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A Misleading Triviality Argument
in the Theory of Conditionals

Abstract. PCCP is the much discussed claim that the probability of a conditional $A \to B$ is conditional probability. Triviality results purport to show that PCCP — as a general claim — is false. A particularly interesting proof has been presented in (Hájek, 2011), who shows that — even if a probability distribution P initially satisfied PCCP — a rational update can produce a non-PCCP probability distribution.

We argue that the notion of rational update in this argumentation is construed in much too broad a way. In order to make the argumentation precise, we discuss the general rules for modeling conditionals in probability spaces and present formalized version(s) of PCCP and of minimal assumptions concerning the appropriate spaces. Using the introduced apparatus we give a detailed analysis of Hájek's (2011) triviality proof and show that it is based on an application of revision rules which allow one to construct probability distributions violating not only PCCP, but also fundamental properties of conditionals. This means that they do not really provide arguments against PCCP, properly formalized.

We also discuss a Dutch Book argument which shows that the updated belief system is not coherent. This gives an additional, strong argument against accepting the update rules. We also discuss the Converse Dutch Book theorem and argue, that even if the produced probability measure seems to violate it, it cannot serve as the counterexample, as it is not an appropriate model for conditionals. Ultimately, we show that important arguments against PCCP fail.

Keywords: probability of conditionals; PCCP; Dutch Book; triviality results; Hájek's update rules

1. Introduction

PCCP is the much debated claim that the probability of a conditional $A \to B$ is conditional probability. There is an intense discussion on the plausibility of its diverse variants.[1] It is intuitive and easy to formulate — as Hájek puts it, PCCP "rolls easily off the tongue" (Hájek, 2011, p. 7).[2] Numerous examples suggest its truth, and Ramsey's test is an important argument in favor of it.[3] PCCP is often called "Adams' thesis" or "Stalnaker's thesis": both are authors of versions of PCCP. (Stalnaker later withdrew his support for PCCP, but we shall not discuss historical details here.)[4]

However, there are also important arguments against PCCP based on triviality results which purport to show that PCCP is false. The seminal papers are (Lewis, 1976, 1986), generalizations are presented, for instance, in (Hall, 1994; Hájek and Hall, 1994; Hájek, 2011) (see also Milne, 2003; Fitelson, 2004), and an illuminating discussion can be found in (Hájek, 2011, 2012).

We consider Hájek's triviality proof given in (2011) to be particularly interesting in this context: it is simple, intuitive, general, and its

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[1] See, for instance, (Hájek, 2011, 2012) or (Khoo and Santorio, 2018) for a presentation and discussion. Stalnaker (2009) gives a general presentation. For a more general discussion, see, for instance, (van Fraassen, 1976; Bennett, 2003; Edgington, 1995; Rehder, 1982; Stalnaker, 2019), and many others. (Edgington, 2020) contains a thorough discussion on indicative conditionals with an extensive bibliography. PCCP is valid in McGee's model (1989) and in Bernoulli-Stalnaker spaces, see (van Fraassen, 1976; Kaufmann, 2004, 2005, 2009, 2015, 2023).

[2] "What is the probability that I throw a six if I throw an even number, if not the probability that if I throw an even number, it will be a six?" (van Fraassen, 1976, p. 273). Examples are abundant: consider a standard pack of cards and conditionals: *If it is a spade, it is an ace*; *If it is red, it is diamonds*; *If it is an ace, it is a spade*. What are the probabilities of these sentences? The obvious intuitive answers are: $\frac{1}{13}$, $\frac{1}{2}$, $\frac{1}{4}$. What else could they be?

[3] "If two people are arguing 'If p will q?' and both are in doubt as to p, they are adding p hypothetically to their stock of knowledge and arguing on that basis about q.... We can say that they are fixing their degrees of belief in q given p" (Ramsey, 1990, p. 247).

[4] Adams' important papers are (1965; 1970; 1975; 1998); in (Adams, 1965) the term "assertability" is used. Stalnaker's seminal paper is (1968). However, there are important differences between them: Stalnaker speaks of conditional degrees of belief, while Adams' original formulation refers to assertability. Stalnaker's thesis applies to compound conditionals, while Adams' definition only applies to simple conditionals, i.e., $A \to B$, with A, B being factual sentences not containing a conditional connective.

assumptions are listed in a clear way. Hájek's proof is intended to show then even if PCCP is initially accepted, after a rational update U, the agent might arrive at a new probability distribution P^U, which violates PCCP. This means that accepting PCCP is – more or less – the matter of coincidence. As soon as new, relevant information is available, PCCP might be abandoned. According to Hájek, his argument shows

> [...] how precarious PCCP is: while it may hold for a single probability function (for all that the theorem tells us), it is easily torn asunder. If you are a Bayesian agent who seeks to conform to PCCP at all times, you are apparently unable to revise boldly and moderately your opinions regarding certain propositions.

The consequence is as follows:

> There is no binary operator #, conditional-like or otherwise, such that the equation $P(A\#B) = P(B|A)$ survives all revisions by a given bold and moderate revision rule. (Hájek, 2011, p. 13)

Hájek's proof is claimed to show an inherent weakness and instability in PCCP. It also inspires a more general discussion concerning the status of revision rules in the context of conditionals. Therefore, we focus on Hájek's proof in the present paper. We claim that Hájek's construction does not really provide an argument against PCCP, properly formulated. Triviality results — in general – show that in every class of probability functions fulfilling certain closure conditions there must be a function violating PCCP. This is true — but this does not refute PCCP (at least — not in its reasonable, non-extreme version) because these probability distributions serving as counterexamples are not reasonable formalizations of our system of beliefs. So the complaint that they do not satisfy PCCP is void of argumentative power.

Hájek analyzed bold and moderate update rules (conditionalization is an example, the general definition is reminded in Section 3). Such rules seem perfectly suited to update factual knowledge. However, our analysis shows that when beliefs involving conditionals are updated, it is not enough to assume that the update rule is bold and moderate. The very special and intricate character of conditionals and conditional knowledge makes the boldness and moderateness condition insufficient. Even conditionalization must be handled with care in such situations.

In the present paper we focus on two issues:

1. We provide an analysis in terms of a Dutch Book and show, that the agent who follows Hájek's update can be Dutch Booked.

2. We give an analysis in terms of formal properties of probability spaces modeling conditionals, and show, that even a very weak and obvious meaning postulate concerning conditionals is violated in the course of Hájek's update. So, even if we neglect arguments involving Dutch Books (some authors are skeptical about their force), there are fundamental reasons which make the update method inacceptable.

The structure of the paper is as follows:

In Section 2, *Modeling factual and conditional beliefs in probability spaces*, we discuss the assumptions which must be met when we are modeling beliefs about probabilities of conditionals $A \to B$. We also formulate some reasonable expectations concerning spaces that are appropriate for interpreting conditionals and present the formalized version(s) of PCCP.

In Section 3, *Hájek's update method* we present a slightly simplified and modified proof of how the update works.

In Section 4, *Hájek's update and Dutch Books* we first discuss an attempt to directly justify PCCP by a Dutch Book argumentation. Afterwards we show that the agent who uses Hájek's update method can be Dutch Booked.

In Section 5, *Hájek's update produces a space violating* (IMP^*) we present another argument showing that Hájek's update method leads to bizarre results.

In Section 6, *The Fall of Converse Dutch Book theorem?* we discuss the Converse Dutch Book Theorem. Our result shows that there is a probability distribution P^{*U}, such that the agent using P^{*U} can be Dutch Booked. This might suggest that the Converse Dutch Book theorem is false. However, we argue that it is not the case – it indicates a problem with the update U which produces P^{*U}.

Section 7 is a short *Conclusion*.

2. Modeling factual and conditional beliefs in probability spaces

We start with a set of factual beliefs expressed in the factual language L_{FACT}. It is sufficient to assume that it is a fragment of the standard propositional calculus (for simplicity we can take the language to be finite, but this is not crucial). The underlying logic is classical. We assume that factual beliefs can be modeled in a standard probability space $S = (\Omega, \Sigma, P)$, where Ω is the set of elementary events, Σ is the

algebra (σ-field) of events, and P is the probability distribution. This means in particular that sentences from L_{FACT} are interpreted in S as events. For any sentence $A \in L_{FACT}$, we shall use the symbol $[A]$ for its interpretation in $S = (\Omega, \Sigma, P)$, i.e., for the corresponding event $[A] \subseteq \Omega$.[5]

Let L_{COND} be a language containing (apart from factual sentences) also (some) simple conditionals of the form $A \to B$, for $A, B \in L_{FACT}$, i.e., A, B being factual. We assume that L_{COND} is closed under Boolean connectives, i.e., if $(A \to B) \in L_{COND}$, utterances like $\neg(A \to B)$ or $C \wedge (A \to B)$ can be made.[6]

2.1. The probability space S^*

We assume that systems of beliefs — both factual and conditional — of the agents are modeled in probability spaces.[7] In particular, the factual system of beliefs is modeled in a probability space $S = (\Omega, \Sigma, P)$. If the agent also holds conditional beliefs, the space S might be not appropriate to model them — and a new probability space $S^* = (\Omega^*, \Sigma^*, P^*)$ is needed. Its form depends on the language L_{COND} — i.e., on which conditionals we need to model. Obviously, the original space S and the "conditional space" S^* must be connected in a reasonable way.

Throughout the text we will use a simple toy example as an illustration. This allows us to exhibit the relevant phenomena. Consider an agent who has beliefs concerning a die. The sample space $S = (\Omega, \Sigma, P)$ is very simple: $\Omega = \{1, 2, 3, 4, 5, 6\}$ and the σ-field Σ consists of 64 events (all subsets of Ω). The probability distribution P reflect the properties of the die (for instance whether it is fair). The factual language L_{FACT} contains claims like: $A = $ *The number is Even*; $B = $ *It is a Six*; $C = $ *It is Prime*, etc. These sentences have obvious interpretations in S as events:

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[5] Formally, we demand that $[A]$ is the element of the σ-field Σ.

[6] In this paper we only consider simple conditionals $A \to B$, so we do not discuss the potential hierarchy of languages comprising nested conditionals and more complex constructions. For the purpose of argumentation, it is enough to assume that it contains only one such conditional. So, we do not need even to assume that L_{COND} contains **all** conditionals $A \to B$, for $A, B \in L_{FACT}$.

[7] In our opinion formalizing the discussion using the tools of standard probability theory allows to elucidate several issues. However, many of the arguments and analyses given in the present text have their counterparts if we only use the notion of credence, i.e., an assignment $Cr \colon L_{COND} \to [0, 1]$, which has the needed formal features of probability theory (but no probability space is postulated). We prefer to follow the "probability-space paradigm".

$[A] = \{2, 4, 6\}$, $[B] = \{6\}$, $[C] = \{2, 3, 5\}$. Of course, the Boolean structure of the language is preserved in the Boolean structure of the events, which means that $[\neg A] = \Omega \backslash [A]$ (complement), $[A \wedge B] = [A] \cap [B]$, etc.

We also want to express conditional claims like *If it is Even, it is a Six* (*Even → Six*) and evaluate their probabilities. Obviously, the judgments should be based on our factual beliefs — for instance, whether the die is fair. It is also clear that if our factual beliefs change (we learn something new about the die), at least some of our beliefs concerning the probability of conditionals will change. For instance, if we learn that the numbers on the die are 1,2,3,4,5,5 (the 6 has been replaced by a 5), our judgment concerning *Even → Six* will change rather dramatically. Indeed, it would be very strange to assign a positive probability to *Even→ Six* in a situation when a 6 cannot occur at all.

In general, the conditional $A \rightarrow B$ might have **no** interpretation in sample space S. Consider *Even → Six*. Which of the 64 subsets of $\Omega = \{1, 2, 3, 4, 5, 6\}$ might be its interpretation? It is not $\{2, 4, 6\}$; it is not $\{6\}$ either — and obviously none of the 64 subsets of Ω are appropriate.[8] The sample space $S = (\Omega, \Sigma, P)$ is designed to model **factual** beliefs, and it rarely even has the opportunity to feature an event that is an interpretation of conditional $A \rightarrow B$. This means that we need another probability space $S^* = (\Omega^*, \Sigma^*, P^*)$, in which the conditional $\alpha = A \rightarrow B$ has an interpretation as an event.[9]

It is important to be precise about the notation:

1. A, B, C, ... (with italics) refer to factual sentences.
2. $[A]$, $[B]$, $[C]$ are events representing A, B, C in sample space S.

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[8] Consider the fair die, and the conditional *If it is not a Six, it is a Five*. Its intuitive probability is $\frac{1}{5}$ — but the space S contains no event of this probability. Another example is given by a sample space with three elementary events $\Omega = A, B, C$ with probabilities $P(A) = P(B) = P(C) = \frac{1}{3}$. If we agree that the probability of the conditional $(A \vee B) \rightarrow A$ is $\frac{1}{2}$, then it becomes clear that we need another probability space, as there is no event with a probability of $\frac{1}{2}$ within Ω (this example is presented in (Hájek, 2012)). Hájek (1989) shows that *any* non-trivial finite-ranged probability function has more distinct conditional probability values than distinct unconditional probability values. This means that if PCCP holds, the original probability space is not the right one. But even without PCCP it is clear, that Ω is not well suited to model conditionals.

[9] We use the symbol S^* to indicate the general character of the space: the particular class of sentences which are interpreted within S^* is not relevant. For the sake of our analysis, we only need to assume that $\alpha = A \rightarrow B$ has an interpretation in the probability space S^* as an event $[A \rightarrow B]^*$.

3. The stars indicate that the given entity is associated with space S*. This means that $[A]^*$, $[B]^*$, $[C]^*$, $[A \to B]^*$ are events in S* representing the corresponding sentences.

In order to construct the probability space S* we have to determine the set of elementary events Ω^*. Every elementary event $\omega \in \Omega^*$ has to decide whether it supports the given $\alpha \in L_{COND}$ or not. In other words, when specifying the suitable Ω^*, we have to define the relation "$\omega \vDash \alpha$". Only then we will be able to identify the semantic correlate of α as the set $[\alpha]^* = \{\omega \in \Omega^* : \omega \vDash \alpha\}$. This means that if we want to model the conditional $A \to B$ as an event in a probability space, we need to accept the notion of circumstances in which the conditional $A \to B$ is true and in which it is false. Given this, we can say that the probability P* of a conditional is the probability of its truth in the space S*.

In general, $[A]$ and $[A]^*$ might be very different objects — the first being the interpretation of the factual sentence A in sample space S, the second being the interpretation of the very same sentence A in S*. Nevertheless, we expect that in both spaces S and S*, the probability assignments made on factual sentences coincide and that the structure of factual knowledge is preserved. Indeed, the agent who wants to model conditional claims does it on the base of factual knowledge, which is not forgotten. This is assured by the postulate according to which there is a homomorphic imbedding of S into S*. Intuitively, an imbedding might be imagined as presenting a copy of the space S $= (\Omega, \Sigma, P)$ within S* $= (\Omega^*, \Sigma^*, P^*)$, which preserves the essential features of S and assures that factual knowledge is properly modeled in S*.

Formally, a homomorphic imbedding of S $= (\Omega, \Sigma, P)$ into S* $= (\Omega^*, \Sigma^*, P^*)$ is a function $\iota : \Sigma \to \Sigma^*$ satisfying the following conditions:

a. $\iota(\Omega) = \Omega^*$;
b. $\iota(X \cap Y) = \iota(X) \cap \iota(Y)$, for $X, Y \subseteq \Omega$;
c. $\iota(X^c) = (\iota(X))^c$, for $X \subseteq \Omega$;
d. $P^*(\iota(X)) = P(X)$, for $X \subseteq \Omega$.

Conditions (a)–(d) apply to any two probability spaces S and S*. Here we are concerned with two spaces modeling L_{FACT} and L_{COND} so we add one more important condition:

(e) $\iota([A]) = [A]^*$, for $A \in L_{FACT}$.

The last condition states, that the imbedding ι is faithful to the interpretations of the factual sentences. We interpret the factual sentence A

as $[A]$ within S, and then "transfer" this interpretation into S^*, using the imbedding ι. We can also give a direct interpretation of A within S^* which is the event $[A]^*$. Condition (e) states that these two methods coincide. This means that the space $S^* = (\Omega^*, \Sigma^*, P^*)$ preserves the structure of interpretation of L_{FACT} within S. Of course, the probabilities of factual sentences $A \in L_{FACT}$ given in S, are preserved in S^*, i.e. $P^*([A]^*) = P([A])$ (by (d) and (e)).

2.2. A minimal meaning postulate (IMP)

So far we have made no assumptions whatsoever concerning the interpretation of the conditional connective $\to$. The class of meaning postulates we accept depends on our interpretation of the conditional. They express our intuitions concerning $\to$ and are later formalized as logical or probabilistic principles.

We want to make our assumptions as weak and uncontroversial as possible.[10] The following principle seems rather undisputable:

(IMP) If A is possible, and B is impossible, then $A \to B$ is impossible.

It motivates the principle according to which it is not rational to hold probabilistic beliefs such that:

(i) $P(A) > 0$;
(ii) $P(B) = 0$ and;
(iii) $P(A \to B) > 0$.

Indeed, the agent, who thinks that A is be possible, B is **impossible**, however is willing to assign positive probability to $A \to B$ is not rational. Consider someone who observes a fake die, where the 6 **never** comes up (the agent knows this!) and assigns positive probability to the conditional *If it is Even, than it is an Six* — knowing perfectly well that the probability of *It is a Six* is 0. Such a system of beliefs would be quite bizzare.[11]

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[10] We believe that there are many reasonable and well-motivated meaning postulates concerning the conditional connective $\to$. However, here we only want to discuss the "update phenomena" and we show that even a very weak meaning postulate shows that the discussed update method is problematic.

[11] Similarly, we would hardly assign positive probability to the claim *If John comes to the party, his wife will come too*, knowing that John is quite likely to come, but his wife never goes to parties (or that John has no wife).

The principle (IMP) might be considered to be a minimal postulate concerning the coherence between the factual beliefs and the beliefs concerning conditionals. It is important to formalize it, as it allows to make the analysis precise.

We give its probabilistic version, worded in terms of a relationship between P and P*:

(IMP(P;P*)) If P($[A]$) > 0 and P($[B]$) = 0, then P*($[A \to B]^*$) = 0.

This assumption stresses the balance between the probabilities of the factual sentence A, B and the conditional $A \to B$. Observe that IMP(P;P*) excludes only the extreme case, but imposes no restrictions on P*($[A \to B]^*$) when P($[A]$) > 0 and P($[B]$) > 0. It is really very weak and in no sense it assumes PCCP.

As we assumed that there is an imbedding of S into S*, IMP(P;P*) is equivalent to a version mentioning only the space S*:

(IMP*(P*)) If P*($[A]^*$) > 0 and P*($[B]^*$) = 0, then P*($[A \to B]^*$) = 0.

The version IMP*(P*) might be considered to be an internal version of the rationality/coherence postulate concerning any probability space S* = $(\Omega^*, \Sigma^*, P^*)$ which is intended to model conditional knowledge. It is a criterion for whether a probability space S* has any chance of being a reasonable model for a coherent, rational system of beliefs — which include also conditional propositions. If S* does not satisfy IMP*(P*), then it is not a reasonable candidate at all.[12]

We will use the term "L$_{COND}$-space" for any probability space S* extending S (via imbedding) and satisfying IMP(P;P*). For any factual probability space S there might be many admissible L$_{COND}$-spaces S*, the conditions do not specify one single "canonical" construction of S* from S. For convenience, let L$_{COND}$-SPACES(S) denote the class of L$_{COND}$-spaces S* suitable for extending S. So S* ∈ L$_{COND}$-SPACES(S) iff: (i) S is imbeddable in S*; and (ii) S and S* satisfy (IMP*).

2.3. The formalized version(s) of PCCP

PCCP is the general claim that probability of conditionals is conditional probability. In discussions the notion of probability is often taken to be

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[12] All these considerations apply also to the "credence version". If someone rejects the idea of modeling conditionals in a probability space and is interested only in direct credence assignments to sentences, i.e. Cr: L$_{COND}$ → $[0, 1]$, then (Cr-IMP) has the form: (Cr-IMP) If Cr(A) > 0 and Cr(B) = 0, then Cr($A \to B$) = 0.

informal (i.e. as subjective probability, degree of belief etc) — and it is not always clear what exactly the thesis is meant to express. Once we have the spaces S and S* at disposal, we can present the formal counterpart(s). We consider PCCP to express a kind of harmony between the factual beliefs and conditional beliefs. The following formulation seems to express it in the most natural way:

(P*CCP(S; S*)) $P^*([A \to B]^*) = P([B]|[A])$ for all $A, B \in L_{COND}$ with $(A \to B) \in L_{COND}$ and $P([A]) > 0$.

Given the imbedding of S into S*, it is equivalent to the "S*-internal" form:

((P*CCP*)(S*)) $P^*([A \to B]^*) = P^*([B]^*|[A]^*)$ for all $A, B \in L_{FACT}$ with $(A \to B) \in L_{COND}$ and $P^*([A]^*) > 0$.[13]

Regardless of the formulation, these principles mention — apart from the factual space S in the first version — a particular probability space S* with the corresponding probability distribution P^*. It might be one of many possible L_{COND}-spaces, so it is important to stress, that this version has a "local" character, relativized to a given S*. PCCP as a general claim would be a general statement concerning probability spaces from a certain class **K**. PCCP in its strongest possible form would be the claim that PCCP holds in in **all** L_{COND}-spaces S*, i.e. (in semiformal notation) $\forall S^* \in L_{COND}(S)$-CLASS, PCCP(S*). This principle is false and hardly worth discussing: there are L_{COND}-spaces violating PCCP. So, a reasonable form of PCCP is relativized to a class of spaces $K \subseteq L_{COND}(S)$-CLASS.

Discussions concerning the justification of PCCP might have a twofold character. Very broadly, we might engage in:

1. Philosophical discussion justifying PCCP as a intuitive claim.
2. Formal analysis, concerning mathematical, formal versions of PCCP.

Ad 1. "Philosophical PCCP" states, that the only reasonable probability assignment to conditionals satisfies PCCP. The potential justification would — generally speaking — in a philosophical (semantic, linguistic, epistemological etc.) analysis of concepts. It would involve identifying meaning postulates adequately characterizing the conditional con-

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[13] The general (much stronger) version of this postulate is $P^*([\alpha \to \beta]^*) = P^*([\beta]^*|[\alpha]^*)$ for all $\alpha, \beta \in L_{COND}$ such that $(\alpha \to \beta) \in L_{COND}$ and $P^*([\alpha]^*) > 0$. Here we only need the simplest version.

nective $\rightarrow$. Depending on the results of this analysis, we would consider diverse forms of PCCP to be justified or not.

Ad 2. As formal claims, formalized version(s) of PCCP concern mathematical structures. Their proofs would — in general — have a relative character: if a probability space S* modeling conditionals satisfies certain assumptions **A**, then S* satisfies PCCP. These assumptions **A** might be formalizations of meaning postulates or logical principles concerning conditionals. Constructing the formal proof of "**A** implies PCCP" might be interesting from the mathematical point of view. Obviously, it could not be in any sense an "absolute proof of PCCP".

In the present paper we do not discuss the problem of presenting a formal proof of PCCP in the second sense.[14] However, we think that it is possible to give a positive argument in favor of PCCP, based on Dutch Book argumentation. We address this issue in Section 4.2.

It is worth observing that apart from its intrinsically interesting nature, resolving this problem also has important consequences for the discussion concerning causal versus evidential decision theory.[15]

2.4. The existence of a L$_{\text{COND}}$-space

The definition of a L$_{\text{COND}}$-space does not prove its existence but L$_{\text{COND}}$-spaces exist and examples can be given. One of them is Stalnaker-Bernoulli spaces (cf. van Fraassen, 1976; Kaufmann, 2004, 2005, 2009, 2015, 2023). The elementary events are infinite sequences and the probability measure P* is defined on (a kind of) cylindric sets first and then extended to the appropriate σ-field generated by these sets. The set of elementary events has the power of the continuum. A different example is provided by spaces based on Markov chains (graphs) (see Wójtowicz and Wójtowicz, 2021a,b, 2022). Elementary events are finite sequences (generated by graphs) and the probability measure P* is given in a simple way. These spaces are countably infinite. An interesting example is

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[14] For instance, if we assume the formalized version of the Independence Principle and some intuitive assumptions concerning the conditional connective, we can prove PCCP.

[15] "After all, if probabilities of conditionals really are conditional probabilities, then the Gibbard-Harper theory is equivalent to Jeffrey's; but since the latter is simpler, then the former is otiose, as are all its allegedly equivalent reformulations. Thirty-five years of literature on causal decision theory would then appear to be a big red herring" (Hájek, 2011, p. 7).

given in (Węgrecki and Wroński, 2023), where the elementary events are constructed — broadly speaking — as a kind of formal expressions over possible words. In the model Wójtowicz and Wójtowicz (2023), elementary events are permutations of the set of possible worlds.[16] It is interesting to observe that in all these spaces PCCP holds. However, it is possible to transform them into non-PCCP spaces (by formally redefining the probability measures). PCCP is in no way, even tacitly assumed in the definition of L_{COND}-space.

3. Hájek's update method

Triviality proofs purport to show that there is a fundamental problem with PCCP.[17] In particular, Hájek's proof suggests that there is some fundamental conflict between accepting PCCP and accepting update rules which seem to be perfectly rational and justified. Indeed, Hájek shows how to transform a probability distribution P^* (which satisfies PCCP) into a non-PCCP probability distribution P^{*U}, by updating on information U.

Hájek's focus is on bold and moderate update rules. Being bold and moderate are — as Hájek argues — sufficient conditions for an update method to be considered rational. Conditionalization satisfies these conditions, but Hájek also mentions several other examples of update methods.[18]

Assume the agents models their system of beliefs (including conditionals) using the probability distribution P^* (the star indicates that we work with a probability space where conditionals are modeled). The agent wants to modify the initial probability distribution P^*, taking into account the information U and obtaining a new probability distribution P^{*U}. Two conditions are relevant in this context:

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[16] Both these examples provide interpretation for the language L_{COND} containing all simple conditionals and their Boolean combinations; the constructions can be iterated.

[17] The *locus classicus* is (Lewis, 1976). Lewis shows that if a suitable class of probability measures is closed under conditionalization and PCCP holds, then the probability measure for PCCP is trivial

[18] If conditionalization as the update method is rejected, Lewis' original proofs are blocked. However, Hájek's proof works for a wide class of update methods: he mentions imaging, blurred imaging, maximum entropy, minimum cross entropy — but stresses that the proof works for a wide class of such rules.

(**Boldness**) $P^{*U}(U) = 1$.

(**Moderateness**) For any A such that the truth of A implies the truth of U: If $P^*(A) \neq 0$, then $P^{*U}(A) \neq 0$.

Boldness means, that if you learn U, you fully believe it. Moderateness is justified by Hájek as follows:

> Moderation is surely a desideratum of a revision rule: your belief system would be highly fragile if you could suddenly fully disbelieve some proposition A that previously you gave some credence, when you learned something implied by A. By the lights of your original credence unction, A is confirmed (assuming the usual Bayesian account of confirmation); but by the lights of your new credence function, it is maximally discon-firmed. (Hájek, 2011, p. 10)

We present a slightly modified (and simplified) version of Hájek's proof. We make a simplifying (but safe) assumption, which shortens the proof — but also reveals an important problem with the update.

A, B are factual sentences, $A \rightarrow B$ is the conditional in question. P^* is the initial probability distribution. The "updating information" U is $[\neg(A \wedge B)]^*$. P^{*U} is the updated probability distribution. We assume:

1. $[B]^* \subseteq [A]^*$
2. $0 < P^*([B]^*) < P^*([A]^*) < 1$.
3. PCCP(P^*) holds for $A \rightarrow B$, i.e. $P^*([A \rightarrow B]^*) = P^*([B]^*|[A]^*)$.

As $[B]^* \subseteq [A]^*$, $[(A \wedge B)]^* = [B]^*$, i.e. $[A]^* \cap [B]^* = [B]^*$. So the update is by $U = [\neg B]^*$.

The proof proceeds in a few easy steps:

1. By PCCP(P^*), $P^*([A \rightarrow B]^*) = P^*([B]^*|[A]^*) = \frac{P^*([A]^* \cap [B]^*)}{P^*([A]^*)} = \frac{P^*([B]^*)}{P^*([A]^*)}$
2. As $0 < P^*([A]^*) < 1$, we have: $P^*([B]^*) < P^*([A \rightarrow B]^*)$. This implies that $P^*([A \rightarrow B]^* \backslash [B]^*) > 0$, i.e. $P^*([A \rightarrow B]^* \cap [\neg B]^*) > 0$
3. Obviously, $[A \rightarrow B]^* \cap [\neg B]^* \subseteq [\neg B]^*$. By moderateness, $P^{*U}([A \rightarrow B]^* \cap [\neg B]^*) > 0$.
4. But then obviously $P^{*U}([A \rightarrow B]^*) > 0$.
5. By boldness, $P^{*U}([\neg B]^*) = 1$. This means that $P^{*U}([B]^*) = 0$.
6. As $P^*([B]^*) < P^*([A]^*)$, this means that $P^*([A]^* \backslash [B]^*) > 0$. As $[A]^* \backslash [B]^* = [A]^* \cap [\neg B]^*$, we have $P^*([A]^* \cap [\neg B]^*) > 0$. Obviously, $[A \wedge \neg B]^* \subseteq [\neg B]^*$, so, by moderateness, we have $P^{*U}([A]^* \cap [\neg B]^*) > 0$.
7. This means in particular that $P^{*U}([A]^*) > 0$.

We have the required conclusion:
- $P^{*U}([A]^*) > 0$; $P^{*U}([B]^*) = 0$; $P^{*U}([A \to B]^*) > 0$; and
  $P^{*U}([B]^*|[A]^*) = 0$,

which means that PCCP is false for P^{*U}. According to Hájek's argument, this shows that PCCP is very unstable: even if you accept it, a rational update U might easily force you to abandon PCCP.[19]

In Hájek's original proof no conditions are imposed on the sentences A and B, apart from $P^*([\neg(A \wedge B)]) \neq 0$. Using this assumption, he proves that $P^{*U}([A \to B]^*) > 0$ and $P^{*U}([B]^*|[A]^*) = 0$. But it is sufficient to assume that $[B]^* \subseteq [A]^*$ to produce a probability distribution which not only shows that PCCP is false, but also has the property that $P^{*U}([A]^*) > 0$, $P^{*U}([B]^*) = 0$ and $P^{*U}([A \to B]^*) > 0$. But this means that (IMP*), which is a fundamental condition, is violated.[20]

4. Hájek's update and Dutch-Books

It is commonly assumed that the beliefs of a rational agent can be modeled in a probability space.[21] Agents who violate the rules of probability calculus are threatened by a Dutch Book, which means that they can suffer a financial or intellectual loss.[22] We consider Dutch Book argumentation to be a very powerful tool for discovering incoherence in the agents systems of beliefs. This applies also to a system of beliefs including conditionals.[23]

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[19] Consider our toy example, taking $A = Even$ and $B = Six$ and U = $[\neg(Even \wedge Six)]^*$. From the general proof it follows, that: (i) $P^{*U}([Even]^*) > 0$; (ii) $P^{*U}([Six]^*) = 0$ and (iii) $P^{*U}([Even \to Six]^*) > 0$. So indeed, P^*CCP^* does not hold for P^{*U} and $Even \to Six$.

[20] Observe, that our proof is blocked if there are no events $[A]^*, [B]^*$ with $0 < P^*([B]^*) < P^*([A]^*) < 1$, i.e. when the probability space is trivial (using Lewis's terminology).

[21] An interesting discussion can be found in (Hájek, 2008).

[22] We agree that the subjective probabilities ascribed by the agents is connected with the kind of bet the agents are going to make. This is a very common assumption in the literature (see, e.g., Easwaran, 2011a,b, for a general presentation). For a general presentation of Dutch Books see, e.g., (Vineberg, 2016), where extensive references can be found.

[23] The Dutch Book is based on the internal relationships between the agent's beliefs. It is the incoherence in the agent's beliefs which leads to the diachronic Dutch Book and an inevitable loss. The question whether they are adequate in any sense to the empirical situation is irrelevant. Dutch Book is the coherence test, not the test

It is obvious what the rules of the betting game are in case of a factual sentence, like *The number on the die is even.* But what are the rules for betting when a conditional is involved? Consider a fair die and the bet on the conditional *Even → Six.* The game consists in rolling a die (i.e. — formally — choosing an event from the probability space $S = (\Omega, \Sigma, P)$). It is fair to assume that the bet is:

- Won if we see a 6.
- Lost if we see a 2 or 4.
- Cancelled if we see a 1 or 3 or 5 (in this case the money is refunded).[24]

    In general, the bet $Bet(A \to B)$ on the conditional $A \to B$ is:

- Won when $A \wedge B$ happens
- Lost when $A \wedge \neg B$ happens
- Cancelled when $\neg A$ happens (in this case the money is refunded).[25]

These stipulations are entirely natural — and they are counterparts of the following two natural assumptions concerning conditionals:

(1) The truth of $A \wedge B$ guarantees the truth of the conditional $A \to B$. In terms of bets this means, that if you bet on $A \to B$ and an $A \wedge B$-event occurs, you win.[26]

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of empirical adequacy of the system of beliefs. So if the agent sets for instance \$0.50 on *It is a six* and \$0.90 on *It is Even*, they cannot be Dutch-booked (however, in the long run the agent will probably lose the money).

[24] Cancelling the bet is not a very special requirement. Consider a fair coin, you bet \$0.5 on Heads, and agree, that — apart from the standard rules — if is the coin is lost (or evaporates), the money is refunded. This is very natural, as in this situation we cannot decide the bet. No problems arise with this bet; in particular, no Dutch Book can be constructed.

[25] This is assumed for instance by McGee (1989) in his fair bet analysis concerning also conjoined conditionals of the form $(A \to B) \wedge (C \to D)$. The idea is not new — in fact, the idea of the conditional bet dates back at least to (de Finetti, 1937).

[26] (1) is a special case of the general rule, which — in its most general sense — states that if we accept $\alpha \to \beta$, then we accept $\alpha \wedge \beta$ (for any α and β). In the literature it is known, for instance, as *Conjunctive Sufficiency* (Egré and Rott, 2021), *Centering*, or *And-to-If* (Cruz et al., 2016; Berto and Özgün, 2021). In the present paper we only need to assume that both arguments of $\to$ are factual sentences (which means that we need the weakest version of this principle).

    Potential counterarguments against (1) rest on the assumption that the contents of the antecedent and consequent in the conditional are somehow relevant to each other. For a discussion see (Berto and Özgün, 2021). The discussed principle is $A \wedge \beta \vDash A \to \beta$, so it is a logical principle, using the notion of logical consequence.

(2) The truth of $A \wedge \neg B$ guarantees the falsity of the conditional $A \to B$. In terms of bets this means, that if you bet on $A \to B$ and an $A \wedge \neg B$-event occurs, you lose.

In all the examples we think of standard bets: the agent sells of buys bets for some price p (with $0 < p < 1$), and the payout is \$1.

4.1. Dutch Book and PCCP[27]

Consider an agent who accepts the aforementioned betting rules on conditionals. Also, the agent is willing to engage in diachronic Dutch Book, i.e. to make new bets when new information is introduced. The agent does not accept PCCP, i.e. thinks that the probability of $A \to B$ is **not** P(B|A). For simplicity assume that the agent ascribes probability $p > $ P(B|A) to $A \to B$. Also, after the update U, the agent is willing to use the new probability distribution P^{*U} and is ready to make new bets according to the new belief system.

For simplicity, consider our toy example $Even \to Six$ (the die is fair).

The agent overestimates $P^*(Even \to Six)$ and is willing to pay \$0.50 for the bet (in the game in which the die is rolled). The diachronic Dutch Book against the agent consists of two steps:

Before the die roll.
1. The Bookkeeper sells four bets Bet($Even \to Six$) for \$0.50 each (i.e. gets \$2 from the agent);
2. The Bookkeeper sells Bet($Even$) for \$0.50.

After the die roll. The die is rolled.
1. If it is odd, Bet($Even$) is lost, and Bet($Even \to Six$) is cancelled.

Berto and Özgün note that "And-to-If" holds in the similarity-based possible world semantics of (Stalnaker, 1968). They also contend that "A number of mainstream theories of indicatives validate And-to-If: the material conditional view (Jackson, 1997; Grice, 1989) and the probabilistic-suppositional view (Adams, 1975; Edgington, 1995; Evans and Over, 2004), for instance, have it." (Berto and Özgün, 2021, p. 3701).

It exceeds the scope of this paper to discuss the problem in detail. However it is important to stress that (1) and (2) are assumed, among others, by Hájek and Lewis. The main purpose of the paper is to discuss their arguments against PCCP, which means that we are allowed (and even obliged) to accept the basic principles concerning conditionals which are accepted by them.

[27] In this section we make use of the results presented in (Wójtowicz and Wójtowicz, 2021b).

2. If it is even, the agent updates the probability distribution —
   now it is P_{Even}.[28] The price for the fair bet on *It is not a Six* is
   $\$\frac{2}{3}$ (the agent accepts such bets). So **after** learning that Even
   has occurred, the Bookkeeper sells four $\text{Bet}_{\text{Even}}(Not\ a\ Six)$ for
   $\$\frac{2}{3}$ each.

After this bet is made, full information is revealed. The outcomes —
from the point of view of the agent — are presented in the table:

	Bet(*Even*)	Bet(*Even* → *Six*)	Bet_{Even} (*Not a Six*)	The total outcome
Odd	$-\$\frac{1}{2}$	0 (the bet was cancelled)	0 (the bet has not been placed)	$-\$\frac{1}{2}$
Six	$+\$\frac{1}{2}$	$+\$2$	$-\$\frac{8}{3}$	$-\$\frac{1}{6}$
Two or four	$-\$\frac{1}{2}$	$-\$2$	$-\$\frac{4}{3}$	$-\$\frac{1}{6}$

The agent has been Dutch Booked. The cause of the disaster is
accepting the bet $\text{Bet}(A \to B)$ for a price p **different** from $P(B|A)$.[29]
This means the only rational choice for the agent who is willing to engage
in bets, is to assume that the probability of $A \to B$ is $P(B|A)$. Acting
in accordance with PCCP generates no unwanted consequences.

In the case of factual beliefs, if you violate probability calculus, you
will be punished. Form the considerations above it follows, that in case
of conditionals, if you violate PCCP you will be punished.[30]

Regardless of the general argument, we will analyze the betting be-
havior of two agents, Alice and Bob, who perform different updates.

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[28] The agent performs the update by conditionalizing on A, and is willing a
bet on B in light of the new knowledge. Now the probability estimate for B is the
conditional probability $P(B|A)$ — and the fair price for $\text{Bet}_A(B)$ is $P(B|A)$.

[29] Why exactly four bets? We need to make use of the difference between the
subjective evaluations of $P(A \to B)$ and $P(B|A)$. Let the difference be $\gamma = P(A \to B) - P(B|A)$. We need to make N bets so that $N \cdot \gamma > \frac{1}{2}$. In our toy example $\gamma = \frac{1}{6}$,
so N = 4 suffices.

[30] Observe, that under these conditions, the bet on $Even \to Six$ for a price of $\$\frac{1}{3}$
is fair. Indeed, the payouts are: (i) 1 if 6 occurs; (ii) 0 if 2 or 4 occur; (iii) $\frac{1}{3}$ (refund)
if 1 or 3 or 5 occurs. The expected value is $\frac{1}{6} \cdot 1 + \frac{2}{6} \cdot 0 + \frac{3}{6} \cdot \frac{1}{3} = \frac{1}{3}$, so the bet is fair.

4.2. Alice and Bob — two different update methods

Alice and Bob are two agents who have beliefs concerning a fair die, and model their beliefs in the appropriate probability space S. Alice and Bob wholly agree with regard to the probabilities of factual sentences, i.e., they assign the same probabilities to *One*, *Two*, *Three*, *Four*, *Five*, *Six*[31] and they also have an initial opinion on the probability of *Even → Six*. Moreover they use the probability space S* with P(*Even → Six*) = $\frac{1}{3}$, i.e. they both accept PCCP(S*). They are happy to bet on *Even → Six*, making use of the betting scheme in Section 4.1. We have seen, they are not vulnerable to a Dutch Book. This means simply, that in the game which consists in rolling the die, there is no collection of bets which assures their loss (independently of the result of the die roll).

Both Alice and Bob believe that after performing a rational update, which leads to a new probability distribution, bets can be safely made according to this new distribution. After all — if the update is rational, the new distribution is rational as well!

However, their opinions on updating methods are different. Alice makes her updates in what she considers to be the most common-sense way, focusing on updating factual knowledge. Bob is glad to use Hájek's update method and works directly in the space S*. They both believe that making bets is a good and safe thing: indeed, they used their initial P* with good results. So they are willing to make bets also after the rational update has been made.

We analyze Bob's behavior first.

4.3. Bob's betting behavior

Bob accepts Hájek's proviso that bold and moderate update rules work properly (i.e. are rational) also for conditionals. Bob considers both the initial P*, and the update method to be rational. He had no qualms about using P* in his bets: in fact, as we have seen no Dutch Book can be constructed against Bob who uses the probability distribution P*. Bob also accepts bold and moderate updating, so he is happy to use the new probability distribution P^*_{Bob} when making bets. So Bob works in S* and performs a bold and moderate update on U = [¬*Six*]*, arriving

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[31] If the die is fair, all these probabilities are $\frac{1}{6}$, but the argumentation does not depend in any way on this particular probability distribution. If the die is not fair, we have $p_1, \ldots, p_6$ as the corresponding probabilities.

at a new system of beliefs with $P^*_{Bob} = (P^*)^U$ as the new probability distribution.[32]

We have seen that Hájek's update rule leads to accepting:

(i) $P^*_{Bob}([Even]^*) > 0$;
(ii) $P^*_{Bob}([Six]^*) = 0$;
(iii) $P^*_{Bob}([Even \rightarrow Six]^*) > 0$.

In Bob's new system of beliefs the probability $P^*_{Bob}([Even]^*)$ is some γ, with $0 < \gamma < 1$.

Two bets concerning the die roll are made, both considered by Bob to be fair.

1. The Bookmaker sells the bet Bet($Even$) for γ, i.e. gets \$$\gamma$ from Bob. If the number is even, Bob gets \$1; if it is odd — he loses the \$$\gamma$.
2. The Bookmaker sells n bets on $Even \rightarrow Six$. Bob believes that the probability of this conditional is $p > 0$ so he is willing to buy n standard bets for \$$p$ each (Bob considers this bet to be fair). If the result is a 6, the Bookmaker will have to pay \$$n$ to Bob. If the result is a 2 or 4 — Bob looses the money. And if the result is an odd number, the bet is cancelled (and the \$$np$ is refunded to Bob).

After the bets have been made, full information is revealed. The possible outcomes of the game are given in the table (from Bob's point of view):

The result	Bet($Even$)	Bet($Even \rightarrow Six$)	The total outcome
1 or 3 or 5	$-\gamma$	cancelled	$-\gamma$
2 or 4	$+(1-\gamma)$	$-np$	$(1-\gamma) - np$
6	$+(1-\gamma)$	$n(1-p)$	Will never occur

We already know that the probability of 6 is 0 (indeed: we updated on $[\neg Six]^*$), which means that the last possibility surely does not occur.

This means, that if only we take n big enough so that $np > (1 - \gamma)$, then regardless of the result on the die, Bob loses:

(i) If it is 1 or 3 or 5 — he loses Bet($Even$) and Bet($Even \to Six$) is cancelled, so his total loss is \$$(1 - \gamma)$.

(ii) If it is 2 or 4 — he wins Bet($Even$) and loses Bet($Even \to Six$), which means that he gets \$$(\gamma - np)$ in total. This is negative, so Bob loses.

So we have a situation in which Bob is willing to accept two bets, considering both to be fair: the bets correspond to the subjective probability P^*_{Bob} which was obtained by using a bold a moderate update rule of the sentences in question. But this leads to a Dutch Book against Bob, which means that his belief system is not consistent (not rational).

4.4. Alice's betting behavior

Alice has never heard about being bold an moderate as the sufficient condition for rational updates. She learns that Six is false, and updates her beliefs in what she considers to be the most common-sense way, taking care of updating factual knowledge first. This is because Alice believes that all reasonable conditional beliefs are anchored in factual knowledge. From her point of view, the new probabilities of factual sentences are:

$P_{Alice}([One]) = P_{Alice}([Two]) = P_{Alice}([Three]) = P_{Alice}([Four]) = P_{Alice}([Five]) = \frac{1}{5}$;
$P_{Alice}([Six]) = 0$.

Observe that Alice works in the factual space S, making use of the information $\neg Six$. So she arrives at the new probability distribution P_{Alice}. She then updates her beliefs on conditionals using this knowledge, by choosing her favorite L_{COND}-space $(S_{Alice})^*$. Of course IMP*(P*) holds there. This means in particular that $(P_{Alice})^*([Even \to Six]^*) = 0$. So this is exactly how Alice proceeds:

1. She learns that $\neg Six$ is true.
2. She updates her factual space S, obtaining S_{Alice}.
3. She chooses a L_{COND}-space $(S_{Alice})^*$. Here she makes use of the obvious and commonsense (IMP*) principle. Alice is free to choose **any** of the L_{COND}-spaces $(S_{Alice})^*$ — which are suitable for the **updated** factual space S_{Alice}.

It is clear that she cannot be Dutch booked by using bets on $Even \to Six$: she knows that $(P_{Alice})^*([Even \to Six]^*) = 0$, and is not going to bet on it.

However, Alice's update is not moderate with respect to the initial P^*. Indeed: the truth of $(Even \to Six) \wedge \neg Six$ implies the truth of $\neg Six$ and $P^*([(Even \to Six) \wedge \neg Six]^*) \neq 0$, but $(P_{Alice})^*([(Even \to Six) \wedge \neg Six]^*) = 0$.

We do not know whether in the new $(S_{Alice})^*$ PCCP holds in general or not. But our aim was to show that the proper update method does not generate a counterexample (i.e. we only discuss $Even \to Six$). And obviously, Alice is free to choose a "PCCP-friendly" L_{COND}-space $(S_{Alice})^*$: in Section 2.4 we mentioned several examples of such spaces, which Alice has at her disposal.

4.5. Would you prefer to behave like Alice or like Bob?

We have arrived at the following observation:

Observation. There are functions P^*_{Bob} and $(P_{Alice})^*$ which are modifications of the probability distribution P^* (both making use of the information that $\neg Six$ is true), such that:
(a) $P^*_{Bob}([X]^*) = P^*_{Alice}([X]^*)$, for every factual sentence $0X$;
(b) $P^*_{Bob}([Even \to Six]^*) \neq 0$;
(c) $(P_{Alice})^*([Even \to Six]^*) = 0$;
(d) P^*_{Bob} has been obtained from P^* by using a bold and moderate update rule (i.e. by updating on $[\neg Six]^*$). Bob can be Dutch Booked.
(e) $(P_{Alice})^*$ has been obtained in a "naïve" way: she steps back into the factual space S, performs the update there (i.e. she conditionalizes **within S** by $[\neg Six]$), obtaining a new factual space S_{Alice}. Only later she produces the new space $(S_{Alice})^*$. She violates the moderateness rule — but cannot be Dutch Booked (at least when the bet on $Even \to Six$ is concerned)!

Observe the difference in notation (i.e. where the * is placed). P^*_{Bob} indicates that the update was performed directly in S^*. $(P_{Alice})^*$ indicates, that first a factual update has been performed within S — producing a new factual space S_{Alice} — and after that a new L_{COND}-space $(S_{Alice})^*$ has been chosen.[33]

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[33] Schematically, we can present the update scheme in the following way:

Importantly, before the update neither Alice nor Bob could be Dutch Booked. They both trust their common probability distributions $P_{Bob} = P_{Alice}$. They both believe, that rational updates keeps them safe from Dutch-Booking: it is sufficient to stick to the values of the updated probability distribution. However, they interpret the notion of rational update method in a different way.

After the update, Alice's system of beliefs is still Dutch Book resistant. However, Bob can be Dutch Booked. This is **not** the problem with the system of bets or with the Dutch Book as the "coherence test". It is rather the problem with the update, which produces Dutch Bookable systems of beliefs. We will discuss this issue in more detail in Section 6.[34]

5. Hájek's update produces a space violating (IMP*)

We have seen that Bob can be Dutch Booked. In our opinion, this is a very important argument against P^*_{Bob} — i.e. in particular against the update method. However, there is a discussion concerning the viability of Dutch Book arguments.[35] According to critical voices the Dutch Book argument is not considered to be decisive. Consequently, if we reject Dutch Book arguments, the results of the previous sections are not convincing.

However, the presented Dutch Book phenomenon is a reflection of a much deeper problem with the updates. And even if we disregard Dutch Books, the problem remains. The update has produced a space violating (IMP*). Indeed, $(S^*)^U = (\Omega^*, \Sigma^*, (P^*)^U)$ (with U = $[\neg(Six)]^*$) has the property that:

Bob: $P^* \xrightarrow[\text{update within } S^*]{} P^*_{Bob}$

Alice: $P^* \xrightarrow[\text{focusing on fact}]{} P \xrightarrow[\text{update within } S]{} (P_{Alice}) \xrightarrow[\text{choosing an } L_{COND} - space]{} (P_{Alice})^*$

[34] Dutch Books are coherence tests for systems of beliefs — regardless of whether the agent models them in a mathematical probability space, or whether the agent just assigns some credences (degrees of belief) to sentences and acts — i.e. bets — accordingly. If the agent's beliefs are mirrored in the betting behavior, then they can be tested for coherence. So the argumentation remains valid also in the context of credence functions.

[35] For instance, it might be questioned that the betting prices match subjective probabilities or that even if a collection of bets is made, they are valued individually, or that there are no "interference phenomena" between the bets — and so on. See, e.g., (Hájek, 2009) for a presentation.

(i) $(P^*)^U([Even]^*) > 0$;
(ii) $(P^*)^U([Six]^*) = 0$ and
(iii) $(P^*)^U([Even \rightarrow Six]^*) > 0$.

This means that either:

(1) we abandon $(IM(P^*)^U)$, or
(2) we agree that being bold and moderate is **not** a sufficient condition of being a rational update method for conditionals. In particular, not all conditionalizations might be appropriate.

We definitely opt for the second variant. Indeed, (IMP^*) is so connected with our interpretation of the conditional connective that it is virtually impossible to reject. Moreover, it is natural to suppose that belief systems involving conditionals have some special features, and updating has to take into account the specific character and meaning of conditionals.

Update procedures need to take into account the complex and intricate character of the conditional connective $\rightarrow$. Bold and moderate rules (in particular standard conditionalization) operate in the "Boolean environment" — the only connectives (i.e. negation and conjunction) are modeled as set-theoretic operations, i.e. as the complement and intersection of sets (events). Indeed, standard Dutch Books against someone who violates for instance additivity is based on the correspondence between the Boolean functors and the Boolean set-theoretic operations. But when the conditional connective is involved, probabilistic modeling should satisfy additional requirements. The proper modeling of $\wedge$ and $\neg$ (which — by definition — holds in **every** probability space) is not a sufficient condition.[36]

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[36] Hájek makes an interesting comment: "If you are a Bayesian agent who seeks to conform to PCCP at all times, you are apparently unable to revise boldly and moderately your opinions regarding certain propositions. These propositions then have a curious status for you: you give them positive credence, but you can never learn them — where learning is modeled by a bold and moderate rule. Borrowing terminology from (van Fraassen, 1984), they are 'Moore propositions' — propositions that you cannot learn without violating a structural constraint that is imposed on you (in this case, the upholding of PCCP). There is something Moore paradoxical about your saying, or thinking: "p has positive probability, but it is impossible for me to learn it", where p may be easily accessible to your enquiry." (Hájek, 2011, p. 12–13). He is certainly aware of the strange situation. Indeed, you ascribe positive credence to $A \rightarrow B$, but you will never witness a situation making it true (i.e. you will never

It is an interesting general question how to identify the criteria for proper update rules for conditionals. However, the problem is not easier than the problem of identifying the proper logical or probabilistic rules for conditionals. The general discussion exceeds the scope of the present paper.

6. The Fall of Converse Dutch Book theorem?[37]

Two important claims concerning Dutch Books are found in the literature:

> **Dutch Book Theorem.** If your belief system violates the probability calculus, you can be Dutch Booked.
>
> **Converse Dutch Book Theorem.** If your belief system obeys the probability calculus, you cannot be Dutch Booked.

The first theorem predicts a penalty for irrational belief systems (given that "being rational" means "obeying the probability calculus"). The Converse Dutch Book Theorem assures you that rational behavior keeps you safe.

Now we have a situation, in which the agent obeys the probability calculus (indeed, P^*_{Bob} is a mathematically correct probability distribution) — but can be Dutch Booked. Have we shown that the Converse Dutch Book Theorem is false?

We strongly disbelieve this. We have a situation in which a Dutch Book resistant probability distribution P^* has been updated, and a Dutch Bookable probability distribution P^*_{Bob} has been produced. But in our opinion this suggests a problem with the update method, not with the Converse Dutch Book theorem.

The explanation is hinted at in the previous section and indicates, that probabilistic modeling of a rational belief system, when conditionals are involved, differs from the Boolean case and imposes additional restrictions on both the probability measures and the update methods.

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witness $A \wedge B$). Our explanation is simple: in fact, there is no paradox. The credence assignment is wrong, as it is obtained with the aid of a defect update rule.

[37] The title of the section was inspired by Hájek's paper *The Fall of Adams' Thesis?*, (2012). Hájek's paper contains a highly interesting analysis of PCCP and its diverse versions.

Not every probability distribution is acceptable — and this is for profound reasons, concerning the very interpretation of conditionals. P^*_{Bob} is **not** appropriate for modeling conditionals, and this is simply because it violates (IMP*), which is a very fundamental assumption. So surely it cannot serve as a counterexample to the Converse Dutch Book theorem.

Informally speaking, ordinary probability assignments are consistent with the meaning of $\wedge$ and $\neg$. This is why following rules of probability is sufficient to be Dutch Book resistant. However, now we have the new connective $\rightarrow$ and in order to be safe from Dutch Booking, it is not sufficient to give any probability assignment. It has to take into account the specific meaning of $\rightarrow$. To make the point more clear, we consider the example of a new connective #.

6.1. The strange

Consider the following example: we have a new connective # in the language, and we have some rules concerning the bets, i.e. how the bets on A#B are decided. Everyone using the connective accepts the rules. Here they come:

If $A \wedge B$ occurs — we win.
If $\neg A \wedge B$ occurs — we lose.
If $\neg B$ occurs — the bet is cancelled and the money is refunded.

These betting rules impose some restrictions on what # is: for instance it is neither the conjunction nor the disjunction.

We also have some probability distribution P* in a space S*, which assigns probabilities to the events, in particular to the event $[A\#B]^*$. For instance, if $A = Even$ and $B = Six$, the probability of $[A\#B]^*$ is $\frac{1}{3}$.

It turns out that the agent who uses P* can be Dutch Booked. Does this mean that we have found a counterexample to the Reverse Dutch Book theorem? This would be a very strange corollary: the very first objection that comes to mind is that $A\#B$ is simply $B \rightarrow A$, and the probability of $Six \rightarrow Even$ is not $\frac{1}{3}$! This probability modeling is obviously wrong! But this does not indicate that the Converse Dutch Book theorem is wrong: it only indicates, that P* is simply inadequate. So the fact, that P* is a probability distribution is not sufficient: it has to model $A\#B$ in the appropriate way — i.e. so that the meaning postulates (encoded — at least partially — in the betting rules) are satisfied.

This phenomenon does not arise in the "Boolean environment", as the interpretation of functors are rigid: $\wedge$ corresponds to $\cap$, and $\neg$ corresponds to the complement. The betting rules are also clear. But when some new, non Boolean functor like $\#$ or $\rightarrow$ appears, we have to be very careful with probability distributions: they must correspond to the intended meaning of the functors – especially, if this intended meaning is somehow reflected in the betting rules.

These observations remains valid also outside of the Dutch Book paradigm. They show perfectly well that we are allowed to impose conditions on the class of acceptable probability functions, if they are intended to model the connective in question in accordance with the intended meaning of the connective. And an acceptable update method should not produce a probability distribution violating these conditions.

This toy example situation explains the sources of Bob's loss. P^*_{Bob} is **not** the appropriate modeling of $A \rightarrow B$.

Finally, observe that the classical Reverse Dutch Book theorem operates in the "Boolean environment" and is precise. We might formulate its counterpart concerning conditionals. It is just a very tentative hypothesis, not mathematically precise, as it contains an informal (and vague) notion. However, we think that it worth discussing:

> **Conditional Dutch Book Resistance Hypothesis.** Assume your conditional system of beliefs is Dutch Book resistant. If your update method is **suitable for conditionals** it remains Dutch Book resistant.

Alice's update is suitable. Bob's update is not.

7. Conclusion

I. We assume (which is standard) that the probability of the conditional $A \rightarrow B$ should be evaluated in the light of our factual knowledge. Also, updating factual beliefs (usually) leads to an update of our beliefs concerning conditionals.

II. The factual language L_{FACT} is modeled in the sample space $S = (\Omega, \Sigma, P)$. We also have the conditional language L_{COND}, which is an extension of L_{FACT}.

III. Usually, the sample space S does not allow one to interpret the conditional $A \rightarrow B$. We need a different space $S^* = (\Omega^*, \Sigma^*, P^*)$, in

which the conditional $A \to B$ has an interpretation as an event $[A \to B]^* \subseteq \Omega^*$. We call such a space "L$_{COND}$-space".

IV. In general, L$_{COND}$-spaces are not uniquely determined, and they might exhibit various properties (for instance — PCCP might be true in some of them and false in others). But any L$_{COND}$-space S* has to satisfy some fundamental assumptions concerning conditionals. In particular, the probability space S should be imbeddable in S* — i.e., factual sentences from L$_{FACT}$ should have interpretations in S* as events, and their probabilities from S should be preserved in S*.

V. We assume only one meaning postulate concerning conditionals:

IMP(P; P*) If $P([A]) > 0$ and $P([B]) = 0$, then
P$^*([A \to B]^*) = 0$

It corresponds to the intuitive principle (IMP), which states that if A is possible and B is impossible, then $A \to B$ is impossible. This is rather undisputable, and we expect the L$_{COND}$-spaces to satisfy (IMP*).

VI. PCCP is the general claim that probability of conditionals is conditional probability. It expresses coherence between factual beliefs and conditional beliefs. We have considered two intimately connected formalized versions of PCCP:

P*CCP(S; S*). $P^*([A \to B]^*) = P([B]|[A]))$.
P*CCP(S*). $P^*([A \to B]^*) = P^*([B]^*|[A]^*)$.

VII. We give a version (simplification) of Hájek's proof, and show that it produces a probability distribution which cannot represent the system of beliefs of a rational agent. To show this, we analyze the behavior of two agents, Alice and Bob, and a toy example of a fair die. They perform different updates, arriving at different probability assignments of the conditional *Even → Six*.

VIII. Bob and Alice start with the same probability distribution P*, which is Dutch Book resistant. Later they perform two different updates (both make use of a piece of information, "translated" into an update procedure in a different way). They arrive at P$^*_{Bob}$ and (P$_{Alice}$)*. It turns out that Alice — using (P$_{Alice}$)* — cannot be Dutch Booked, whereas Bob — using P$^*_{Bob}$ — can!

IX. Following Hájek's update rule leads to the conclusion that the updated probability of the conditional *Even → Six* is positive, even if the probability of *Even* is positive and probability of *Six* is 0! This violates one of the obvious properties of conditionals, i.e. (IMP*).

X. This shows, that not every probability distribution is suited to model conditionals — and not every update method is admissible. In particular, being bold and moderate is neither a necessary nor a sufficient condition of being Dutch Book resistant.

XI. Our findings might suggest, that the Converse Dutch Book is false. Indeed, there is a Dutch Bookable probability distribution P^*_{Bob}. However, the problem is that this probability distribution is not adequate because it has been obtained in a wrong way. The classic Converse Dutch Book theorem operates in the "Boolean environment", and is not affected in any way by our findings.

XII. Alice's update method is safe: after her update, if she uses her $(P_{Alice})^*$ she cannot be Dutch booked. The explanation is that conditionals should be evaluated in the light of factual knowledge: if new factual knowledge is obtained, the factual probability distribution is updated (and P_{Alice} is produced), and only then the appropriate L_{COND}-space $(S_{Alice})^*$ is chosen. So, contrary to Hájek's claim, if you are a Bayesian agent who seeks to conform to PCCP at all times, you **are able to** rationally (even if not moderately from the point of view of the initial distribution P^*) revise your opinions regarding certain propositions.

References

Adams, E. W., 1965, "On the logic of conditionals", *Inquiry* 8: 166–197. DOI: 10.1080/00201746508601430

Adams, E. W., 1970, "Subjunctive and indicative conditionals", *Foundations of Language* 6: 89–94. DOI: 10.2307/2272204

Adams, E. W., 1975, *The Logic of Conditionals*, Dordrecht: D. Reidel.

Adams, E. W., 1998, "A useful four-valued logic", pages 7–37 in J. M. Dunn and G. Epstein (eds.), *Modern Uses of Multiple-Valued Logic*, Boston: Reidel Publishing Company. DOI: 10.1007/978-94-010-1161-7_2

Bennett, J., 2003, *A Philosophical Guide to Conditionals*, Oxford: Clarendon Press.

Berto, R., and A. Özgün, 2021, "Indicative conditionals: Probabilities and relevance", *Philosophical Studies* 178: 3697–3730. DOI: s11098-021-01622-3

Cruz, N., D. Over, M. Oaksford, and J. Baratgin, 2016, "Centering and the meaning of conditionals", pages 1104–1109 in A. Papafragou, D. Grodner, D. Mirman, and J. C. Trueswell (eds.), *Proceedings of the 38th Annual Conference of the Cognitive Science Society*, Cognitive Science Society.

de Finetti, B., 1937, "La Prévision: ses lois logiques, ses sources subjectives", *Annales de l'Institut Henri Poincaré* 7: 1–68; trans. as "Foresight: Its Logical Laws, its Subjective Sources", pages 93–159 in H. E. Kyburg and H. E. Smokler (eds.), *Studies in Subjective Probability*, New York: Wiley, 1980.

Easwaran, K., 2011a, "Bayesianism I: Introduction and arguments in favor", *Philosophy Compass* 6(5): 312–320. DOI: 10.1111/j.1747-9991.2011.00399.x

Easwaran, K., 2011b, "Bayesianism II: Criticisms and applications", *Philosophy Compass* 6(5): 321–332. DOI: 10.1111/j.1747-9991.2011.00398.x

Edgington, D., 1995, "On conditionals", *Mind* 104: 235–329. DOI: 10.1093/mind/104.414.235

Edgington, D., 2020, "Indicative conditionals", in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy* (Fall 2020 Edition), https://plato.stanford.edu/archives/fall2020/entries/conditionals/

Egré, P., and H. Rott, 2021, "The logic of conditionals", in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy* (Winter 2021 Edition). https://plato.stanford.edu/archives/win2021/entries/logic-conditionals/

Evans, J. St. B. T., and D. E. Over, 2004, *If: Supposition, Pragmatics, and Dual Processes*, Oxford: Oxford University Press.

Fitelson, B., 2015, "The strongest possible Lewisian triviality result", *Thought* 4: 69–74. DOI: 10.1002/tht3.159

Grice, H. P., 1989, *Studies in the Way of Words*, Cambridge: Harvard University Press.

Hájek, A., 1989, "Probabilities of conditionals — revisited", *Journal of Philosophical Logic* 18: 423–428.

Hájek, A., 2008, "Arguments for – or against – probabilism?", *British Journal for the Philsophy of Science* 59(4): 793–819. DOI: 10.1093/bjps/axn045

Hájek, A., 2009, "Dutch Book arguments", pages 173–195 in P. Anand, P. K. Pattanaik, C. Puppe (eds.), *The Handbook of rational and social choice*, Oxford University Press. DOI: 10.1093/acprof:oso/9780199290420.001.0001

Hájek, A., 2011,"Triviality pursuit", *Topoi* 30(1): 3–15. DOI: 10.1007/s11245-010-9083-2

Hájek, A., 2012, "The Fall of "Adams' Thesis"?", *Journal of Logic, Language and Information* 21: 145–161. DOI: 10.1007/s10849-012-9157-1

Hájek, A., and N. Hall, 1994, "The hypothesis of the conditional construal of conditional probability", pages: 75–110 in E. Eells and B. Skyrms (eds.), *Probabilities and Conditionals: Belief Revision and Rational Decision*, Cambridge: Cambridge University Press.

Hall, N., 1994, "Back in the (CCCP)", pages 141–160 in E. Eells and B. Skyrms (eds.), *Probabilities and Conditionals: Belief Revision and Rational Decision*, Cambridge: Cambridge University Press.

Jackson, F., 1987, *Conditionals*, New York: Blackwell.

Kaufmann, S., 2004, "Conditioning against the grain: Abduction and indicative conditionals", *Journal of Philosophical Logic* 33(6): 583–606. DOI: 10.1023/B:LOGI.0000046142.51136.bf

Kaufmann, S. 2005, "Conditional predictions: A probabilistic account", *Linguistics and Philosophy* 28(2): 181–231. DOI: 10.1007/s10988-005-3731-9

Kaufmann, S., 2009, "Conditionals right and left: Probabilities for the whole family", *Journal of Philosophical Logic* 38: 1–53. DOI: 10.1007/s10992-008-9088-0

Kaufmann, S., 2015, "Conditionals, conditional probability, and conditionalization", pages 71–94 in H. Zeevat and H. C. Schmitz (eds.), *Bayesian Natural Language Semantics and Pragmatics*, New York, Heidelberg: Springer.

Kaufmann, S., 2023, "Bernoulli semantics and ordinal semantics for conditionals", *Journal of Philosophical Logic* 52: 199–220. DOI: 10.1007/s10992-022-09670-8

Khoo, J., and P. Santorio, 2018, *Lecture Notes: Probability of Conditionals in Modal Semantics*. http://paolosantorio.net/teaching.html

Lewis, D., 1976, "Probabilities of conditionals and conditional probabilities", *Philosophical Review* 85: 297–315. DOI: `10.2307/2184045`

Lewis, D., 1986, "Probabilities of conditionals and conditional rrobabilities II", *Philosophical Review* 95: 581–589. DOI: `10.2307/2185051`

McGee, V., 1989, "Conditional probabilities and compounds of conditionals", *Philosophical Review* 98(4): 485–541. DOI: `10.2307/2185116`

Milne, P., 2003, "The simplest Lewis-style triviality proof yet?", *Analysis* 63(4): 300–303. DOI: `10.1093/analys/63.4.300`

Ramsey, F. P., 1990, "General propositions and causality", pages 145–163 in D. H. Mellor (ed.), *Philosophical Papers*, Cambridge: Cambridge University Press.

Rehder, W., 1982, "Conditions for probabilities of conditionals to be conditional probabilities", *Synthese* 53: 439–443.

Stalnaker, R., 1968, "A theory of conditionals", *Studies in Logical Theory, American Philosophical Quarterly* 2: 98–112.

Stalnaker, R., 2009, "Conditional propositions and conditional assertions", pages 277–248 in A. Egan and B. Weatherson (eds.), *Epistemic Modality*, Oxford: Oxford University Press. DOI: `10.1093/acprof:oso/9780199591596.003.0008`

Stalnaker, R., 2019, *Knowledge and Conditionals*, Oxford: Oxford University Press.

van Fraassen, B. C., 1976, "Probabilities of conditionals", pages 261–308 in W. L. Harper, R. Stalnaker, and G. Pearce (eds.), *Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science*, Dordrecht: D. Reidel.

van Fraassen, B. C., 1984, "Belief and the will", *The Journal of Philosophy* 81(5): 235–256. DOI: `10.2307/2026388`

Vineberg, S., 2016, "Dutch Book arguments", in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy* (Spring 2016 Edition). `https://plato.stanford.edu/archives/spr2016/entries/dutch-book/`

Węgrecki, J., and L. Wroński, 2023, "Revisiting the conditional construal of conditional probability", *Logic and Logical Philosophy* 32(2): 261–268. DOI: `10.12775/LLP.2022.024`

Wójtowicz, A., and K. Wójtowicz, 2021a, "A stochastic graphs semantics for conditionals", *Erkenntnis* 86,: 1071–1105. DOI: `10.1007/s10670-019-00144-z`

Wójtowicz, A., and K. Wójtowicz, 2021b, "Dutch Book against Lewis", *Synthese* 199: 9185–9217. DOI: 10.1007/s11229-021-03199-0

Wójtowicz, K., and A. Wójtowicz, 2022, "A graph model for probabilities of nested conditionals", *Linguistics and Philosophy* 45(3): 1–48. DOI: 10.1007/s10988-021-09324-z

Wójtowicz, A., and K. Wójtowicz, 2023, "A minimal probability space for conditionals", *Journal of Philosophical Logic* 52(3): 1385–1415. DOI: 10.1007/s10992-023-09710-x

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