Logic and Logical Philosophy
Volume 34 (2025), 631–675
DOI: 10.12775/LLP.2024.002

Michał Walicki

Paradoxes versus Contradictions in Logic of Sentential Operators*

Abstract. Classical logic, of first or higher order, is extended with sentential operators and quantifiers, interpreted substitutionally over unrestricted substitution class. Operators mark a single layered, consistent metalanguage. Self-reference, arising from substitutional quantification over sentences, allows to express paradoxes which, unlike contradictions, do not lead to explosion. Semantics of the resulting language, using semi-kernels of digraphs, is non-explosive yet two-valued and has classical semantics as a special case for clasically consistent theories. A complete reasoning is obtained by extending LK with two rules for sentential quantifiers. Adding (cut) yields a complete system for the explosive semantics.

Keywords: sentential operators; semantic and intensional paradoxes; classical logic; paraconsistent semantics; (semi)kernels of digraphs

1. Introduction

Much of the appeal of dialetheism arises from the liar and his likes appearing true and false, true iff false, or something of the sort. Confronted with the liar, we note the contradiction and ... continue reasoning the way we did before, as if locking it in a drawer to be reopened only for a renewed contemplation of its oddity. We do not explode, which seems an over-reaction. But declaring contradictions true is no less drastic. The notion of contradiction becomes abstruse, if at all survives. Unsatisfiable sentences disappear, at least in LP, where being only false is inexpressible and each falsifiable sentence is also satisfiable.

* The work was partially supported by a grant from L. Meltzers Høyskolefond.

Special Issue on Paraconsistency. Guest Editors: D. Fazio, T. Jarmużek, J. Malinowski
Received April 27, 2023. *Accepted* July 27, 2023. *Published online* January 2, 2024

Logic of sentential operators, LSO, avoids both extremes. It distinguishes contradictions from paradoxical claims which only lead to them, enabling classical reasoning (except explosion) in the face of the latter. The difference may be illustrated by the scenario from [4]:

> Imagine a situation in which many clubs have hired secretaries but have established rules excluding such secretaries from membership. Suppose that these secretaries form their own club, the rules of which state: 'A person is eligible to join this club if, and only if, he (she) is secretary of a club which he (she) is not eligible to join.' All goes well for the club until it hires itself a secretary [...] who has the misfortune of being secretary of no other club.

Possible occurrences of such situations are not prevented by the implied contradiction. It does not affect the world and clubs can function with secretaries until the unfortunate hiring, upon which one simply adjusts the rule, e.g., excepting the secretaries' club. The possibility of actually occurring distinguishes such situations from plain contradictions. One can not both be and not be a secretary (putting possible ambiguities aside) — presumably, even for the dialetheists.

Another difference is that paradoxes, although implying contradictions, are not merely hidden or unnoticed ones. Contradictions, when identified, are simply recognized as false, while paradox can not be assigned any boolean value. This difference disappears when metalanguage is internalized in the object-language via Gödelization or similar means. Paradox becomes then a mere contradiction involving perhaps metalanguage, but only informally, since metalanguage became part of the object-language. The main medicines in classical context are restrictions on convention (T) that ensure consistency by preventing interpretation of some numbers as statements, which they actually code. But convention (T) provides only one example of problems. When metapredicates, internalized as predicates on arithmetized syntax, reproduce some basic modal properties, elements of temporality, or just negation, the diagonalization lemma yields paradoxes without convention (T) [14, 24, 12, 6].

Corresponding paradoxes do not arise with operators, suggesting that they may be less paradox prone. Advantages of modal operators are reviewed in [11], while this paper develops such suggestions into a general formalization of sentential operators, not limited to modalities. We view an operator applied to a sentence S as a metastatement about S. It may concern syntax of S, its semantics or mark its intensional context, e.g., of being said or claimed. Appropriate axiomatizations can turn operators into usual modalities, but we do not address special cases and view operators here primarily as devices of metalanguage. Of course, with operators handling enough of syntax, especially substitution, the usual paradoxes return [17, 9], but this requires operators on open formulas, which are left for future work. In the present paper operators and quantification over sentences (closed formulas) give self-reference without diagonalization lemma nor unavoidable paradoxes. Reflecting a language model different from that arising with predicates over arithmetized syntax, sentential operators offer a different view of paradoxes and different ways to avoid them:

- Paradoxes are not mere contradictions but form locally coherent wholes. Unlike contradictions, they can occur in the world without forcing any explosion.
- They arise only in the metalanguage, by unfortunate definitions of sentential operators (restriction to sentences reflects only restrictions of the present formalism, not of this claim which extends to operators on open formulas).
- Paradoxical claims imply only specific contradictions. The liar implies that he is lying and not lying, but nothing about the actual world or statements of others. In particular, no object-level statements follow from such metalevel claims.
- Paradoxes are avoided by not making such claims, formally, excluding certain valuations of sentential operators, which form atomic statements of the metalanguage.
- The difference between merely implying a contradiction and explosion is captured in reasoning by inadmissibility of (cut). It is admissible only at the object-level, while adding its unrestricted version turns paraconsistent logic into explosive one.

Paraconsistency of LSO is thus extremely weak. No model satisfies $A \wedge \neg A$, but paradoxes of metalanguage, implying contradictions, have locally coherent models. Contradictions implied concern only metalanguage, statements under operators, e.g., that the liar lies and tells the truth. Like in informal reasoning, virtually nothing else is implied by the liar. Affecting thus only metalanguage, paradoxes do not force explosion and allow classically consistent interaction with the rest of the world, e.g., in the object-language.

Section 2 introduces an extension of any classical, first or higher order, language with sentential quantifiers and operators and Section 3 presents the corresponding semantic notions. Any classical language $\mathcal{L}$ is first extended with quantification over sentences, *s-quantification*, that is not propositional (ranging only over truth-values) but substitutional with the unrestricted substitution class, containing all sentences of the extended language. This extension does not introduce any paradoxes, as it does not even increase expressive power of the language. Then, as indicated, we do not internalize metalanguage, but use instead sentential operators (*sentential predicates* or *s-predicates* are synonyms), obtaining the full language $\mathcal{L}^+$. Operators distinguish the metalevel from the object-level. Unrestricted s-quantification extends to the full language $\mathcal{L}^+$ which remains consistent, that is, paradoxes can be avoided, but can now occur. Just like informally they occur due to some maliciously formulated claims, they appear formally due to unfortunate atomic claims, valuations of s-predicates. Reading $K(S)$ as Karen saying sentence S, $Kl = \forall\phi(K\phi \to \neg\phi)$ states that every sentence Karen says is false. $K(Kl)$ is not paradoxical, as Karen can also say some true things. However, Karen saying only Kl is paradoxical, implying $Kl \wedge \neg Kl$, and this is caused only by what Karen is saying, valuation of the s-predicate K.

S-predicates need not be truth-functional and can treat arguments purely syntactically, but do not aim at any deeper analysis of intensional phenomena. LSO is an intensional logic only in so far as s-predicates can be opaque, failing to preserve logical equivalence of arguments. However, it neither provides any intensional semantics nor considers the status of propositions or propositional attitudes. Propositions appear at most as mere sentences, while examples blur easily borders between "says ϕ", "thinks ϕ", "intends ϕ", etc. The significant distinction is that between statements with and without s-predicates, between the metalevel and the object-level. Modalities, attitudes or intensions can be handled by further axiomatizations of s-predicates.

Although intensional paradoxes are not addressed explicitly, they arise and can be treated in LSO in the same way as the semantic ones. The liar is significantly different neither from one not believing any of his beliefs nor from a club whose members are people not belonging to any club. Problems are caused by the same patterns involving primarily vicious circularity, captured precisely by our semantics utilizing graphs. Truth of sentence $\forall\phi.\phi$ requires truth of each instance, in particular, of this very sentence. It is false due to existence of other false sentences, but the graph semantics substantiates such dependencies and allows to handle related circularity and impredicativity, marginalizing for instance the issue of (un)groundedness. Vicious circles, that is, unresolvable odd cycles in language graphs, are the prime reasons for paradoxes, distinguishing them also from contradictions. (Also Yablo-like paradoxes can be expressed in LSO, but we do not discuss them separately here.)

Another central feature of our semantics is that interpretation of consistent theories, coinciding with the classical semantics, arises as a special case of the non-explosive one. Relations between the two are summarized in Section 3.5. Informally, exclusive claim of always lying does not entail much beyond the contradiction that one is lying and not lying. LSO reflects this limited consequence. Nothing follows about what others may be saying, nor about whiteness of snow. Semantics of such local coherence, admitting nonsense but circumscribing its effects, utilizes the graph-theoretic generalization of kernels (providing the classical, explosive semantics) to semi-kernels. These are like locally coherent situations, allowing classical valuation of the pronounced sentences but disallowing its extension to the whole language, when contradiction is implied.

These close connections between explosive and paraconsistent semantics are reflected in reasoning system LSO, extending LK with two rules for s-quantifiers. S-predicates and s-quantification bring flavor of higher order, but the operator form of the former and substitutional interpretation of the latter allow LSO to be sound and complete for the paraconsistent semantics of FOL^+. This reasoning is classical in that it retains all rules of LK, except for (cut). Explosive — one might want to say, fully classical — semantics is reflected by reasoning in LSO extended with (cut). It conjoins a specific contradiction, implied by a paradox, with all statements, implied by any contradiction. Thus also (cut) reflects the difference between contradictions and paradoxes, or between the object-level, where it is admissible, and the metalevel, where it makes paradoxes explode.

Although LSO allows thus paradoxes to occur, providing the means for their analysis as well as for functioning in their presence, it does not imply any paradoxes. Central theorems ensure consistency of the whole language, relatively to the choice of the atomic claims.

Section 4 collects earlier examples into a bigger one, illustrating also connections to the motivations (but not the formalism) of Jaśkowski's discussive logic. Section 5 summarizes the main threads, while Appendices A and B contains the proofs and needed technicalities.

2. Reasoning about sentences

A classical (propositional, first or higher order) language $\mathcal{L}$ is extended in two steps. First, $\mathcal{L}^{\Phi}$ is obtained by adding free sentential variables, *s-variables* Φ, as well as bound s-variables Ψ which are *s-quantified* in the usual manner, so that $\forall\phi\forall x(A(x)\vee\phi)$ is a sentence.[1] To this we add *operators* (or *s-predicates*), applicable to sentences, so that $\forall x\forall\phi(A(x)\vee\phi\vee P(\phi))$ is a sentence. In the resulting language $\mathcal{L}^+$, the substitution class for the interpretation of s-quantifiers comprises all sentences of $\mathcal{L}^+$. The grammar below contains these extensions of FOL to FOL$^+$ in the underlined productions for atoms $\mathbf{A}^+_{X,\Phi}$ and formulas $\mathbf{F}^+_{X,\Phi}$

$$
\begin{array}{lcl}
\mathbf{T}_X & ::= & X \mid Y \mid \mathit{Const} \mid \mathit{Func}(\mathbf{T}_X,\ldots,\mathbf{T}_X) \\
\mathbf{A}^+_{X,\Phi} & ::= & \mathbf{P}1(\mathbf{T}_X,\ldots,\mathbf{T}_X) \mid \underline{\mathbf{C} \mid \mathbf{P}2(\mathbf{F}^+_{X,\Phi},\ldots,\mathbf{F}^+_{X,\Phi},\mathbf{T}_X,\ldots,\mathbf{T}_X)} \\
\mathbf{F}^+_{X,\Phi} & ::= & \mathbf{A}^+_{X,\Phi} \mid \neg\mathbf{F}^+_{X,\Phi} \mid \mathbf{F}^+_{X,\Phi}\wedge\mathbf{F}^+_{X,\Phi} \mid \forall Y.\mathbf{F}^+_{X,\Phi} \mid \underline{\Phi \mid \Psi \mid \forall\Psi.\mathbf{F}^+_{X,\Phi}}
\end{array}
$$

Sentential constants $\mathbf{C}$ can be applied for naming sentences. S-predicates $\mathbf{P}2$ can also have terms $\mathbf{T}_X$ as arguments, but these are handled in the expected way, mostly, without explicit mention. Not assuming any semantic restrictions, s-predicates treat their arguments purely syntactically, acting possibly as metapredicates in a theory of syntax, although the grammar above restricts their application to formulas and terms. In this paper, we restrict them even further, essentially to sentential operators, defining their semantics only in contexts where their arguments have no free variables, primarily, when applied to sentences. Like all formulas, atoms are divided into

(a) $\mathcal{L}$ atoms, $\mathbf{A}_X$, e.g., $A(t)$ for $A\in\mathbf{P}1$ and $t\in\mathbf{T}_X$, and
(b) metalevel or s-atoms, $\mathbf{A}^\circ_{X,\Phi}=\mathbf{A}^+_{X,\Phi}\setminus\mathbf{A}_X$, e.g., $C\in\mathbf{C}$, $R(S,t)$ for $R\in\mathbf{P}2$, $S\in\mathbf{F}^+_{X,\Phi}, t\in\mathbf{T}_X$.

For a set M, by $\mathbf{T}_M$ we denote the free term algebra over M, by $\mathbf{S}_M/\mathbf{S}^+_M$ all $\mathcal{L}/\mathcal{L}^+$ sentences over $\mathbf{T}_M$, and by $\mathbf{S}/\mathbf{S}^+$ all $\mathcal{L}/\mathcal{L}^+$ sentences. Superscript $\_^\circ$ marks the metalevel, added to the object-level $\mathcal{L}$ and yielding the resulting extension $\_^+$, e.g., $\mathcal{L}^\circ=\mathcal{L}^+\setminus\mathcal{L}$, $\mathbf{S}^\circ_M=\mathbf{S}^+_M\setminus\mathbf{S}_M$, etc.

Reasoning system LSO for FOL$^+$, given below, extends LK with two rules for s-quantifiers. The basic syntax uses only $\{\wedge,\neg,\forall\}$, with other

[1] Syntactic distinction between free and bound variables ($\Phi\cap\Psi=\emptyset$ for s-variables, and $X\cap Y=\emptyset$ for object-variables, where X/Φ are free and Y/Ψ bound) is a technicality which can be ignored in the text and examples as long as substitutions avoid variable capture. It only simplifies the proof of completeness.

connectives and $\exists$, and rules for them, defined in the classical way. Sequents, written $\Gamma \Rightarrow \Delta$, are formed from countable sets $\Gamma \cup \Delta$ of $\mathbf{F}^+_{X,\Phi}$. By $\Gamma \vdash \Delta$ we denote provability of $\Gamma \Rightarrow \Delta$ in LSO.

(Ax) $\Gamma \vdash \Delta \quad \text{for} \quad \Gamma \cap \Delta \neq \emptyset$

$(\neg_L)\ \dfrac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}$ $\qquad (\neg_R)\ \dfrac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$

$(\wedge_L)\ \dfrac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$ $\qquad (\wedge_R)\ \dfrac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$

$(\forall_L)\ \dfrac{F(t), \Gamma, \forall y F(y) \vdash \Delta}{\Gamma, \forall y F(y) \vdash \Delta}\ t \in \mathbf{T}_X$ $\qquad (\forall_R)\ \dfrac{\Gamma \vdash \Delta, F(x)}{\Gamma \vdash \Delta, \forall y F(y)}\ x \in X$

$(\forall^+_L)\ \dfrac{F(S), \Gamma, \forall \psi F(\psi) \vdash \Delta}{\Gamma, \forall \psi F(\psi) \vdash \Delta}\ S \in \mathbf{F}^+_{X,\Phi}$ $\qquad (\forall^+_R)\ \dfrac{\Gamma \vdash \Delta, F(\alpha)}{\Gamma \vdash \Delta, \forall \psi F(\psi)}\ \alpha \in \Phi$

In $(\forall^+_L), (\forall_L)$ the substituted t, S are arbitrary (S violates typically subformula property), while x, α in $(\forall^+_R), (\forall_R)$ are fresh free variables.

Infinite sequents allow to handle cases of infinite axiomatizations. An example is a typical situation of making only finitely many claims. Karen saying only A or B is not captured by $K(A) \vee K(B)$, but requires in addition explicit $\neg K(S)$ for every S distinct from A and B. Such cases have finite representation using s-predicate for syntactic equality of sentences, *s-equality* $S \doteq Q$, which requires only a trivial check. We consider practically only sequents with no free s-variables, but they are useful as eigenvariables, rule $(\forall^+_R)$, and for handling s-equalities. For instance, 'Karen saying only ϕ' is expressible as $K\phi \wedge \forall\psi(K\psi \rightarrow \psi \doteq \phi)$, abbreviated by $K!\phi$. The following rules suffice, for $S, Q, R \in \mathbf{F}^+_{X,\Phi}$; in (neq) $Q \not\sim R$ denotes that Q and R are not unifiable (with standard first-order unification, only substituting formulas for s-variables):

$$(\text{ref})\ \frac{S \doteq S, \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \qquad (\text{rep})\ \frac{A(S), A(Q), S \doteq Q, \Gamma \vdash \Delta}{A(Q), S \doteq Q, \Gamma \vdash \Delta}$$

$$(\text{neq})\ \frac{\Gamma \vdash \Delta, Q \doteq R}{\Gamma \vdash \Delta}\ Q \not\sim R.$$

Claims like 'for every sentence α except $S_1, \ldots, S_n$: $D(\alpha)$' become finitely expressible as $\forall\alpha(\alpha \not\doteq S_1, \ldots, \alpha \not\doteq S_n \rightarrow D(\alpha))$, allowing sometimes to establish $\forall\phi D(\phi)$ by finite case analysis, introducing fresh $\alpha \in \Phi$:

$$(\forall^{\doteq}_R)\ \frac{\Gamma \vdash \Delta, D(S_1) \ldots \Gamma \vdash \Delta, D(S_n) \qquad \Gamma \vdash \Delta, D(\alpha), \alpha \doteq S_1, \ldots, \alpha \doteq S_n}{\Gamma \vdash \Delta, \forall\phi D(\phi)}$$

Some properties of LSO, like soundness/completeness, the interaction of its object-level and metalevel, the role of (cut), will be discussed along with the explanation of the semantics.

3. Semantics

We keep the presentation focused on FOL, but semantic definitions and results of this section work equally for higher order classical logics. Informally, an interpretation of the object-level sentences $\mathbf{S}$ in a structure M is extended to $\mathbf{S}^+$ by interpreting s-quantifiers substitutionally:

$$M \models \forall\phi F(\phi) \iff \forall S \in \mathbf{S}^+\colon M \models F(S). \tag{3.1}$$

The right side has instances like $F(\forall\phi F(\phi))$ or $F(F(\forall\phi F(\phi)) \wedge \forall\phi F(\phi))$, apparently involving the definiendum. Such circularities are handled by recasting semantics in terms of graph kernels.

Saying "graph" we mean a digraph $G = (\mathbf{V}_G, \mathbf{E}_G)$, with $\mathbf{E}_G \subseteq \mathbf{V}_G \times \mathbf{V}_G$, and $\mathbf{E}_G^-$ denoting the converse of $\mathbf{E}_G$. The subscript $\__G$ is dropped when an arbitrary or fixed graph is addressed. For a binary relation E, we let $E(x) = \{y \mid E(x,y)\}$ and extend function applications pointwise to sets, $E(X) = \bigcup_{x\in X} E(x)$. A *kernel* (or *solution*, introduced in [27]) of G is a $K \subseteq \mathbf{V}$, which is

(a) independent, i.e., $\mathbf{E}^-(K) \subseteq \mathbf{V} \setminus K$, and
(b) absorbing, i.e., $\mathbf{V} \setminus K \subseteq \mathbf{E}^-(K)$,

in short, such that $\mathbf{E}^-(K) = \mathbf{V} \setminus K$. Equivalently, it is an assignment $\kappa \in \mathbf{2}^{\mathbf{V}}$, such that

$$\forall x \in \mathbf{V}\colon \kappa(x) = \mathbf{1} \iff \forall y \in \mathbf{E}(x)\colon \kappa(y) = \mathbf{0}. \tag{3.2}$$

Intuitively, each edge marks negation of its target, and branching stands for conjunction of such negations. Given (3.2), the set $\{x \in \mathbf{V} \mid \kappa(x) = \mathbf{1}\}$ satisfies (a) and (b), while if K satisfies (a), (b) then $\kappa \in \mathbf{2}^{\mathbf{V}}$ given by $\kappa(x) = \mathbf{1} \iff x \in K$ satisfies (3.2). We therefore do not distinguish the two and by $sol(G)$ denote the set of kernels or such assignments. Graph G is *solvable* if $sol(G) \neq \emptyset$.

The equation $\mathbf{E}^-(K) = \mathbf{V} \setminus K$ means that kernel K covers the whole graph, where a subset $L \subseteq \mathbf{V}$ *covers* vertices $L \cup \mathbf{E}^-(L)$, denoted by $\mathbf{E}^-[L]$. A valuation is *coherent* on vertices for which it satisfies (3.2), so a kernel represents a coherent valuation of all sentences. Our semantics is two-valued but admits paraconsistency, that is, only locally coherent valuations with no extension to the whole language. In the absence of a kernel, containing all sentences required to be true, a relevant part of the graph may still be covered by a *semi-kernel*, [15], namely, an $L \subseteq \mathbf{V}$ such that

$$\mathbf{E}(L) \subseteq \mathbf{E}^-(L) \subseteq \mathbf{V} \setminus L. \tag{3.3}$$

The set of semi-kernels of G is denoted by $SK(G)$. A kernel is a semi-kernel covering the whole graph. In a kernel, falsehood of every vertex in $\mathbf{V} \setminus K$ is justified by an edge it has to some (true) vertex in K. In a semi-kernel L, such a justification is required only for vertices which are out-neighbours of L and must have an edge back to L, $\mathbf{E}(L) \subseteq \mathbf{E}^-(L)$. The inclusion $\mathbf{E}^-(L) \subseteq \mathbf{V} \setminus L$ makes L independent. A semi-kernel L satisfies equivalence (3.2) for all $x \in \mathbf{E}^-[L]$. Thus it represents a coherent situation, in the sense that all statements denied by any true one (in L) are false (in $\mathbf{E}^-(L)$), while every false statement denies some true one. We will later see that such a coherent situation, although locally consistent in this sense, can entail inconsistency.

Every graph possesses a semi-kernel, since $\emptyset$ satisfies trivially (3.3). But semi-kernels of interest are nontrivial, also in graphs not possessing any kernel, as we will see in what follows.[2]

3.1. Language graphs

Semantics of a language $\mathcal{L}^+$ is defined by (semi)kernels of *language graphs*. One such graph is constructed for each $\mathcal{L}$-domain: a set M with an interpretation of $\mathcal{L}$-terms $\mathbf{T}_M$ but not of the predicate symbols. Graph's vertices are sentences $\mathbf{S}^+_M$ and outgoing edges amount to conjunction of the negations of their targets. A *sink* is a vertex with no outgoing edges and $\overline{X}$ stands for $\neg X$.

DEFINITION 3.1. The *language graph* $\mathcal{G}_M(\mathcal{L}^+)$, for a language $\mathcal{L}^+$ and domain M, is given by:

1. Vertices $\mathbf{V} = \mathbf{S}^+_M$
2. Each atomic sentence $A \in \mathbf{A}^+_M$, except s-equality, has a 2-cycle to its negation: $A \leftrightarrows \overline{A}$.
3. For each $S \in \mathbf{S}^+_M$, s-equality atom $S \doteq S$ is a sink; for each syntactically distinct $S, Q \in \mathbf{S}^+_M$, vertex $Q \doteq S$ has an edge to the sink $\overline{Q \doteq S}$.
4. Each nonatomic sentence $S \in \mathbf{S}^+_M$ is the root of the subgraph $\mathcal{G}_M(S)$:

$\mathcal{G}_M(S) =$	root	with edges	to the root of:
(a)	$\neg F$	$\to$	$\mathcal{G}_M(F)$,
(b)	$F_1 \wedge F_2$	$\to$	$\mathcal{G}_M(\neg F_i)$, for $i \in \{1, 2\}$,
(c)	$\forall x F x$	$\to$	$\mathcal{G}_M(\neg F(m))$, for each $m \in M$,
(d)	$\forall \phi F \phi$	$\to$	$\mathcal{G}_M(\neg F(S))$, for each $S \in \mathbf{S}^+$.

This definition is for FOL, but when $\mathcal{L}$ is higher-order the only difference is the domain M, containing required sets, with object quantifier(s) in 4.(c) ranging over their required $\mathcal{L}$-domains.

[2] The branch of argumentation theory arising from [7] shares only its origins in a similar reading of digraphs and their (semi)kernels. Links to reference graphs, used in [21, 3] for paradox analysis, although closer, are also insignificant.

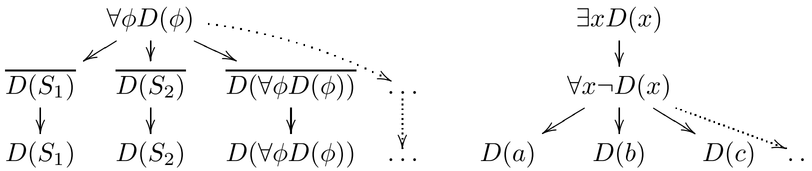

Figure 3.1.

Keeping $\mathcal{L}^+$ implicit, we write usually $\mathcal{G}_M$ instead of $\mathcal{G}_M(\mathcal{L}^+)$, and by $\mathcal{LG}r$ denote the class of all language graphs for a language relevant in any actual context. We drop also M when it is inessential, and write $\mathcal{G}$ for $\mathcal{G}_M(\mathcal{L}^+)$.

Drawing graphs, different vertices are often labelled by the same sentence. Two such vertices, say x and y, have then isomorphic out-neighbourhoods and recursively so, i.e., the subgraphs $\mathbf{E}^*(x)$ and $\mathbf{E}^*(y)$ are isomorphic. Identifying such vertices does not change (semi)kernels in any essential way, as intuition suggests and Fact A.2 shows. AUX denotes the set of such extra vertices serving the presentation only.

For $S \in \mathbf{S}$, the subgraph $\mathcal{G}(S)$ is actually a tree $T(S)$, reminding of S's parse tree but, primarily, reflecting the semantics of the formula constructors $(\neg, \wedge, \forall)$ in terms of kernels. Out-branching represents conjunction (or universal quantification), and each edge negation of its target. The 2-cycles at atoms force, in any kernel, choice of one element from each pair, giving valuations of atoms; sinks are true by (3.2).

The universal and existential quantifiers give rise to branchings, as shown in Figure 3.1, to instantiations of the quantified variables by all elements a, b, c, ... of the domain, and of s-variables by all $\mathbf{S}^+$. (A *double edge* $x \to y \to z$, where x has no other out-neighbours and y no other neighbours, can be contracted by removing y and identifying $x = z$, Fact A.1. This is done for $\exists$-pattern to the right.)

Quantifier prefix is converted to the graph by successively performing such branchings and instantiations, until no quantified variables remain.

Representation of propositional connectives follows the same pattern: conjunction by branching and negation by an edge. Assuming a sentence in prenex form with the matrix in DNF, the quantifier prefix ends with DNF-feet, one for each instantiation of the quantified variables. For example, DNF matrix $D(\phi, x) = (\neg\phi \wedge \neg Q(x)) \vee (\neg P(\phi) \wedge R)$, where R is a ground atom, gives one DNF-foot for each instantiation of ϕ and x, e.g., by $S \in \mathbf{S}^+$ and $a \in M$ in $\mathcal{G}_M(D(S,a))$ (see Figure 3.2).

Vertex $\circ_{\overline{D(S,a)}}$ is the sentence $\neg C_1(S,a) \wedge \neg C_2(S,a) \in \mathbf{S}^+_M$, while the auxiliary $C_1(S,a)$ the sentence $\neg S \wedge \neg Q(a)$. For $K \in sol(\mathcal{G}_M(D(S,a)))$: $D(S,a) \in K$

$$
\begin{tikzcd}
D(S,a) \arrow[rrr] & & & \circ_{\overline{D(S,a)}} \arrow[dll] \arrow[dr] & & \\
& C_1(S,a) \arrow[d] \arrow[dr] & & & C_2(S,a) \arrow[d] \arrow[dr] & \\
& S & Q(a) \arrow[d, shift left] & & P(S) \arrow[d] & \overline{R} \arrow[d, shift left] \\
& \neg S \arrow[u] & \overline{Q(a)} \arrow[u, shift left] & & \overline{P(S)} & R \arrow[u, shift left]
\end{tikzcd}
$$

Figure 3.2. A DNF-foot of matrix $D(\phi, x) = (\neg\phi \wedge \neg Q(x)) \vee (\neg P(\phi) \wedge R)$.

iff $\circ_{\overline{D(S,a)}} \notin K$ iff $C_1(S,a) \in K \vee C_2(S,a) \in K$ iff $\{\neg S, \neg Q(a)\} \subseteq K \vee \{\neg P(S), R\} \subseteq K$, reflecting the expected $D(S,a) = \mathbf{1}$ iff $S = \mathbf{0} = Q(a) \vee P(S) = \mathbf{0} \wedge R = \mathbf{1}$.

Subgraph $\mathcal{G}_M(\mathcal{L})$ described so far captures language $\mathcal{L}$. For each $\mathcal{L}$-sentence A, its subgraph $\mathcal{G}_M(A)$ sketched above is a tree except that, instead of leaves, there are atoms with 2-cycles. Exactly one element from such a cycle can be in any kernel and every such a tree has exactly one kernel for every selection from these cycles. Inclusion of $\overline{P(S)}$ and R from (3.2) in a kernel K forces, by independence, $P(S)$ and $\overline{R}$ out of it. This, in turn, forces $C_2(S,a) \in K$ by absorption, so that $\circ_{\overline{D(S,a)}} \notin K$ and $D(S,a) \in K$. This implication from $\{\overline{P(S)}, R\} \subseteq K$ to $D(S,a) \in K$ reflects the implication from $\neg P(S) \wedge R$ to $D(S,a)$. Kernel K of $\mathcal{G}_M(\mathcal{L})$ represents exactly the satisfied formulas under valuation of atoms $K|_{\mathbf{A}_M}$, given by the selection from atomic 2-cycles.

There is thus a bijection mapping a FOL structure (M, ρ), $\rho \in \mathbf{2}^{\mathbf{A}_M}$, to the language graph with kernel $(\mathcal{G}_M(\mathcal{L}), K_\rho)$, where $A \in K_\rho \iff \rho(A) = \mathbf{1}$ for $\mathcal{L}_M$-atoms A. Then also for all $S \in \mathbf{S}_M$

$$(M, \rho) \models S \iff S \in K_\rho \tag{3.4}$$

and this correspondence underlies the generalization of FOL semantics in what follows. A kernel for a language graph determines boolean values of all sentences, amounting to absence of paradoxes.

The full graph $\mathcal{G}_M(\mathcal{L}^+)$ has, besides the forest $\mathcal{G}_M(\mathcal{L})$ described above, also subgraph $\mathcal{G}_M(\mathcal{L}^\circ)$ containing subgraphs $\mathcal{G}_M(A)$ for sentences with s-quantifiers, $A \in \mathbf{S}^\circ_M$. Such a $\mathcal{G}_M(A)$ is obtained point 4.(d) of Definition 3.1, but substitution into A of each $S \in \mathbf{S}^+$ for s-variable ϕ in a *sentential position*, i.e., not in the scope of any s-predicate, like ϕ in $C_1 = \neg\phi \wedge \neg Q(x)$, becomes so to say postponed. S is not processed during further construction, and the vertex appearing in its position(s) as leaf of the DNF-foot in (3.2) obtains a double edge to the root of the subgraph $\mathcal{G}_M(S)$. (The double edge can be contracted, as explained before but keeping the two separate gives a more intuitive picture.) In particular, also sentence A is thus substituted for s-variable ϕ, and the resulting

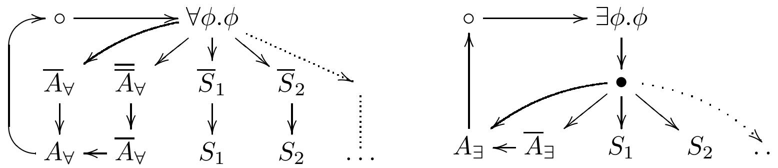

Figure 3.3.

leaf has a double edge to the root of this very $\mathcal{G}_M(A)$. Every $S \in \mathbf{S}^+$, substituted for ϕ in A, either occurs on some path from the root A as an internal node or not. In the former case, the leaf S is called an *internal atom* of $\mathcal{G}_M(A)$, and has a double edge back to its occurrence in $\mathcal{G}_M(A)$ (possibly forming a cycle). In the latter case, when S occurs in $\mathcal{G}_M(A)$ only as a leaf, it is its *external atom*, $ext(\mathcal{G}_M(A))$, and has a double edge to the root of its separate $\mathcal{G}_M(S)$. In this case, if S itself is s-quantified, its subgraph instantiates its s-variables by all sentences, in particular by A, giving paths back to the root of $\mathcal{G}_M(A)$. All s-quantified sentences among $\mathbf{S}^\circ$ form thus one strongly connected component of $\mathcal{G}_M(\mathcal{L}^+)$. Their leaves instantiated with sentences $\mathbf{S}_M$ have double edges to the subgraph $\mathcal{G}_M(\mathcal{L})$, but there are no edges returning thence to $\mathcal{G}_M(\mathcal{L}^\circ)$.

Such double edges, yielding cycles and connecting distinct sentence subgraphs, arise only from sentences substituted for s-variables in sentential positions. Sentences substituted into *nominal positions*, i.e., into the scope of some s-predicate, form atoms with 2-cycles to their duals, like $P(S) \leftrightarrows \overline{P(S)}$ in (3.2), arising from substituting S into $P(\phi)$.[3]

Example 3.1. Let S_1, S_2, ... stand for all $\mathbf{S}^+$, except the leftmost two in each graph sketched below: $\mathcal{G}(A_\forall)$, for sentence $A_\forall = \forall\phi.\phi$, and $\mathcal{G}(A_\exists)$ for $A_\exists = \neg\forall\phi.\neg\phi$. Only the essential aspects are indicated, ignoring other edges and cycles in Figure 3.3.

In the left graph $\mathcal{G}(A_\forall)$, the two vertices $\overline{A}_\forall$ (as well as $\forall\phi.\phi$ and $A_\forall$) could be identified. Any $S_i \in \mathbf{S}^+$ valuated to $\mathbf{0}$ yields $\overline{S}_i = \mathbf{1}$ and $\forall\phi.\phi = \mathbf{0}$, but even if all $S_i = \mathbf{1}$, the mere cycles involving $A_\forall$ and $\overline{A}_\forall$ force $\forall\phi.\phi = \mathbf{0}$. To obtain a kernel, the odd cycle via $\overline{\overline{A}}_\forall$ must be broken, i.e., some of its vertices must have an out-neighbour $= \mathbf{1}$. If all $\overline{S}_i = \mathbf{0}$, this still happens when both $\overline{A}_\forall = \mathbf{1} = \circ$, making $A_\forall = \overline{\overline{A}}_\forall = \mathbf{0} = \forall\phi.\phi$. Thus, $\forall\phi.\phi$ is a counterexample to its own truth.

[3] These 2-cycles are formed only for atoms with the outermost s-predicate. Substituting S into $P(\phi, Q(\phi))$ yields atom $P(S, Q(S))$ with edges to its dual $\overline{P(S, Q(S))}$. The inner $Q(S)$ does not obtain any edges to its dual $\overline{Q(S)}$, which happens only for the atom $Q(S)$ that occurs in sentential position.

A dual situation occurs in $\mathcal{G}(A_\exists)$, where kernel requires breaking the odd cycle via $\bullet$ and $\overline{A}_\exists$. This happens if any $S_i = \mathbf{1}$, but even if they all are $\mathbf{0}$, the odd cycle via $\overline{A}_\exists$ and even one via $A_\exists$ force $\exists\phi.\phi = \mathbf{1}$. The only way to break the odd cycle is then by $\overline{A}_\exists = \mathbf{0}$, which requires $A_\exists = \mathbf{1}$, complying with $\bullet = \mathbf{0} = \circ$ and $\exists\phi.\phi = \mathbf{1}$, which provides thus a witness to its own truth. $\square$

This completes the description of language graphs. They can be seen as consisting of the trees for all object-language sentences and the circular component of the metalanguage, capturing respectively the classical compositional semantics of the former and the holistic self-references of the latter. Before using them to define validity, an important issue should be clarified. One wonders naturally if complexity of language graphs does not hide unavoidable paradoxes. Section 3.2 shows that, for language $\mathcal{L}^\Phi$ with s-quantifiers but no s-predicates, language graphs not only have kernels, but have unique one for every valuation of $\mathcal{L}$ atoms. Section 3.3 shows then that although in the full language $\mathcal{L}^+$ paradoxes become expressible, also its graphs are solvable.

3.2. Sentential quantifiers and solvability of $\mathcal{G}(\mathcal{L}^\Phi)$

In $\mathcal{L}^\Phi$, extending the object-language $\mathcal{L}$ with s-quantifiers but no s-predicates, s-variables occur only in sentential positions. The only atoms are $\mathcal{L}$-atoms $\mathbf{A}$ (and possibly $\mathbf{C}$. When $\mathbf{A} = \emptyset$, the language $\emptyset^\Phi$ is that of quantified boolean sentences, QBS.) Given a domain M and $\rho \in \mathbf{2}^{\mathbf{A}_M}$, all $\mathcal{L}^\Phi$ sentences obtain values under unique extension of ρ to a kernel $\hat{\rho}$ of the graph $\mathcal{G}_M(\mathcal{L}^\Phi)$.

THEOREM 3.1 (A.4). *In any graph $\mathcal{G}_M(\mathcal{L}^\Phi)$, each valuation of atoms $\rho \in \mathbf{2}^{\mathbf{A}_M}$ has a unique extension $\hat{\rho} \in sol(\mathcal{G}_M(\mathcal{L}^\Phi))$ with $\hat{\rho}|_{\mathbf{A}_M} = \rho$.*

Proofs are given in Appendix A (with the corresponding theorem number in parentheses), but we comment briefly that the proof of this theorem relies on the lemma below, showing that for any solution of $\mathcal{G}^-_M(S)$ — denoting, for $S \in \mathbf{S}^\Phi_M \setminus \mathbf{S}_M$, vertices of $\mathcal{G}_M(S)$ without those in its DNF-feet — depends on the valuation of $\mathbf{A}_M$, but not of external atoms $ext(\mathcal{G}_M(S))$, as the second part of the lemma states. In a way, DNF-matrix determines a boolean function, and the value of S depends on this function (and valuation of $\mathbf{A}_M$), rather than on the values of external atoms, which span all possibilities. Valuation of $ext(\mathcal{G}_M(S))$ affects, of course, values in DNF-feet, where they occur. For either A from Example 3.1, the lemma implies that valuation of $\mathcal{G}^-(A)$, i.e., the root vertex with its marked cycles, is independent from valuation of all external atoms among S_1, S_2, etc.

LEMMA 3.2 (A.5). *For every $\mathcal{G}_M(\mathcal{L}^\Phi)$ and sentence $A \in \mathbf{S}^\Phi_M$, each valuation ρ of atoms $\mathbf{A}_M$ and external atoms of $\mathcal{G}_M(A)$, $\rho \in \mathbf{2}^{\mathbf{A}_M \cup ext(\mathcal{G}_M(A))}$, has a unique extension to $\rho_A \in sol(\mathcal{G}_M(A))$. Valuation of atoms, $\rho|_{\mathbf{A}_M}$, determines restriction of ρ_A to $\mathcal{G}^-_M(A)$.*

Valuation of sentences $\mathbf{S}^{\Phi}_M \setminus \mathbf{S}_M$ does not have any standard definition, which is merely suggested by (3.1). By Theorem 3.1, such a valuation $\hat{\rho}$ is determined by $\rho \in \mathbf{2}^{\mathbf{A}_M}$, just as is valuation of $\mathbf{S}_M$. Existence and uniqueness of $\hat{\rho}$ ensure well-definedness of (3.1), given by the following.

DEFINITION 3.2. An $\mathcal{L}^{\Phi}_M$-sentence A is true in an $\mathcal{L}$ domain M under valuation $\rho \in \mathbf{2}^{\mathbf{A}_M}$, $(M,\rho) \models A$, iff $\hat{\rho}(A) = \mathbf{1}$ for the unique solution $\hat{\rho} \in sol(\mathcal{G}_M(\mathcal{L}^{\Phi}))$ with $\hat{\rho}|_{\mathbf{A}_M} = \rho$.

This gives, for any theory $\Gamma \subseteq \mathbf{S}^{\Phi}$ a well-defined class of its models

$$Mod(\Gamma) = \{(M,\rho) \mid \forall A \in \Gamma\colon (M,\rho) \models A\} = \bigcap_{A\in\Gamma} Mod(A).$$

The bijection (3.4) between FOL structures and graphs with kernels, mapping (M,ρ) to $(\mathcal{G}_M, K_\rho)$, extends to FOL$^{\Phi}$ mapping (M,ρ) to $(\mathcal{G}_M, \hat{\rho})$.

Theorem 3.1 implies that extension of a classical language $\mathcal{L}$ with quantification over all sentences to $\mathcal{L}^{\Phi}$ preserves $\mathcal{L}$'s property of having a unique consistent valuation of all sentences for every valuation of atoms: $\mathcal{L}^{\Phi}$ remains free from paradoxes. The same holds for the language $\mathcal{L}^{\doteq}$, extending $\mathcal{L}^{\Phi}$ with syntactic equality $\doteq$, by a non-trivial extension of the proof of Theorem 3.1.

CLAIM 3.3. *In any graph $\mathcal{G}_M(\mathcal{L}^{\doteq})$, each valuation of atoms $\rho \in \mathbf{2}^{\mathbf{A}_M}$ has a unique extension $\hat{\rho} \in sol(\mathcal{G}_M(\mathcal{L}^{\doteq}))$ with $\hat{\rho}|_{\mathbf{A}_M} = \rho$.*

3.3. Sentential predicates and solvability of $\mathcal{G}(\mathcal{L}^{+})$

Predicates applied to sentences provide only fresh atoms, so one might think that everything works unchanged. It does, if only such predicates are introduced without sentential quantifiers. The language graph — which is then, as for the object-language, a forest only with new s-atoms — is uniquely solvable for every valuation of atoms. However, combination of s-predicates with s-quantifiers changes things dramatically. For instance, blind ascriptions of truth, called also infinitary conjunctions, namely claims like "All Ks are true", for $K \in \mathbf{P}2$, become expressible as $\forall\phi(K\phi \to \phi)$.[4] Technically, a more significant novelty is the dependence of valuations of s-predicates on their argument *sentences*, not only boolean values of these sentences, and the possibility of violating semantic equivalence of arguments. Consequently, only even cycles can be broken, without breaking the corresponding odd ones, leading to paradoxes. Unlike valuations of $\mathcal{L}$-atoms in a domain M, which determine unique

[4] Their role for truth-theory has been discussed at least since Quine's [20]. When syntax is arithmetized, they become problematic due to complications in controlling their interaction with the restrictions on convention (T), e.g., [10, 18]. In LSO, a paradox requires a sentence or s-variable to occur in both a sentential and a nominal position, exemplified also by such blind ascriptions.

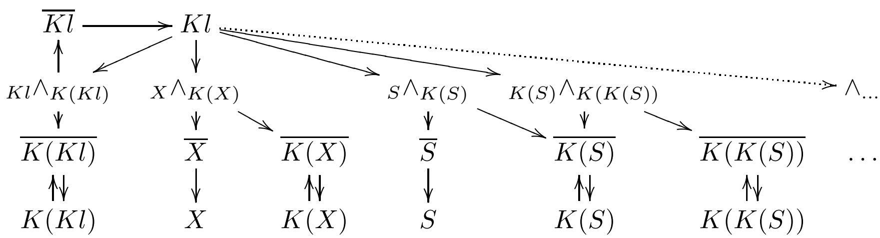

Figure 3.4.

solutions of the graph $\mathcal{G}_M(\mathcal{L}^{\Phi})$, some valuations of s-atoms can make graph $\mathcal{G}_M(\mathcal{L}^{+})$ unsolvable, as illustrated by the example below.

Concerning representation of the liar, its form $L \leftrightarrow \neg L$, or $L \leftrightarrow \neg\mathcal{T}(L)$ with (T) $\forall\phi(\mathcal{T}\phi \leftrightarrow \phi)$, gives a straightforward contradiction, distinguished from paradox in Section 3.5. Self-reference arises in LSO via s-quantification, so "This sentence is false" is recast as saying only "Every sentence I am saying now is false".

Example 3.2. The liar Karen says only that everything she is saying is false, $K!Kl$, where $Kl = \forall\phi(K\phi \to \neg\phi)$. Semi-kernel $L = \{K(Kl)\} \cup \{\overline{K(S)} \mid S \not\doteq Kl\}$ captures this situation, but can not be extended to any kernel because $K(Kl) = \mathbf{1}$ makes $\overline{K(Kl)} = \mathbf{0}$, while each ${}_{X}\wedge_{K(X)} = \mathbf{0}$, for $X \not\doteq Kl$, due to $\overline{K(X)} = \mathbf{1}$. Thus odd cycle $\overline{Kl} - Kl - {}_{Kl}\wedge_{K(Kl)}$ remains unresolved (see Figure 3.4). □

FOL^{+} can thus express some paradoxes, but does not imply any. They appear, as in the example, only due to unfortunate valuations of s-atoms. For a classical language $\mathcal{L}$, $\mathcal{L}^{+}$ remains consistent.

THEOREM 3.4. *Every language graph $\mathcal{G}_M(\mathcal{L}^{+})$ has a kernel.*

A simple way of ensuring solvability of a graph $\mathcal{G}_M(\mathcal{L}^{+})$ is to start with its solvable subgraph $\mathcal{G}_M(\mathcal{L}^{\Phi})$, and to introduce new s-predicate by *definitional extension*, i.e., by a sentence

$$\forall\phi(P(\phi) \leftrightarrow \exists\!\!\!\forall\psi F[\phi,\psi]) \tag{3.5}$$

where F is an $\mathcal{L}^{\Phi}$-formula (possibly with free variables ϕ among those of the left side $P(\phi)$). Theorem 3.4 follows directly from the following. (Kernel models of Γ, explained below, are graph kernels containing Γ.)

THEOREM 3.5 (A.8). *For every $\Gamma \subseteq \mathcal{L}^{+}$ and its definitional extension F, every kernel model of Γ can be extended to a kernel model of $\Gamma \cup F$.*

In practice, s-predicates are introduced also in ways other than definitional extensions, and then consistency has to be verified in each case. Theorem 3.5, applied after 3.1(3.3) guarantees only that no contradiction nor paradox is hiding in the language itself.

3.4. Semi-kernel and kernel semantics

Semantics utilizes (semi)kernels, representing sets of true sentences under valuations of atoms determined by these (semi)kernels. An $\mathcal{L}^+$-sequent $\Gamma \Rightarrow \Delta$ is *valid*, $\Gamma \models \Delta$, iff every relevant situation satisfies it in every language graph $\mathcal{G}_M \in \mathcal{LG}r(\mathcal{L}^\oplus)$, where $\mathcal{L}^\oplus \supseteq \mathcal{L}^+$ is an arbitrary extension of $\mathcal{L}^+$ with fresh s-constants.[5] A *situation* is a semi-kernel L, it is *relevant* if it covers $\Gamma \cup \Delta$, i.e., $\Gamma \cup \Delta \subseteq \mathbf{E}^-[L] = \mathbf{E}^-(L) \cup L$, and it *satisfies* the sequent if some $D \in \Delta$ is true, i.e., $D \in \Delta \cap L$, or some $G \in \Gamma$ is false, i.e., $G \in \mathbf{E}^-(L)$. (This is generalized to valuations $\alpha \in (M \cup \mathbf{S}^+)^{\mathcal{V}(\Gamma,\Delta)}$ of free variables $\mathcal{V}(\Gamma, \Delta)$, with $\alpha(A)$ denoting the result of such a valuation of A's free s-variables in $\mathbf{S}^+$ and its free o-variables in M.)

$$\begin{aligned}
\Gamma \models \Delta &\iff \forall \mathcal{L}^\oplus \supseteq \mathcal{L}^+\ \forall \mathcal{G}_M \in \mathcal{LG}r(\mathcal{L}^\oplus)\ \forall L \in SK(\mathcal{G}_M) \colon L \models \Gamma \Rightarrow \Delta, \\
L \models \Gamma \Rightarrow \Delta &\iff \forall \alpha \in M^{\mathcal{V}(\Gamma,\Delta)} \colon L \models_\alpha \Gamma \Rightarrow \Delta, \\
L \models_\alpha \Gamma \Rightarrow \Delta &\iff \alpha(\Gamma \cup \Delta) \subseteq \mathbf{E}^-[L] \to (\alpha(\Gamma) \cap \mathbf{E}^-(L) \neq \emptyset) \\
&\qquad \vee (\alpha(\Delta) \cap L \neq \emptyset).
\end{aligned} \tag{3.6}$$

For $\Gamma \subseteq \mathbf{S}^+$, *semi-kernel models* are pairs $(\mathcal{G}, L) \in \mathcal{LG}r(\mathcal{L}^+) \times SK(\mathcal{G})$ satisfying $\Rightarrow F$, for all $F \in \Gamma$. Such models allow conundrums among $\mathbf{S}^\circ$ affect their own truth, yielding paradoxes or sentences with undetermined values, even when the object-language and s-atoms are fully interpreted. Nevertheless, each valuation ρ of $\mathcal{L}$-atoms determines interpretation of the object-language, independent from possible paradoxes in the following sense. By Theorem 3.1, the subgraph $\mathcal{G}(\mathcal{L})$ (and $\mathcal{G}(\mathcal{L}^\Phi)$) has a kernel – reflecting simply the standard interpretation of $\mathcal{L}$ under ρ. Due to absence of edges from $\mathcal{G}(\mathcal{L})$ to $\mathcal{G}(\mathcal{L}^\circ)$, this kernel is a semi-kernel of $\mathcal{G}(\mathcal{L}^+)$, independent from valuation of metalevel sentences $\mathbf{S}^\circ$ and from possible nonexistence of a kernel of $\mathcal{G}(\mathcal{L}^+)$ extending ρ.

Metastatements in $\mathbf{S}^\circ$ do not affect facts also in the sense that John saying S excludes John not saying S, but allows John to say not-S and Karen to say anything. Contradicting anybody, even facts, does not affect the object-level, only limits the shared situation to the things agreeable with one's claims, precluding extension of such an agreement (semi-kernel) to the full language (kernel).

Semantics is thus non-explosive, admitting seeds of inconsistency in semi-kernels which can not be extended to kernels, but is two-valued: each semi-kernel determines a unique boolean value of each sentence, perhaps vacuously by not covering it. Semi-kernel models of a theory Γ need not cover the whole

[5] Definition relative to such extensions $\mathcal{L}^\oplus$ of $\mathcal{L}^+$ plays role in the proof of completeness Theorem 3.6 (Fact B.3). Completeness for validity relative only to $\mathcal{L}^+$ could be obtained by replacing $(\forall^+_R)$ by infinitary rule with premises $\Gamma \vdash \Delta, F(S)$ for all $S \in \mathbf{S}^+$.

language, and may exist even if Γ implies a contradiction, like model L for $K!Kl$ in Example 3.2. Even then consequences of Γ are not arbitrary, relying on its semi-kernel models. Syntactic and semantic analyses of FOL$^+$ theories are not separated by any gap. LSO provides a sound and complete reasoning for semi-kernel semantics.

THEOREM 3.6. *For a countable $\Gamma \cup \Delta \subseteq FOL^+$: $\Gamma \vdash \Delta \iff \Gamma \models \Delta$.*

Now, every kernel is a semi-kernel, so narrowing (3.6) to kernels yields a special case of the semantics. An $\mathcal{L}^+$-sequent $\Gamma \Rightarrow \Delta$ is *c-valid*, $\Gamma \models_c \Delta$, iff in every language graph $\mathcal{G}_M \in \mathcal{LG}r(\mathcal{L}^+)$, every kernel satisfies it under every valuation of free variables. The only difference from (3.6) is that in the first line, $L \in SK(\mathcal{G}_M)$ is replaced by $K \in sol(\mathcal{G}_M)$:

$$\Gamma \models_c \Delta \iff \forall \mathcal{L}^{\oplus} \supseteq \mathcal{L}^+ \ \forall \mathcal{G}_M \in \mathcal{LG}r(\mathcal{L}^{\oplus}) \ \forall K \in sol(\mathcal{G}_M)\colon K \models \Gamma \Rightarrow \Delta. \tag{3.7}$$

For $\Gamma \subseteq \mathbf{S}^+$, *kernel models* are pairs $(\mathcal{G}, K) \in \mathcal{LG}r(\mathcal{L}^+) \times sol(\mathcal{G})$ satisfying $\Rightarrow F$, for all $F \in \Gamma$. Such a model covers the whole language graph, since for $K \in sol(G)\colon \mathbf{E}^-(K) = \mathbf{V} \setminus K$. (Hence, the antecedent of implication in the third line of (3.6) is now trivially satisfied.) Consequently, if Γ forms a (locally coherent) situation but implies a contradiction, like $K!Kl$ in Example 3.2, it may have a semi-kernel model but not any kernel model. (Non)explosiveness of (semi)kernel semantics deserves some closer comments.

3.5. Paraconsistency versus (cut)

Although Karen can say whatever she likes, even $K(S \wedge \neg S)$, LSO is not dialetheic, as there is no semi-kernel satisfying $S \wedge \neg S$, that is, containing both S and $\neg S$. Its derivability from some assumptions signals the impossibility of combining them with any coherent valuation of S. Turning this into a definition, we call $S \subseteq \mathbf{S}^+$ a *contradiction*, $S \in \mathbb{C}$, if it is not contained in any semi-kernel, i.e., $S \not\subseteq L$ for every language graph $\mathcal{G}$ and $L \in SK(\mathcal{G})$. It is a *tautology*, $S \in \mathbb{T}$, if it is contained in every semi-kernel covering it, $S \subseteq \mathbf{E}^-[L] \Rightarrow S \subseteq L$. By Theorem 3.6, $\mathbb{C} = \{S \subseteq \mathbf{S}^+ \mid S \vdash \emptyset\}$ and $\mathbb{T} = \{S \subseteq \mathbf{S}^+ \mid \forall S_i \in S\colon \emptyset \vdash S_i\}$ for FOL$^+$. For a single sentence, $S \in \mathbb{C}$ abbreviates $\{S\} \in \mathbb{C}$.

Although not dialetheic, LSO is non-explosive because semi-kernels admit contradictions outside the covered set. Semantics (3.6) is local, checking satisfaction of $\Gamma \Rightarrow \Delta$ only in semi-kernels covering $\Gamma \cup \Delta$. A paradox — apparently meaningful statements which, at a closer analysis (expanding the context), lead to a contradiction — is represented by a set of sentences contained in a semi-kernel which can not be extended to a kernel. Statements implying a contradiction can thus form locally coherent situations that need not imply everything. For a semi-kernel L, truth condition (3.2), restricted to $\mathbf{E}^-[L]$ is the classical (kernel) condition. Consequently, reasoning with the same essentially classical system LSO remains sound. The difference from classical reasoning concerns provable contradictions.

If Karen claims to be always lying then LSO proves that she does not, $K(Kl) \vdash \neg Kl$, but this is no paradox. As noted by Prior, [19], Karen must then sometimes lie, $K(Kl) \vdash \neg\forall\phi(K\phi \to \phi)$. The resulting Prior's theorem, $K(\forall\phi(K\phi \to \neg\phi)) \to (\exists\phi(K\phi \wedge \phi) \wedge \exists\phi(K\phi \wedge \neg\phi))$, is still no paradox, as Karen can also say other things. If this is everything she says, then LSO proves that she is always lying, and $K!Kl \vdash Kl \wedge \neg Kl$ witnesses to a paradox.

Now, a contradiction entails every sentence S reflecting the fact that it does not belong to any semi-kernel. Still, although $Kl \wedge \neg Kl \vdash S$ for all S, so $K!Kl \nvdash S$ for some S, e.g., for $S = J(S_0)$, as witnessed by the following semi-kernel satisfying $K!Kl$ and $\neg S$:

$$Z = \{\neg S, K(Kl), \forall\phi(K\phi \to \phi \doteq Kl)\} \cup \{\neg K(S_i) \mid S_i \not\doteq Kl\}. \tag{3.8}$$

Semi-kernel Z witnesses also to $K!Kl \nvdash \emptyset$, justifying the fact that $K!Kl$ is not a contradiction. Karen can say only that she always lies.

This brings forth the difference between a contradiction, entailing every sentence and belonging to no semi-kernel, and a "half-contradiction" like $K!Kl$ which entails some contradiction, but not every sentence, and can be contained in a semi-kernel. S is an *s-contradiction* (a paradox) if $S \vdash C$ for some $C \in \mathbb{C}$, but there is a semi-kernel $L \supseteq S$. An s-contradiction involves necessarily s-predicates and the contradiction it entails involves these s-predicates. It does not entail most other contradictions nor any contingent object-level sentences.

Since s-contradiction entails some contradiction, like $K!Kl \vdash \neg Kl \wedge Kl$, while contradiction entails every sentence S, like $\neg Kl \wedge Kl \vdash S$, using (cut) would yield $K!Kl \vdash S$. However, semi-kernel in (3.8) provides a countermodel $Z \not\models K!Kl \Rightarrow S$, so (cut) is not sound. It is trivially admissible for the object-language, as long as only LK is used, but changes the semantic for the whole LSO. The contradiction $Kl \wedge \neg Kl$, following from Karen's statement, is not 'discovered' in Z. A semi-kernel that is not a kernel represents a limited context which is only locally consistent. It satisfies the classical condition (3.2) only on vertices it is covering without taking into account the whole language. Z allows thus Karen to say only Kl, but asking whether she is saying truth or not, Kl or $\neg Kl$, expands this context to the point where the paradox — the impossibility of a valuation of Kl coherent with Z — is discovered. This still does not prevent John from saying (or not) S_0, captured by a semi-kernel extending Z with $J(S_0)$ (or $\neg J(S_0)$).

Provability $K!Kl \vdash Kl \wedge \neg Kl$ does not imply nonexistence of a situation where Karen says only Kl, that is, of a semi-kernel containing $K!Kl$, as provability $K!Kl \vdash \emptyset$ would do, but nonexistence of such a semi-kernel covering also $Kl \wedge \neg Kl$. As its graph is $(Kl \wedge \neg Kl) \to \overline{Kl} \to Kl \ldots$, this means that no semi-kernel containing $K!Kl$ can contain Kl or $\neg Kl$. Semi-kernels containing $K!Kl$

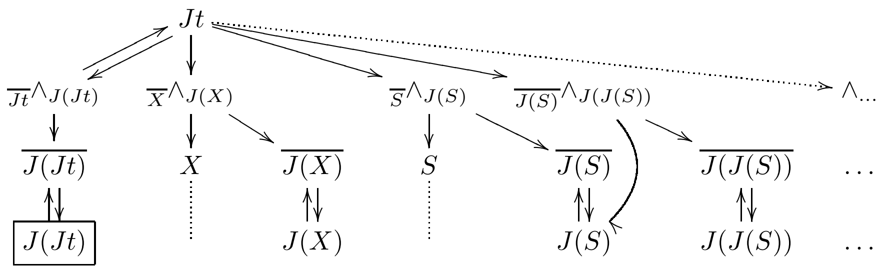

Figure 4.1. Truth-teller $J(Jt)$.

have a special relation to this particular contradiction, like informal reasoning concluding that the liar lies and does not lie, but nothing more. Most contradictions are still not derivable from $K!Kl$. To derive everything from $K!Kl$, via the contradiction it entails, (cut) is needed. It makes derived contradictions explode, bringing us back to the kernel semantics (3.7). Adding (cut) to LSO, yielding $\vdash_c$, does not trivialize logic, as it happens in non-transitive ST-systems [26, 25, 5]. Instead, it turns paraconsistent logic into explosive ones, providing a sound and complete reasoning system for it.

Fact 3.7 (B.4). *For a countable $\Gamma \cup \Delta \subseteq FOL^+$: $\Gamma \models_c \Delta$ iff $\Gamma \vdash_c \Delta$.*

4. An example

Non-explosive paradoxes are one feature distinguishing the metalanguage from the object-language. Another such feature are sentences that remain indeterminate in spite of fully interpreted object-language and s-atoms.

John saying only that he always tells the truth, the framed $J(Jt)$ with $Jt = \forall\phi(J\phi \to \phi)$ on the drawing (see Figure 4.1), is the truth-teller. Each $\overline{X}\wedge_{J(X)}$, for $X \not\doteq Jt$, is false due to John not saying X, while $\overline{J(Jt)} = \mathbf{0}$ leaves 2-cycle $\overline{Jt}\wedge_{J(Jt)} \leftrightarrows Jt$ with one solution $Jt = \mathbf{1}$ and the other $Jt = \mathbf{0}$.

Thus, unlike for the object-language $\mathcal{L}$, valuation of all $\mathcal{L}^+$-atoms may leave values of some $\mathcal{L}^+$-sentences undetermined. Considering this fact a flaw, as sometimes happens, seems due to internalization of the metalanguage in the object-language, which suggests that the former is just part of the latter and should behave in the same way. In LSO this indeterminacy is simply another, besides paradoxes, feature distinguishing the two. The difference between such innocent self-reference of the truth-teller and vicious circularity of paradoxes, reflecting informal indeterminacy of the former versus impossibility of valuating the latter, is captured by even versus odd cycles in language graphs.

The unproblematic status of the truth-teller amounts to the informal observation that it says nothing. Making no real claim, its truth or falsity makes no difference. A difference appears if he says also something else, because then, no matter what else it is, the 2-cycle $Jt \leftrightarrows \overline{{}_{Jt}}\wedge_{J(Jt)}$ can always be solved by $\overline{{}_{Jt}}\wedge_{J(Jt)} = \mathbf{1}$ and $Jt = \mathbf{0}$. The mere claim of telling only truth implies consistency of this claim being false.[6]

As we saw in Section 3.5, the liar K, stating only the single claim $Kl = \forall\phi(K\phi \to \neg\phi)$, does not entail almost anything except the contradiction of K lying and not lying. Semi-kernel $Z \setminus \{\neg S\}$ from (3.8), limits its consequences to this minimum. Generally, semi-kernels make statements of distinct persons independent, so that all can occur simultaneously, even if they contradict each other. In this sense LSO follows the tradition of Jaśkowski's discussive logic. Contradictory statements, confronting each other in the metalanguage, need not cause explosion, unlike contradictions which always do. Thus Karen saying X and Karen not saying X is a contradiction, unlike Karen saying X and $\neg X$, which remains the problem of Karen's or her interlocutors.

Figure 4.2 gives a part of the graph $\mathcal{G}^+$ for a discussive example, with John saying that Karen always lies, $J(Kl)$, and Karen saying that John always tells the truth. It joins the graph for the truth-teller $Jt = \forall\psi(J\psi \to \psi)$ with that for the liar $Kl = \forall\phi(K\phi \to \neg\phi)$ from Example 3.2. The two interact via the semi-kernel containing the framed statements, assumed to hold.

Truths $J(Kl) = \mathbf{1} = K(Jt)$ force $Jt = \mathbf{0}$, reflected by the provability $J(Kl), K(Jt) \vdash \neg Jt$. The situations of John and Karen are not symmetric, since $J(Kl), K(Jt) \nvdash \neg Kl$, with $Kl = \mathbf{1}$, for instance, if Karen says nothing but Jt, making all ${}_S\wedge_{K(S)} = \mathbf{0}$ (${}_{Jt}\wedge_{K(Jt)} = \mathbf{0}$ because $\neg Jt = \mathbf{1}$). Neither Kl is provable, $J(Kl), K(Jt) \nvdash Kl$, since Karen can say any true sentence.

If Jt is the only thing Karen says, $K!Jt$, while John does not say anything else (that is false), e.g., $J!Kl$, a paradox results, with the unresolved odd cycle C (marked with double arrows). This situation is possible, witnessed by the semi-kernel $\{J(Kl), K(Jt)\} \cup \{\overline{J(S)} \mid S \not\equiv Kl\} \cup \{\overline{K(S)} \mid S \not\equiv Jt\}$, but has no extension covering Kl and Jt, hence none to a kernel. The paradox, created thus by Karen and John, can be resolved only if either or both withdraw their claims or make some additional ones. If neither does, both can continue reasoning classically and consistently, recognizing even the facts $J!Kl$ and $K!Jt$. As long as they do not apply (cut), they do not explode and can agree, in particular, on the truths of object-language. But without resolving the paradox, no such consensus can be extended to the full language and deciding whether Karen always lies or John always tells the truth.

[6] Buridan's early proposal, that each statement claims its own truth in addition to whatever it may be saying, provides thus a 'solution' to the liar and similar paradoxes by making them false — but for the price of consistency of all statements being false. Earlier, Bradwardine maintained falsity of paradoxes without this over-generalization, taking only some — self-negating — statements as claiming also their truth [cf. 23, 22].

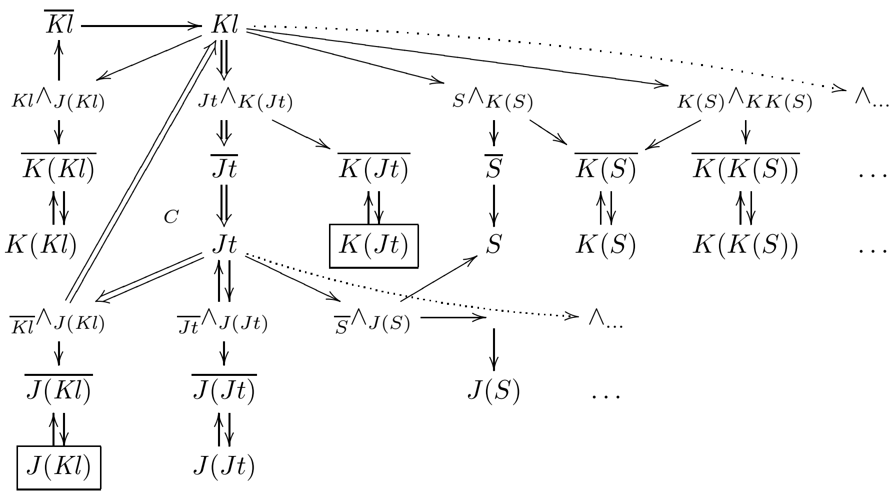

Figure 4.2. A part of graph for $Kl = \forall\phi(K\phi \to \neg\phi)$ and $Jt = \forall\phi(J\phi \to \phi)$.

5. Summary

Paradoxes appearing only in the metalanguage distinguish LSO from all usual approaches, where the metalanguage is internalized in the object-language. Although the distinction can be recovered from such a coding, the view of paradoxes and their sources changes drastically. With arithmetized syntax, they arise due to the diagonalization lemma, primarily, via convention (T). In classical context, a paradox becomes then a contradiction causing explosion and search for restrictions on convention (T), which deny some coded statements the character of truth-bearers. In LSO, paradoxes appear due to valuation of metapredicates and affect only metalevel, leaving all interpretations of the object-language available.

In LSO, the metalanguage differs from the object-language not only by possible paradoxes but also by statements, like the truth-teller, that remain indeterminate even when all atoms are consistently valuated. Uneasiness with such sentences seems to arise from conflating the object-language with the metalanguage or else extrapolating to the latter the unique interpretability of the former under every valuation of atoms. This fundamental distinction between the two language levels is the only one in LSO, and no hierarchy of metalevels arises. There is only one metalevel, at which language can speak about itself. Whether and how this allows extensions of LSO to capture own syntax and semantics is the main question for further work.

Representation of intensional and semantic statements in language graphs is completely identical, the differences amounting at most to differences of various operators. Capturing intensional and modal logics in LSO is a natural topic for future work. So far, its general schema and simple examples suggest structural identity of semantic and intensional paradoxes, arising due to the same, vicious circularity represented by the unresolvable odd cycles in language graphs.

The claim about circularity requires, of course, a comment. Yablo paradox appears noncircular, unless one plays with some esoteric notions of circularity. In LSO circularity is just a graph cycle and Yablo graph $(\omega, \{\langle i, j\rangle \mid i < j\})$ has none. Theorem 3.4 ensures that language graphs are free from paradoxes, including Yablo's, so they do not affect our metatheory. The essential aspects of Yablo can be captured inside LSO, e.g., by the following theory Y from [13]:

(a) a transitive binary relation R on a nonempty subset of sentences,
(b) that has no endpoints, $\forall\alpha\exists\beta R(\alpha,\beta)$, and where
(B) s-predicate P satisfies: $\forall\alpha(P(\alpha) \leftrightarrow \forall\beta(R(\alpha,\beta) \to \neg P(\beta)))$.

A single sentence with a loop provides a model of R, and so do ω sentences with a strict total order without maximum, but no semi-kernel contains Y. Its author observes that its "inconsistency [...] has nothing to do with truth, for it [...] arises irrespective of what P means: other than the Yablo scheme itself (B) and the auxiliary axioms (a), (b), no specific axioms for P are used in the deduction of the inconsistency." Indeed, Y with variables ranging over objects rather than sentences is a contradiction. It has nothing to do with truth *predicate* which only recovers paradoxical effects when the metalanguage is internalized in the object-language. According to our definition of paradox, as inclusion in a semi-kernel which can not be extended to a kernel, this formulation of Yablo is not a paradox but a plain contradiction.

Our model of paradoxes complies with the diagnosis from [4], according to which they arise from taking for granted some assumptions that, on a closer analysis, display a contradiction. Often, such assumptions are hidden behind the name "definition". Restricting Y above to (a) and (b), one might see (B) merely as a definition of P, albeit a self-referential one. Such a view has permeated much of the discussion, e.g., around revision theory. LSO embodies this idea, since paradoxes arise here exactly due to the way s-predicates are defined by their valuations. So understood definitions are, however, just axioms or valuations — far from logically innocent. Unlike nominal definitions or definitional extensions, particular valuations of predicates amount to specific axioms and valuations of metapredicates amount to specific metalinguistic claims.

Granting that paradoxes yield inconsistency due to bad assumptions or 'definitions' does not extend to the extreme cases of the view originating with Tarski's diagnosis and voiced occasionally, if only informally, in recent years, according to which natural language simply is inconsistent [8, 1, 16, 2]. This view, arising from perceiving predicates on arithmetized syntax as an adequate model of (natural) metalanguage, need not be maintained if we replace them by operators. Language modeled in LSO is consistent, and one can expect possibility of its extension to operators on open formulas, needed for internal theory of syntax and semantics, but this remains to be investigated.

LSO reflects the informal intuition that paradoxes do not reside in the language as such, but in the way we talk about it. S-predicates and their valuations/definitions represent just various ways of talking about language in the language. Each paradox makes a claim, involving valuation of the applied s-predicates, and can be avoided by avoiding the unfortunate claims. The fact that a paradoxical s-predicate can be defined in the language does not make the language paradoxical, just as expressibility of a contradiction does not make it inconsistent. Predicates like heterological are disturbing when presented but do not bother us because possibility of defining them, and this means making some claims, leaves also open the possibility of restricting their applicability or not making such claims. If nobody claims to be (always) lying, no liar paradox results and the language remains consistent.

To this, however, one wants to object! Statements like L "This sentence is false" do not have to be claimed, they cause trouble by simply being there. Well, by simply being, the liar sentence makes a claim. In the usual representation, it claims $L \leftrightarrow \neg\mathcal{T}(L)$ or simply $L \leftrightarrow \neg L$, expressing the pretense to truth of this unsatisfiable equivalence, liar's semantic claim. The liar's sentence L is distinct from the claim "L is true iff it is false" which, however, is involved in L. (The name "truth condition" might replace "claim", if it were not overloaded with other associations.) Typically this claim, or condition, amounts to disquotation captured by convention (T). It breaks however for a paradox, which amounts to the nonexistence of a boolean value satisfying its claim. The liar, represented as a mere contradiction, is avoided as contradictions are in general, by withdrawing the unsatisfiable claim, $L \leftrightarrow \neg L$ or $L \leftrightarrow \neg\mathcal{T}(L)$. (LSO can avoid the latter, since it does not enjoy — or rather suffer from — the diagonalization lemma.) More elaborate cases, like somebody claiming only to be always lying, represented explicitly as paradoxes in LSO, show plainly that troubles arise from impossible claims. Therefore, informally we ignore paradoxes so easily: no matter truth-values of the pronounced sentences, the claims hidden behind them are false.

References

[1] Azzouni, J., 2003, 'The strengthened liar, the expressive strength of natural languages, and regimentation', *The Philosophical Forum* 34 (3–4): 329–350. DOI: 10.1111/1467-9191.00142

[2] Båve, A., 2012, "On using inconsistent expressions", *Erkenntins* 77 (1): 133–148. DOI: 10.1007/s10670-011-9310-2

[3] Beringer, T., and T. Schindler, 2017, "A graph-theoretic analysis of the semantic paradoxes", *The Bulletin of Symbolic Logic* 23 (4): 442–492. DOI: 10.1017/bsl.2017.37

[4] Chihara, C., 1974, "The semantic paradoxes: a diagnostic investigation", *The Philosophical Review* 88 (4): 590–618. DOI: 10.2307/2184846

[5] Cobreros, P., P. Égré, D. Ripley, and R. van Rooij, 2013, "Reaching transparent truth", *Mind* 122 (488): 841–866. DOI: 10.1093/mind/fzt110

[6] Deutsch, H., 2010, "Diagnoalization and truth functional operators", *Analysis* 70 (2): 215–217.

[7] Dung, P. M., 1995, "On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games", *Artificial Intelligence* 77 (2): 321–357. DOI: 10.1016/0004-3702(94)00041-X

[8] Eklund, M., 2002, "Inconsistent languages", *Philosophy and Phenomenological Research* 64 (2): 251–275. DOI: 10.1111/j.1933-1592.2002.tb00001.x

[9] Grim, P., 1993, "Operators in the paradox of the knower", *Synthese* 44 (3): 409–428. DOI: 10.1007/BF01064487

[10] Halbach, V., 1999, "Disquotationalism and infinite conjunctions", *Mind* 108: 1–22. DOI: 10.1093/mind/108.429.1

[11] Halbach, V., 2009, "On the benfits of a reduction of modal predicates to modal operators", pages 323–333 in A. Hieke and H. Leitgeb (eds.), *Reduction, Abstraction, Analysis*, Ontos Verlag.

[12] Horsten, L., and H. Leigeb, 2001, "No future", *Journal of Philosophical Logic* 30 (3): 259–265. DOI: 10.1023/A:1017569601150

[13] Ketland, J., 2005, "Yablo's paradox and ω-inconsistency", *Synthese* 145 (3): 295–302.

[14] Montague, R., 1963, "Syntactical treatment of modality, with corollaries on reflection principles and finite axiomatizability", *Acta Philosophica Fennica* 16: 153–167.

[15] Neumann-Lara, V., 1971, "Seminúcleos de una digráfica", Technical report, Anales del Instituto de Matemáticas II, Universidad Nacional Autónoma México.

[16] Patterson, D., 2009, "Inconsistency theory of semantic paradox", *Philosophy and Phenomenological Research* 79 (2): 387–422. DOI: 10.1111/j.1933-1592.2009.00283.x

[17] Perlis, D., 1988, "Languages with self-reference II: knowledge, belief, and modality", *Arificial Intellgence* 34 (2): 179–212. DOI: 10.1016/0004-3702(88)90038-0

[18] Picollo, L., and T. Schindler, 2018, "Disquotation and infinite conjunctions", *Erkenntins* 83 (5): 899–928. DOI: 10.1007/s10670-017-9919-x

[19] Prior, A. N., 1961, "On a family of paradoxes", *Notre Dame Journal of Formal Logic* 2 (1): 16–32. DOI: 10.1305/ndjfl/1093956750

[20] Quine, W. V. O., 1970, *Philosophy of logic*. Harvard University Press, Cambridge.

[21] Rabern, L., B. Rabern, and M. Macauley, 2013, "Dangerous reference graphs and semantic paradoxes", *Journal of Philosophical Logic* 42 (5): 727–765. DOI: 10.1007/s10992-012-9246-2

[22] Rahman, S., T. Tulenheimo, and E. Genot (eds.), 2008, *Unity, Truth and the Liar*, Springer. DOI: 10.1007/978-1-4020-8468-3

[23] Read, S., 2002, "The liar paradox from John Buridan back to Thomas Bradwardine", *Vivarium* 40 (2): 189–218. DOI: 10.1163/156853402320901812

[24] Reinhardt, W. N., 1980, "Necessity predicates and operators", *Journal of Philosophical Logic* 9 (4): 437–450. DOI: 10.1007/BF00262865

[25] Ripley, D., 2012, "Conservatively extending classical logic with transparent truth", *The Review of Symbolic Logic* 5 (2): 354–378. DOI: 10.1017/S1755020312000056

[26] Ripley, D., 2013, "Paradoxes and failures of cut", *Australasian Journal of Philosophy* 91 (1): 139–164. DOI: 10.1080/00048402.2011.630010

[27] von Neumann, J., and O. Morgenstern, 1944, *Theory of Games and Economic Behavior*, Princeton University Press.

Appendices

A. (Semi)kernels and solvability of language graphs

A.1. Some facts about (semi)kernels

The two facts below, applied implicitly on drawings, imply equisolvability of graphs, showing actually that the two have *essentially the same solutions*: each solution of one can be expanded to a solution of the other, and each solution of the other, restricted to the first, is its solution.

A path $a_0 \dots a_k$ is *isolated* if $\mathbf{E}_G(a_i) = \{a_{i+1}\}$ for $0 \leqslant i < k$ and $\mathbf{E}^-_G(a_i) = \{a_{i-1}\}$ for $0 < i < k$. A *double edge*, introduced earlier, is an isolated path of length 2. *Contraction* of such an isolated path amounts to identifying the first and the last vertex, joining their neighbourhoods and removing the intermediate vertices, i.e., obtaining graph G' where $\mathbf{V}_{G'} = \mathbf{V}_G \setminus \{a_1 \dots a_k\}$, $\mathbf{E}_{G'}(a_0) = \mathbf{E}_G(a_k)$ and $\mathbf{E}^-_{G'}(a_0) = \mathbf{E}^-_G(a_0) \cup \mathbf{E}^-_G(a_k) \setminus \{a_{k-1}\}$. The first fact is a trivial observation.

FACT A.1. *If G' results from G by contracting an isolated path of even length, then $\forall K' \in sol(G') \exists! K \in sol(G)$: $K' \subseteq K$, and $\forall K \in sol(G)$: $K \cap \mathbf{V}_{G'} \in sol(G')$.*

The same holds if G' results from a transfinite number of such contractions, provided that no *ray*, i.e., an infinite outgoing path with no repeated vertex, is contracted to a finite path.

The second fact justifies duplication of vertices $\mathbf{S}^+_M$ as AUX, without affecting solutions. It shows that identifying vertices with identical out-neighbourhoods preserves and reflects (semi)kernels. To define this operation, let $R_G \subseteq \mathbf{V}_G \times \mathbf{V}_G$ relate two vertices in G with identical out-neighbourhoods, i.e., $R_G(a,b) \iff \mathbf{E}_G(a) = \mathbf{E}_G(b)$. It is an equivalence, so let $G^{\downarrow}$ denote the quotient graph over equivalence classes, $[v] = \{u \in \mathbf{V}_G \mid R_G(v,u)\}$, with edges $\mathbf{E}_{G^{\downarrow}}([v],[u]) \iff \exists v \in [v], u \in [u] : \mathbf{E}_G(v,u)$. The operation can be iterated any number n of times, denoted by $G^{\downarrow n}$ and defined by:

- $G^{\downarrow 1} = G^{\downarrow}$ and $G^{\downarrow(n+1)} = (G^{\downarrow n})^{\downarrow}$.

Vertices of $G^{\downarrow n}$ are taken as subsets of $\mathbf{V}_G$, $[u]^n = \{v \in \mathbf{V}_G \mid \exists i \leqslant n : R_{G^{\downarrow i}}([v]^i, [u]^i)\}$. For limit ordinals λ, $G^{\downarrow\lambda}$ is given by

- $\mathbf{V}_{G^{\downarrow\lambda}} = \{[u]^\lambda \mid u \in \mathbf{V}_G\}$,

where $[u]^\lambda = \bigcup_{i<\lambda} [u]^i = \{v \in \mathbf{V}_G \mid \exists i < \lambda : R_{G^{\downarrow i}}([u]^i, [v]^i)\}$ and

- $\mathbf{E}_{G^{\downarrow\lambda}}([v]^\lambda, [u]^\lambda) \iff \exists n \in \lambda \colon \mathbf{E}_{G^{\downarrow n}}([v]^n, [u]^n)$.

FACT A.2. *For every ordinal n, and SKr denoting either kernels or semi-kernels (sol or SK):*

(a) $K \in SKr(G) \Rightarrow \{[v]^n \mid v \in K\} \in SKr(G^{\downarrow n})$, *and*
(b) $K^{\downarrow n} \in SKr(G^{\downarrow n}) \Rightarrow \bigcup K^{\downarrow n} \in SKr(G)$.

PROOF. (1) The proof for $n = 1$ shows the claim also for every $n = n' + 1$.
(a) $K^{\downarrow} = \{[v] \mid v \in K\}$ is independent, for if $\mathbf{E}_{G^{\downarrow}}([v],[w])$ for some $[v],[w] \in K^{\downarrow}$, then $\mathbf{E}_G(v,w)$ for some $v \in [v], w \in [w]$. But then $v, w \in K$ contradicting independence of K — if $x \in K$ then $[x] \subseteq K$, since $\forall x, y \in [v] \colon \mathbf{E}_G(x) = \mathbf{E}_G(y)$, so $\mathbf{E}_G(x) \cap K = \emptyset \iff \mathbf{E}_G(y) \cap K = \emptyset$.

If $[v] \in \mathbf{V}_{G^{\downarrow}} \setminus K^{\downarrow}$, then $[v] \subseteq \mathbf{V}_G \setminus K \subseteq \mathbf{E}^-_G(K)$, so $\forall v \in [v]\ \exists w \in K \colon \mathbf{E}_G(v,w)$. Then $[w] \in K^{\downarrow}$ and $[v] \in \mathbf{E}^-_{G^{\downarrow}}([w]) \subseteq \mathbf{E}^-_{G^{\downarrow}}(K^{\downarrow})$. Thus $\mathbf{V}_{G^{\downarrow}} \setminus K^{\downarrow} \subseteq \mathbf{E}^-_{G^{\downarrow}}(K^{\downarrow})$, so $K^{\downarrow} \in sol(G^{\downarrow})$.

If $K \in SK(G)$ and $[v] \in \mathbf{E}^-_{G^\downarrow}(K^\downarrow)$, i.e., for some $v \in [v], w \in K\colon v \in \mathbf{E}^-_G(w)$, then $[v] \subseteq \mathbf{E}^-_G(w)$ and $[w] \in K^\downarrow$, so $[v] \in \mathbf{E}^-_{G^\downarrow}(K^\downarrow)$, i.e., $\mathbf{E}_{G^\downarrow}(K^\downarrow) \subseteq \mathbf{E}^-_{G^\downarrow}(K^\downarrow)$, so $K^\downarrow \in SK(G^\downarrow)$.

(b) $K = \bigcup K^\downarrow = \{v \in \mathbf{V}_G \mid [v] \in K^\downarrow\}$ is independent, for if $\mathbf{E}_G(v,u)$ for some $v,u \in K$, then also $\mathbf{E}_{G^\downarrow}([v],[u])$ contradicting independence of $K^\downarrow$. If $x \notin K$ then $[x] \notin K^\downarrow$, and since $\mathbf{E}_{G^\downarrow}([x],[v])$ for some $[v] \in K^\downarrow$, so for some $y \in [x]$ and $v \in [v] \subseteq K$, $\mathbf{E}_G(y,v)$. But since $\mathbf{E}_G(y) = \mathbf{E}_G(x)$, so also $\mathbf{E}_G(x,v)$, i.e., $x \in \mathbf{E}^-_G(K)$. Thus $\mathbf{V}_G \setminus K \subseteq \mathbf{E}^-_G(K)$, and $K \in sol(G)$.

If $K^\downarrow \in SK(G^\downarrow)$ and $v \in \mathbf{E}_G(K)$, then $[v] \in \mathbf{E}_{G^\downarrow}(K^\downarrow) \subseteq \mathbf{E}^-_{G^\downarrow}(K^\downarrow)$, i.e., $[v] \in \mathbf{E}^-_{G^\downarrow}([w])$ for some $[w] \in K^\downarrow$. Then $[w] \subseteq K$ and $[v] \subseteq \mathbf{E}^-_G([w])$, so that $\mathbf{E}_G(K) \subseteq \mathbf{E}^-_G(K)$.

(2) We show the claim for limit λ.

(a) If $K \in sol(G)$, let $K^{\downarrow\lambda} = \{[v]^\lambda \mid v \in K\}$. If $\mathbf{E}_{G^\lambda}([v]^\lambda, [u]^\lambda)$ for some $[v]^\lambda, [u]^\lambda \in K^{\downarrow\lambda}$, i.e., $v,u \in K$, then for some $n \in \lambda$: $\mathbf{E}_{G^n}([v]^n,[u]^n)$, which means that $K^{\downarrow n} = \{[x]^n \mid x \in K\}$ is not a kernel of G^n, contrary to point (1). Hence $K^{\downarrow\lambda}$ is independent. If $[v]^\lambda \in \mathbf{V}_{G^\lambda} \setminus K^{\downarrow\lambda}$, then $[v]^\lambda \subseteq \mathbf{V}_G \setminus K$, so for any $v \in [v]$, there is a $u \in \mathbf{E}_G(v) \cap K$. Then $[u]^\lambda \in \mathbf{E}_{G^\lambda}([v]^\lambda) \cap K^{\downarrow\lambda}$, hence $\mathbf{V}_{G^\lambda} \setminus K^{\downarrow\lambda} \subseteq \mathbf{E}^-_{G^\lambda}(K^{\downarrow\lambda})$, and $K^{\downarrow\lambda} \in sol(G^{\downarrow\lambda})$.

If $K \in SK(G)$, i.e., $\mathbf{E}_G(K) \subseteq \mathbf{E}^-_G(K)$ and $[v]^\lambda \in \mathbf{E}_{G^{\downarrow\lambda}}(K^{\downarrow\lambda})$, then for some $n \in \lambda$: $[v]^n \in \mathbf{E}_{G^{\downarrow n}}([w]^n)$ for some $[w]^n \in K^{\downarrow n}$. Then also $[w]^n \in \mathbf{E}^-_{G^{\downarrow n}}(K^{\downarrow n})$, as $K^{\downarrow n} \in SK(G^{\downarrow n})$ by IH, but then also $[w]^\lambda \in \mathbf{E}^-_{G^{\downarrow\lambda}}(K^{\downarrow\lambda})$. Thus $\mathbf{E}_{G^{\downarrow\lambda}}(K^{\downarrow\lambda}) \subseteq \mathbf{E}^-_{\downarrow\lambda}(K^{\downarrow\lambda})$.

(b) For a kernel $K^{\downarrow\lambda}$ of $G^{\downarrow\lambda}$, let $K = \bigcup K^{\downarrow\lambda} = \{v \in \mathbf{V}_G \mid [v]^\lambda \in K^{\downarrow\lambda}\}$. If $v \in \mathbf{E}^-_G(x)$ for some $x \in K$, then $v \notin K$ for if $v \in K$, i.e., $[v]^\lambda \subseteq K$, then $[v]^\lambda \in \mathbf{E}^-_{G^{\downarrow\lambda}}([x]^\lambda) \cap K^{\downarrow\lambda} \subseteq \mathbf{E}^-_{G^{\downarrow\lambda}}(K^{\downarrow\lambda}) \cap K^{\downarrow\lambda}$ contradicting independence of $K^{\downarrow\lambda}$. If $v \in \mathbf{V}_G \setminus K$, i.e., $[v]^\lambda \notin K^{\downarrow\lambda}$, then there is some $[u]^\lambda \in \mathbf{E}_{G^{\downarrow\lambda}}([v]^\lambda) \cap K^{\downarrow\lambda}$. Since $[u]^\lambda \in \mathbf{E}_{G^{\downarrow\lambda}}([v]^\lambda)$, so for some $n < \lambda, [u]^n \in \mathbf{E}_{G^{\downarrow n}}([v]^n)$, that is, for some $u' \in [u]^n, u' \in \mathbf{E}_G(v)$. Since $[u]^\lambda \in K^{\downarrow\lambda}$, so $[u]^n \subseteq [u]^\lambda \subseteq K$, hence $v \in \mathbf{E}^-_G(K)$ and $K \in sol(G)$.

If $K^{\downarrow\lambda} \in SK(G^{\downarrow\lambda})$, independence of K follows as above. If $v \in \mathbf{E}_G(K)$, then $[v]^\lambda \in \mathbf{E}_{G^{\downarrow\lambda}}(K^{\downarrow\lambda}) \subseteq \mathbf{E}^-_{G^{\downarrow\lambda}}(K^{\downarrow\lambda})$, i.e., $\exists n \in \lambda$: $[v]^n \in \mathbf{E}^-_{G^{\downarrow n}}(K^{\downarrow n})$. By IH, $[v]^n \subseteq \mathbf{E}^-_G(K^{\downarrow n}) \subseteq \mathbf{E}^-_G(K)$. Hence $\mathbf{E}_G(K) \subseteq \mathbf{E}^-_G(K)$. □

A.2. Logical and graph equivalences

We formulate logical and some other notions of equivalence in terms of graphs. Two $\mathcal{L}^+_M$ sentences are equivalent, in $\mathcal{G}_M(\mathcal{L}^+)$, if for every semi-kernel L covering both, either both belong to L or neither does. $\mathcal{L}^+$ sentences are (logically) equivalent if they are so in every language graph, according to the third line below:

for $A, B \in \mathbf{V}_G$: $A \overset{G}{\Longleftrightarrow} B$ iff

$$\forall L \in SK(G)\colon \{A,B\} \subseteq \mathbf{E}^-[L] \to (A \in L \leftrightarrow B \in L),$$

$$\text{for } A,B \in \mathbf{S}^+_M\colon A \overset{\mathcal{L}^+_M}{\Longleftrightarrow} B \text{ iff } A \overset{\mathcal{G}_M(\mathcal{L}^+)}{\Longleftrightarrow} B,$$
$$\text{for } A,B \in \mathbf{S}^+\colon A \overset{\mathcal{L}^+}{\Longleftrightarrow} B \text{ iff } \forall M\colon A \overset{\mathcal{L}^+_M}{\Longleftrightarrow} B.$$

A more specific equivalence will be used, corresponding to prenex operations. Each sentence can be written in PDNF, that is, prenex normal form with matrix in DNF. Two $\mathcal{L}^+_M$ sentences are *PDNF equivalent*, denoted by $A \overset{P}{\Longleftrightarrow} B$, if they have (also) identical PDNFs. To show that PDNF equivalence implies $\mathcal{L}^+$ equivalence, we use a more structural notion of equivalence in a graph.

By $\mathbf{E}^*_G$ we denote the reflexive and transitive closure of $\mathbf{E}_G$ and by $\mathbf{E}^*_G(S)$, for $S \in \mathbf{V}_G$, the subgraph of G induced by all vertices reachable from S. A *common cut* of $A, B \in \mathbf{V}_G$ is a set of vertices $C \subseteq \mathbf{E}^*_G(A) \cap \mathbf{E}^*_G(B)$, such that every path leaving A and prolonged sufficiently far crosses C and so does every path leaving B. (C may intersect A and B and contain vertices on various cycles intersecting A and B.) We say that A and B are *cut equivalent*, $A \overset{c}{\Longleftrightarrow} B$, if there is a common cut C such that for every correct (not falsifying (3.2)) valuation of C, every correct extension to $\{A, B\}$ forces identical value of A and B. Obviously, if $A \overset{c}{\Longleftrightarrow} B$ in a graph G, then also $A \overset{G}{\Longleftrightarrow} B$, as each $K \in sol(G)$ determines a correct valuation of every common cut of A and B.

FACT A.3. *For $A, B \in \mathbf{S}^+_M$ in $\mathcal{G}_M(\mathcal{L}^+)$, if $A \overset{P}{\Longleftrightarrow} B$ then $A \overset{c}{\Longleftrightarrow} B$, hence $A \overset{\mathcal{L}^+_M}{\Longleftrightarrow} B$.*

PROOF. For $\mathcal{G} = \mathcal{G}_M(\mathcal{L}^+)$ with $\{L \in SK(\mathcal{G}) \mid \{A, B\} \subseteq L\} \neq \emptyset$, we verify standard prenex transformations, considering only s-quantifiers, as o-quantifiers can be treated in the same way.

1. The claim holds trivially for B renaming bound s-variables (without name clashes) of A, as the two have the same subgraph. This is also the case for the subgraphs of $A = \neg\forall\phi D[\phi]$ and $B = \exists\phi\neg D[\phi]$.

2. $A = (\forall\phi D[\phi]) \wedge C \overset{\mathcal{G}}{\Longleftrightarrow} \forall\phi(D[\phi] \wedge C) = B$, with no free occurrences of ϕ in C. On the schematic subgraph in Figure A.1 below, X_i, X_j … stand for all $\mathbf{S}^+_M$ and common cut is marked by the waved line.

Inspecting the graph, we see that, for any semi-kernel $\mathbf{E}^-[L] \supseteq \{A, B\}$: $B \in L$ iff $(D[X_i] \wedge C) \in L$ for all X_i iff $C \in L$ and $D[X_i] \in L$ for all X_i iff $A \in L$.

3. For $A = \neg\exists\phi D[\phi] \overset{\mathcal{G}}{\Longleftrightarrow} \forall\phi\neg D[\phi] = B$ the schema is as Figure A.2.

Trivially, for any semi-kernel $\mathbf{E}^-[L] \supseteq \{A, B\}$: $A \in L \iff B \in L$. □

Thus, every sentence in $\mathcal{L}^+$ has an $\overset{\mathcal{L}^+}{\Longleftrightarrow}$-equivalent PDNF sentence. A useful consequence is that, considering below solvability of $\mathcal{G}_M(\mathcal{L}^\Phi)$ or $\mathcal{G}_M(\mathcal{L}^+)$, we can limit attention to sentences in PDNF.

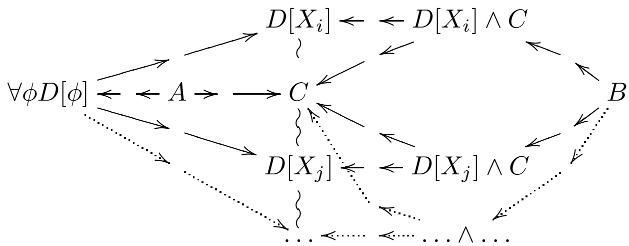

Figure A.1.

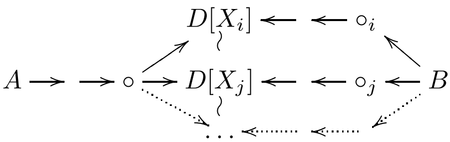

Figure A.2.

A.3. No paradoxes in $\mathcal{L}^{\Phi}$ — solvability of $\mathcal{G}(\mathcal{L}^{\Phi})$

Extending classical language $\mathcal{L}$ with s-quantifiers to $\mathcal{L}^{\Phi}$ does not introduce any paradoxes. The following theorem shows a stronger claim that, in a domain M, all $\mathcal{L}^{\Phi}_M$ sentences obtain unique values under every valuation of $\mathcal{L}_M$ sentences. Throughout, valuation of atoms $\mathbf{A}_M$ is assumed to determine a unique valuation of object-level sentences $\mathbf{S}_M$, so the two are practically identifiable.

THEOREM A.4 (3.1). *In any $\mathcal{G}_M(\mathcal{L}^{\Phi})$, each valuation of atoms $\rho \in \mathbf{2}^{\mathbf{A}_M}$ has a unique extension $\hat{\rho} \in sol(\mathcal{G}_M(\mathcal{L}^{\Phi}))$ with $\hat{\rho}|_{\mathbf{A}_M} = \rho$.*

PROOF. Graph $\mathcal{G}_M(\mathcal{L}^{\Phi})$ consists of two subgraphs, the strong component with all s-quantified sentences, $\mathcal{G}_M(\mathcal{L}^{\Phi} \setminus \mathcal{L}) = \bigcup_{A \in \mathbf{S}^{\Phi}_M \setminus \mathbf{S}_M} \mathcal{G}_M(A)$, and the forest $\mathcal{G}_M(\mathcal{L}) = \bigcup_{B \in \mathbf{S}_M} T_M(B)$ of trees for sentences without s-quantifiers, with no edges from the latter to the former. For each $A \in \mathbf{S}^{\Phi}_M \setminus \mathbf{S}_M$ in the former, there are (single or double) edges from external atoms $V \in ext(\mathcal{G}_M(A))$, to the roots of $\mathcal{G}_M(V)$, that are trees $T_M(V)$ when $V \in \mathbf{S}_M$. By Lemma 3.2 below, valuation ρ of $\mathbf{S}_M = \mathbf{V}_{\mathcal{G}_M(\mathcal{L})}$, determines a solution ρ^-_A of each $\mathcal{G}^-_M(A)$ (subgraph of $\mathcal{G}_M(A)$ without its DNF-feet), compatible with every valuation of $ext(\mathcal{G}_M(A))$. Hence, these can be combined into $\rho \cup \bigcup_{A \in \mathbf{S}^{\Phi}_M \setminus \mathbf{S}_M} \rho^-_A$ forcing value $\rho^-_V(V)$ at each $V \in ext(\mathcal{G}_M(A))$, and thus determining solutions of all DNF-feet. Each $\mathcal{G}_M(A)$ obtains thus a solution $\rho_A \supset \rho^-_A$, yielding a unique $\hat{\rho} = (\rho \cup \bigcup_{A \in \mathbf{S}^{\Phi}_M \setminus \mathbf{S}_M} \rho_A) \in sol(\mathcal{G}_M(\mathcal{L}))$, extending ρ. $\square$

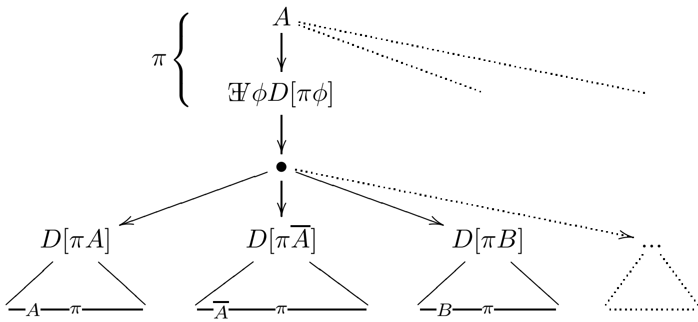

Figure A.3.

The missing lemma shows that for each sentence $A \in \mathbf{S}_M^{\Phi} \setminus \mathbf{S}_M$, solution of the subgraph of $\mathcal{G}_M(A)$ without its DNF-feet, denoted by $\mathcal{G}_M^{-}(A)$, depends on the valuation of $\mathbf{S}_M$, but not of external atoms $ext(\mathcal{G}_M(A))$, as the second part of the lemma states. Valuation of $ext(\mathcal{G}_M(A))$ affects, of course, values in DNF-feet in which they occur.

LEMMA A.5 (3.2). *For every graph $\mathcal{G}_M(\mathcal{L}^{\Phi})$ and sentence $A \in \mathbf{S}_M^{\Phi}$, each valuation ρ of $\mathbf{A}_M$ and of external atoms of $\mathcal{G}_M(A)$, $\rho \in \mathbf{2}^{\mathbf{A}_M \cup ext(\mathcal{G}_M(A))}$, has a unique extension to $\rho_A \in sol(\mathcal{G}_M(A))$. Restriction $\rho|_{\mathbf{S}_M}$, determines restriction of ρ_A to $\mathcal{G}_M^{-}(A)$: if $\rho|_{\mathbf{S}_M} = \sigma|_{\mathbf{S}_M}$ then $\rho_A|_{\mathcal{G}_M^{-}(A)} = \sigma_A|_{\mathcal{G}_M^{-}(A)}$.*

PROOF. By Fact A.3, we can limit attention to sentences in PDNF.

Figure A.3 illustrates the general situation which we first describe. For $A \in \mathbf{S}_M^{\Phi}$, with the number $q(A) = n + 1 \geqslant 1$ of s-quantifiers and s-variables, and for n-sequence of sentences $\pi \in (\mathbf{S}_M^{\Phi})^n$ substituted for the n s-variables of A bound by its first n quantifiers, the roots of all feet, $A(\pi S) = D[\pi S]$, $S \in \mathbf{S}_M^{\Phi}$, are grandchildren of vertex $A(\pi) = \exists\!\!\!\forall \phi D[\pi\phi]$. (On the drawing, $\exists\!\!\!\forall = \exists$ and all feet have the common parent $\bullet$; when $\exists\!\!\!\forall = \forall$, their distinct parents are children of $A(\pi)$.) Each foot $A(\pi S)$ represents an application of the same boolean function $d^{\pi}(\phi) = D[\pi\phi]$, evaluating $D[\pi\phi]$ given valuation of its parameters π, ϕ and, possibly, some atoms $L_A \subset \mathbf{S}_M$ occurring in the original matrix $D[\ldots]$. For any $\rho \in \mathbf{2}^{\mathbf{S}_M}$, L_A obtain fixed values so, considering d^{π}, we assume the effects of $\rho(L_A)$ taken into account.

1. The *internal* vertices of π, $int(\pi)$ are sentences occurring on the path after substitutions, and *external* ones are those which do not, $ext(\pi) = \mathbf{S}_M^{\Phi} \setminus int(\pi)$. Some 'sinks' of the feet have single or double edges to vertices from π, which are $int(\pi)$, including $\pi_0 = A$ and $\exists\!\!\!\forall \phi D[\pi\phi]$ (when this is substituted for ϕ in $D[\pi\phi]$.) As branches from $\bullet$ instantiate ϕ with every sentence $S \in \mathbf{S}_M^{\Phi}$, all sentences from $int(\pi)$ occur in some feet.

2. Depending on whether $\forall\!\!\!\exists$ is $\forall$ or $\exists$, the value at vertex $\forall\!\!\!\exists\phi D[\pi\phi]$, as a function of values of its grandchildren, is either

$$\exists\phi D[\pi\phi] = \bigvee_{S\in\mathbf{S}_M} d^{\pi}(S) \text{ or } \forall\phi D[\pi\phi] = \bigwedge_{S\in\mathbf{S}_M} d^{\pi}(S). \qquad (*)$$

We consider first the case when $|\pi| = q(A) - 1$, i.e., $A(\pi) = \forall\!\!\!\exists\phi D[\pi\phi]$ is the grandparent of the completely substituted (roots of) DNF-feet ($D[\pi A], D[\pi B]$, etc., on the drawing).

Each valuation of sentences from π, abbreviated as $\alpha \in \mathbf{2}^{\pi}$, specializes $d^{\pi}(\phi)$ to a unary boolean function $d^{\alpha(\pi)}(\phi) = D[\alpha(\pi)\phi]$, and ($*$) to

$$\exists\phi D[\alpha(\pi)\phi] = \bigvee_{S\in\mathbf{S}_M} d^{\alpha(\pi)}(S) \text{ or } \forall\phi D[\alpha(\pi)\phi] = \bigwedge_{S\in\mathbf{S}_M} d^{\alpha(\pi)}(S). \qquad (**)$$

3. As a boolean function of one variable, $d^{\alpha(\pi)}(\phi)$ is either constant or not. If it is constant, i.e., $d^{\alpha(\pi)}(\phi) = d^{\alpha(\pi)}(\neg\phi)$, then $\forall\!\!\!\exists\phi D[\alpha(\pi)\phi]$ obtains the same value in either case of ($**$). Otherwise, $d^{\alpha(\pi)}(\neg\phi) = \neg d^{\alpha(\pi)}(\phi)$ and, since for each $S \in \mathbf{S}_M^{\Phi}$ both $d^{\alpha(\pi)}(S)$ and $d^{\alpha(\pi)}(\neg S)$ enter evaluation of ($**$), this yields constant $\mathbf{0}$ at their least common predecessor ($\bullet$ when $\forall\!\!\!\exists = \exists$ and $A(\pi)$ when $\forall\!\!\!\exists = \forall$). In this way, for every $\alpha \in \mathbf{2}^{\pi}$, $A(\pi)$ obtains a unique value $\overleftarrow{\alpha}(A(\pi))$, induced from all $D[\alpha(\pi)S]$ by ($**$), but determined already by $d^{\alpha(\pi)}(\phi)$, independently from

(a) valuation $\alpha(A(\pi))$, i.e., if $\alpha_{\mathbf{0}}, \alpha_{\mathbf{1}} \in \mathbf{2}^{\pi}$ differ only at $A(\pi)$, then $\overleftarrow{\alpha}_{\mathbf{0}}(A(\pi)) = \overleftarrow{\alpha}_{\mathbf{1}}(A(\pi))$, and
(b) independently from valuation of $ext(\pi)$, since each external vertex S enters both evaluation of $d^{\alpha(\pi)}(S)$ and of $d^{\alpha(\pi)}(\neg S)$, with jointly constant contribution to ($**$) as just explained.

Point (a) means that cycles from the feet to $A(\pi)$ admit a unique solution $\rho_{A(\pi),\alpha}$ to the subgraph $\mathcal{G}_M(A(\pi))$ of $\mathcal{G}_M(A)$, given any $\rho \in \mathbf{2}^{\mathbf{S}_M \cup ext(\pi)}$ and $\alpha \in \mathbf{2}^{\pi^-}$, where π^- is π without its last element $A(\pi)$. By point (b), $\rho|_{ext(\pi)}$ is inessential, so if $\rho|_{\mathbf{S}_M^-} = \sigma|_{\mathbf{S}_M^-}$ then $\rho_{A(\pi),\alpha}(A) = \sigma_{A(\pi),\alpha}(A)$.

4. This is the basis for the claim that for each A with $q(A) \geqslant 1$ and each path π from the root A with $|\pi| < q(A)$, each valuation of π^- and $\mathbf{S}_M$ determines a unique value of $A(\pi)$. We use its formulation above, i.e., for each $\rho \in \mathbf{2}^{\mathbf{S}_M \cup ext(\pi)}$ and each $\alpha \in \mathbf{2}^{\pi}$, vertex $V = A(\pi)$ (above the roots of the feet) obtains a unique value $\overleftarrow{\alpha}(V)$, which depends at most on valuation of vertices on π^- (above V), but neither on the value (a) of $\alpha(V)$ nor (b) of $\rho(X)$, for any $X \in ext(\pi)$.

The claim is shown by induction on $h - l$, where $h \geqslant 1$ is the distance of the root A from the roots of the feet and l is the distance of V from the root A, $h > l \geqslant 0$. The basis $h - l = 1$ is 3.

5. The argument from 3 works also in the induction step. In Figure A.4 for $0 \leqslant |\pi| = l < h - 1$, we have the following counterpart of the drawing

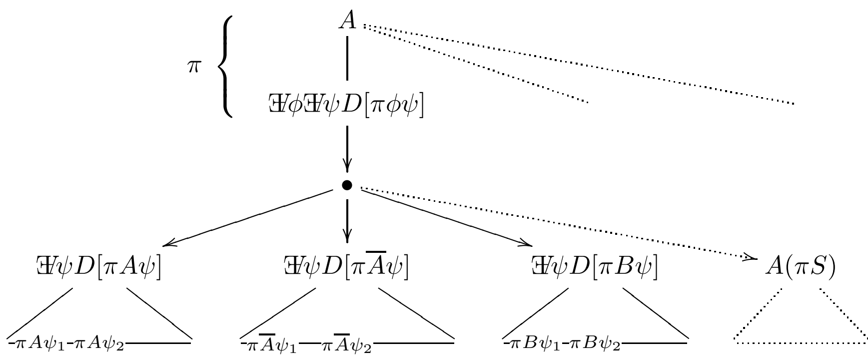

Figure A.4.

from 3, with $A(\pi) = \text{Ⅎ}\phi\text{Ⅎ}\psi D[\pi\phi\psi]$, where $\text{Ⅎ}\psi$ is the sequence of remaining quantifiers, and ψ_1, ψ_2 at the bottom signal various substitutions for ψ.
Given $\alpha \in \mathbf{2}^{\pi}$, IH applied to the lowest triangles on the drawing, i.e., subgraphs $\mathcal{G}_M(A(\pi S))$ with roots $A(\pi S)$ for $S \in \mathbf{S}_M^{\Phi}$, gives to each $A(\pi S)$ a unique value, independent of valuation of $ext(\pi S)$. Consequently $A(\pi\phi)$ is a function of only π and ϕ, so that for any $\alpha \in \mathbf{2}^{\pi}$, it represents a function $d^{\alpha(\pi)}$ of ϕ. The same argument and cases for $d^{\alpha(\pi)}$ as in 3 show that the value $\overleftarrow{\alpha}(A(\pi))$, induced to the common grandparent of all $A(\pi S)$ under valuation $\alpha \in \mathbf{2}^{\pi}$, is equal whether $\alpha(A(\pi)) = \mathbf{1}$ or $\alpha(A(\pi)) = \mathbf{0}$, giving point a) of induction. As for each $A(\pi S)$ its value under $\overleftarrow{\alpha}$ is independent from valuation of $ext(\pi S)$ by IH, the induced value $\overleftarrow{\alpha}(A(\pi))$ is independent from $ext(\pi) = \bigcap_{S \in \mathbf{S}_M} ext(\pi S)$, giving point (b) of induction. Consequently, $\overleftarrow{\alpha}(A(\pi))$ is unique and independent of valuations of $ext(\pi)$ and of $A(\pi)$, which establishes the induction step.

6. Thus, the value of the root A is determined, for each $\rho \in \mathbf{2}^{\mathbf{S}_M}$, independently from valuation of $ext(\mathcal{G}_M(A))$. Starting now from A and using claim 4 downwards, the value of $A(S)$, for each $S \in \mathbf{S}_M^{\Phi}$, is determined by ρ and value of A (independently from valuation of $ext(\mathcal{G}_M(A))$). Since A is determined by ρ, so is the value of $A(S)$. Proceeding inductively down the tree $T_M(A)$, valuation ρ_A^- of $T_M(A)^-$ is seen determined by ρ, independently from valuation of $ext(\mathcal{G}_M(A))$. The latter determines then values in all feet of $\mathcal{G}_M(A)$, yielding a unique solution ρ_A of $\mathcal{G}_M(A)$, with $\rho_A^- \subset \rho_A$ and $\rho_A|_{\mathbf{S}_M \cup ext(\mathcal{G}_M(A))} = \rho$. □

A.4. Solvability of $\mathcal{G}(\mathcal{L}^+)$

We show that any solution of a graph for any language $\mathcal{L}^+$ can be extended to a solution of a graph for $\mathcal{L}^P$, i.e., $\mathcal{L}^+$ augmented with a countable number of new s-predicates. Starting then with a solvable graph for $\mathcal{L}^{\Phi}$ (or for $\mathcal{L}^{\doteq}$), yields the general claim. We show this first for adding only a single predicate P by definitional extension, that is, by a sentence

$$\forall\phi(P(\phi) \leftrightarrow \text{Ⅎ}\psi F[\phi,\psi]) \tag{3.5}$$

where F is an $\mathcal{L}^+$-formula, with no P and free variables ϕ among those of the left side $P(\phi)$. In the standard graph $\mathcal{G}_M(\mathcal{L}^P)$, we can then replace, for every $S \in \mathbf{S}^P_M$, edge $P(S) \to \neg P(S)$ in the 2-cycle $P(S) \leftrightarrows \neg P(S)$ by edge from $P(S)$ to its defining sentence $\text{Ⅎ}\psi F[S,\psi]$, obtaining graph G^P. Each solution K of G^P determines a solution of the standard language graph for $\mathcal{L}^P$, with value of each atom $P(S)$ determined by K. (The standard graph may also have other solutions, not respecting (3.5).) Lemma A.7 below, giving immediately Theorem 3.5, shows that G^P preserves solutions of the underlying graph G for $\mathcal{L}^+$. Its proof amounts to elimination of symbol P, replacing each $P(S)$ by its definiens $\text{Ⅎ}\psi F[S,\psi]$. Such a replacement, trivial in FOL, must proceed recursively on a cyclic graph (e.g., $P(P(S))$ needs repeated replacements) and involves some technicalities. These end with the paragraph before Lemma A.7.

The proof assumes a solvable language graph G (for any language $\mathcal{L}$ or $\mathcal{L}^+$) over some domain M, in which no two vertices have equal out-neighbourhoods. (If G contains such vertices, as language graphs with auxiliary vertices do, their identification preserves essentially the solutions by Fact A.2, and we apply the construction and fact below to the so quotiented G.) The graph G^P for $\mathcal{L}^P$ contains G as an induced subgraph.

As the first step, we quotient atoms of G^P containing P. Let $\simeq$ be congruence on $\mathcal{L}^P_M$-sentences induced by the basic reflexive relation $P(S) \simeq_0 \text{Ⅎ}\psi F[S,\psi]$, for every $\mathcal{L}^P_M$-sentence S. For every s-predicate Q distinct from P, we identify every two atoms $Q(A_1 \dots A_n) \simeq Q(B_1 \dots B_n)$ when $A_i \simeq B_i$ for $1 \leqslant i \leqslant n$. Each equivalence class contains an atom $Q(S_1 \dots S_n)$ for some $S_i \in \mathbf{S}^+_M$, not containing any P, so in the following we can assume only such atoms present, as vertices of the resulting graph H. It is a simple observation that quotient $q\colon G^P \to H$, where $\mathbf{E}_H(q(x)) = \{q(y) \mid y \in \mathbf{E}_{G^P}(x)\}$ in the resulting graph H, reflects kernels, so the preimage of any kernel of H is a kernel of G^P.

We now map $\gamma\colon H \to G$, performing a sequence of identifications $\gamma_i\colon H_{i-1} \to H_i$, for $0 < i \in \omega$ and $H_0 = H$. Each γ_i is identity on the subgraph G of H_i, identifying some vertices from $\mathbf{V}_i \setminus \mathbf{V}_G$ with some in $\mathbf{V}_G$. First, we identify $\gamma_1(P(S)) = \text{Ⅎ}\psi F[S,\psi]$, removing the double edge and the intermediate vertex $\bullet_{P(S)}$ between $P(S)$ and its definiens $\text{Ⅎ}\psi F[S,\psi]$, for $S \in \mathbf{S}^P_M$. Then $\gamma_{i+1}(v) = w$ when vertices $v \in \mathbf{V}_i \setminus \mathbf{V}_G$ and $w \in \mathbf{V}_G$ have the same out-neighbourhood. More precisely, let $\mathbf{V}_0 = \mathbf{V}_H, \mathbf{E}_0 = \mathbf{E}_H$ and define γ as shown in Figure A.5.

Function γ is well-defined by the assumption that G has no pair of vertices with identical out-neighbourhoods. For $A, B \in \mathbf{V}_H$ and $n \in \omega$, we denote

$$
\begin{array}{rl}
i=1, & \text{let } Re_0 = \bigcup\{\{P(S), \bullet_{P(S)}\} \mid S \in \mathbf{S}_M^P, \{\bullet_{P(S)}\} = \mathbf{E}_0(P(S))\} \text{ and:} \\
& \gamma_1(v) = \begin{cases} \exists\!\!\!/\psi F[S,\psi], & \text{if } v = P(S) \text{ for any } S \in \mathbf{S}_M^P \\ v & \text{if } v \notin Re_0 \end{cases} \\
& \text{The resulting graph } H_1 \text{ is given by:} \\
& \mathbf{V}_1 = \mathbf{V}_0 \setminus Re_0, \text{ and } \mathbf{E}_1(v) = \{\gamma_1(w) \mid w \in \mathbf{E}_0(v)\} \setminus Re_0 \\
i+1, & \text{let } Re_i = \{v \in \mathbf{V}_i \setminus \mathbf{V}_G \mid \exists w \in \mathbf{V}_G \colon \mathbf{E}_i(v) = \mathbf{E}_i(w)\} \text{ and:} \\
& \gamma_{i+1}(v) = \begin{cases} w \in \mathbf{V}_G \text{ so that } \mathbf{E}_i(v) = \mathbf{E}_i(w) & \text{if } v \in Re_i \\ v & \text{if } v \notin Re_i \end{cases} \\
& \text{The resulting graph } H_{i+1} \text{ is given by:} \\
& \mathbf{V}_{i+1} = \mathbf{V}_i \setminus Re_i \text{ and } \mathbf{E}_{i+1}(v) = \mathbf{E}_i(\gamma_{i+1}(v)) \setminus Re_i \\
\gamma(v) & = \ \gamma_n(v), \text{ for } v \in \mathbf{V}_H, \text{ and the least } n \in \omega \text{ such that} \\
& \qquad\qquad\qquad\qquad \forall m > n \colon \gamma_m(v) = \gamma_n(v).
\end{array}
$$

Figure A.5. Function $\gamma\colon \mathbf{V}_H \to \mathbf{V}_G$

by $A \sim_n B$ that $\gamma_n(A) = \gamma_n(B)$, and by $A \sim B$ that $\gamma(A) = \gamma(B)$, i.e., $\exists n \in \omega \colon A \sim_n B$.

Example A.1. Let $P(\phi) \leftrightarrow \exists\psi(\phi \wedge \psi)$ and, for some $S \in \mathbf{S}_M$, consider vertex $P(P(S)) \in \mathbf{V}_H$. The relevant parts of the graph H are sketched in Figure A.6, with $A/_X$, $B/_X$, ... denoting vertices with X substituted for the $\exists$-quantified ψ. The subscripts $_{L}, _{R}$ mark these instantiations in the respective subgraphs, e.g., $A/_L = P(S) \wedge A$ and $A/_R = \exists\phi(S \wedge \phi) \wedge A$. Sentences $A, B, \ldots$ (and $\overline{A}, \overline{B}, \ldots$) are duplicated in both subgraphs to increase readability, but they are actually the same vertices.

1. $P(P(S)) \sim_1 \exists\psi(P(S) \wedge \psi)$ and $P(S) \sim_1 \exists\psi(S \wedge \psi)$, hence $\mathbf{E}_1(\overline{P(S)}) = \{\gamma_1(P(S))\} = \{\exists\psi(S \wedge \psi)\} = \mathbf{E}_1(\overline{\exists\psi(S \wedge \psi)})$ and, consequently,

2. $\overline{P(S)} \sim_2 \overline{\exists\psi(S \wedge \psi)}$. Then, for each $A \in \mathbf{S}_M^P$ we have $\mathbf{E}_2(A/_L) = \{\overline{\exists\psi(S \wedge \psi)}, \overline{A}\} = \mathbf{E}_2(A/_R)$, so

3. $A/_L \sim_3 A/_R$, for every $A \in \mathbf{S}_M^P$.

4. Consequently, $\bullet_L \sim_4 \bullet_R$ and then

5. $\exists\psi(\exists\phi(S \wedge \phi) \wedge \psi)) \sim_5 \exists\psi(P(S) \wedge \psi) \sim_1 P(P(S))$, leaving only G's subgraph to the right. □

The equivalence $\sim$ is a congruence on $\mathbf{V}_H$ in the sense that if all out-neighbours of A and B are $\sim$-equivalent then also $A \sim B$, i.e., for $\mathbf{E}_H(A) = \{A_i \mid i \in I\}$ and $\mathbf{E}_H(B) = \{B_i \mid i \in I\}$:

$$\text{if } (\forall i \in I \colon A_i \sim B_i) \text{ then } A \sim B. \tag{A.1}$$

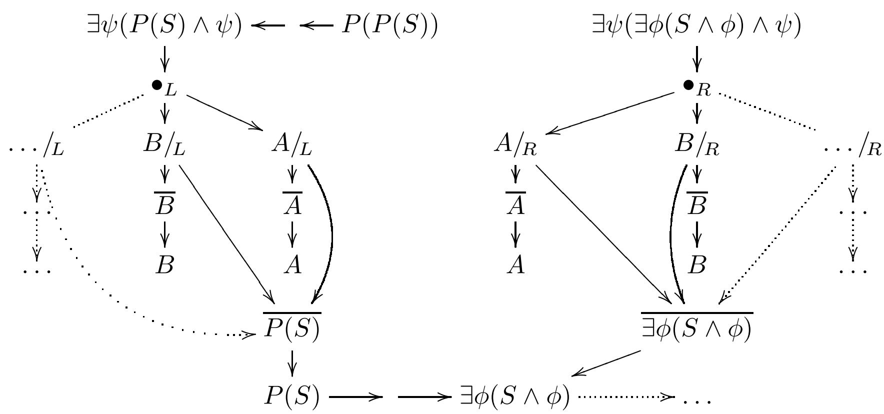

Figure A.6.

This holds since each sentence subgraph $\mathcal{G}_M(A)$ (tree $T_M(A)$) has finite height $h(A)$, in particular distance from the root A to atoms $P(S)$ of $\mathcal{G}_M(A)$ is at most $h(A)$. Hence, if $\forall i \in I\colon A_i \sim B_i$ then $\exists n \leqslant \max\{h(A), h(B)\}\forall i \in I\colon A_i \sim_n B_i$.[7] The equality $\gamma_n(A_i) = \gamma_n(B_i)$ implies, in turn, that $\mathbf{E}_n(A) = \{\gamma_n(A_i) \mid i \in I\} = \{\gamma_n(B_i) \mid i \in I\} = \mathbf{E}_n(B)$, which yields $A \sim_{n+1} B$.

FACT A.6. (a) *$\forall S \in \mathbf{S}^P_M \setminus \mathbf{S}_M\ \exists Q \in \mathbf{S}_M\colon Q \sim S$, hence $\gamma(H) = G$.*
(b) *H and G have essentially the same solutions.*
(c) *Every solution of G extends to a unique solution of G^P.*

PROOF. Point (a) is shown by induction on the number p of Ps in a sentence $S \in \mathbf{S}^P_M \setminus \mathbf{S}_M$.

1. If $p = 1$ and S is atomic, then $S = P(R)$ for some $R \in \mathbf{S}_M$, so $S \sim_1 \exists\!\!\!\forall\psi F[R,\psi] \in \mathbf{S}_M$.

2. If $p = 1$ and S is not atomic, we proceed by structural induction on S, with point 1 providing the basis and induction hypothesis IH_2:

(i) $\bigwedge_{i\in I} S_i$, for finite I. By IH_2, for each S_i there is $Q_i \in \mathbf{S}_M$ with $S_i \sim Q_i$, so $\bigwedge_{i\in I} S_i \sim \bigwedge_{i\in I} Q_i$ by (A.1), and $\bigwedge_{i\in I} Q_i \in \mathbf{S}_M$.

(ii) $\neg A$. By IH_2, $A \sim Q$ for some $Q \in \mathbf{S}_M$, so $\neg A \sim \neg Q$ by (A.1), while $\neg Q \in \mathbf{S}_M$.

(iii) $S = \exists\!\!\!\forall\phi A[\phi]$, where ϕ does not occur under P, so that, for some $R \in \mathbf{S}_M$ and context $A[\phi, \_]$ with no P, we have $S = \exists\!\!\!\forall\phi A[\phi, P(R)]$. Since

[7] This implication fails in general graphs for $\sim$ defined in (A.5) from some basis $\sim_1$, when I is infinite and distance from $A_i, B_i, i \in I$, to relevant pairs $X \sim_1 Y$ is unbounded.

$P(R) \sim_1 \text{Ɐ}\psi F[R] \in \mathbf{S}_M$, taking $Q = \text{Ɐ}\phi A[\phi, \text{Ɐ}\psi F[R]] \in \mathbf{S}_M$, we obtain $A[T, P(R)] \sim A[T, \text{Ɐ}\psi F[R]]$ for every $T \in \mathbf{S}_M^P$ by (A.1), i.e., for all grandchildren of S and Q. By (A.1), this yields $S \sim Q$.

(iv) $S = \text{Ɐ}\phi A[P(C[\phi])]$, i.e., S contains quantification into P, for some contexts $A[\_], C[\_]$ without any P, as $p = 1$. For grandchildren of S, namely, $A[P(C[T])]$ for all $T \in \mathbf{S}_M^P$, the equivalence $P(C[T]) \sim_1 \text{Ɐ}\psi F[C[T], \psi]$ gives $A[P(C[T])] \sim A[\text{Ɐ}\psi F[C[T], \psi]]$ by (A.1). Sentences on the left, for all $T \in \mathbf{S}_M^P$, comprise all grandchildren of S, and those on the right all grandchildren of $Q = \text{Ɐ}\phi A[\text{Ɐ}\psi F[C[\phi], \psi]] \in \mathbf{S}_M$, so $S \sim Q$ by (A.1).

3. For the induction step for $p > 1$, the two cases depend on whether P is nested or not.

(i) If the number of Ps not nested under others is $n > 1$, consider all these highest Ps in $T_M(S)$, i.e., $S = C[P(A_1), \ldots, P(A_n)]$, where $C[\_]$ contains no Ps. For $R = C[\text{Ɐ}\psi F[A_1, \psi], \ldots, \text{Ɐ}\psi F[A_n, \psi]]$, $S \sim R$ by (A.1). R has $p-n < p$ Ps so, by IH, $R \sim Q$ for some $Q \in \mathbf{S}_M$. Hence $S \sim Q$.

(ii) If all Ps are nested under each other, then $S = C[P(A)]$ for some context $C[\_]$ without any Ps, and with $p-1$ occurrences of P in A. $P(A) \sim_1 \text{Ɐ}\psi F[A, \psi]$ and, by IH, $\text{Ɐ}\psi F[A, \psi] \sim R$ for some $R \in \mathbf{S}_M$, so that also $P(A) \sim R$. Then $C[P(A)] \sim C[R]$, by (A.1) and $C[R] \in \mathbf{S}_M$, as required.

The equality $\gamma(H) = G$ follows since each $S \in \mathbf{V}_H \setminus \mathbf{V}_G$ represents a sentence in $\mathbf{S}_M^P \setminus \mathbf{S}_M$.

(b) For $i \geqslant 0$, H_i is the quotient of H by $\sim_1, \ldots, \sim_i$. By Fact A.1, H_1 has essentially the same solutions as H. (No ray is contracted to a finite path, because the case $P(S) \sim_1 \text{Ɐ}\psi F[S, \psi]$ is applied at most finitely many times along each path under each sentence Q, since Q contains at most finitely many nested Ps.) By Fact A.2, the same holds for H_1 and every H_i, $i > 1$, including limits H_λ. Thus, H and $\gamma(H) = G$ have essentially the same solutions.

(c) By the observation before this fact, quotient $G^P \to H$ reflects solutions, so that the preimage of every solution of H is a solution of G^P. Using (b), each solution of G extends to one for G^P. $\square$

Let *definitional extension* refer to any well-ordered chain starting with any theory $\mathit{\Gamma}_0 \subseteq \mathcal{L}_0 \subseteq \mathcal{L}^+$ and adding, at step $i+1$, axiom (3.5) with a fresh predicate $P \notin \mathcal{L}_i$ and $F(\phi, \psi) \in \mathcal{L}_i$, for language $\mathcal{L}_i$ of theory $\mathit{\Gamma}_i$ obtained at step i. In the limits, the language and theory are unions of all steps. The following counterpart of model theoretic conservativity of usual definitional extensions holds.

LEMMA A.7. *Each solution of a language graph $G_0 = \mathcal{G}_M(\mathcal{L}_0)$ extends to a solution of the graph of its definitional extension.*

PROOF. Fact A.6.(c) gives the claim for an extension with a single predicate. By IH, definitional extension G_i of G_0 with $P_1, \ldots, P_i$, preserves all solutions of G. Graph G_{i+1}, obtained now by adding P_{i+1}, whose definiens F_{i+1} can utilize $P_j, j \leqslant i$, preserves by Fact A.6 solutions of G_i, and hence of G. This establishes successor step.

For any limit, the language $\mathcal{L}^\omega_M = \bigcup_{i\in\omega} \mathcal{L}^i_M$ extends the initial language $\mathcal{L}^0_M$ with all ω predicates $P_1, P_2, \ldots$ introduced on the way. Its graph $G_\omega = \bigcup_{i\in\omega} G_i$, with unions taken on vertices and on edges, contains all double edges from the new predicate's instances to their definienses. We repeat the proof with the unions of all equivalences used along the way. As the first step, let $\simeq^\omega$ be a congruence on $\mathcal{L}^\omega_M$-sentences induced from the relation $A \simeq^\omega B \iff \exists n \in \omega$: $A \simeq^n B$, where $\simeq^n$ is the congruence $\simeq$ on $\mathcal{L}^n_M$-sentences from step n. Identification of all atoms $Q(A_1 \ldots A_k) \simeq^\omega Q(B_1 \ldots B_k)$ when $A_i \simeq^\omega B_i$ for $1 \leqslant i \leqslant k$ gives a quotient H reflecting kernels as before. Each equivalence class contains an atom from $\mathcal{L}^0_M$. Let H denote the resulting graph, and H_i its restriction to the subgraph induced by vertices of G_i (with the atoms identified as just described), so that $H = \bigcup_{i\in\omega} H_i$.

In the chain $G_0 = H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots$, for each pair of subsequent $H_{i-1} \subseteq H_i$, the construction from (A.5) yields $\gamma^i\colon H_i \to H_{i-1}$ satisfying Fact A.6. Composing $\gamma^1(\gamma^2(\ldots(\gamma^{i-1}(\gamma^i(H_i)))\ldots))$ gives surjective $\overline{\gamma}^i\colon H_i \to G_0$, where $\overline{\gamma}^j(H_i) = \overline{\gamma}^i(H_i)$ for any $j \geqslant i$. Hence, the union $\gamma^\omega = \bigcup_{i\in\omega} \gamma^i$ gives a surjective quotient $\gamma^\omega\colon H \to G_0$, reflecting solutions. □

A non-paradoxical language $\mathcal{L}^+$ is one having a solvable graph $\mathcal{G}_M(\mathcal{L}^+)$ so, by this lemma, its definitional extension remains non-paradoxical.

Theorem A.8 (3.5). *For every $\Gamma \subseteq \mathcal{L}^+$ and its definitional extension F, every kernel model of Γ can be extended to a kernel model of $\Gamma \cup F$.*

B. Soundness and completeness

Facts B.1 and B.3 below show soundness and completeness of LSO for semi-kernel semantics from (3.6), establishing Theorem 3.6. Fact 3.7 shows then soundness and completeness of LSO with (cut) for kernel semantics (3.7).

Fact B.1. *The rules of* LSO *are sound and invertible for semi-kernel semantics from* (3.6).

Proof. Given an arbitrary language graph $\mathcal{G} \in \mathcal{LG}r(\mathcal{L}^+)$ (over an arbitrary domain M), soundness for each rule follows by showing that every semi-kernel L covering the conclusion satisfies it, assuming validity of the premise(s), while invertibility by showing that every semi-kernel L covering (each) premise satisfies it, assuming validity of the rule's conclusion.

1. $(\wedge_R)$. For soundness, assume $\Gamma \models \Delta, A_1$ and $\Gamma \models \Delta, A_2$, and let semi-kernel L cover the rule's conclusion, under a given $\alpha \in (M \cup \mathbf{S}^+)^{\mathcal{V}(\Gamma,\Delta)}$. Assume that $\alpha(\Gamma) \subseteq L$, $\alpha(\Delta) \subseteq \mathbf{E}^-(L)$ and $\alpha(A_1 \wedge A_2) \in \mathbf{E}^-(L)$ — if not,

then $L \models_\alpha \mathit{\Gamma} \Rightarrow \mathit{\Delta}, A_1 \wedge A_2$, as desired. Since $\alpha(A_1 \wedge A_2) \in \mathbf{E}^-(L)$ and $\mathbf{E}(\alpha(A_1 \wedge A_2)) = \{\neg\alpha(A_1), \neg\alpha(A_2)\}$ so, for some $i \in \{1,2\}$, $\neg\alpha(A_i) \in L$, and then $\alpha(A_i) \in \mathbf{E}^-(L)$, contradicting the assumption $\mathit{\Gamma} \models \mathit{\Delta}, A_i$.

For invertibility, let $\mathit{\Gamma} \models \mathit{\Delta}, A_1 \wedge A_2$ and L cover A_1 (or A_2) under α. If (*) $\alpha(\mathit{\Gamma}) \subseteq L$ and $\alpha(\mathit{\Delta} \cup \{A_1\}) \subseteq \mathbf{E}^-(L)$, then $L' = L \cup \{\neg\alpha(A_1)\}$ is a semi-kernel, since $\mathbf{E}(\neg\alpha(A_1)) = \{\alpha(A_1)\} \subseteq \mathbf{E}^-(L) \subseteq \mathbf{E}^-(L')$. L' covers also $\alpha(A_1 \wedge A_2) \in \mathbf{E}^-(\neg\alpha(A_1))$. Thus L' covers the conclusion, while $\alpha(\mathit{\Gamma}) \cap \mathbf{E}^-(L') = \emptyset$ and $\alpha(\mathit{\Delta} \cup \{A_1 \wedge A_2\}) \cap L' = \emptyset$, so $L' \not\models \mathit{\Gamma} \Rightarrow \mathit{\Delta}, A_1 \wedge A_2$, contrary to $\mathit{\Gamma} \models \mathit{\Delta}, A_1 \wedge A_2$. Hence (*) fails, so $\alpha(\mathit{\Gamma}) \cap \mathbf{E}^-(L) \neq \emptyset$ or $\alpha(\mathit{\Delta} \cup \{A_1\}) \cap L \neq \emptyset$, yielding the claim.

Assignments α to free variables do not affect the argument, so covering by L below is to be taken relatively to a given α, which we do not mention, except for $(\forall_R)$ and $(\forall_R^+)$.

2. $(\wedge_L)$. For soundness, assume $\mathit{\Gamma}, A_1, A_2 \models \mathit{\Delta}$, let semi-kernel L cover the rule's conclusion, $\mathit{\Gamma} \subseteq L$ and $\mathit{\Delta} \subseteq \mathbf{E}^-(L)$. If $A_1 \wedge A_2 \in L$, then $\mathbf{E}(A_1 \wedge A_2) = \{\neg A_1, \neg A_2\} \subseteq \mathbf{E}^-(L)$, so $\mathbf{E}(\{\neg A_1, \neg A_2\}) = \{A_1, A_2\} \subseteq L$, contradicting $\mathit{\Gamma}, A_1, A_2 \models \mathit{\Delta}$. Thus $A_1 \wedge A_2 \in \mathbf{E}^-(L)$ and $L \models \mathit{\Gamma}, A_1 \wedge A_2 \Rightarrow \mathit{\Delta}$.

For invertibility, assume $\mathit{\Gamma}, A_1 \wedge A_2 \models \mathit{\Delta}$, let semi-kernel L cover the rule's premise, and assume $\mathit{\Gamma} \subseteq L$ and $\mathit{\Delta} \subseteq \mathbf{E}^-(L)$. If $A_1, A_2 \in L$, which is the only way L can contradict $\mathit{\Gamma}, A_1, A_2 \models \mathit{\Delta}$, then $\{\neg A_1, \neg A_2\} \subseteq \mathbf{E}^-(L)$, and $L' = L \cup \{A_1 \wedge A_2\}$ is also a semi-kernel:

$$\begin{aligned}\mathbf{E}(L') = \mathbf{E}(L \cup \{A_1 \wedge A_2\}) = \mathbf{E}(L) \cup \mathbf{E}(\{A_1 \wedge A_2\}) \ \subseteq \\ \mathbf{E}^-(L) \cup \{\neg A_1, \neg A_2\} \subseteq \ \mathbf{E}^-(L) \ \subseteq \ \mathbf{V} \setminus (L \cup \{A_1 \wedge A_2\}).\end{aligned}$$

The last inclusion follows because $\mathbf{E}^-(L) \subseteq \mathbf{V} \setminus L$ and $A_1 \wedge A_2 \notin \mathbf{E}^-[L]$, since $A_1 \wedge A_2 \in L$ contradicts $\mathit{\Gamma}, A_1 \wedge A_2 \models \mathit{\Delta}$ (as $\mathit{\Gamma} \subseteq L$ and $\mathit{\Delta} \subseteq \mathbf{E}^-(L)$), while $A_1 \wedge A_2 \in \mathbf{E}^-(L)$ contradicts independence of L, implying $\neg A_i \in L$ (for $i = 1$ or $i = 2$), while $\neg A_i \in \mathbf{E}^-(L)$ since $A_i \in L$.

Since $L' \not\models \mathit{\Gamma}, A_1 \wedge A_2 \Rightarrow \mathit{\Delta}$ contradicts the assumption, either $A_1 \notin L$ or $A_2 \notin L$, and $L \models \mathit{\Gamma}, A_1 \wedge A_2 \Rightarrow \mathit{\Delta}$ as desired.

3. $(\neg_R)$. For soundness, assume $\mathit{\Gamma}, A \models \mathit{\Delta}$, let semi-kernel L cover the rule's conclusion, and assume $\mathit{\Gamma} \subseteq L$ and $\mathit{\Delta} \subseteq \mathbf{E}^-(L)$. If $\neg A \in L$, we are done, while if $\neg A \in \mathbf{E}^-(L)$ then $A \in L$, which contradicts the assumption, since now $\mathit{\Gamma} \cup \{A\} \subseteq L$ and $\mathit{\Delta} \subseteq \mathbf{E}^-(L)$.

For invertibility, assuming $\mathit{\Gamma} \models \mathit{\Delta}, \neg A$, let L cover the rule's premise, $\mathit{\Gamma} \subseteq L$ and $\mathit{\Delta} \subseteq \mathbf{E}^-(L)$. If $A \in L$ then $\neg A \in \mathbf{E}^-(L)$ and $L \not\models \mathit{\Gamma} \Rightarrow \mathit{\Delta}, \neg A$, contradicting the assumption. So $A \in \mathbf{E}^-(L)$, as required for $L \models \mathit{\Gamma}, A \Rightarrow \mathit{\Delta}$.

4. $(\neg_L)$. For soundness, assume $\mathit{\Gamma} \models \mathit{\Delta}, A$, let L cover the rule's conclusion, $\mathit{\Gamma} \subseteq L$ and $\mathit{\Delta} \subseteq \mathbf{E}^-(L)$. If $\neg A \in \mathbf{E}^-(L)$, we are done, while if $\neg A \in L$ then $A \in \mathbf{E}(\neg A) \subseteq \mathbf{E}^-(L)$, contradicting the assumption, since now $\mathit{\Gamma} \cup \{A\} \subseteq L$ and $(\mathit{\Delta} \cup \{A\}) \subseteq \mathbf{E}^-(L)$.

For invertibility, assume $\mathit{\Gamma}, \neg A \models \mathit{\Delta}$, let L cover the rule's premise, $\mathit{\Gamma} \subseteq L$ and $\mathit{\Delta} \subseteq \mathbf{E}^-(L)$. If $A \in \mathbf{E}^-(L)$ then $L' = L \cup \{\neg A\}$ is a semi-kernel, because L is and $\mathbf{E}(\neg A) = \{A\} \subseteq \mathbf{E}^-(L)$. But L' contradicts the assumption, so $A \in L$, as required for $L \models \mathit{\Gamma} \Rightarrow \mathit{\Delta}, A$.

5. $(\forall_L)$. For soundness, assume $F(t), \mathit{\Gamma}, \forall x F(x) \models \mathit{\Delta}$ and let L cover the rule's conclusion. If $\forall x F(x) \notin L$, i.e., $\forall x F(x) \in \mathbf{E}^-(L)$, then $L \models \mathit{\Gamma}, \forall x F(x) \Rightarrow \mathit{\Delta}$. If $\forall x F(x) \in L$ then also $F(t) \in L$, since $\neg F(t) \in \mathbf{E}(\forall x F(x)) \subseteq \mathbf{E}^-(L)$ and $\mathbf{E}(\neg F(t)) = \{F(t)\}$. As L covers the premise, either $\mathit{\Gamma} \cap \mathbf{E}^-(L) \neq \emptyset$, since $F(t) \notin \mathbf{E}^-(L)$, or $\mathit{\Delta} \cap L \neq \emptyset$. Either case yields the claim for L, which was arbitrary, so $\mathit{\Gamma}, \forall x F(x) \models \mathit{\Delta}$.

For invertibility, if $\mathit{\Gamma}, \forall x F(x) \models \mathit{\Delta}$ and L covers the rule's premise, it covers also this conclusion. Satisfying it, L trivially satisfies the premise.

6. $(\forall_R)$. For soundness, let (*) $\mathit{\Gamma} \models \mathit{\Delta}, F(y)$ and L cover the rule's conclusion under a given assignment α to $\mathcal{V}(\mathit{\Gamma}, \mathit{\Delta}, \forall x F(x)) \not\ni y$. Assume also $\alpha(\mathit{\Gamma}) \subseteq L$ and $\alpha(\mathit{\Delta}) \subseteq \mathbf{E}^-(L)$. If $\alpha(\forall x F(x)) \notin L$ then $\alpha(\forall x F(x)) \in \mathbf{E}^-(L)$ and some $\alpha(\neg F(m)) \in L$, since $\mathbf{E}(\alpha(\forall x F(x))) = \{\alpha(\neg F(m)) \mid m \in M\}$. Extending α with $\alpha(y) = m$, we obtain $L \not\models_\alpha \mathit{\Gamma} \Rightarrow \mathit{\Delta}, F(y)$, contrary to (*). Thus, $\alpha(\forall x F(x)) \in L$ and $L \models_\alpha \mathit{\Gamma} \Rightarrow \mathit{\Delta}, \forall x F(x)$.

For invertibility, if $L \not\models_\alpha \mathit{\Gamma} \Rightarrow \mathit{\Delta}, F(y)$, for $\alpha(y) = m$, i.e., $\alpha(\mathit{\Gamma}) \subseteq L$, $\alpha(\mathit{\Delta}) \subseteq \mathbf{E}^-(L)$ and $\alpha(F(m)) \in \mathbf{E}^-(L)$, then $L' = L \cup \{\alpha(\neg F(m))\}$ is a semi-kernel, because L is and $\mathbf{E}(\alpha(\neg F(m))) = \{\alpha(F(m))\} \subseteq \mathbf{E}^-(L) \subseteq \mathbf{E}^-(L')$. L' covers the conclusion since $\alpha(\forall x F(x)) \in \mathbf{E}^-(\alpha(\neg F(m)))$, but $L' \not\models_\alpha \mathit{\Gamma} \Rightarrow \mathit{\Delta}, \forall x F(x)$.

7. $(\forall_L^+)$. The argument repeats that for $(\forall_L)$. For soundness, let $\mathit{\Gamma}$, $F(S)$, $\forall \phi F(\phi) \models \mathit{\Delta}$ and L cover the rule's conclusion (under a fixed α). If $\forall \phi F(\phi) \notin L$ then $\forall \phi F(\phi) \in \mathbf{E}^-(L)$, yielding $L \models \mathit{\Gamma}, \forall \phi F(\phi) \Rightarrow \mathit{\Delta}$. If $\forall \phi F(\phi) \in L$ then also $F(S') \in L$, for each sentence instantiating S, since $\neg F(S') \in \mathbf{E}(\forall \phi F(\phi)) \subseteq \mathbf{E}^-(L)$ and $\mathbf{E}(\neg F(S')) = \{F(S')\}$. Thus L covers also the premise, hence, either $\mathit{\Gamma} \cap \mathbf{E}^-(L) \neq \emptyset$, since $F(S) \notin \mathbf{E}^-(L)$, or $\mathit{\Delta} \cap L \neq \emptyset$. Either case yields the claim for L, which was arbitrary (as was α), so $\mathit{\Gamma}, \forall \phi F(\phi) \models \mathit{\Delta}$.

For invertibility, if $\mathit{\Gamma}, \forall \phi F(\phi) \models \mathit{\Delta}$ and L covers the rule's premise, it covers also this conclusion. Satisfying it, L trivially satisfies the premise.

8. $(\forall_R^+)$. For soundness, $\mathit{\Gamma} \models \mathit{\Delta}, F(\psi)$, with a fresh $\psi \in \mathit{\Phi}$, iff $\mathit{\Gamma} \models \mathit{\Delta}, F(S)$ for every $S \in \mathbf{S}^+$. Let L cover the rule's conclusion. If $\forall \phi F(\phi) \in L$ then L satisfies the rule's conclusion. If $\forall \phi F(\phi) \notin L$ then $\forall \phi F(\phi) \in \mathbf{E}^-(L)$, so some $\neg F(S) \in L$, since $\mathbf{E}(\forall \phi F(\phi)) = \{\neg F(S) \mid S \in \mathbf{S}^+\}$. Now L covers also $\mathit{\Gamma} \Rightarrow \mathit{\Delta}, F(S)$ and $F(S) \notin L$. Since $\mathit{\Gamma} \models \mathit{\Delta}, F(S)$, either $\mathit{\Gamma} \cap \mathbf{E}^-(L) \neq \emptyset$ or $\mathit{\Delta} \cap L \neq \emptyset$. In each case L satisfies the conclusion.

For invertibility, assume $\mathit{\Gamma} \models \mathit{\Delta}, \forall \phi F(\phi)$, and let L cover $\mathit{\Gamma} \Rightarrow \mathit{\Delta}, F(\psi)$ under $\alpha(\psi) = S$. Assume $\alpha(\mathit{\Gamma}) \subseteq L$ and $\alpha(\mathit{\Delta}) \subseteq \mathbf{E}^-(L)$, since otherwise $L \models_\alpha \mathit{\Gamma} \Rightarrow \mathit{\Delta}, F(\psi)$. Then $\forall \phi F(\phi) \in L$ and, as L is a semi-kernel, $\neg F(S) \in \mathbf{E}^-(L)$, so $F(S) \in L$ and $L \models_\alpha \mathit{\Gamma} \Rightarrow \mathit{\Delta}, F(\psi)$. Since α was arbitrary, $L \models \mathit{\Gamma} \Rightarrow \mathit{\Delta}, F(\psi)$.

9. S-equality rules are sound and invertible, because atoms introduced in premises are redundant (due to point 3 of Definition 3.1 of language graph). E.g., $S \doteq S$ in the premise of (ref) is satisfied (being a sink) for each sentence S, hence satisfaction of the premise implies satisfaction of the conclusion. Conversely, satisfaction of the conclusion by any semi-kernel allows its extension with any sink, in particular, with $S \doteq S$. Analogous argument works for (rep) and (neq). ☐

The following simple consequence of Definition 3.1 is used in the completeness proof below.

FACT B.2. *In any graph $\mathcal{G}_M \in \mathcal{LG}r(\mathcal{L}^+)$, the following relations hold between the form of a nonatomic sentence $X \in \mathbf{S}_M^+$ and forms of its out- and in-neighbours*:

(a) $\mathbf{E}^-(X) = \{\neg X\}$ — *when X does not start with $\neg$,*
(b) $\mathbf{E}^-(\neg X) = \{\neg\neg X\} \cup \{X \wedge S \mid S \in \mathbf{S}_M^+\} \cup \cdots \cup \{\forall\phi.D(\phi) \mid \exists S \in \mathbf{S}_M^+ : D(S) = X\} \cup \{\forall x.D(x) \mid \exists t \in \mathbf{T}_M : D(t) = X\}$,
(c) *when X does not start with $\neg$, then each out-neighbour of X does,*
(d) $\mathbf{E}(\neg X) = \{X\}$.

For atomic X, $\mathbf{E}^-(X) = \{\neg X\} = \mathbf{E}(X)$ and $\mathbf{E}^-(\neg X) = \{X\} = \mathbf{E}(\neg X)$.

The proof of completeness can apply the standard techniques because proofs in LSO, even if infinite, are well-founded trees with axioms as leaves. A few adjustments are needed for handling deviations from LK. We must ensure not only that all formulas are processed and all terms are substituted by $(\forall_L)$, but also that all sentences are substituted by $(\forall_L^+)$. Missing subformula property, due to substitution of all sentences for s-variables, is handled by retaining the principal formula from the conclusion in all its premises, in a bottom-up construction of a derivation tree. Any nonaxiomatic branch (either infinite or terminating with a non-axiom) provides a countermodel.

FACT B.3. *For a countable $\Gamma \cup \Delta \subseteq FOL^+$:*

$$\Gamma \nvdash \Delta \implies \exists\mathcal{G}\exists L \in SK(\mathcal{G}) : L \not\models \Gamma \Rightarrow \Delta.$$

PROOF. We assume an enumeration E_F of all formulas $\mathbf{F}_{X,\Phi}^+$, where each formula occurs infinitely often, and an enumeration $E_T = t_1, t_2 \ldots$ of terms $\mathbf{T}_X$ so that each occurs infinitely often. (FOL variables, requiring special care, are treated in the standard way and ignored below. Recall that we assume disjoint sets of free and bound variables.) We enumerate all triples $\langle S_i, t_j, S_k\rangle \in E_F \times E_T \times E_F$, with each $\langle S_i, t_j, \_\rangle$ and $\langle S_i, \_, S_k\rangle$ occurring infinitely often.

This is interleaved with an enumeration of all pairs $E_F \times E_F$, where each pair contains either identical or non-unifiable formulas and occurs infinitely often.

1. We construct a derivation tree, starting with the root $\Gamma \vdash \Delta$, which is to be proven. An *active* sequent — initially, only the root — is a nonaxiomatic leaf of the tree constructed bottom-up so far. We proceed along the enumeration of the triples and pairs. For each $\langle S_i, t_j, S_k \rangle$, we find the active occurrences (in the active sequents) of S_i. Pairs $\langle S_i, S_j \rangle$ serve treatment of $\doteq$ atoms.

(ii) Encountering a pair $\langle S_i, S_j \rangle$, we apply rules for $\doteq$. If $S_i \doteq S_j$, we add it to the antecedent of each active sequent. For each active sequent containing $S_i \doteq Q$ in its antecedent, along with any formula $A(S_i)$, we add to it $A(Q)$. If S_i, S_j are not unifiable, we add atom $S_i \doteq S_j$ to the consequent of every active sequent.

The remaining cases address triples $\langle S_i, t_j, S_k \rangle$ encountered in the enumeration.

(ii) If $S_i \in \mathbf{A}^+ \cup \Phi$, or S_i has no active occurrences, proceed to the next item in the enumeration.

(iii) Otherwise, proceed retaining S_i from the active sequent, which instantiates the conclusion of the relevant rule, in the new leaves obtained from the rule's premises. E.g., if $S_i = A \wedge B$ then every active sequent of the form $\Gamma', A \wedge B, \Gamma'' \vdash \Delta$ is replaced by

$$\frac{A, B, \Gamma', A \wedge B, \Gamma'' \vdash \Delta}{\Gamma', A \wedge B, \Gamma'' \vdash \Delta}$$

while every active sequent of the form $\Gamma \vdash \Delta', A \wedge B, \Delta''$ by

$$\frac{\Gamma \vdash A, \Delta', A \wedge B, \Delta'' \qquad \Gamma \vdash B, \Delta', A \wedge B, \Delta''}{\Gamma \vdash \Delta', A \wedge B, \Delta''}$$

Analogously, for negation — other elements of the triple do not matter here.

(iv) If $S_i = \forall x D(x)$, each active sequent of the form $\Gamma', \forall x D(x), \Gamma'' \vdash \Delta$, is replaced by the derivation with a new leaf adding $D(t_j)$ to its antecedent

$$\frac{D(t_j), \Gamma', \forall x D(x), \Gamma'' \vdash \Delta}{\Gamma', \forall x D(x), \Gamma'' \vdash \Delta}\,.$$

Every active sequent of the form $\Gamma \vdash \Delta', \forall x D(x), \Delta''$ is replaced by

$$\frac{\Gamma \vdash D(x), \Delta', \forall y D(y), \Delta''}{\Gamma \vdash \Delta', \forall y D(y), \Delta''}$$

for a fresh eigenvariable $x \in X$.

(v) If $S_i = \forall \phi D(\phi)$, then replace every active sequent of the form Γ', $\forall \phi D(\phi)$, $\Gamma'' \vdash \Delta$ by

$$\frac{D(S_k), \Gamma', \forall \phi D(\phi), \Gamma'' \vdash \Delta}{\Gamma', \forall \phi D(\phi), \Gamma'' \vdash \Delta}$$

while every active sequent of the form $\Gamma \vdash \Delta', \forall \phi D(\phi), \Delta''$ by

$$\frac{\Gamma \vdash \Delta', D(\alpha), \Delta''}{\Gamma \vdash \Delta', \forall \phi D(\phi), \Delta''}$$

for a fresh s-eigenvariable $\alpha \in \Phi$.

2. A branch gets closed when its leaf is an axiom, and the tree is obtained as the ω-limit of this process. If all branches are closed (finite), the derivation yields a proof of the root.

Otherwise, a finite nonaxiomatic branch gives easily a countermodel. We show that also infinite branch gives a countermodel of all sequents on this branch, including the root sequent.

3. The claim is that if β is an infinite branch, with β'_L/β'_R all formulas occurring in β on the left/right of $\vdash$, then there is a language graph $\mathcal{G}$ with a semi-kernel L' such that $\beta'_L \subseteq L'$ and $\beta'_R \subseteq \mathbf{E}^-(L')$. The rest of the proof establishes this claim.

Absence of any axiom in β implies that $\beta'_L \cap \beta'_R = \emptyset$, which is often applied implicitly. $\beta'_L = \beta_L \cup Eq_L$, where Eq_L are $\doteq$-atoms $S \doteq S$ occurring on the left. $\beta'_R = \beta_R \cup Eq_R$, where Eq_R are $\doteq$-atoms occurring on the right, with $\overline{Eq}_R$ denoting the set of their negations.

If β contains any FOL-atoms, construct first a FOL-structure M, giving a countermodel to $(\beta_L \cap \mathbf{S}_M) \Rightarrow (\beta_R \cap \mathbf{S}_M)$, in the standard way. Otherwise, set $M = \emptyset$. Let $\mathcal{G} = \mathcal{G}_M(\mathcal{L}^\oplus)$, for the language $\mathcal{L}^\oplus$ obtained from the original $\mathcal{L}^+$ by adding each free s-variable occurring on the branch as a fresh s-constant $\mathbf{C}$. (This is the reason for the extension $\forall\mathcal{L}^\oplus \supseteq \mathcal{L}^+$ in definitions of the semantics (3.6) and (3.7).) By $\mathbf{S}^+_M$ we denote sentences of $\mathcal{L}^\oplus$ over terms $\mathbf{T}_M$. We show that (def) $L = \beta_L \cup (\mathbf{E}(\beta_R) \cap \mathbf{E}^-(\beta_R))$ is a semi-kernel of $\mathcal{G}$, with $\beta_R \subseteq \mathbf{E}^-(L)$ and $\beta_L \subseteq L$. Then $L' = L \cup Eq_L \cup \overline{Eq}_R$ is a required semi-kernel of $\mathcal{G}$.

4. First, $\doteq$-atoms can be treated separately. Since $\beta'_L \cap \beta'_R = \emptyset$, each $S \doteq T \in Eq_R$ has syntactically distinct sentences, while each such atom in Eq_L has the form $S \doteq S$. Any semi-kernel of $\mathcal{G}$, in particular L, can be extended to semi-kernel $L' = L \cup Eq_L \cup \overline{Eq}_R$, as the added vertices are sinks of $\mathcal{G}$, by Definition 3.1. Thus $\mathbf{E}(Eq_L \cup \overline{Eq}_R) = \emptyset$, while $Eq_R \subseteq \mathbf{E}^-(L') \cap (\mathbf{V} \setminus L')$.

5 To show $L \in SK(\mathcal{G})$, we show first $\beta_R \subseteq \mathbf{E}^-(L)$, which follows from definitions of L and $\mathcal{G}$ by considering the cases for $A \in \beta_R$. Use of Fact B.2 (Definition 3.1) is marked by superscript $\_^{\text{B.2}}$.

(i) If $A \in \mathbf{A}^+ \cup \mathbf{C}$ then $\mathbf{E}(A) \overset{\text{B.2}}{=} \{\neg A\} \overset{\text{B.2}}{=} \mathbf{E}^-(A)$, so $\neg A \in L$ by (def) and $A \overset{\text{B.2}}{\in} \mathbf{E}^-(L)$.

(ii) If $A = \neg C$ then $C \in \beta_L \subseteq L$, so $A \overset{\text{B.2}}{\in} \mathbf{E}^-(L)$.

(iii) If $A = C \wedge D$ then $C \in \beta_R$ (or $D \in \beta_R$), so $\neg C \overset{\text{B.2}}{\in} \mathbf{E}^-(C) \cap \mathbf{E}(C \wedge D) \subseteq \mathbf{E}^-(\beta_R) \cap \mathbf{E}(\beta_R) \subseteq L$, and thus $A = C \wedge D \overset{\text{B.2}}{\in} \mathbf{E}^-(\neg C) \subseteq \mathbf{E}^-(L)$. (The case of $D \in \beta_R$ is analogous.)

(iv) If $A = \forall x.D(x)$ then $D(c) \in \beta_R$, for some $c \in M$, so $\neg D(c) \overset{\text{B.2}}{\in} \mathbf{E}^-(D(c)) \cap \mathbf{E}(\forall x.D(x)) \subseteq L$, and $A \overset{\text{B.2}}{\in} \mathbf{E}^-(\neg D(c)) \subseteq \mathbf{E}^-(L)$.

(v) If $A = \forall \phi.D(\phi)$ then $D(\psi) \in \beta_R$ for some $\psi \in \mathbf{C}$, so $\neg D(\psi) \overset{\text{B.2}}{\in} \mathbf{E}^-(D(\psi)) \cap E(\forall \phi.D(\phi)) \subseteq L$, and $A \in \mathbf{E}^-(\neg D(\psi)) \subseteq \mathbf{E}^-(L)$.

6. We show $\mathbf{E}(L) \subseteq \mathbf{E}^-(L) \cap (\mathbf{V} \setminus L)$, partitioning $L = \beta_L \cup Z$, where $Z = (\mathbf{E}(\beta_R) \cap \mathbf{E}^-(\beta_R)) \setminus \beta_L$, and establish first $\mathbf{E}(\beta_L) \subseteq \mathbf{E}^-(L) \cap (\mathbf{V} \setminus L)$, considering cases of $A \in \beta_L$.

(i) For atoms $A \in \mathbf{A}^+ \cup \mathbf{C}$, $A \in \beta_L \subseteq L$ and $A \notin \beta_R$ imply $\neg A \notin \beta_L$ and, since $\mathbf{E}(\neg A) \stackrel{\text{B.2}}{=} \{A\}$, $\neg A \notin \mathbf{E}^-(\beta_R)$. Thus $\mathbf{E}(A) \stackrel{\text{B.2}}{=} \{\neg A\} \subseteq \mathbf{E}^-(A) \cap \mathbf{V} \setminus L \subseteq \mathbf{E}^-(L) \cap \mathbf{V} \setminus L$.

(ii) $A = \neg C \in \beta_L$ implies $C \in \beta_R$, so $\mathbf{E}(A) \stackrel{\text{B.2}}{=} \{C\} \subseteq \beta_R \subseteq \mathbf{E}^-(L)$ by point 5.

We show $\mathbf{E}(A) \subset \mathbf{V} \setminus L$. $C \notin \beta_L$ since $\beta_L \cap \beta_R = \emptyset$. Suppose $C \in \mathbf{E}(\beta_R) \cap \mathbf{E}^-(\beta_R)$. If $C = \neg D$ then $\neg D \in \mathbf{E}^-(\beta_R)$, i.e., $\mathbf{E}(\neg D) \stackrel{\text{B.2}}{=} \{D\} \subset \beta_R$, while $A = \neg C = \neg\neg D \in \beta_L$ implies also $\neg D \in \beta_R$ and $D \in \beta_L$, contradicting $\beta_L \cap \beta_R = \emptyset$.

Otherwise, i.e., if C does not start with $\neg$, then for any $F \in \beta_R$ for which $C \in \mathbf{E}(F)$, Fact B.2.(c–d) forces $F = \neg C = A$, contradicting $\beta_R \cap \beta_L = \emptyset$.

(iii) $A = B \wedge C \in \beta_L$ implies $\{B, C\} \subset \beta_L$ and $\{\neg B, \neg C\} \cap \beta_L = \emptyset$, so $\mathbf{E}(B \wedge C) \stackrel{\text{B.2}}{=} \{\neg B, \neg C\} \subseteq \mathbf{V} \setminus \beta_L$ and $\mathbf{E}(B \wedge C) = \{\neg B, \neg C\} \stackrel{\text{B.2}}{\subseteq} \mathbf{E}^-(\{B, C\}) \subseteq \mathbf{E}^-(\beta_L)$. If, say, $\neg B \in \mathbf{E}^-(\beta_R)$, then $B \in \beta_R$ would contradict $\beta_L \cap \beta_R = \emptyset$. The same if $\neg C \in \mathbf{E}^-(\beta_R)$. Thus, $\mathbf{E}(B \wedge C) \subseteq \mathbf{E}^-(L) \cap \mathbf{V} \setminus L$.

(iv) $A = \forall \phi D(\phi) \in \beta_L \Rightarrow \{D(S) \mid S \in \mathbf{S}^+_M\} \subseteq \beta_L$, so $\mathbf{E}(\forall \phi D(\phi)) \stackrel{\text{B.2}}{=} \{\neg D(S) \mid S \in \mathbf{S}^+_M\} \stackrel{\text{B.2}}{\subseteq} \mathbf{E}^-(\{D(S) \mid S \in \mathbf{S}^+_M\}) \subseteq \mathbf{E}^-(\beta_L) \subseteq \mathbf{E}^-(L)$.

If any $\neg D(S) \in L$ then either $\neg D(S) \in \beta_L$, so $D(S) \in \beta_R$, or $\neg D(S) \in \mathbf{E}(\beta_R) \cap \mathbf{E}^-(\beta_R)$, which implies $D(S) \in \beta_R$, since $\mathbf{E}(\neg D(S)) \stackrel{\text{B.2}}{=} \{D(S)\}$. In either case, $D(S) \in \beta_R$ contradicts $\beta_L \cap \beta_R = \emptyset$. Thus $\mathbf{E}(\forall \phi . D(\phi)) \subseteq \mathbf{V} \setminus L$.

(v) For $A = \forall x . D(x)$, the argument is as in (iv). $\forall x . D(x) \in \beta_L$ implies $\{D(t) \mid t \in \mathbf{T}_M\} \subseteq \beta_L$, so $\mathbf{E}(\forall x D(x)) \stackrel{\text{B.2}}{=} \{\neg D(t) \mid t \in \mathbf{T}_M\} \stackrel{\text{B.2}}{\subseteq} \mathbf{E}^-(\{D(t) \mid t \in \mathbf{T}_M\}) \subseteq \mathbf{E}^-(\beta_L) \subseteq \mathbf{E}^-(L)$.

If any $\neg D(t) \in L$, then either $\neg D(t) \in \beta_L$, so $D(t) \in \beta_R$, or $\neg D(t) \in \mathbf{E}(\beta_R) \cap \mathbf{E}^-(\beta_R)$, which implies $D(t) \in \beta_R$, since $\mathbf{E}(\neg D(t)) \stackrel{\text{B.2}}{=} \{D(t)\}$. In either case, $D(t) \in \beta_R$ contradicts $\beta_L \cap \beta_R = \emptyset$. Thus $\mathbf{E}(\forall x . D(x)) \subseteq \mathbf{V} \setminus L$.

7. Also each sentence $S \in Z = (\mathbf{E}(\beta_R) \cap \mathbf{E}^-(\beta_R)) \setminus \beta_L$ satisfies $\mathbf{E}(S) \subseteq \mathbf{E}^-(L) \cap (\mathbf{V} \setminus L)$:

(i) If $S \in Z$ does not start with $\neg$, then $\mathbf{E}^-(S) \stackrel{\text{B.2}}{=} \{\neg S\}$, so $\neg S \in \beta_R$, implying $S \in \beta_L$, so $S \notin Z$.

(ii) If $S = \neg A \in Z \subseteq \mathbf{E}^-(\beta_R)$ then $\mathbf{E}(\neg A) \stackrel{\text{B.2}}{=} \{A\} \subset \beta_R \stackrel{5}{\subseteq} \mathbf{E}^-(L)$. If $A \in Z$, then it starts with $\neg$ by point 7(i), i.e., $A = \neg B$ and $\mathbf{E}(\neg B) \stackrel{\text{B.2}}{=} \{B\} \subset \beta_R$. Since also $A \in \beta_R$ so $B \in \beta_L$, contradicting $\beta_L \cap \beta_R = \emptyset$. Hence $A \notin Z$ and $A \notin \beta_L$ (since $A \in \beta_R$), i.e., $A \notin L = Z \cup \beta_L$, so that $\mathbf{E}(\neg A) = \{A\} \subseteq \mathbf{V} \setminus L$.

By points 6 and 7, we haver $\mathbf{E}(L) = \mathbf{E}(\beta_L) \cup \mathbf{E}(Z) \subseteq \mathbf{E}^-(L) \cap (\mathbf{V} \setminus L)$, so $L \in SK(\mathcal{G})$. $\square$

Soundness and completeness with (cut) for the explosive kernel semantics (3.7) follow by adjusting the above proofs.

Fact B.4 (3.7). *For a countable $\Gamma \cup \Delta \subseteq FOL^{+}$: $\Gamma \models_c \Delta$ iff $\Gamma \vdash_c \Delta$.*

Proof. Soundness and invertibility follow by the same argument as in Fact B.1, with some simplifications due to each kernel $K \in sol(G)$ covering the whole graph, $\mathbf{E}^{-}_{G}(K) = \mathbf{V}_G \setminus K$.

For completeness, we modify the construction from the proof of Fact B.3, by interleaving the enumeration of all triples $E_F \times E_T \times E_F$ and pairs $E_F \times E_F$ with enumeration E'_F of single formulas $\mathbf{F}^{+}_{X,\Phi}$, where each formula occurs only once. Following this interleaved enumeration yields now a new case **1.vi** of an $S_i \in E'_F$ in constructing a derivation, in which we expand each active sequent $\Gamma \Rightarrow \Delta$ with the premises of (cut) over S_i, i.e., with $\Gamma \Rightarrow \Delta, S_i$ and $\Gamma, S_i \Rightarrow \Delta$. A semi-kernel falsifying any one of them, falsifies the conclusion. Given an infinite nonaxiomatic branch β, a language graph $\mathcal{G}_M$ is obtained as in the proof of Fact B.3, over domain M consisting of free variables and ground terms used in the standard construction of a FOL countermodel for $\beta \cap \mathbf{S}_M$. Point 3 of the proof of Fact B.3 shows also now β to determine a semi-kernel K of $\mathcal{G}_M$, falsifying each sequent on β. Now, β contains one of the premises of an application of (cut) for each $S_i \in E'_F = \mathbf{S}^{+}_{M}$. As every sentence $\mathbf{S}^{+}_{M}$ occurs thus in β_L or β_R, semi-kernel K covers all $\mathbf{S}^{+}_{M}$, so it is a kernel of $\mathcal{G}_M$. □

Michał Walicki
Department of Informatics
University of Bergen, Norway
Michal.Walicki@uib.no