcal{C}_1 \nabla \mathcal{C}_2$ is a consequence system. We start by showing monotonicity of $\rhd_{12}$. Assume that $\Gamma, \Delta \subseteq L_{12}(\Xi)$ are such that $\Gamma \subseteq \Delta$. Then $\Gamma|^k \subseteq \Delta|^k$ for $k = 1, 2$. Let $\Gamma \rhd_{12} \varphi$. Hence, by definition, $\Gamma|^k \rhd_k \varphi|^k$ for $k = 1, 2$. Therefore, $\Delta|^k \rhd_k \varphi|^k$, by monotonicity of $\rhd_k$ for $k = 1, 2$, and so by definition $\Delta \rhd_{12} \varphi$.

We now prove idempotence. Suppose that $\Gamma \rhd_{12} \Lambda$ and $\Phi \rhd_{12} \varphi$ with $\Phi \subseteq \Lambda$. Then $\Gamma|^k \rhd_k \Lambda|^k$ and $\Phi|^k \rhd_k \varphi|^k$ with $\Phi \subseteq \Pi$ for $k = 1, 2$.

Hence $\Gamma|^k \rhd_k \varphi|^k$ by idempotence over $\mathcal{C}_k$ for $k = 1, 2$ and so $\Gamma \rhd_{12} \varphi$.
(2) We now show that $\mathcal{C}_1 \nabla \mathcal{C}_2$ is suitable. Take $\mathtt{tt}^n_{12} = \langle \mathtt{tt}^n_1 \mathtt{tt}^n_2 \rangle$. Hence $\emptyset \rhd_{12} \mathtt{tt}^n_{12}(\varphi_1, \ldots, \varphi_n)$ by definition since $\emptyset \rhd_k \mathtt{tt}^n_k(\varphi_1|^k, \ldots, \varphi_n|^k)$ for $k = 1, 2$. ⊣

Example 1.4. Recall Examples 1.2 and 1.3. Observe that

$$\emptyset \ \rhd_{\mathsf{J}} \xi_1 \supset_{\mathsf{J}} (\xi_2 \supset_{\mathsf{J}} \xi_1) \text{ and } \emptyset \ \rhd_{\mathsf{K}} \xi_1 \supset_{\mathsf{K}} (\xi_2 \supset_{\mathsf{K}} \xi_1).$$

Then $\emptyset \ \rhd_{\mathsf{JK}} \xi_1 \supset_{\mathsf{JK}} (\xi_2 \supset_{\mathsf{JK}} \xi_1)$. Moreover

$$\emptyset \not\rhd_{\mathsf{J}} ((\neg_{\mathsf{J}}\xi_1) \supset_{\mathsf{J}} (\neg_{\mathsf{J}}\xi_2)) \supset_{\mathsf{J}} (\xi_2 \supset_{\mathsf{J}} \xi_1).$$

Hence, $\emptyset \not\rhd_{\mathsf{JK}} ((\neg_{\mathsf{JK}}\xi_1) \supset_{\mathsf{JK}} (\neg_{\mathsf{JK}}\xi_2)) \supset_{\mathsf{JK}} (\xi_2 \supset_{\mathsf{JK}} \xi_1)$. ⊣

We now show that inconsistency is preserved and reflected by meet-combination.

Proposition 1.2. *A set is inconsistent in the meet-combination of consequence systems if and only if its projections are inconsistent.*

Proof. Let $\Gamma \subseteq L_{12}(\Xi)$.
($\rightarrow$) Suppose by contraposition that $\Gamma|^1$ is consistent. Then there is $\varphi_1 \in L_1(\Xi)$ such that $\varphi_1 \notin (\Gamma|^1)^{\rhd_1}$. Consider two cases. (1) $\varphi_1 \notin \Xi$. Thus $\eta_1(\varphi_1) \notin \Gamma^{\rhd_{12}}$ and so Γ is consistent. (2) φ_1 is $\xi \in \Xi$. Hence, $\xi \notin \Gamma^{\rhd_{12}}$ and so Γ is consistent.

($\leftarrow$) Assume that Γ is consistent. Let $\varphi \notin \Gamma^{\rhd_{12}}$. Then either $\varphi|^1 \notin (\Gamma|^1)^{\rhd_1}$ and so $\Gamma|^1$ is consistent or $\varphi|^2 \notin (\Gamma|^2)^{\rhd_2}$ and so $\Gamma|^2$ is consistent. ⊣

The following properties are preserved by meet-combination.

Proposition 1.3. *The meet-combination of finitary consequence systems is a finitary consequence system.*

Proof. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be finitary consequence systems. We must show that

$$\Gamma^{\rhd_{12}} = \bigcup_{\Psi \subseteq_{\wp\text{fin}} \Gamma} \Psi^{\rhd_{12}} \quad \text{for } \Gamma \subseteq L_{12}(\Xi).$$

($\subseteq$) Assume that $\Gamma \rhd_{12} \varphi$. Then $\Gamma|^k \rhd_k \varphi|^k$ for $k = 1, 2$. Let $\Psi' \subseteq \Gamma$ and $\Psi'' \subseteq \Gamma$ be finite sets such that $\Psi'|^1 \rhd_1 \varphi|^1$ and $\Psi''|^2 \rhd_2 \varphi|^2$, respectively. Take Ψ as $\Psi' \cup \Psi''$. Then $\Psi|^1 \rhd_1 \varphi|^1$ and $\Psi|^2 \rhd_2 \varphi|^2$ and so $\Psi \rhd_{12} \varphi$.

($\supseteq$) Suppose that $\Psi \rhd_{12} \varphi$ for some finite $\Psi \subseteq \Gamma$. The result follows by monotonicity of $\rhd_{12}$. $\dashv$

Our objective now is to show that meet-combination of consequence systems preserves structurality (that is, closure for substitution). Given $\sigma : \Xi \to L_{12}(\Xi)$, we denote by $\sigma_k : \Xi \to L_k(\Xi)$ the substitution such that $\sigma_k(\xi) = \sigma(\xi)|^k$. We start by showing a preliminary result.

LEMMA 1.1. *Let $\theta \in L_{12}(\Xi)$ and $\sigma : \Xi \to L_{12}(\Xi)$ a substitution. Then*

$$\sigma_k(\theta|^k) = \sigma(\theta)|^k \quad \text{for } k = 1, 2.$$

PROOF. The proof follows by induction on the structure of θ.
(Base) θ is ξ. Then $\sigma_k(\xi|^k) = \sigma(\xi)|^k$ because $\xi|^k$ is ξ.
(Step) θ is $\langle c_1 c_2\rangle(\varphi_1, \ldots, \varphi_n)$ with $c_1 \in C_{1,n}$, $c_2 \in C_{2,n}$. So, for $k = 1, 2$,

$$\begin{array}{rcl} \sigma_k(\langle c_1 c_2\rangle(\varphi_1, \ldots, \varphi_n)|^k) & = & \sigma_k(c_k(\varphi_1|^k, \ldots, \varphi_n|^k)) \\ & = & c_k(\sigma_k(\varphi_1|^k), \ldots, \sigma_k(\varphi_n|^k)) \text{ (IH)} \\ & = & c_k(\sigma(\varphi_1)|^k, \ldots, \sigma(\varphi_n)|^k) \\ & = & \sigma(\langle c_1 c_2\rangle(\varphi_1, \ldots, \varphi_n))|^k. \quad \dashv \end{array}$$

PROPOSITION 1.4. *The meet-combination of structural consequence systems is a structural consequence system.*

PROOF. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be structural consequence systems. We show that

$$\text{if } \Gamma \rhd_{12} \varphi \text{ then } \sigma(\Gamma) \rhd_{12} \sigma(\varphi) \text{ for every substitution } \sigma.$$

Let σ be a substitution. Assume that $\Gamma \rhd_{12} \varphi$. Then $\Gamma|^k \rhd_k \varphi|^k$ for $k = 1, 2$. Thus, by hypothesis, $\sigma_k(\Gamma|^k) \rhd_k \sigma_k(\varphi|^k)$ for $k = 1, 2$. Therefore, by Lemma 1.1, $\sigma(\Gamma)|^k \rhd_k \sigma(\varphi)|^k$ for $k = 1, 2$. Thus, $\sigma(\Gamma) \rhd_{12} \sigma(\varphi)$. $\dashv$

We now investigate preservation of paraconsistent properties.

PROPOSITION 1.5. *The meet-combination $\mathcal{C}_1 \nabla \mathcal{C}_2$ of consequence systems $\mathcal{C}_1$ and $\mathcal{C}_2$ with $\supset_1 \in C_{1,2}$, $\supset_2 \in C_{2,2}$, $\neg_1 \in C_{1,1}$, and $\neg_2 \in C_{2,1}$ where either $\mathcal{C}_1$ or $\mathcal{C}_2$ is paraconsistent is a paraconsistent consequence system.*

PROOF. Assume without loss of generality that $\mathcal{C}_1$ is paraconsistent. Then $\xi \supset_1 ((\neg_1 \xi) \supset_1 \xi_1) \notin \emptyset^{\rhd_1}$. Hence, $\xi \supset_{12} ((\neg_{12} \xi) \supset_{12} \xi_1) \notin \emptyset^{\rhd_{12}}$ by definition of meet-combination. So, $\mathcal{C}_1 \nabla \mathcal{C}_2$ is paraconsistent. $\dashv$

Proposition 1.6. *The meet-combination of consequence systems of formal inconsistency is a consequence system of formal inconsistency.*

Proof. Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are consequence systems of formal inconsistency with $\supset_1 \in C_{1,2}$, $\supset_2 \in C_{2,2}$, $\neg_1, \circ_1 \in C_{1,1}$, $\neg_2, \circ_2 \in C_{2,1}$. Observe that $\circ_{12} \in C_{12,1}$. Hence $\rhd_{12} (\circ_{12}\, \xi) \supset_{12} (\xi \supset_{12} ((\neg_{12}\, \xi) \supset_{12} \xi_1))$ by definition of meet-combination of consequence systems. Thus, $\mathcal{C}_1 \nabla \mathcal{C}_2$ is a consequence system of formal inconsistency. ⊣

Below we discuss consequence systems and their meet-combination when they are presented either by a matrix semantics or by a Hilbert calculus.

2. Meet-combination of matrix semantics

We begin by presenting consequence systems generated by a matrix semantics. A *matrix semantics* (see Blok and Pigozzi, 1989; Wójcicki, 1988) is a clean, uniform and algebraic way of defining the semantics of a logic. Moreover, it is general enough to provide the semantics of a wide variety of logics ranging from multi-valued to intuitionistic and modal logics (see Rybakov, 1997) and even some paraconsistent logics (see Bolc and Borowik, 1992). The adoption of a matrix semantics starts with the definition of (logical) matrix that was introduced by Łukasiewicz and Tarski (see Tarski, 1956, and Cocchiarella and Freund, 2008, Ch. 3). A *matrix* is a pair $M = (\mathfrak{A}, D)$ where

$$\mathfrak{A} = (A, \{\overline{c}^M : A^n \to A \mid c \in C_n\}_{n \in \mathbb{N}})$$

is an algebra (A is the carrier of the algebra and $\overline{c}^M$ is the *denotation* of $c \in C_n$) and $D \subseteq A$. The elements of A are known as *truth values* and those of D are the *distinguished* or *designated* ones. The map $\overline{c}^M$ is the denotation of constructor c. Observe that the definition of matrix does not state the values of the schema variables in Ξ. These are imposed by assignments. An *assignment over* M is a map $\rho : \Xi \to A$. The *denotation* of a formula over M and ρ is a map

$$[\![\cdot]\!]^{M\rho} : L(\Xi) \to A$$

inductively defined as follows:

$$[\![\xi]\!]^{M\rho} = \rho(\xi) \text{ and } [\![c(\varphi_1, \ldots, \varphi_n)]\!]^{M\rho} = \overline{c}^M([\![\varphi_1]\!]^{M\rho}, \ldots, [\![\varphi_n]\!]^{M\rho}).$$

Thus ρ induces an homomorphism between the algebras $L(\Xi)$ and $\mathcal{A}$. Moreover, this homomorphism is uniquely determined by the family of truth-values $\{\rho(\xi) : \xi \in \Xi\}$.

Hence, the denotation of a formula is always a truth value in the carrier of the matrix. Moreover, the denotation of a complex formula starting with a constructor c is always the denotation of c applied to the denotation of the subformulas of the complex formula. Thus we assume a truth functional interpretation of each constructor. In the sequel we may omit the reference to M in $\overline{c}^M$.

We say that a matrix M and an assignment ρ *satisfy* formula φ, denoted by $M, \rho \Vdash \varphi$, whenever $[\![\varphi]\!]^{M\rho} \in D$. Moreover, given a set Γ of formulas, we write $M, \rho \Vdash \Gamma$ whenever $M, \rho \Vdash \gamma$ for every $\gamma \in \Gamma$. A *matrix semantics* $\mathcal{M}$ is a non-empty class of matrices. Furthermore, Γ *entails* φ in $\mathcal{M}$, denoted by $\Gamma \vDash_{\mathcal{M}} \varphi$, whenever for every matrix $M \in \mathcal{M}$ and assignment ρ over M if $M, \rho \Vdash \Gamma$ then $M, \rho \Vdash \varphi$. When $\emptyset \vDash_{\mathcal{M}} \varphi$ we say that φ is *valid*.

We now illustrate matrix semantics for several logics. These logics differ in the way the matrices are induced from the usual semantics and in the properties they have.

Example 2.1. Consider classical propositional logic CP. Let $C_{\mathsf{CP},1} = \{\neg_{\mathsf{CP}}\}$ and $C_{\mathsf{CP},2} = \{\supset_{\mathsf{CP}}\}$. We can define $\wedge_{\mathsf{CP}}$ and $\vee_{\mathsf{CP}}$ as abbreviations. A matrix semantics for CP, denoted by $\mathcal{M}_{\mathsf{CP}}$, is composed of the matrix M_{CP} with the algebra $\mathfrak{A}_{\mathsf{CP}} = (\{0,1\}, \{\overline{\neg}_{\mathsf{CP}}, \overline{\supset}_{\mathsf{CP}}\})$ having $\{1\}$ as the set of distinguished values where

$$\overline{\neg}_{\mathsf{CP}}(b) = 1 - b \text{ and } \overline{\supset}_{\mathsf{CP}}(b_1, b_2) = 0 \text{ if and only if } b_1 = 1 \text{ and } b_2 = 0.$$

We can define $\overline{\wedge}_{\mathsf{CP}}$ and $\overline{\vee}_{\mathsf{CP}}$ using the abbreviations. In the sequel we use $\neg_{\mathsf{CP}}\xi_2, \xi_1 \supset_{\mathsf{CP}} \xi_2 \vDash_{\mathcal{M}_{\mathsf{CP}}} \neg_{\mathsf{CP}}\xi_1$ and $\xi_1, \xi_1 \supset_{\mathsf{CP}} \xi_2 \vDash_{\mathcal{M}_{\mathsf{CP}}} \xi_2$ which is straightforward to show. $\dashv$

Example 2.2. Consider a normal modal logic N with Kripke semantics (see Rybakov, 1997). Let $C_{\mathsf{N},1} = \{\neg_{\mathsf{N}}, \Box_{\mathsf{N}}\}$ and $C_{\mathsf{N},2} = \{\supset_{\mathsf{N}}\}$. We consider the usual abbreviations for $\wedge_{\mathsf{N}}$ and $\vee_{\mathsf{N}}$. Then the matrix $M_{(W,S)}$ induced by the Kripke frame (W, S) for N is composed of the algebra $\mathfrak{A}_{(W,S)} = (\wp W, \{\overline{\neg}_{\mathsf{N}}, \overline{\Box}_{\mathsf{N}}, \overline{\supset}_{\mathsf{N}}\})$ with the set of distinguished values $\{W\}$ where

- $\overline{\neg}_{\mathsf{N}}(U) = W \setminus U$
- $\overline{\supset}_{\mathsf{N}}(U_1, U_2) = (W \setminus U_1) \cup U_2$
- $\overline{\Box}_{\mathsf{N}}(U) = \{w \in W : \text{if } wSw' \text{ then } w' \in U, \text{for each } w' \in W\}$.

The denotations $\overline{\wedge}_{\mathsf{N}}$ and $\overline{\vee}_{\mathsf{N}}$ are defined according to the abbreviations. Let $\mathcal{M}_{\mathsf{N}} = \{M_{(W,S)} : (W,S) \in \mathfrak{F}_{\mathsf{N}}\}$ where $\mathfrak{F}_{\mathsf{N}}$ is the class of all Kripke frames for N. We now show that

$$\xi_1,\ \xi_1 \supset_{\mathsf{N}} \xi_2 \vDash_{\mathcal{M}_{\mathsf{N}}} \xi_2.$$

Let $M_{(W,S)} \in \mathcal{M}_{\mathsf{N}}$ and ρ be an assignment over $M_{(W,S)}$. Assume that $M_{(W,S)}, \rho \Vdash \xi_1$ and $M_{(W,S)}, \rho \Vdash \xi_1 \supset_{\mathsf{N}} \xi_2$. Then we have $[\![\xi_1]\!]^{M_{(W,S)}\rho}$, $[\![\xi_1 \supset_{\mathsf{N}} \xi_2]\!]^{M_{(W,S)}\rho} \in D$, that is, $[\![\xi_1]\!]^{M_{(W,S)}\rho} = [\![\xi_1 \supset_{\mathsf{N}} \xi_2]\!]^{M_{(W,S)}\rho} = W$. Moreover

$$[\![\xi_1 \supset_{\mathsf{N}} \xi_2]\!]^{M_{(W,S)}\rho} = (W \setminus \rho(\xi_1)) \cup \rho(\xi_2) = (W \setminus W) \cup \rho(\xi_2) = \rho(\xi_2).$$

Therefore, $\rho(\xi_2) = [\![\xi_2]\!]^{M_{(W,S)}\rho} = W$ and so $M_{(W,S)}, \rho \Vdash \xi_2$.

We consider normal modal logics K, T and $\mathsf{4}$. Let $\mathcal{M}_{\mathsf{K}} = \{M_{(W,S)} : (W,S) \in \mathfrak{F}_{\mathsf{K}}\}$, $\mathcal{M}_{\mathsf{T}} = \{M_{(W,S)} : (W,S) \in \mathfrak{F}_{\mathsf{T}}\}$ and $\mathcal{M}_{\mathsf{4}} = \{M_{(W,S)} : (W,S) \in \mathfrak{F}_{\mathsf{4}}\}$ where $\mathfrak{F}_{\mathsf{K}}$ is the class of all Kripke frames, $\mathfrak{F}_{\mathsf{T}}$ is the class of all reflexive frames and $\mathfrak{F}_{\mathsf{4}}$ is the class of all transitive Kripke frames. We can use $\vDash_{\mathsf{K}}$, $\vDash_{\mathsf{T}}$ and $\vDash_{\mathsf{4}}$ instead of $\vDash_{\mathcal{M}_{\mathsf{K}}}$, $\vDash_{\mathcal{M}_{\mathsf{T}}}$ and $\vDash_{\mathcal{M}_{\mathsf{4}}}$, respectively. Observe that $\vDash_{\mathsf{T}} (\Box_{\mathsf{T}}\xi) \supset_{\mathsf{T}} \xi$ and $\nvDash_{\mathsf{4}} (\Box_{\mathsf{4}}\xi) \supset_{\mathsf{4}} \xi$. $\dashv$

Example 2.3. Consider intuitionistic logic J endowed with Heyting algebra semantics (see Rybakov, 1997). A matrix semantics for J, denoted by $\mathcal{M}_{\mathsf{J}}$, is composed of the matrices induced by Heyting algebras. Given a Heyting algebra $\mathfrak{H} = (A, \sqcap, \sqcup, \rightarrow, 0)$ where $0 \in A$, the matrix induced by $\mathfrak{H}$ is $M_{\mathfrak{H}}$ with the algebra $\mathfrak{A}_{\mathfrak{H}} = (A, \{\overline{\neg}_{\mathsf{J}}, \overline{\supset}_{\mathsf{J}}, \overline{\wedge}_{\mathsf{J}}, \overline{\vee}_{\mathsf{J}}\})$ where

- $\overline{\neg}_{\mathsf{J}}(a) = a \rightarrow 0$
- $\overline{\supset}_{\mathsf{J}}(a_1, a_2) = a_1 \rightarrow a_2$, $\overline{\wedge}_{\mathsf{J}}(a_1, a_2) = a_1 \sqcap a_2$ and $\overline{\vee}_{\mathsf{J}}(a_1, a_2) = a_1 \sqcup a_2$

having $\{1\}$ as the set of distinguished values where 1 is $a \rightarrow a$. $\dashv$

Example 2.4. Consider Łukasiewicz logic Ł_3 (for details see Bolc and Borowik, 1992; Łukasiewicz, 1970) with family of constructors $C_{\text{Ł}_3}$ with $C_{\text{Ł}_3,1} = \{\neg_{\text{Ł}_3}\}$ and $C_{\text{Ł}_3,2} = \{\supset_{\text{Ł}_3}\}$. A matrix semantics for Ł_3, denoted by $\mathcal{M}_{\text{Ł}_3}$ is composed of the matrix $M_{\text{Ł}_3}$ with the algebra $\mathfrak{A}_{\text{Ł}_3} = (\{0,1,2\}, \{\overline{\neg}_{\text{Ł}_3}, \overline{\supset}_{\text{Ł}_3}\})$ having $\{2\}$ as the set of distingusihed values where

- $\overline{\neg}_{\text{Ł}_3}(0) = 2$, $\overline{\neg}_{\text{Ł}_3}(1) = 1$, $\overline{\neg}_{\text{Ł}_3}(2) = 0$
- $\overline{\supset}_{\text{Ł}_3}(0, u) = 2$ for $u \in \{0,1,2\}$
- $\overline{\supset}_{\text{Ł}_3}(1, 0) = 1$ and $\overline{\supset}_{\text{Ł}_3}(1, u) = 2$ for $u \in \{1, 2\}$
- $\overline{\supset}_{\text{Ł}_3}(2, u) = u$ for $u \in \{0,1,2\}$.

Constructors $\wedge_{\text{Ł}_3}$, $\vee_{\text{Ł}_3}$ are defined as abbreviations: $\xi_1 \wedge_{\text{Ł}_3} \xi_2$ and $\xi_1 \vee_{\text{Ł}_3} \xi_2$ stand for $\neg_{\text{Ł}_3}((\neg_{\text{Ł}_3} \xi_1) \supset_{\text{Ł}_3} (\neg_{\text{Ł}_3} \xi_2))$ and $(\xi_1 \supset_{\text{Ł}_3} \xi_2) \supset_{\text{Ł}_3} \xi_2$, respectively. $\dashv$

Example 2.5. Consider the logic of formal inconsistency $\mathsf{LFI1}$ as introduced in (Carnielli and Coniglio, 2016; Feitosa et al., 2015). Let $C_{\mathsf{LFI1},1} = \{\neg_{\mathsf{LFI1}}, \circ_{\mathsf{LFI1}}\}$ and $C_{\mathsf{LFI1},2} = \{\supset_{\mathsf{LFI1}}, \wedge_{\mathsf{LFI1}}, \vee_{\mathsf{LFI1}}\}$. A matrix semantics for $\mathsf{LFI1}$, denoted by $\mathcal{M}_{\mathsf{LFI1}}$, is composed of the matrix M_{LFI1} with the algebra $\mathfrak{A}_{\mathsf{LFI1}} = (\{0, \frac{1}{2}, 1\}, \{\overline{\neg}_{\mathsf{LFI1}}, \overline{\circ}_{\mathsf{LFI1}}, \overline{\supset}_{\mathsf{LFI1}}, \overline{\wedge}_{,\mathsf{LFI1}}, \overline{\vee}_{\mathsf{LFI1}}\})$ having $\{\frac{1}{2}, 1\}$ as the set of distinguished values where

- $\overline{\neg}_{\mathsf{LFI1}}(b) = 1 - b$
- $\overline{\circ}_{\mathsf{LFI1}}(b) = 1$ for $b \in \{0, 1\}$ and $\overline{\circ}_{\mathsf{LFI1}}(\frac{1}{2}) = 0$
- $\overline{\supset}_{\mathsf{LFI1}}(b_1, b_2) = 1$ if $b_1 < b_2$ and $\overline{\supset}_{\mathsf{LFI1}}(b_1, b_2) = b_2$ if $b_1 > b_2$
- $\overline{\supset}_{\mathsf{LFI1}}(b, b) = 1$ if $b \in \{0, 1\}$ and $\overline{\supset}_{\mathsf{LFI1}}(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$
- $\overline{\wedge}_{\mathsf{LFI1}}(b_1, b_2) = \min\{b_1, b_2\}$
- $\overline{\vee}_{\mathsf{LFI1}}(b_1, b_2) = \max\{b_1, b_2\}$

Let $\equiv_{\mathsf{LFI1}}$ be the usual abbreviation and ρ such that $\rho(\xi_1) = \frac{1}{2}$ and $\rho(\xi_2) = 0$. Then

- $[\![(\neg_{\mathsf{LFI1}}\, \xi_1) \supset \xi_2]\!]^{M_{\mathsf{LFI1}}\rho} = 0$ and
- $[\![\xi_1 \supset_{\mathsf{LFI1}} ((\neg_{\mathsf{LFI1}}\, \xi_1) \supset \xi_2)]\!]^{M_{\mathsf{LFI1}}\rho} = 0 \notin \{\frac{1}{2}, 1\}$.

Hence, the *explosion formula* $\xi_1 \supset_{\mathsf{LFI1}} ((\neg_{\mathsf{LFI1}}\xi_1) \supset \xi_2)$ is not a validity. Thus, $\mathsf{LFI1}$ is a paraconsistent logic. Furthermore,

- $(\circ_{\mathsf{LFI1}}\, \xi) \supset_{\mathsf{LFI1}} (\xi \supset ((\neg_{\mathsf{LFI1}}\xi) \supset_{\mathsf{LFI1}} \xi_1))$

is a validity. So, $\mathsf{LFI1}$ is indeed a logic of formal inconsistency. Note that

- $[\![\xi \vee_{\mathsf{LFI1}} (\neg_{\mathsf{LFI1}}\xi)]\!]^{M_{\mathsf{LFI1}}\rho} \in \{\frac{1}{2}, 1\}$.

Indeed let ρ be such that

(1) $\rho(\xi) = 1$. Then $[\![\neg_{\mathsf{LFI1}}\xi]\!]^{M_{\mathsf{LFI1}}\rho} = 0$ and so $[\![\xi \vee_{\mathsf{LFI1}} (\neg_{\mathsf{LFI1}}\, \xi)]\!]^{M_{\mathsf{LFI1}}\rho} = 1$.
(2) $\rho(\xi) = \frac{1}{2}$. Then $[\![\neg_{\mathsf{LFI1}}\xi]\!]^{M_{\mathsf{LFI1}}\rho} = \frac{1}{2}$ and so $[\![\xi \vee_{\mathsf{LFI1}} (\neg_{\mathsf{LFI1}}\, \xi)]\!]^{M_{\mathsf{LFI1}}\rho} = \frac{1}{2}$.
(3) $\rho(\xi) = 0$. Then $[\![\neg_{\mathsf{LFI1}}\xi]\!]^{M_{\mathsf{LFI1}}\rho} = 1$; so $[\![\xi \vee_{\mathsf{LFI1}} (\neg_{\mathsf{LFI1}}\, \xi)]\!]^{M_{\mathsf{LFI1}}\rho} = 1$. $\dashv$

Example 2.6. Consider Bochvar 3-valued logic B_3 described in (Bolc and Borowik, 1992). Let $C_{\mathsf{B}_3,1} = \{\sim_{\mathsf{B}_3}\}$ and $C_{\mathsf{B}_3,2} = \{\wedge_{\mathsf{B}_3}, \vee_{\mathsf{B}_3}, \supset_{\mathsf{B}_3}, \equiv_{\mathsf{B}_3}\}$. A matrix semantics for B_3, denoted by $\mathcal{M}_{\mathsf{B}_3}$, is composed of the matrix M_{B_3} with the algebra $\mathcal{A}_{\mathsf{B}_3} = (\{0, 1, 2\}, \{\overline{\sim}_{\mathsf{B}_3}, \overline{\wedge}_{\mathsf{B}_3}, \overline{\vee}_{\mathsf{B}_3}, \overline{\supset}_{\mathsf{B}_3}, \overline{\equiv}_{\mathsf{B}_3}\})$ having $\{2\}$ as the set of distinguished values where

- $\overline{\sim}_{\mathsf{B}_3}(b) = 1 - b$ whenever $b \in \{0, 2\}$ and $\overline{\sim}_{\mathsf{B}_3}(1) = 1$
- $\overline{\wedge}_{\mathsf{B}_3}(b_1, b_2) = \min\{b_1, b_2\}$ whenever $b_1, b_2 \in \{0, 2\}$ and $\overline{\wedge}_{\mathsf{B}_3}(b_1, b_2) = 1$ otherwise
- $\overline{\vee}_{\mathsf{B}_3}(b_1, b_2) = \max\{b_1, b_2\}$ whenever $b_1, b_2 \in \{0, 2\}$ and $\overline{\vee}_{\mathsf{B}_3}(b_1, b_2) = 1$ otherwise
- $\overline{\supset}_{\mathsf{B}_3}(b_1, b_2) = 2$ whenever $b_1, b_2 \in \{0, 2\}$ and $b_1 \leq b_2$, $\overline{\supset}_{\mathsf{B}_3}(2, 0) = 0$ and $\overline{\wedge}_{\mathsf{B}_3}(b_1, b_2) = 1$ otherwise

- $\equiv_{\mathsf{B}_3}(b,b) = 2$ and $\equiv_{\mathsf{B}_3}(b_1,b_2) = 0$ whenever $b, b_1, b_2 \in \{0,2\}$ and $b_1 \neq b_2$ and $\equiv_{\mathsf{B}_3}(b_1,b_2) = 1$ otherwise.

As pointed out in (Bolc and Borowik, 1992) this logic does not have tautologies. $\dashv$

Observe that we can give different matrix semantics for a given logic. Nevertheless the entailment should always be the same. For example, instead of considering the matrix semantics presented in Example 2.2 for modal logic K we could adopt the matrix semantics induced by modal algebras (Kracht, 1999). In the same way instead of giving a matrix semantics based on Heyting algebras for intuitionistic logic we could present a matrix semantics induced by intuitionistic Kripke frames.

Any matrix semantics induces a consequence system based on semantic entailment as we now state.

Proposition 2.1. *The pair $\mathcal{C}(\mathcal{M}) = (C, \vDash_{\mathcal{M}})$ is a consequence system induced by the matrix semantics $\mathcal{M}$ where $\Gamma^{\vDash_{\mathcal{M}}} = \{\varphi \in L(\Xi) : \Gamma \vDash_{\mathcal{M}} \varphi\}$ for every $\Gamma \subseteq L(\Xi)$.*

Proof. We only prove idempotence of $\vDash_{\mathcal{M}}$. Assume that $\Gamma^{\vDash_{\mathcal{M}}} \vDash_{\mathcal{M}} \varphi$. Let $M \in \mathcal{M}$ and ρ an assignment over M such $M, \rho \Vdash \Gamma$. Thus, $M, \rho \Vdash \Gamma^{\vDash_{\mathcal{M}}}$ and therefore $M, \rho \Vdash \varphi$. $\dashv$

Remark 2.1. The reader may wonder what happens when starting with a non suitable consequence system $\mathcal{C}(\mathcal{M}) = (C, \vDash_{\mathcal{M}})$. We show how to proceed to get a suitable consequence system. The enriched family of constructors was introduced in Remark 1.1. Given a matrix $M \in \mathcal{M}$, we define a matrix $M^{\mathsf{tt}} = (\mathfrak{A}^{\mathsf{tt}}, D)$, where

$$\mathfrak{A}^{\mathsf{tt}} = (A, \{\overline{c}^{M^{\mathsf{tt}}} : A^n \to A \mid c \in C\}_{n \in \mathbb{N}})$$

is such that $\overline{\mathsf{tt}^0}^{M^{\mathsf{tt}}}, \overline{\mathsf{tt}^n}^{M^{\mathsf{tt}}}(a_1, \dots, a_n) \in D$ and $\overline{c}^{M^{\mathsf{tt}}} = \overline{c}^M$ for the other constructors. Note that $[\![\mathsf{tt}^0]\!]^{M^{\mathsf{tt}}\rho} = [\![\mathsf{tt}^0]\!]^{M^{\mathsf{tt}}\rho'}$ for all assignments ρ and ρ' over M^{tt}. Thus, we can write $[\![\mathsf{tt}^0]\!]^{M^{\mathsf{tt}}}$. Observe that $\vDash_{\mathcal{M}^{\mathsf{tt}}} \mathsf{tt}(\varphi_1, \dots, \varphi_n)$.

Moreover, for any set of formulas $\Gamma \cup \{\varphi\}$ without occurrences of tt's

$$\Gamma \vDash_{\mathcal{M}} \varphi \text{ if and only if } \Gamma \vDash_{\mathcal{M}^{\mathsf{tt}}} \varphi.$$

That is, there is preservation and reflection of entailment by the enrichment.

In the sequel we also use $\mathcal{M}$ for denoting the matrix semantics enriched with the denotation of constructor tt^n for every $n \in \mathbb{N}$ when such constructors are not present. $\dashv$

Note that in the case of B_3 (see Example 2.6) the only tautologies after the enrichment are the $\mathtt{tt}$'s formulas since the original logic does not have tautologies.

We omit the proof of the next result because it is standard and valid for every matrix semantics (see Font, 2016; Wójcicki, 1973, 1988).

PROPOSITION 2.2. *The consequence system $\mathcal{C}(\mathcal{M})$ is closed for substitution.*

Meet-combination is product. The objective now is to analyze meet-combination from the point of view of matrix semantics.

DEFINITION 2.1. Given a matrix semantics $\mathcal{M}_k$ for $k = 1, 2$, the *product matrix semantics* of $\mathcal{M}_1$ and $\mathcal{M}_2$ over C_{12}, written $\mathcal{M}_1 \times \mathcal{M}_2$, is the class of matrices

$$\{M_1 \times M_2 \mid M_1 \in \mathcal{M}_1 \text{ and } M_2 \in \mathcal{M}_2\}$$

such that each $M_1 \times M_2$ is $(\mathfrak{A}_1 \times \mathfrak{A}_2, D_1 \times D_2)$ where

$$\mathfrak{A}_1 \times \mathfrak{A}_2 = (A_1 \times A_2, \{\overline{\langle c_1 c_2 \rangle}^{M_1 \times M_2} : (A_1 \times A_2)^n \to A_1 \times A_2 | \langle c_1 c_2 \rangle \in C_{12,n}\}_{n \in \mathbb{N}})$$

with

$$\overline{\langle c_1 c_2 \rangle}^{M_1 \times M_2}((a_1, b_1), \ldots, (a_n, b_n)) = (\overline{c_1}^{M_1}(a_1, \ldots, a_n), \overline{c_2}^{M_2}(b_1, \ldots, b_n)).$$

Below we omit the reference to the matrix in the denotation of constructors.

Remark 2.2. In the sequel, we denote by $\Vdash_k$ and $\vDash_k$ the satisfaction and entailment in $\mathcal{C}(\mathcal{M}_k)$ for $k = 1, 2$, respectively and by $\Vdash_{12}$ and $\vDash_{12}$ the satisfaction and entailment in $\mathcal{C}(\mathcal{M}_1 \times \mathcal{M}_2)$, respectively. Furthermore, given $M_1 \times M_2 \in \mathcal{M}_1 \times \mathcal{M}_2$ and an assignment $\rho : \Xi \to A_1 \times A_2$ over $M_1 \times M_2$, we denote by $\rho_1 : \Xi \to A_1$ and $\rho_2 : \Xi \to A_2$ the unique assignments over M_1 and M_2, respectively such that $\rho(\xi) = (\rho_1(\xi), \rho_2(\xi))$. $\dashv$

The next result relates the denotation of a formula in the product with the denotation of its components.

PROPOSITION 2.3. Let $\psi \in L_{12}(\Xi)$, $M_1 \in \mathcal{M}_1$, $M_2 \in \mathcal{M}_2$ and ρ an assignment over $M_1 \times M_2$. Then

$$[\![\psi]\!]^{M_1 \times M_2 \rho} = \left([\![\psi|^1]\!]^{M_1 \rho_1}, [\![\psi|^2]\!]^{M_2 \rho_2}\right).$$

Proof. The proof is straightforward by induction on ψ. We just consider the base case. Suppose that ψ is $\xi \in \Xi$. Hence, $[\![\xi]\!]^{M_1 \times M_2 \rho} = (\rho_1(\xi), \rho_2(\xi))$ because $[\![\xi|^k]\!]^{M_k \rho_k} = \rho_k(\xi)$ for $k = 1, 2$. ⊣

Example 2.7. Recall Example 1.2 and Examples 2.2 and 2.3. The product matrix semantics $\mathcal{M}_{\mathsf{J}} \times \mathcal{M}_{\mathsf{K}}$ is the class of matrices of the form

$$M_{\mathfrak{H}} \times M_{(W,S)} = (\mathfrak{A}_{\mathfrak{H}} \times \mathfrak{A}_{(W,S)}, \{(1, W)\}).$$

For instance, $\neg_{\mathsf{JK}}(a, U) = (\neg_{\mathsf{J}}(a), \neg_{\mathsf{K}}(U)) = (a \to 0, W \setminus U)$. Note that, using Proposition 2.3,

$$\begin{aligned} &[\![\xi_1 \supset_{\mathsf{JK}} (\xi_2 \supset_{\mathsf{JK}} \xi_1)]\!]^{M_{\mathfrak{H}} \times M_{(W,S)} \rho} \\ &\qquad = ([\![\xi_1 \supset_{\mathsf{J}} (\xi_2 \supset_{\mathsf{J}} \xi_1)]\!]^{M_{\mathfrak{H}} \rho_1}, [\![\xi_1 \supset_{\mathsf{K}} (\xi_2 \supset_{\mathsf{K}} \xi_1)]\!]^{M_{(W,S)} \rho_2}) = (1, W) \end{aligned}$$

and $\{(1, W)\} = D_{\mathfrak{H}} \times D_{(W,S)}$. Hence, $\vDash_{\mathsf{JK}} \xi_1 \supset_{\mathsf{JK}} (\xi_2 \supset_{\mathsf{JK}} \xi_1)$. On the other hand,

$$\vDash_{\mathsf{K}} ((\neg_{\mathsf{K}} \xi_1) \supset_{\mathsf{K}} (\neg_{\mathsf{K}} \xi_2)) \supset_{\mathsf{K}} (\xi_2 \supset_{\mathsf{K}} \xi_1).$$

However $[\![((\neg_{\mathsf{J}} \xi_1) \supset_{\mathsf{J}} (\neg_{\mathsf{J}} \xi_2)) \supset_{\mathsf{J}} (\xi_2 \supset_{\mathsf{J}} \xi_1)]\!]^{M_{\mathfrak{H}} \rho_1}$ is not always in $D_{\mathfrak{H}}$ that is,

$$\nvDash_{\mathsf{J}} ((\neg_{\mathsf{J}} \xi_1) \supset_{\mathsf{J}} (\neg_{\mathsf{J}} \xi_2)) \supset_{\mathsf{J}} (\xi_2 \supset_{\mathsf{J}} \xi_1). \tag{†}$$

Therefore, $[\![((\neg_{\mathsf{JK}} \xi_1) \supset_{\mathsf{JK}} (\neg_{\mathsf{JK}} \xi_2)) \supset_{\mathsf{JK}} (\xi_2 \supset_{\mathsf{JK}} \xi_1)]\!]^{M_{\mathfrak{H}} \times M_{(W,S)} \rho}$ is not always in $D_{\mathfrak{H}} \times D_{(W,S)}$ and so $\nvDash_{\mathsf{JK}} ((\neg_{\mathsf{JK}} \xi_1) \supset_{\mathsf{JK}} (\neg_{\mathsf{JK}} \xi_2)) \supset_{\mathsf{JK}} (\xi_2 \supset_{\mathsf{JK}} \xi_1)$. ⊣

We relate satisfaction of $\psi \in L_{12}(\Xi)$ with satisfaction of its projections.

Lemma 2.1. *Let $\psi \in L_{12}(\Xi)$, $M_1 \in \mathcal{M}_1$, $M_2 \in \mathcal{M}_2$ and ρ an assignment over $M_1 \times M_2$. Then $M_1 \times M_2, \rho \Vdash_{12} \psi$ if and only if $M_k, \rho_k \Vdash_k \psi|^k$ for $k = 1, 2$.*

Proof. Note that $M_1 \times M_2, \rho \Vdash_{12} \psi$ if and only if $[\![\psi]\!]^{M_1 \times M_2 \rho}$ in $D_1 \times D_2$ if and only if $[\![\psi|^k]\!]^{M_k \rho_k}$ in D_k for $k = 1, 2$, by Proposition 2.3, if and only if $M_k, \rho_k \Vdash_k \psi|^k$ for $k = 1, 2$. ⊣

Proposition 2.4. *Let $\Gamma \cup \{\varphi\} \subseteq L_{12}(\Xi)$. Then*

$$\Gamma \vDash_{12} \varphi \ \text{ if and only if } \ \Gamma|^1 \vDash_1 \varphi|^1 \ \text{ and } \ \Gamma|^2 \vDash_2 \varphi|^2.$$

Proof. ($\to$) Assume that $M_1, \rho_1 \Vdash_1 \Gamma|^1$ and $M_2, \rho_2 \Vdash_2 \Gamma|^2$. Thus, by Lemma 2.1, $M_1 \times M_2, \rho \Vdash_{12} \Gamma$. Then, by hypothesis, $M_1 \times M_2, \rho \Vdash_{12} \varphi$. So, once again by Lemma 2.1, $M_1, \rho_1 \Vdash_1 \varphi|^1$ and $M_2, \rho_2 \Vdash_2 \varphi|^2$.

($\leftarrow$) Suppose $M_1 \times M_2, \rho \Vdash_{12} \Gamma$. Thus, by Lemma 2.1, $M_k, \rho_k \Vdash_k \Gamma|^k$ for $k = 1, 2$. Hence, by hypothesis, $M_k, \rho_k \Vdash_k \varphi|^k$ for $k = 1, 2$. Therefore, once again by Lemma 2.1, $M_1 \times M_2, \rho \Vdash_{12} \varphi$. $\dashv$

Note that the meet-combination of B_3 (see Example 2.6) with any other logic satisfies Proposition 2.1 because we enriched B_3 with $\mathtt{tt}$'s.

COROLLARY 2.1. *Let $\Gamma \cup \{\varphi\} \subseteq L_1(\Xi)$. Then*

$$\Gamma \vDash_1 \varphi \text{ implies } \Gamma^* \vDash_{12} \varphi^*$$

where $\Gamma^* \cup \{\varphi^*\}$ is obtained from $\Gamma \cup \{\varphi\}$ by replacing every constructor c by the constructor $\langle c, \mathtt{tt}_2^n \rangle$. Similarly for $\vDash_2$.

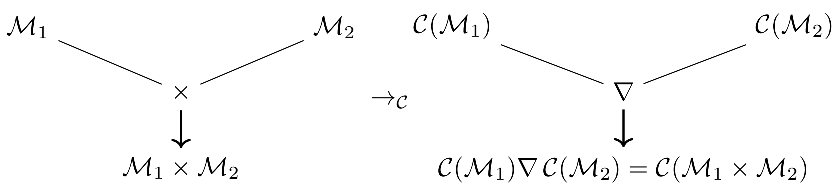

Figure 2. Consequence system of product is the meet-combination

As a consequence of Proposition 2.4 also using Proposition 1.1, we have:

PROPOSITION 2.5. *The consequence system $\mathcal{C}(\mathcal{M}_1 \times \mathcal{M}_2)$ generated by the product $\mathcal{M}_1 \times \mathcal{M}_2$ of matrix semantics is $\mathcal{C}(\mathcal{M}_1) \nabla \mathcal{C}(\mathcal{M}_2)$, that is, the meet-combination of the consequence systems $\mathcal{C}(\mathcal{M}_1)$ and $\mathcal{C}(\mathcal{M}_2)$. Furthermore, if $\mathcal{C}(\mathcal{M}_k)$ is suitable for $k = 1, 2$ then $\mathcal{C}(\mathcal{M}_1 \times \mathcal{M}_2)$ is also suitable.*

Example 2.8. Recall Examples 2.4 and 2.2. Consider the family of constructors $C_{\text{Ł}_3\mathsf{K}}$ of the meet-combination of $C_{\text{Ł}_3}$ and C_{K}. The product matrix semantics $\mathcal{M}_{\text{Ł}_3} \times \mathcal{M}_{\mathsf{K}}$ is the class of matrices of the form

$$M_{\text{Ł}_3} \times M_{(W,S)} = (\mathfrak{A}_{\text{Ł}_3} \times \mathfrak{A}_{(W,S)}, \{(2, W)\}).$$

Note that $[\![\xi \vee_{\text{Ł}_3} (\neg_{\text{Ł}_3} \xi)]\!]^{M_{\text{Ł}_3} \rho} = 1$ when $\rho(\xi) = 1$. So $\nvDash_{\text{Ł}_3} \xi \vee_{\text{Ł}_3} (\neg_{\text{Ł}_3} \xi)$ because the only distinguished value in the matrix $M_{\text{Ł}_3}$ is 2. Therefore, $\nvDash_{\text{Ł}_3\mathsf{K}} \xi \vee_{\text{Ł}_3\mathsf{K}} (\neg_{\text{Ł}_3\mathsf{K}} \xi)$ by Proposition 2.4. $\dashv$

Example 2.9. Recall Examples 2.5 and 2.1. Let $C_{\mathsf{LFI1\,CP}}$ be the meet-combination of C_{LFI1} and C_{CP}. The product matrix semantics $\mathcal{M}_{\mathsf{LFI1}} \times \mathcal{M}_{\mathsf{CP}}$ is a singleton set composed of the matrix

$$M_{\mathsf{LFI1}} \times M_{\mathsf{CP}} = (\mathfrak{A}_{\mathsf{LFI1}} \times \mathfrak{A}_{\mathsf{CP}}, \{(\frac{1}{2}, 1), (1,1)\}).$$

Observe that $[\![(\neg_{\mathsf{LFI1}}\xi_1) \supset_{\mathsf{LFI1}} (\xi_1 \supset_{\mathsf{LFI1}} \xi_2)]\!]^{M_{\mathsf{LFI1}}\rho} \notin \{\frac{1}{2}, 1\}$ when $\rho(\xi_1) = \frac{1}{2}$ and $\rho(\xi_2) = 0$. Indeed, $[\![\neg_{\mathsf{LFI1}}\xi_1]\!]^{M_{\mathsf{LFI1}}\rho} = \frac{1}{2}$, $[\![\xi_1 \supset_{\mathsf{LFI1}} \xi_2]\!]^{M_{\mathsf{LFI1}}\rho} = 0$ and so $[\![(\neg_{\mathsf{LFI1}}\xi_1) \supset_{\mathsf{LFI1}} (\xi_1 \supset_{\mathsf{LFI1}} \xi_2)]\!]^{M_{\mathsf{LFI1}}\rho} = 0$. Thus, by Proposition 2.4,

$$\not\vDash_{\mathsf{LFI1\,CP}} (\neg_{\mathsf{LFI1\,CP}}\xi_1) \supset_{\mathsf{LFI1\,CP}} (\xi_1 \supset_{\mathsf{LFI1\,CP}} \xi_2).$$

Hence, $\mathcal{C}(\mathcal{M}_{\mathsf{LFI1}})\nabla\mathcal{C}(\mathcal{M}_{\mathsf{CP}})$ is paraconsistent, that is the explosion law does not always hold. So paraconsistency of LFI1 is preserved by the meet-combination of consequence systems induced by matrix semantics.

On the other hand, it is known that for every assignment ρ_1 over M_{LFI1}

$$[\![(\circ_{\mathsf{LFI1}}\,\xi_1) \supset_{\mathsf{LFI1}} ((\neg_{\mathsf{LFI1}}\xi_1) \supset_{\mathsf{LFI1}} (\xi_1 \supset_{\mathsf{LFI1}} \xi_2))]\!]^{M_{\mathsf{LFI1}}\rho_1} \in \{\tfrac{1}{2}, 1\}.$$

So, $[\![(\langle \circ_{\mathsf{LFI1}}\, \mathsf{tt}_{\mathsf{CP}}\rangle\xi_1) \supset_{\mathsf{LFI1\,CP}} ((\neg_{\mathsf{LFI1\,CP}}\,\xi_1) \supset_{\mathsf{LFI1\,CP}} (\xi_1 \supset_{\mathsf{LFI1\,CP}} \xi_2))]\!]^{M_{\mathsf{LFI1}} \times M_{\mathsf{CP}}\rho}$ is in $\{(\frac{1}{2}, 1), (1,1)\}$ because for eqch assignment ρ_2 over M_{CP} with $\rho(\xi) = (\rho_1(\xi), \rho_2(\xi))$ we have $[\![(\mathsf{tt}_{\mathsf{CP}}(\xi_1)) \supset_{\mathsf{CP}} ((\neg_{\mathsf{CP}}\,\xi_1) \supset_{\mathsf{CP}} (\xi_1 \supset_{\mathsf{CP}} \xi_2))]\!]^{M_{\mathsf{CP}}\rho_2} \in \{1\}$. Thus, by Proposition 2.4,

$$\vDash_{\mathsf{LFI1\,CP}} (\langle \circ_{\mathsf{LFI1}}\, \mathsf{tt}_{\mathsf{CP}}\rangle\xi_1) \supset_{\mathsf{LFI1\,CP}} ((\neg_{\mathsf{LFI1\,CP}}\,\xi_1) \supset_{\mathsf{LFI1\,CP}} (\xi_1 \supset_{\mathsf{LFI1\,CP}} \xi_2)).$$

So when we impose that ξ_1 is explosive by the formula $\circ_{\mathsf{LFI1}}\xi_1$ we recover gentle explosion in the meet-combination of consequence systems induced by matrix semantics, that is, the meet-combination is a consequence system of formal inconsistency. Similarly for the meet $\mathcal{C}(\mathcal{M}_{\mathsf{LFI1}})\nabla\mathcal{C}(\mathcal{M}_{\mathsf{J}})$. ⊣

Example 2.10. Recall Examples 2.4 and 2.3. Consider the family of constructors $C_{\text{Ł}_3\mathsf{J}}$ of the meet-combination of $C_{\text{Ł}_3}$ and C_{J}. The product matrix semantics $\mathcal{M}_{\text{Ł}_3} \times \mathcal{M}_{\mathsf{J}}$ is the class of matrices of the form

$$M_{\text{Ł}_3} \times M_{\mathfrak{H}} = (\mathfrak{A}_{\text{Ł}_3} \times \mathfrak{A}_{\mathfrak{H}}, \{(2,1)\}).$$

Note that for every assignment ρ_1 over $M_{\text{Ł}_3}$

$$[\![(\xi_1 \vee_{\text{Ł}_3} \xi_2) \supset_{\text{Ł}_3} (\xi_2 \vee_{\text{Ł}_3} \xi_1)]\!]^{M_{\text{Ł}_3}\rho_1} = 2.$$

Hence $\vDash_{\text{Ł}_3} (\xi_1 \vee_{\text{Ł}_3} \xi_2) \supset_{\text{Ł}_3} (\xi_2 \vee_{\text{Ł}_3} \xi_1)$. On the other hand for every assignment ρ_2 over M_{J}

$$[\![(\xi_1 \wedge_{\mathsf{J}} \xi_2) \supset_{\mathsf{J}} (\xi_2 \wedge_{\mathsf{J}} \xi_1)]\!]^{M_{\mathsf{J}}\rho_2} = 1$$

since $\rho_2(\xi_1) \sqcap \rho_2(\xi_2) = \rho_2(\xi_2) \sqcap \rho_2(\xi_1)$. Thus, $\vDash_{\mathsf{J}} (\xi_1 \wedge_{\mathsf{J}} \xi_2) \supset_{\mathsf{J}} (\xi_2 \wedge_{\mathsf{J}} \xi_1)$. Therefore, $\vDash_{\text{Ł}_3\,\mathsf{J}} (\xi_1 \langle \vee_{\text{Ł}_3} \wedge_{\mathsf{J}} \rangle \xi_2) \supset_{\text{Ł}_3\mathsf{J}} (\xi_2 \langle \vee_{\text{Ł}_3} \wedge_{\mathsf{J}} \rangle \xi_1)$ by Proposition 2.4. $\dashv$

Example 2.11. Recall Example 2.9. Note that

$$\varphi,\ \varphi \supset_{\mathsf{LFI1\,CP}} \psi \vDash_{\mathsf{LFI1\,CP}} \psi$$

with $\varphi, \psi \in L_{\mathsf{LFI1\,CP}}(\Xi)$. Indeed, $\xi_1, \xi_1 \supset_{\mathsf{LFI1}} \xi_2 \vDash_{\mathsf{LFI1}} \xi_2$ and $\xi_1, \xi_1 \supset_{\mathsf{CP}} \xi_2 \vDash_{\mathsf{CP}} \xi_2$. Therefore, $\xi_1, \xi_1 \supset_{\mathsf{LFI1\,CP}} \xi_2 \vDash_{\mathsf{LFI1\,CP}} \xi_2$ by Proposition 2.4. Given that $\mathcal{C}(\mathcal{M}_{\mathsf{LFI1}})$ and $\mathcal{C}(\mathcal{M}_{\mathsf{CP}})$ are closed for substitution so is

$$\mathcal{C}(\mathcal{M}_{\mathsf{LFI1}}) \mathbin{\nabla} \mathcal{C}(\mathcal{M}_{\mathsf{CP}})$$

(see Proposition 1.4). Hence the thesis follows. $\dashv$

Example 2.12. Recall Example 2.7. Then

$$\xi_1,\ \xi_1 \supset_{\mathsf{J}} \xi_2 \vDash_{\mathsf{J}} \xi_2 \ \text{ and } \ \xi_1,\ \xi_1 \langle \supset_{\mathsf{J}} \mathsf{tt}_{\mathsf{K}} \rangle \xi_2 \nvDash_{\mathsf{JK}} \xi_2.$$

Indeed it is not the case that $\xi_1 \vDash_{\mathsf{K}} \xi_2$. Therefore, $\xi_1,\ \xi_1 \langle \supset_{\mathsf{J}} \mathsf{tt}_{\mathsf{K}} \rangle\, \xi_2 \nvDash_{\mathsf{JK}} \xi_2$ by Proposition 2.4. $\dashv$

We end this section with the analysis of the preservation of the finite model property by meet-combination. We say that a consequence system induced by a matrix semantics has the *finite model property* whenever it is the case that if there are $M \in \mathcal{M}$ and assignment ρ over M such that $M, \rho \Vdash \Gamma$ then there are $M' \in \mathcal{M}$ with a finite set A' of truth values and an assignment ρ' over M' such that $M', \rho' \Vdash \Gamma$, for every $\Gamma \subseteq L(\Xi)$. In this context we say that M' is a *finite model* of Γ.

PROPOSITION 2.6. *If $\mathcal{C}(\mathcal{M}_1)$ and $\mathcal{C}(\mathcal{M}_2)$ have the finite model property so does $\mathcal{C}(\mathcal{M}_1) \mathbin{\nabla} \mathcal{C}(\mathcal{M}_2)$.*

PROOF. Let $M_1 \times M_2 \in \mathcal{C}(\mathcal{M}_1) \mathbin{\nabla} \mathcal{C}(\mathcal{M}_2)$ and ρ be an assignment over $M_1 \times M_2$ such that $M_1 \times M_2, \rho \Vdash_{12} \Gamma$. Hence, $M_1, \rho_1 \Vdash_1 \Gamma|^1$ and $M_2, \rho_2 \Vdash_2 \Gamma|^2$ by Lemma 2.1. Let $M'_1 \in \mathcal{M}_1$ and $M'_2 \in \mathcal{M}_2$ be finite models and ρ'_1 and ρ'_2 be assignments over M'_1 and M'_2, respectively such that $M'_1, \rho'_1 \Vdash_1 \Gamma|^1$ and $M'_2, \rho'_2 \Vdash_2 \Gamma|^2$. Therefore, $M'_1 \times M'_2$ is a finite model in $\mathcal{C}(\mathcal{M}_1) \mathbin{\nabla} \mathcal{C}(\mathcal{M}_2)$ such that $M'_1 \times M'_2, \rho' \Vdash_{12} \Gamma$ again by Lemma 2.1. $\dashv$

Example 2.13. Recall Example 2.7. Then $\mathcal{C}(\mathcal{M}_{\mathsf{J}})$ and $\mathcal{C}(\mathcal{M}_{\mathsf{K}})$ have the finite model property (see Rybakov, 1997, pp. 224 and 222). Thus, by Proposition 2.6, $\mathcal{C}(\mathcal{M}_{\mathsf{J}}) \,\nabla\, \mathcal{C}(\mathcal{M}_{\mathsf{K}})$ has the finite model property. $\dashv$

The reader may wonder if the converse implication also holds. Indeed this is the case.

Proposition 2.7. *If $\mathcal{C}(\mathcal{M}_1) \,\nabla\, \mathcal{C}(\mathcal{M}_2)$ has the finite model property so does $\mathcal{C}(\mathcal{M}_i)$ for $i = 1, 2$.*

Proof. Let $i = 1$, $M_1 \in \mathcal{M}_1$ and ρ_1 be an assignment over M_1 such that $M_1, \rho_1 \Vdash_1 \varGamma_1$. Let $\varGamma_1^*$ be obtained from $\varGamma_1$ by replacing every constructor c_1 by the constructor $\langle c_1, \mathtt{tt}_2^n \rangle$ and let $M_2 \in \mathcal{M}_2$ and ρ_2 be an assignment over M_2 such that $\rho_2(\xi) = \overline{\mathtt{tt}_2^0}^{M_2}$. Then $M_1 \times M_2 \in \mathcal{C}(\mathcal{M}_1) \,\nabla\, \mathcal{C}(\mathcal{M}_2)$. Let $\rho_1^* : \varXi \to A_1 \times A_2$ be such that $\rho_1^*(\xi) = (\rho_1(\xi), \rho_2(\xi))$. Hence, by Proposition 2.1, $M_1 \times M_2, \rho_1^* \Vdash_{12} \varGamma_1^*$. Then, because by hypothesis $\mathcal{C}(\mathcal{M}_1) \,\nabla\, \mathcal{C}(\mathcal{M}_2)$ has the finite model property, there are finite models $M_1' \in \mathcal{M}_1$ and $M_2' \in \mathcal{M}_2$ and ρ' an assignment over $M_1' \times M_2'$ such that $M_1' \times M_2', \rho' \Vdash_{12} \varGamma_1^*$. Therefore $M_1', \rho_1' \Vdash_1 \varGamma_1$. Similarly when $i = 2$. $\dashv$

3. Meet-combination of Hilbert calculi

It is common to deal with consequence systems generated by a deductive calculus namely by a Hilbert calculus.

A *Hilbert calculus* H is a triple (C, Ax, R) such that $Ax \subseteq L(\varXi)$ is the nonempty set of *axioms* and R is the set of *rules* where each rule is a pair (Θ, δ), $\Theta \subseteq L(\varXi)$ is a finite set of *premises* and $\delta \in L(\varXi)$ is the *conclusion*.

We present a rule by a fraction where the numerator is composed of the premises and the denominator is the conclusion. We say that $\beta \in L(\varXi)$ is an *instance of an axiom* α whenever there is a substitution $\sigma : \varXi \to L(\varXi)$ such that $\sigma(\alpha) = \beta$. Moreover, we say that (Ω, μ) is an *instance of a rule* (Θ, δ) whenever there is a substitution $\sigma : \varXi \to L(\varXi)$ such that $\sigma(\Theta) = \Omega$ and $\sigma(\delta) = \mu$. Let $\varGamma \cup \{\varphi\} \subseteq L(\varXi)$. We say that $\varphi \in L(\varXi)$ is *derived* from $\varGamma \subseteq L(\varXi)$ in H, denoted by $\varGamma \vdash_H \varphi$ whenever there is a derivation of φ from $\varGamma$, that is, a finite sequence of formulas $\psi_1 \dots \psi_n$ in $L(\varXi)$ where ψ_n is φ and each formula ψ_j is either an element of $\varGamma$ or is an instance of an axiom in Ax or is the conclusion of an instance of a rule in R such that the instances of the premises appear in $\psi_1 \dots \psi_{j-1}$. We say that φ is a *theorem* in H, denoted by

$\vdash_H \varphi$ whenever $\emptyset \vdash_H \varphi$. Observe that if we have a schema variable as an axiom then $\emptyset^{\vdash_H} = L(\Xi)$. So we assume that $\xi \notin Ax$ for each $\xi \in \Xi$.

PROPOSITION 3.1. *The pair $\mathcal{C}(H) = (C, \vdash_H)$ is a consequence system induced by the Hilbert calculus H where $\Gamma^{\vdash_H} = \{\varphi \in L(\Xi) : \Gamma \vdash_H \varphi\}$ for every $\Gamma \subseteq L(\Xi)$.*

PROOF. We only prove monotonicity of $\vdash_H$. Assume that $\Gamma_1 \subseteq \Gamma_2$ and $\Gamma_1 \vdash_H \varphi$. Let $\psi_1 \dots \psi_n$ be a derivation of φ from Γ_1. Then $\psi_1 \dots \psi_n$ is also a derivation of φ from Γ_2 since all hypotheses in the sequence are in Γ_1 and so also in Γ_2. $\dashv$

Remark 3.1. The reader may wonder what happens when starting with a non suitable consequence system $\mathcal{C}(H) = (C, \vdash_H)$. We show how to proceed to get a suitable consequence system. The family of constructors was introduced in Remark 1.1. Define the Hilbert calculus $H^{\mathtt{tt}} = (C, Ax^{\mathtt{tt}}, R)$ where $Ax^{\mathtt{tt}} = Ax \cup \{\mathtt{tt}^0, \mathtt{tt}^n(\xi_1, \dots, \xi_n) : \xi_1, \dots, \xi_n \in \Xi\}$.

Moreover, for any set of formulas $\Gamma \cup \{\varphi\}$ without occurrences of $\mathtt{tt}$'s

$$\Gamma \vdash_H \varphi \text{ if and only if } \Gamma \vdash_{H^{\mathtt{tt}}} \varphi.$$

Hence there is preservation and reflection of derivation by the enrichment.

In the sequel we use H for denoting the enriched Hilbert calculus $H^{\mathtt{tt}}$, that is we assume that all Hilbert calculi are suitable. $\dashv$

The proof of the following result follows immediately since we work with schemas of axioms.

PROPOSITION 3.2. *The consequence system $\mathcal{C}(H)$ is structural.*

In the sequel we need the concepts of soundness and completeness of Hilbert calculi. We say that a Hilbert calculus H is *sound* for a matrix semantics $\mathcal{M}$ whenever $\Gamma \vdash_H \varphi$ implies $\Gamma \vDash_{\mathcal{M}} \varphi$ for every $\Gamma \cup \{\varphi\} \subseteq L(\Xi)$. Furthermore, we say that a Hilbert calculus H is *complete* for a matrix semantics $\mathcal{M}$ whenever $\Gamma \vDash_{\mathcal{M}} \varphi$ implies $\Gamma \vdash_H \varphi$ for every $\Gamma \cup \{\varphi\} \subseteq L(\Xi)$.

Example 3.1. Recall logic CP presented in Example 2.1. The Hilbert calculus $H^{\mathrm{MT}}_{\mathsf{CP}}$ for CP is composed of the set of axioms Ax_{CP}

- $\xi_1 \supset_{\mathsf{CP}} (\xi_2 \supset_{\mathsf{CP}} \xi_1)$
- $(\xi_1 \supset_{\mathsf{CP}} (\xi_2 \supset_{\mathsf{CP}} \xi_3)) \supset_{\mathsf{CP}} ((\xi_1 \supset_{\mathsf{CP}} \xi_2) \supset_{\mathsf{CP}} (\xi_1 \supset_{\mathsf{CP}} \xi_3))$
- $((\neg_{\mathsf{CP}} \xi_1) \supset_{\mathsf{CP}} (\neg_{\mathsf{CP}} \xi_2)) \supset_{\mathsf{CP}} (\xi_2 \supset_{\mathsf{CP}} \xi_1)$

and the rule Modus Tollens

$$\mathrm{MT}_{\mathsf{CP}} \quad \frac{\neg\,\xi_2 \quad \xi_1 \supset_{\mathsf{CP}} \xi_2}{\neg\,\xi_1}$$

is the unique rule in R_{CP}. ⊣

Example 3.2. Consider logic J as in Example 2.3. The Hilbert calculus H_{J} for J is composed of the following set of axioms Ax_{J}

- $\xi_1 \supset_{\mathsf{J}} (\xi_2 \supset_{\mathsf{J}} \xi_1)$
- $(\xi_1 \supset_{\mathsf{J}} \xi_2) \supset_{\mathsf{J}} ((\xi_1 \supset_{\mathsf{J}} (\xi_2 \supset_{\mathsf{J}} \xi_3)) \supset_{\mathsf{J}} (\xi_1 \supset_{\mathsf{J}} \xi_3))$
- $(\xi_1 \wedge_{\mathsf{J}} \xi_2) \supset_{\mathsf{J}} \xi_1 \qquad (\xi_1 \wedge_{\mathsf{J}} \xi_2) \supset_{\mathsf{J}} \xi_2 \qquad \xi_1 \supset_{\mathsf{J}} (\xi_2 \supset_{\mathsf{J}} (\xi_1 \wedge_{\mathsf{J}} \xi_2))$
- $\xi_1 \supset_{\mathsf{J}} (\xi_1 \vee_{\mathsf{J}} \xi_2) \qquad \xi_2 \supset_{\mathsf{J}} (\xi_1 \vee_{\mathsf{J}} \xi_2)$
- $(\xi_1 \supset_{\mathsf{J}} \xi_3) \supset_{\mathsf{J}} ((\xi_2 \supset_{\mathsf{J}} \xi_3) \supset_{\mathsf{J}} ((\xi_1 \vee_{\mathsf{J}} \xi_2) \supset_{\mathsf{J}} \xi_3))$
- $(\xi_1 \supset_{\mathsf{J}} \xi_2) \supset_{\mathsf{J}} ((\xi_1 \supset_{\mathsf{J}} (\neg_{\mathsf{J}}\, \xi_2)) \supset_{\mathsf{J}} (\neg_{\mathsf{J}}\, \xi_1)) \qquad (\neg_{\mathsf{J}}\, \xi_1) \supset_{\mathsf{J}} (\xi_1 \supset_{\mathsf{J}} \xi_2)$

The only rule in R_{J} is:

$$\mathrm{MP}_{\mathsf{J}} \quad \frac{\xi_1 \quad \xi_1 \supset_{\mathsf{J}} \xi_2}{\xi_2}$$

Example 3.3. Recall logic K presented in Example 2.2. The Hilbert calculus H_{K} for K is composed of the set of axioms Ax_{K}

- $\xi_1 \supset_{\mathsf{K}} (\xi_2 \supset_{\mathsf{K}} \xi_1) \qquad (\xi_1 \supset_{\mathsf{K}} (\xi_2 \supset_{\mathsf{K}} \xi_3)) \supset_{\mathsf{K}} ((\xi_1 \supset_{\mathsf{K}} \xi_2) \supset_{\mathsf{K}} (\xi_1 \supset_{\mathsf{K}} \xi_3))$
- $((\neg_{\mathsf{K}}\, \xi_1) \supset_{\mathsf{K}} (\neg_{\mathsf{K}}\, \xi_2)) \supset_{\mathsf{K}} (\xi_2 \supset_{\mathsf{K}} \xi_1)$
- $(\Box_{\mathsf{K}}(\xi_1 \supset_{\mathsf{K}} \xi_2)) \supset_{\mathsf{K}} ((\Box_{\mathsf{K}}\xi_1) \supset_{\mathsf{K}} (\Box_{\mathsf{K}}\xi_2))$

and R_{K} is composed of Modus Ponens MP_{K} and the necessitation rule

$$\mathrm{NEC}_{\mathsf{K}} \quad \frac{\xi}{\Box_{\mathsf{K}}\xi}$$

Example 3.4. Recall modal logics T and $\mathsf{4}$ (see Example 2.2). The Hilbert calculus H_{T} for T is such that Ax_{T} is composed of the axioms in Ax_{K} by replacing K for T plus the axiom $(\Box_{\mathsf{T}}\xi) \supset_{\mathsf{T}} \xi$ (called T) and R_{T} is composed of rules Modus Ponens MP_{T} and necessitation $\mathrm{NEC}_{\mathsf{T}}$. Moreover, consider the Hilbert calculus H_{4} for modal logic $\mathsf{4}$ similar to H_{K} including the new axiom $(\Box_{\mathsf{4}}\xi) \supset_{\mathsf{4}} (\Box_{\mathsf{4}}\Box_{\mathsf{4}}\xi)$ (called $\mathsf{4}$). ⊣

Example 3.5. Consider logic $\mathsf{Ł}_3$ in Example 2.4. The Hilbert calculus $H_{\mathsf{Ł}_3}$ for $\mathsf{Ł}_3$ (see Gottwald, 2001) is composed of the set $Ax_{\mathsf{Ł}_3}$ of axioms

- $\xi_1 \supset_{\mathsf{Ł}_3} (\xi_2 \supset_{\mathsf{Ł}_3} \xi_1)$
- $(\xi_1 \supset_{\mathsf{Ł}_3} \xi_2) \supset_{\mathsf{Ł}_3} ((\xi_2 \supset_{\mathsf{Ł}_3} \xi_3) \supset_{\mathsf{Ł}_3} (\xi_1 \supset_{\mathsf{Ł}_3} \xi_3))$
- $((\neg_{\mathsf{Ł}_3}\, \xi_1) \supset_{\mathsf{Ł}_3} (\neg_{\mathsf{Ł}_3}\, \xi_2)) \supset_{\mathsf{Ł}_3} (\xi_2 \supset_{\mathsf{Ł}_3} \xi_1)$
- $((\xi \supset_{\mathsf{Ł}_3} (\neg_{\mathsf{Ł}_3}\, \xi)) \supset_{\mathsf{Ł}_3} \xi) \supset_{\mathsf{Ł}_3} \xi$

and the set $R_{\mathsf{Ł}_3}$ is composed of the rule Modus Ponens $\mathrm{MP}_{\mathsf{Ł}_3}$. ⊣

Example 3.6. Let logic $\mathsf{LFI1}$ be as in Example 2.5. The Hilbert calculus H_{LFI1} for $\mathsf{LFI1}$ is composed of the set of axioms Ax_{LFI1} containing all the axioms in Example 3.2 with the exception of those for $\neg_{\mathsf{J}}$ as well as

- $\xi \equiv_{\mathsf{LFI1}} (\neg_{\mathsf{LFI1}} \neg_{\mathsf{LFI1}} \xi)$
- $\xi_1 \vee_{\mathsf{LFI1}} (\xi_1 \supset_{\mathsf{LFI1}} \xi_2)$
- $(\circ_{\mathsf{LFI1}} \xi_1) \supset_{\mathsf{LFI1}} ((\neg_{\mathsf{LFI1}} \xi_1) \supset_{\mathsf{LFI1}} (\xi_1 \supset_{\mathsf{LFI1}} \xi_2))$
- $(\neg_{\mathsf{LFI1}} \circ_{\mathsf{LFI1}} \xi) \supset_{\mathsf{LFI1}} (\xi \wedge_{\mathsf{LFI1}} (\neg_{\mathsf{LFI1}} \xi))$
- $(\circ_{\mathsf{LFI1}} \xi) \supset_{\mathsf{LFI1}} (\circ_{\mathsf{LFI1}} \neg_{\mathsf{LFI1}} \xi)$
- $((\circ_{\mathsf{LFI1}} \xi_1) \wedge_{\mathsf{LFI1}} (\circ_{\mathsf{LFI1}} \xi_2)) \supset_{\mathsf{LFI1}} (\circ_{\mathsf{LFI1}} (\xi_1 \supset_{\mathsf{LFI1}} \xi_2))$
- $((\circ_{\mathsf{LFI1}} \xi_1) \wedge_{\mathsf{LFI1}} (\circ_{\mathsf{LFI1}} \xi_2)) \supset_{\mathsf{LFI1}} (\circ_{\mathsf{LFI1}} (\xi_1 \vee_{\mathsf{LFI1}} \xi_2))$

and the set R_{LFI1} is composed of the rule Modus Ponens $\mathrm{MP}_{\mathsf{LFI1}}$. ⊣

Meet-combination is interconnection. In (Sernadas et al., 2012), the Hilbert calculus corresponding to the meet-combination is defined by putting together the axioms and the rules of the Hilbert calculi for the components restricting the rules that have a schema variable as conclusion. In that paper we only use instances of such rules when the conclusion starts with a constructor. One of the consequences of this restriction was that we were only able to prove preservation of completeness for concrete formulas (that is, formulas without schema variables). Herein, we are able to cope with such rules in a different way.

We start by introducing compatibility between Hilbert calculi.

Definition 3.1. We say that Hilbert calculi $H_1 = (C_1, Ax_1, R_1)$ and $H_2 = (C_2, Ax_2, R_2)$ are *compatible* whenever

- if $(\Delta_1, \xi) \in R_1$ then there is $\Theta \subseteq L_{12}(\Xi)$ such that $\Theta|^1$ is Δ_1, $\Theta|^2$ is Δ_2, and $\Delta_2 \vdash_2 \xi$
- if $(\Delta_2, \xi) \in R_2$ then there is $\Theta \subseteq L_{12}(\Xi)$ such that $\Theta|^2$ is Δ_2, $\Theta|^1$ is Δ_1, and $\Delta_1 \vdash_1 \xi$.

Definition 3.2. The *interconnection of compatible Hilbert calculi* H_1 and H_2 is the Hilbert calculus $H_1 \bowtie H_2 = (C_{12}, Ax_{12}, R_{12})$ such that

- $Ax_{12} = Ax_1 \cup Ax_2$
- the set of rules R_{12} is $R_1 \cup R_2$ plus the *lifting* and the *co-lifting rules*

$$\mathrm{LFT}\quad \frac{\varphi|^1 \quad \varphi|^2}{\varphi} \qquad \text{and} \qquad \mathrm{cLFT}_k\quad \frac{\varphi}{\varphi|^k} \quad \text{for } k = 1, 2$$

where $\varphi \in L_{12}(\Xi)$.

In the sequel it may be useful to use as a rule in the interconnection a pair $r_2 = (\Delta_2, \xi)$ such that there are $r_1 = (\Delta_1, \xi) \in R_1$ and $\Theta \subseteq L_{12}(\Xi)$ such that $\Theta|^1$ is Δ_1, $\Theta|^2$ is Δ_2. Similarly for the other component.

We denote by $\vdash_{12} \subseteq \wp L_{12}(\Xi) \times L_{12}(\Xi)$ the derivation in $H_1 \bowtie H_2$. A typical derivation of $\varphi \in L_{12}(\Xi)$ from $\Gamma \subseteq L_{12}(\Xi)$ is depicted in Figure 3. The first step is to project hypotheses in Γ to hypotheses in both components using rule cLFT_k for $k = 1, 2$. Afterwards we derive the projections $\varphi|^1$ and $\varphi|^2$ in the corresponding component. Finally we obtain φ using rule LFT.

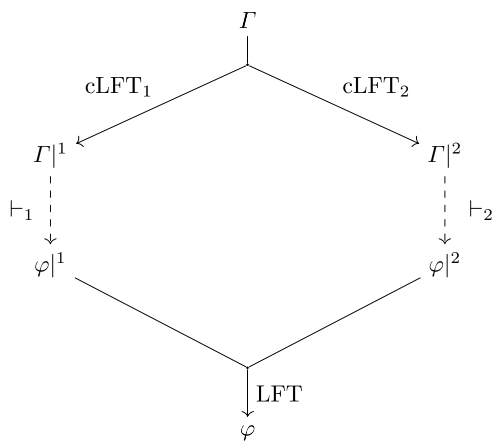

Figure 3. Typical derivation in $H_1 \bowtie H_2$

In the sequel we use HS_{X} and $\mathrm{Thm}_{\mathrm{X}}$ to indicate in a derivation the application of hypothetical syllogism in logic X and that a formula is a theorem of X, respectively. Moreover, we use HYP to indicate an hypothesis.

Example 3.7. Recall Examples 3.2 and 3.3. Note that the only rules with conclusion in Ξ are MP_{J} and MP_{K}. Therefore H_{J} and H_{K} are compatible Hilbert calculi. In $H_{\mathsf{J}} \bowtie H_{\mathsf{K}}$ the set Ax_{JK} is $Ax_{\mathsf{J}} \cup Ax_{\mathsf{K}}$ and the set R_{JK} is $\{\mathrm{MP}_{\mathsf{J}}, \mathrm{MP}_{\mathsf{K}}, \mathrm{NEC}_{\mathsf{K}}, \mathrm{LFT}, \mathrm{cLFT}_{\mathsf{J}}, \mathrm{cLFT}_{\mathsf{K}}\}$. In particular

$$\xi_1 \supset_{\mathsf{J}} (\xi_2 \supset_{\mathsf{J}} \xi_1) \text{ and } \xi_1 \supset_{\mathsf{K}} (\xi_2 \supset_{\mathsf{K}} \xi_1)$$

are axioms in $H_{\mathsf{J}} \bowtie H_{\mathsf{K}}$ and so $\vdash_{\mathsf{JK}} \xi_1 \supset_{\mathsf{JK}} (\xi_2 \supset_{\mathsf{JK}} \xi_1)$ by LFT.

Moreover, $\vdash_{\mathsf{JK}} \xi_1 \supset_{\mathsf{JK}} ((\neg_{\mathsf{JK}} \xi_1) \supset_{\mathsf{JK}} \xi_2)$. Indeed, consider the sequence

1.	$\xi_1 \supset_{\mathsf{J}} ((\neg_{\mathsf{J}}\, \xi_1) \supset_{\mathsf{J}} \xi_2)$	Ax_{J}
2.	$\xi_1 \supset_{\mathsf{K}} ((\neg_{\mathsf{K}}\, \xi_1) \supset_{\mathsf{K}} \xi_2)$	$\mathrm{Thm}_{\mathsf{K}}$
3.	$\xi_1 \supset_{\mathsf{JK}} ((\neg_{\mathsf{JK}} \xi_1) \supset_{\mathsf{JK}} \xi_2)$	LFT 1,2

Furthermore, we have

$$\vdash_{\mathsf{J}} (\neg_{\mathsf{J}}(\xi_1 \vee_{\mathsf{J}} \xi_2)) \supset_{\mathsf{J}} ((\neg_{\mathsf{J}}\, \xi_1) \wedge_{\mathsf{J}} (\neg_{\mathsf{J}}\, \xi_2))$$
$$\vdash_{\mathsf{K}} (\neg_{\mathsf{K}}(\xi_1 \vee_{\mathsf{K}} \xi_2)) \supset_{\mathsf{K}} ((\neg_{\mathsf{K}}\, \xi_1) \wedge_{\mathsf{K}} (\neg_{\mathsf{K}}\, \xi_2)).$$

Then, $\vdash_{\mathsf{JK}} (\neg_{\mathsf{JK}}(\xi_1 \vee_{\mathsf{JK}} \xi_2)) \supset_{\mathsf{JK}} ((\neg_{\mathsf{JK}}\, \xi_1) \wedge_{\mathsf{JK}} (\neg_{\mathsf{JK}}\, \xi_2))$. Moreover we consider a derivation of a formula in $H_{\mathsf{J}} \bowtie H_{\mathsf{K}}$ involving unexpected pairs of constructors.

1.	$\xi_1 \langle \wedge_{\mathsf{J}} \vee_{\mathsf{K}} \rangle \xi_2$	HYP
2.	$\xi_1 \wedge_{\mathsf{J}} \xi_2$	$\mathrm{cLFT}_{\mathsf{J}}$ 1
3.	$(\xi_1 \wedge_{\mathsf{J}} \xi_2) \supset_{\mathsf{J}} (\xi_2 \wedge_{\mathsf{J}} \xi_1)$	$\mathrm{Thm}_{\mathsf{J}}$
4.	$\xi_2 \wedge_{\mathsf{J}} \xi_1$	MP_{J} 2,3
5.	$\xi_1 \vee_{\mathsf{K}} \xi_2$	$\mathrm{cLFT}_{\mathsf{K}}$ 1
6.	$(\xi_1 \vee_{\mathsf{K}} \xi_2) \supset_{\mathsf{K}} (\xi_2 \vee_{\mathsf{K}} \xi_1)$	$\mathrm{Thm}_{\mathsf{K}}$
7.	$\xi_2 \vee_{\mathsf{K}} \xi_1$	MP_{K} 5,6
8.	$\xi_2 \langle \wedge_{\mathsf{J}} \vee_{\mathsf{K}} \rangle \xi_1$	LFT 4,7

Finally, the sequence

1.	ξ_1	HYP
2.	$\xi_1 \langle \wedge_{\mathsf{J}} \supset_{\mathsf{K}} \rangle \xi_2$	HYP
3.	$\xi_1 \wedge_{\mathsf{J}} \xi_2$	$\mathrm{cLFT}_{\mathsf{J}}$ 2
4.	$(\xi_1 \wedge_{\mathsf{J}} \xi_2) \supset_{\mathsf{J}} \xi_2$	$\mathrm{Thm}_{\mathsf{J}}$
5.	ξ_2	MP_{J} 3,4
6.	ξ_1	$\mathrm{cLFT}_{\mathsf{K}}$ 1
7.	$\xi_1 \supset_{\mathsf{K}} \xi_2$	$\mathrm{cLFT}_{\mathsf{K}}$ 2
8.	ξ_2	MP_{K} 6,7
9.	ξ_2	LFT 5,8

shows that

$$\xi_1, \xi_1 \langle \wedge_{\mathsf{J}} \supset_{\mathsf{K}} \rangle \xi_2 \vdash_{\mathsf{JK}} \xi_2.$$

Other properties of $\wedge_{\mathsf{J}}$ and $\vee_{\mathsf{K}}$ are given in (Marcelino, 2022). $\dashv$

Example 3.8. Recall Example 3.4. Observe that H_{T} and H_{4} are compatible Hilbert calculi. In $H_{\mathsf{T}} \bowtie H_{\mathsf{4}}$ the set Ax_{T4} is $Ax_{\mathsf{T}} \cup Ax_{\mathsf{4}}$ and the set R_{T4} is $\{\mathrm{MP}_{\mathsf{T}}, \mathrm{MP}_{\mathsf{4}}, \mathrm{NEC}_{\mathsf{T}}, \mathrm{NEC}_{\mathsf{4}}, \mathrm{LFT}, \mathrm{cLFT}_{\mathsf{T}}, \mathrm{cLFT}_{\mathsf{4}}\}$. The following sequence

1.	$(\Box_{\mathsf{T}}\xi) \supset_{\mathsf{T}} \xi$	Ax_{T}
2.	$(\Box_{\mathsf{T}}\Box_{\mathsf{T}}\xi) \supset_{\mathsf{T}} \Box_{\mathsf{T}}\xi$	Ax_{T}
3.	$(\Box_{\mathsf{T}}\Box_{\mathsf{T}}\xi) \supset_{\mathsf{T}} \xi$	HS_{T} 2,1
4.	$(\neg_{\mathsf{4}}\neg_{\mathsf{4}}\xi) \supset_{\mathsf{4}} \xi$	$\mathrm{Thm}_{\mathsf{4}}$
5.	$(\langle\Box_{\mathsf{T}}\neg_{\mathsf{4}}\rangle\langle\Box_{\mathsf{T}}\neg_{\mathsf{4}}\rangle\xi) \supset_{\mathsf{T4}} \xi$	LFT 3,4

is a derivation for $\vdash_{\mathsf{T4}} (\langle\Box_{\mathsf{T}}\neg_{\mathsf{4}}\rangle\langle\Box_{\mathsf{T}}\neg_{\mathsf{4}}\rangle\xi) \supset_{\mathsf{T4}} \xi$. Finally, the sequence

1.	ξ	HYP
2.	$\Box_{\mathsf{T}}\xi$	$\mathrm{NEC}_{\mathsf{T}}$ 1
3.	$\Box_{\mathsf{4}}\xi$	$\mathrm{NEC}_{\mathsf{4}}$ 1
4.	$\langle\Box_{\mathsf{T}}\Box_{\mathsf{4}}\rangle\xi$	LFT 2,3

is a derivation for $\xi \vdash_{\mathsf{T4}} \langle\Box_{\mathsf{T}}\Box_{\mathsf{4}}\rangle\xi$. ⊣

Example 3.9. Recall Example 2.8. Note that $H_{\text{Ł}_3}$ and H_{K} are compatible. Note that $\neg_{\text{Ł}_3\mathsf{K}}\,\xi \vdash_{\text{Ł}_3\mathsf{K}} \xi \supset_{\text{Ł}_3\mathsf{K}} \xi_1$ with the following derivation

1.	$\neg_{\text{Ł}_3\mathsf{K}}\,\xi$	HYP
2.	$\neg_{\text{Ł}_3}\xi$	$\mathrm{cLFT}_{\text{Ł}_3}$ 1
3.	$\neg_{\text{Ł}_3}\,\xi \supset_{\text{Ł}_3} (\xi \supset_{\text{Ł}_3} \xi_1)$	$\mathrm{Thm}_{\text{Ł}_3}$
4.	$\xi \supset_{\text{Ł}_3} \xi_1$	$\mathrm{MP}_{\text{Ł}_3}$ 2,3
5.	$\neg_{\mathsf{K}}\xi$	$\mathrm{cLFT}_{\mathsf{K}}$ 1
6.	$\neg_{\mathsf{K}}\,\xi \supset_{\mathsf{K}} (\xi \supset_{\mathsf{K}} \xi_1)$	$\mathrm{Thm}_{\mathsf{K}}$
7.	$\xi \supset_{\mathsf{K}} \xi_1$	MP_{K} 5,6
8.	$\xi \supset_{\text{Ł}_3\mathsf{K}} \xi_1$	LFT 4,7

Hence explosion still holds in $H_{\text{Ł}_3} \bowtie H_{\mathsf{K}}$. ⊣

Example 3.10. Recall Examples 3.6 and 3.1. Assume that $H^{\mathrm{MT}}_{\mathsf{CP}}$ is sound and complete. Then we show that H_{LFI1} and $H^{\mathrm{MT}}_{\mathsf{CP}}$ are compatible. It is enough to show that

$$\xi_1,\, \xi_1 \supset_{\mathsf{CP}} \xi_2 \vdash_{H^{\mathrm{MT}}_{\mathsf{CP}}} \xi_2.$$

Observe that $\xi_1,\, \xi_1 \supset_{\mathsf{CP}} \xi_2 \vDash_{\mathcal{M}_{\mathsf{CP}}} \xi_2$ (see Example 2.1). Therefore by completeness of $H^{\mathrm{MT}}_{\mathsf{CP}}$ we conclude $\xi_1, \xi_1 \supset_{\mathsf{CP}} \xi_2 \vdash_{H^{\mathrm{MT}}_{\mathsf{CP}}} \xi_2$. This allow us to use $(\{\xi_1,\, \xi_1 \supset_{\mathsf{CP}} \xi_2\}, \xi_2)$ as a rule that we call $\mathrm{MP}_{\mathsf{CP}}$. Moreover $\mathrm{MT}_{\mathsf{CP}}, \mathrm{MP}_{\mathsf{LFI1}}$ are in R_{LFI1CP}. Furthermore

$$\vdash_{\mathsf{LFI1\,CP}} (\langle \circ_{\mathsf{LFI1}} \mathtt{tt}_{\mathsf{CP}} \rangle\, \xi_1) \supset_{\mathsf{LFI1\,CP}} ((\neg_{\mathsf{LFI1\,CP}}\, \xi_1) \supset_{\mathsf{LFI1\,CP}} (\xi_1 \supset_{\mathsf{LFI1\,CP}} \xi_2)). \tag{‡}$$

Indeed the sequence

1. $(\circ_{\mathsf{LFI1}}\, \xi_1) \supset_{\mathsf{LFI1}} ((\neg_{\mathsf{LFI1}} \xi_1) \supset_{\mathsf{LFI1}} (\xi_1 \supset_{\mathsf{LFI1}} \xi_2))$ $\quad \mathrm{Ax}_{\mathsf{LFI1}}$
2. $(\neg_{\mathsf{CP}}\, \xi_1) \supset_{\mathsf{CP}} (\xi_1 \supset_{\mathsf{CP}} \xi_2)$ $\quad \mathrm{Thm}_{\mathsf{CP}}$
3. $((\neg_{\mathsf{CP}}\, \xi_1) \supset_{\mathsf{CP}} (\xi_1 \supset_{\mathsf{CP}} \xi_2))$
   $\supset_{\mathsf{CP}} (\mathtt{tt}_{\mathsf{CP}}(\xi_1) \supset_{\mathsf{CP}} ((\neg_{\mathsf{CP}}\, \xi_1) \supset_{\mathsf{CP}} (\xi_1 \supset_{\mathsf{CP}} \xi_2)))$ $\quad \mathrm{Ax}_{\mathsf{CP}}$
4. $\mathtt{tt}_{\mathsf{CP}}(\xi_1) \supset_{\mathsf{CP}} ((\neg_{\mathsf{CP}}\, \xi_1) \supset_{\mathsf{CP}} (\xi_1 \supset_{\mathsf{CP}} \xi_2))$ $\quad \mathrm{MP}_{\mathsf{CP}}$ 2,3
5. $(\langle \circ_{\mathsf{LFI1}} \mathtt{tt}_{\mathsf{CP}} \rangle\, \xi_1)$
   $\supset_{\mathsf{LFI1\,CP}} ((\neg_{\mathsf{LFI1\,CP}}\, \xi_1) \supset_{\mathsf{LFI1\,CP}} (\xi_1 \supset_{\mathsf{LFI1\,CP}} \xi_2))$ $\quad$ LFT 1,4

is a derivation for (‡). A similar reasoning can be used in $H_{\mathsf{LFI1}} \bowtie H_{\mathsf{J}}$ to show

$$\vdash_{\mathsf{LFI1\,J}} (\langle \circ_{\mathsf{LFI1}}\, \mathtt{tt}_{\mathsf{J}} \rangle \xi_1) \supset_{\mathsf{LFI1\,J}} ((\neg_{\mathsf{LFI1\,J}}\, \xi_1) \supset_{\mathsf{LFI1\,J}} (\xi_1 \supset_{\mathsf{LFI1\,J}} \xi_2)).$$

Hence gentle explosion is preserved by interconnection. $\dashv$

PROPOSITION 3.3. *Let H_1 and H_2 be compatible Hilbert calculi. Then, for every $\Gamma \cup \{\varphi\} \subseteq L_{12}(\Xi)$,*

$$\Gamma \vdash_{12} \varphi \;\; \textit{if and only if} \;\; \Gamma|^1 \vdash_1 \varphi|^1 \;\; \textit{and} \;\; \Gamma|^2 \vdash_2 \varphi|^2.$$

PROOF.
$(\rightarrow)$ Suppose that $\Gamma \vdash_{12} \varphi$. There are two possibilities.

(1) $\varphi \in \Gamma$. There are two subcases.

(a) φ is $\xi \in \Xi$. Hence $\xi \in \Gamma|^k$ for $k = 1, 2$.

(b) φ is $\langle c_1 c_2 \rangle (\varphi_1, \dots, \varphi_n)$. So $c_k(\varphi_1|^k, \dots, \varphi_n|^k) \in \Gamma|^k$ for $k = 1, 2$.

(2) φ is justified by LFT from $\Gamma|^1 \vdash_1 \varphi|^1$ and $\Gamma|^2 \vdash_2 \varphi|^2$. Thus the thesis follows. $(\leftarrow)$ Assume that $\Gamma|^1 \vdash_1 \varphi|^1$ and $\Gamma|^2 \vdash_2 \varphi|^2$. Let $\Psi = \{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ be a finite set such that $\Psi|^1 \vdash_1 \varphi|^1$ and $\Psi|^2 \vdash_2 \varphi|^2$ with derivations $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. Hence the sequence

$$
\begin{array}{ll}
\psi_1 & \text{HYP} \\
\ldots & \\
\psi_n & \text{HYP} \\
\mathcal{D}_1 & \\
\varphi|^1 & \\
\mathcal{D}_2 & \\
\varphi|^2 & \\
\varphi & \text{LFT}
\end{array}
$$

is a derivation of φ from Ψ and so also a derivation of φ from Γ. ⊣

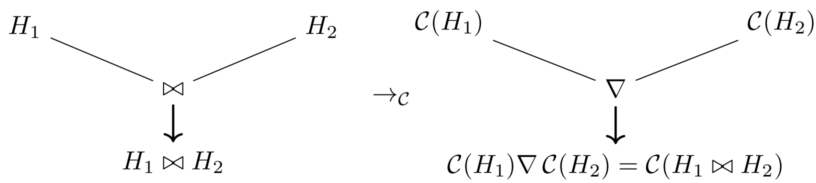

Figure 4. Consequence system of interconnection is the meet-combination

Proposition 3.4. *The consequence system $\mathcal{C}(H_1 \bowtie H_2)$ is the meet-combination $\mathcal{C}(H_1)\nabla\,\mathcal{C}(H_2)$. Furthermore, $\mathcal{C}(H_1 \bowtie H_2)$ is suitable whenever $\mathcal{C}(H_k)$ is suitable for $k = 1, 2$.*

Proof. The first assertion is a consequence of Proposition 3.3 taking into account Proposition 1.1. The proof of the second assertion is depicted in Figure 5. ⊣

So meet-combination of consequence systems generated by Hilbert calculi contains the common consequences of the argument consequence systems as stated in Proposition 3.3.

We now provide illustrations where we can conclude that certain consequences are not present in the interconnection of sound Hilbert calculi.

Example 3.11. Recall Example 3.7. We show that

$$\not\vdash_{\mathsf{JK}} (\neg_{\mathsf{JK}}(\xi_1 \wedge_{\mathsf{JK}} \xi_2)) \supset_{\mathsf{JK}} ((\neg_{\mathsf{JK}}\,\xi_1) \vee_{\mathsf{JK}} (\neg_{\mathsf{JK}}\,\xi_2)).$$

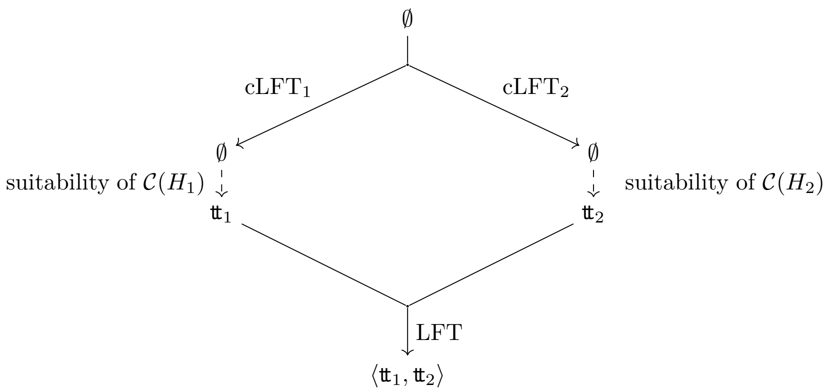

Figure 5. Suitability of $\mathcal{C}(H_1 \bowtie H_2)$

Indeed $\nvdash_{\mathsf{J}} (\neg_{\mathsf{J}}(\xi_1 \wedge_{\mathsf{J}} \xi_2)) \supset_{\mathsf{J}} ((\neg_{\mathsf{J}} \xi_1) \vee_{\mathsf{J}} (\neg_{\mathsf{J}} \xi_2))$ because this formula is not valid and H_{J} is sound for $\mathcal{M}_{\mathsf{J}}$. Hence the thesis follows by Proposition 3.3. Observe also that it is not the case

$$\xi_1,\, \xi_1 \langle \supset_{\mathsf{J}} \mathsf{tt}_{\mathsf{K}} \rangle \xi_2 \vdash_{\mathsf{JK}} \xi_2.$$

Indeed it is the case that $\xi_1,\, \mathsf{tt}_{\mathsf{K}}(\xi_1, \xi_2) \nvDash_{\mathsf{K}} \xi_2$ and since H_{K} is sound for $\mathcal{M}_{\mathsf{K}}$ then $\xi_1,\, \mathsf{tt}_{\mathsf{K}}(\xi_1, \xi_2) \nvdash_{\mathsf{K}} \xi_2$. The thesis follows by Proposition 3.3. Finally consider Example 3.9. Then $\nvdash_{\mathsf{Ł}_3} \xi \vee_{\mathsf{Ł}_3} (\neg_{\mathsf{Ł}_3} \xi)$ because $H_{\mathsf{Ł}_3}$ is sound for $\mathcal{M}_{\mathsf{Ł}_3}$ (see Bolc and Borowik, 1992) and $\nvDash_{\mathsf{Ł}_3} \xi \vee_{\mathsf{Ł}_3} (\neg_{\mathsf{Ł}_3} \xi)$ (see Example 2.8). So by Proposition 3.3, $\nvdash_{\mathsf{Ł}_3\mathsf{K}} \xi \vee_{\mathsf{Ł}_3\mathsf{K}} (\neg_{\mathsf{Ł}_3\mathsf{K}} \xi)$. $\dashv$

For showing Proposition 3.5 we need the following concept. The formula tt^{φ} is inductively defined on φ as follows: tt^{ξ} is tt^0 and $\mathsf{tt}^{c(\varphi_1,\dots,\varphi_n)}$ is $\mathsf{tt}^n(\mathsf{tt}^{\varphi_1}, \dots, \mathsf{tt}^{\varphi_n})$.

PROPOSITION 3.5. *Let $\langle c_1 c_2 \rangle(\varphi_1, \dots, \varphi_n) \in L_{12}(\Xi)$ and $\Gamma \subseteq L_{12}(\Xi)$. Then $\Gamma \vdash_{12} \langle c_1 c_2 \rangle(\varphi_1, \dots, \varphi_n)$ if and only if $\Gamma \vdash_{12} \eta_k(c_k(\varphi_1|^k, \dots, \varphi_n|^k))$ for $k = 1, 2$.*

PROOF. Suppose without loss of generality that Γ is a finite set.

($\rightarrow$) Assume that $\Gamma \vdash_{12} \langle c_1 c_2 \rangle(\varphi_1, \dots, \varphi_n)$. There are two cases to consider.

(1) $\langle c_1 c_2 \rangle(\varphi_1, \dots, \varphi_n) \in \Gamma$. Thus $c_1(\varphi_1|^1, \dots, \varphi_n|^1) \in \Gamma|^1$. Then the sequence

$$\begin{array}{ll}
1.\ \langle c_1c_2\rangle(\varphi_1,\dots,\varphi_n) & \text{HYP}\\
2.\ c_1(\varphi_1|^1,\dots,\varphi_n|^1) & \text{cLFT}_1\ 1\\
3.\ \mathtt{tt}_2^n(\mathtt{tt}_2^{\varphi_1},\dots,\mathtt{tt}_2^{\varphi_n}) & \text{Ax}_2\\
4.\ \eta_1(c_1(\varphi_1|^1,\dots,\varphi_n|^1)) & \text{LFT } 2{,}3
\end{array}$$

is a derivation for $\eta_1(c_1(\varphi_1|^1,\dots,\varphi_n|^1))$ from Γ. Similarly we can prove the result for $k=2$.

(2) Otherwise, $\langle c_1c_2\rangle(\varphi_1,\dots,\varphi_n)$ is obtained from $c_k(\varphi_1|^k,\dots,\varphi_n|^k)$ by LFT for $k=1,2$. Therefore $\Gamma|^1\vdash_1 c_1(\varphi_1|^1,\dots,\varphi_n|^1)$ with a derivation $\mathcal{D}_1$. Then the sequence

$$\begin{array}{ll}
\Gamma & \text{HYP}\\
\mathcal{D}_1 & \\
c_1(\varphi_1|^1,\dots,\varphi_n|^1) & \\
\mathtt{tt}_2^n(\mathtt{tt}_2^{\varphi_1},\dots,\mathtt{tt}_2^{\varphi_n}) & \text{Ax}_2\\
\eta_1(c_1(\varphi_1|^1,\dots,\varphi_n|^1)) & \text{LFT } 3{,}4
\end{array}$$

is a derivation for $\eta_1(c_1(\varphi_1|^1,\dots,\varphi_n|^1))$ from Γ. Similarly we can prove the result for $k=2$.

($\leftarrow$) Suppose that $\Gamma\vdash_{12}\eta_k(c_k(\varphi_1|^k,\dots,\varphi_n|^k))$ for $k=1,2$. So formula $\eta_1(c_1(\varphi_1|^1,\dots,\varphi_n|^1))$ follows by rule LFT from $c_1(\varphi_1|^1,\dots,\varphi_n|^1)$ and $\mathtt{tt}_2(\mathtt{tt}_2^{\varphi_1},\dots,\mathtt{tt}_2^{\varphi_n})$. So $\Gamma|^1\vdash_1 c_1(\varphi_1|^1,\dots,\varphi_n|^1)$. In the same way we can conclude that $\Gamma|^2\vdash_2 c_2(\varphi_1|^2,\dots,\varphi_n|^2)$. Hence the thesis follows. $\dashv$

We now analyze preservation of two metatheorems by interconnection of Hilbert calculi. We say that a Hilbert calculus H has the *metatheorem of deduction* (MTD) whenever $\supset\in C_2$ and for every $\Gamma\cup\{\psi,\varphi\}\subseteq L(\Xi)$

$$\text{if } \Gamma,\psi\vdash_H\varphi \text{ then } \Gamma\vdash_H\psi\supset\varphi.$$

Proposition 3.6. *If H_k has the metatheorem of deduction for $k=1,2$ then $H_1\bowtie H_2$ also has the metatheorem of deduction.*

Proof. Assume that $\Gamma,\psi\vdash_{12}\varphi$ where $\Gamma\cup\{\psi,\varphi\}\subseteq L_{12}(\Xi)$. Therefore by Proposition 3.3, $\Gamma|^1,\psi|^1\vdash_1\varphi|^1$ and $\Gamma|^2,\psi|^2\vdash_2\varphi|^2$. Hence, by MTD in H_1 and H_2, $\Gamma|^1\vdash_1\psi|^1\supset_1\varphi|^1$ and $\Gamma|^2\vdash_2\psi|^2\supset_2\varphi|^2$. Thus, $\Gamma\vdash_{12}\psi\supset_{12}\varphi$, by Proposition 3.3. $\dashv$

Example 3.12. Recall Example 3.10. The Hilbert calculi H_{LFI1} and H_{CP} have the MTD. Hence, by Proposition 3.6, $H_{\mathsf{LFI1}}\bowtie H_{\mathsf{CP}}$ has the MTD. $\dashv$

We say that a Hilbert calculus H has the *metatheorem of proof by cases* whenever there is a constructor $\neg \in C_1$ and

$$\text{if } \Gamma, \psi \vdash_H \varphi \text{ and } \Gamma, \neg\, \psi \vdash_H \varphi \text{ then } \Gamma \vdash_H \varphi.$$

Proposition 3.7. *Let H be a Hilbert calculus such that there are constructors $\neg \in C_1$ and $\supset, \vee \in C_2$ such that*

(1) *H has MTD and MP*
(2) $\xi_1 \supset \xi, \xi_2 \supset \xi \vdash_H (\xi_1 \vee \xi_2) \supset \xi$
(3) $\vdash_H \xi \vee (\neg\, \xi)$.

Then H has the metatheorem of proof by cases.

Proof. Assume that $\Gamma, \psi \vdash_H \varphi$ and $\Gamma, \neg\, \psi \vdash_H \varphi$. Hence by (1), $\Gamma \vdash_H \psi \supset \varphi$ and $\Gamma \vdash_H (\neg\, \psi) \supset \varphi$. Thus, by (2), $\Gamma \vdash_H (\psi \vee (\neg\, \psi)) \supset \varphi$. On the other hand by (3), $\vdash_H \psi \vee (\neg\, \psi)$ and so by (1) the thesis follows. ⊣

Example 3.13. Recall Examples 2.5 and 3.6. We show that H_{LFI1} has the metatheorem of proof by cases using Proposition 3.7. Observe that H_{LFI1} has MP and MTD. Note that condition (2) of Proposition 3.7 follows from axiom

$$(\xi_1 \supset_{\mathsf{LFI1}} \xi) \supset_{\mathsf{LFI1}} ((\xi_2 \supset_{\mathsf{LFI1}} \xi) \supset_{\mathsf{LFI1}} ((\xi_1 \vee_{\mathsf{LFI1}} \xi_2) \supset_{\mathsf{LFI1}} \xi))$$

of $\mathsf{LFI1}$ using $\mathrm{MP}_{\mathsf{LFI1}}$ twice. Finally (3) holds by completeness of H_{LFI1} (see Carnielli and Coniglio, 2016) because $\xi \vee_{\mathsf{LFI1}} (\neg_{\mathsf{LFI1}} \xi)$ is valid. ⊣

Proposition 3.8. *If H_k is a Hilbert calculus with the metatheorem of proof by cases for $k = 1, 2$ then $H_1 \bowtie H_2$ has the metatheorem of proof by cases.*

Proof. We must show that

$$\text{if } \Gamma, \psi \vdash_{12} \varphi \text{ and } \Gamma, \neg_{12}\, \psi \vdash_{12} \varphi \text{ then } \Gamma \vdash_{12} \varphi.$$

Suppose that $\Gamma, \psi \vdash_{12} \varphi$ and $\Gamma, \neg_{12}\, \psi \vdash_{12} \varphi$. Then by Proposition 3.3, $\Gamma|^k, \psi|^k \vdash_k \varphi|^k$ and $\Gamma|^k, \neg_k \psi|^k \vdash_k \varphi|^k$ for $k = 1, 2$. Therefore, $\Gamma|^k \vdash_k \varphi|^k$ for $k = 1, 2$ by hypothesis. So by Proposition 3.3, $\Gamma \vdash_{12} \varphi$. ⊣

Example 3.14. Consider Examples 3.6 and 3.1. Observe that H_{LFI1} has the metaheorem of proof by cases (see Example 3.13). It is very easy to show that $H^{\mathrm{MT}}_{\mathsf{CP}}$ also has the metaheorem of proof by cases. Hence $H_{\mathsf{LFI1}} \bowtie H^{\mathrm{MT}}_{\mathsf{CP}}$ has the metatheorem of proof by cases by Proposition 3.8. ⊣

4. Interconnection is product

We now concentrate on preservation by interconnection of soundness and completeness of compatible Hilbert calculi with respect to matrix semantics.

Proposition 4.1. *The Hilbert calculus $H_1 \bowtie H_2$ is sound for $\mathcal{M}_1 \times \mathcal{M}_2$ whenever H_k is sound for $\mathcal{M}_k$ with $k = 1, 2$.*

Proof. We show that if $\Gamma \vdash_{12} \varphi$ then $\Gamma \vDash_{12} \varphi$ for $\Gamma \cup \{\varphi\} \subseteq L_{12}(\Xi)$. Suppose that $\Gamma \vdash_{12} \varphi$. Let $M_1 \times M_2 \in \mathcal{M}_1 \times \mathcal{M}_2$ and ρ an assignment over $M_1 \times M_2$ such that $M_1 \times M_2, \rho \Vdash_{12} \Gamma$. Consider the derivation

1.	Γ	HYP
2.	$\Gamma\|^1$	cLFT$_1$ 1
3.	$\Gamma\|^2$	cLFT$_2$ 1
4.	$\varphi\|^1$	$\Gamma\|^1 \vdash_1 \varphi\|^1$
5.	$\varphi\|^2$	$\Gamma\|^2 \vdash_2 \varphi\|^2$
6.	φ	LFT 4,5

for $\Gamma \vdash_{12} \varphi$ of φ from Γ in $H_1 \bowtie H_2$ where we assume without loss of generality that Γ is finite. Thus, $M_1, \rho_1 \Vdash_1 \Gamma|^1$ and $M_2, \rho_2 \Vdash_2 \Gamma|^2$ by Lemma 2.1. Hence $M_k, \rho_k \Vdash_k \varphi|^k$ since $\Gamma|^k \vdash_k \varphi|^k$ for $k = 1, 2$ and so, by soundness of H_k, $\Gamma|^k \vDash_k \varphi|^k$ for $k = 1, 2$. So, by Lemma 2.1, $M_1 \times M_2, \rho \Vdash_{12} \varphi$. $\dashv$

Proposition 4.2. *The Hilbert calculus $H_1 \bowtie H_2$ is complete for $\mathcal{M}_1 \times \mathcal{M}_2$ whenever H_k is complete for $\mathcal{M}_k$ with $k = 1, 2$.*

Proof. We show that if $\Gamma \vDash_{12} \varphi$ then $\Gamma \vdash_{12} \varphi$ for $\Gamma \cup \{\varphi\} \subseteq L_{12}(\Xi)$. Assume that $\Gamma \vDash_{12} \varphi$. Then, by Proposition 2.4, $\Gamma|^k \vDash_k \varphi|^k$ for $k = 1, 2$. Hence, $\Gamma|^1 \vdash_1 \varphi|^1$ and $\Gamma|^2 \vdash_2 \varphi|^2$ by completeness of H_1 and H_2 with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. So, by Proposition 3.3, $\Gamma \vdash_{12} \varphi$. $\dashv$

Proposition 4.3. *Given Hilbert calculi H_1 and H_2 sound and complete with respect to matrix semantics $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, we have*

$$\mathcal{C}(H_1 \bowtie H_2) = \mathcal{C}(\mathcal{M}_1 \times \mathcal{M}_2).$$

5. Concluding remarks

We revisited meet-combination of logics with the objective of capturing the consequences that hold in both argument logics. For that we introduced meet-combination of consequence systems. This allows us to abstract away from the particular way of presenting logics. We established results regarding preservation by meet-combination of paraconsistency, formal inconsistency, finitariness and structurality.

We investigated consequence systems generated by matrix semantics and showed that their meet-combination is the consequence system generated by the product of the given semantics. We also analyzed consequence systems generated by Hilbert calculi and established that their meet-combination is the consequence system generated by the interconnection of the given calculi. We obtained preservation of finite model property as well as preservation of some metatheorems such as metatheorems of deduction and proof by cases. We also proved preservation of soundness and completeness and showed that, in this case, the consequence system generated by the product of matrix semantics is the consequence system generated by the interconnection of Hilbert calculi.

In this paper we assumed that the matrices were deterministic, that is, the denotation for any constructor assigns to each tuple of truth values a unique truth value. Hence, we cannot cope with the paraconsistent logic mbC (see Carnielli and Coniglio, 2016) because the semantics of $\neg_{\mathsf{mbC}}$ is as follows: if $V(\varphi) = 0$ then $V(\neg_{\mathsf{mbC}}\, \varphi) = 1$ and nothing is said when $V(\varphi) = 1$. That is, when $V(\varphi) = 1$ then either $V(\neg_{\mathsf{mbC}}\, \varphi) = 1$ or $V(\neg_{\mathsf{mbC}}\, \varphi) = 0$. We intend to define product of nondeterministic matrices. The concept was introduced in (Rescher, 1962) under the name quasi-truth functional constructors and later on developed in (Avron and Lev, 2005; Avron and Zohar, 2019; Filipe et al., 2022). It seems worthwhile to investigate preservation of other meta properties by meet-combination, namely decidability, algebraizability, protoalgebraicity and amalgamation (see Blok and Pigozzi, 1989; Czelakowski, 1982, 2001).

We would like to investigate the categorical characterization of meet-combination of consequence systems extending the work in (Voutsadakis, 2013) for meet-combination of logics as in (Sernadas et al., 2012).

As we said before meet-combination captures the common consequences of the argument logics. This property is reminescent of conservative translations (maps between two logics that preserve properties, see

(Feitosa and D'Ottaviano, 2001)). It seems natural to generalize the notion of conservative translation in order to cope with meet-combination.

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